Lattice sub-tilings and frames in LCA groups

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Abstract

Given a lattice Λ in a locally compact abelian group G and a measurable subset Ω with finite and positive measure, then the set of characters associated to the dual lattice form a frame for $L^2(\Omega)$ if and only if the distinct translates by Λ of Ω have almost empty intersections. Some consequences of this results are the well-known Fuglede theorem for lattices, as well as a simple characterization for frames of modulates.

1 Introduction

Let G denote a locally compact and second countable abelian group. A closed subgroup Λ of G is called a *lattice* if it is discrete and co-compact, i.e, the quotient group G/Λ is compact. Recall that, since G is second countable, then any discrete subgroup of G is also countable (see e.g. [18, Section 12, Example 17]). Assume that G is abelian, and denote the dual group by $\widehat{G}$. The dual lattice of Λ is defined as follows:

$$\Lambda^\perp = \{\xi \in \widehat{G} : \ \langle \xi, \lambda\rangle = 1 \ \forall \lambda \in \Lambda\}, \tag{1}$$

where $\langle \xi, \lambda\rangle$ indicates the action of character ξ on the group element λ.

We recall that, by the duality theorem between subgroups and quotient groups (see e.g. [20, Lemma 2.1.2]), the dual lattice $\Lambda^\perp$ is a subgroup of $\widehat{G}$ that is topologically isomorphic to the dual group of G/Λ, i.e., $\Lambda^\perp \cong \widehat{G/\Lambda}$. Moreover, since G/Λ is compact, the dual lattice $\Lambda^\perp$ is discrete. Notice that also that $\widehat{G}/\Lambda^\perp \cong \widehat{\Lambda}$, which implies that $\Lambda^\perp$ is co-compact, hence it is a lattice.

Let dg denote a Haar measure on G. For a function f in $L^1(G)$, the Fourier transform of f is defined by

$$\mathcal{F}_G(f)(\chi) = \int_G f(g)\overline{\langle \chi, g\rangle}\, dg\,, \qquad \chi \in \widehat{G},$$

where $\langle \chi, g\rangle$ denotes the action of the character χ on g. By the inversion theorem [20, Section 1.5.1], a Haar measure $d\chi$ can be chosen on $\widehat{G}$ so that the Fourier transform $\mathcal{F}_G$ is an isometry from $L^2(G)$ onto $L^2(\widehat{G})$.

For any $\chi \in \widehat{G}$, we define the *exponential function* e_χ by

$$e_\chi : G \to \mathbb{C}, \qquad e_\chi(g) := \langle \chi, g \rangle.$$

For any measurable subset Ω of G, we let $|\Omega|$ denote the Haar measure of Ω. Throughout this paper, we let $\mathbf{1}_\Omega$ denote the characteristic function of set Ω. We shall also use the addition symbol '+' for the group action, and 0 for the neutral element, since G is abelian.

Definition 1.1 (Sub-Tiling)**.** *Let $\Omega \subset G$ be a measurable set with finite and positive Haar measure, and let Λ be a lattice subgroup of G. We say that (Ω, Λ) is a sub-tiling pair for G if*

$$\sum_{\lambda \in \Lambda} \mathbf{1}_\Omega(g - \lambda) \leq 1 \quad a.e.\ g \in G\,. \tag{2}$$

By replacing the inequality with an equality, the definition is that of a *tiling* pair. In this weaker form, it is equivalent to say that the translates of Ω by elements of Λ are disjoint, i.e. (Ω, Λ) is a sub-tiling pair for G if and only if

$$|\Omega \cap (\Omega + \lambda)| = 0 \quad \forall\, \lambda \in \Lambda\,,\ \lambda \neq 0.$$

Observe also that any sub-tiling set is a subset of a tiling set.

Definition 1.2 (Frame spectrum)**.** *Let $\widetilde{\Lambda}$ be a countable subset of $\widehat{G}$. We say $\widetilde{\Lambda}$ is a frame spectrum for Ω, if the exponentials $\{e_{\tilde{\lambda}} : \tilde{\lambda} \in \widetilde{\Lambda}\}$ form a frame for $L^2(\Omega)$.*

Our main result is the following.

Theorem 1.3 (Main Result)**.** *Let Λ be a lattice in G, let Ω be a set with finite and positive measure, and let Q_Λ be a cross section for G/Λ. Then the following are equivalent.*

1) *The pair (Ω, Λ) is sub-tiling for G.*

2) *For a.e. $\chi \in \widehat{G}$ it holds*

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\mathcal{F}_G(\mathbf{1}_\Omega)(\chi + \tilde{\lambda})|^2 = |Q_\Lambda|\, |\Omega|.$$

3) *The system of translates $\{\sqrt{|\Omega|}^{\,-1} \mathbf{1}_\Omega(\cdot - \lambda) :\ \lambda \in \Lambda\}$ is orthonormal in $L^2(G)$.*

4) *The exponential set $E_{\Lambda^\perp} = \{e_{\tilde{\lambda}} : \tilde{\lambda} \in \Lambda^\perp\}$ is a frame for $L^2(\Omega)$.*

Moreover, if any of the above conditions holds, then the frame in point 4) is tight, with constant $|Q_\Lambda|$.

As a first corollary we can obtain the following result, which was proved by B. Fuglede in the Euclidean setting [4], and in the present setting by S. Pedersen ([17]) with topological arguments.

Corollary 1.4. *A set of finite and positive measure* Ω *tiles* G *with translations by* Λ *if and only if the exponential set* $E_{\Lambda^\perp}$ *is an orthogonal basis for* $L^2(\Omega)$*.*

Let us now denote with $M : \Lambda^\perp \to \mathcal{U}(L^2(\Omega))$ the modulations $M_{\tilde{\lambda}} f(g) = e_{\tilde{\lambda}}(g) f(g)$. As a second consequence of Theorem 1.3 we obtain the following.

Corollary 1.5. *Conditions 1) - 4) of Theorem 1.3 are equivalent to*

5) The system of modulates $\Psi_{\Lambda^\perp} = \{M_{\tilde{\lambda}}\psi : \tilde{\lambda} \in \Lambda^\perp\}$ *is a frame for* $L^2(\Omega)$*, with frame bounds* $0 < A|Q_\Lambda| \leq B|Q_\Lambda| < \infty$*, for any* $\psi \in L^2(\Omega)$ *satisfying*

$$0 < A \leq \operatorname{ess\,inf} |\psi|^2 \leq \operatorname{ess\,sup} |\psi|^2 \leq B < \infty.$$

The motivation for this paper comes from the problem of studying the relationship between spectrum sets and tiling pairs, whose roots dates back to a 1974 paper of B. Fuglede ([4]). There he proved that a set $E \subset \mathbb{R}^d$, $d \geq 1$, of positive Lebesgue measure, tiles $\mathbb{R}^d$ by translations with a lattice Λ if and only if $L^2(E)$ has an orthogonal basis of exponentials indexed by the annihilator of Λ. A more general statement in $\mathbb{R}^d$, which says that if $E \subset \mathbb{R}^d$, $d \geq 1$, has positive Lebesgue measure, then $L^2(E)$ has an orthogonal basis of exponentials (not necessary indexed by a lattice) if and only if E tiles $\mathbb{R}^d$ by translations, has been known as the Fuglede Conjecture.
A variety of results were proved establishing connections between tiling and orthogonal exponential bases. See, for example, [16], [10], [15], [11] and [12]. In 2001, I. Laba proved that the Fuglede conjecture is true for the union of two intervals in the plane ([14]). In 2003, A. Iosevich, N. Katz and T. Tao ([8]) proved that the Fuglede conjecture holds for convex planar domains. The conjecture was also proved for the unit cube of $\mathbb{R}^d$ in [10] and [16]. In 2004, T. Tao ([21]) disproved the Fuglede Conjecture in dimension $d = 5$ and larger, by exhibiting a spectral set in $\mathbb{R}^5$ which does not tile the space by translations. In [13], M. Kolountzakis and M. Matolcsi also disproved the reverse implication of the Fuglede Conjecture for dimensions $d = 4$ and higher. In [2] and [1], the dimension of counter-examples was further reduced. In fact, B. Farkas, M. Matolcsi and P. Mora show in [1] that the Fuglede conjecture is false in $\mathbb{R}^3$. The general feeling in the field is that sooner or later the counter-examples of both implications will cover all dimensions. However, in [9] the authors showed that the Fuglede Conjecture holds in two-dimensional vector spaces over prime fields.

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2 Notations and Preliminaries

Let Λ be a lattice in a second countable LCA group G. Denote by $Q_\Lambda \subset G$ a measurable cross section of G/Λ. By definition, a cross section is a set of representatives of all cosets in G/Λ such that the intersection of Q_Λ with any coset $g+\Lambda$ has only one element. The existence of a Borel measurable cross section is guaranteed by [3, Theorem 1]. Moreover, it is evident that (Q_Λ, Λ) is a tiling pair for G, while any tiling set Ω differs from a cross section at most for a zero measure set.

Let $d\dot{g}$ be a normalized Haar measure for G/Λ. Then the relation between Haar measure on G and Haar measure for G/Λ is given by the following *Weil's formula*: for any function $f \in L^1(G)$, the periodization map $\Phi(\dot{g}) = \sum_{\lambda\in\Lambda} f(g+\lambda),\ \dot{g} \in G/\Lambda$ is well defined almost everywhere in G/Λ, belongs to $L^1(G/\Lambda)$, and

$$\int_G f(g)dg = |Q_\Lambda| \int_{G/\Lambda} \sum_{\lambda\in\Lambda} f(g+\lambda)d\dot{g}. \tag{3}$$

This formula is a special case of [19, Theorem 3.4.6]. The constant $|Q_\Lambda|$, called the *lattice size*, appears in (3) because G/Λ is equipped with the normalized Haar measure $d\dot{g}$.

Definition 2.1 (Dual integrable representations ([6])). *Let G be an LCA group, and let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. We say π is dual integrable if there exists a sesquilinear map $[\cdot,\cdot]_\pi : \mathcal{H}\times\mathcal{H} \to L^1(\widehat{G})$, called bracket map for π, such that*

$$\langle \phi, \pi(g)\psi\rangle_{\mathcal{H}} = \int_{\widehat{G}} [\phi,\psi]_\pi(\chi) e_{-g}(\chi)d\chi \quad \forall\ g\in G\,, \quad \forall\ \phi,\psi \in \mathcal{H}.$$

Example 2.2. *Let Λ be a lattice in a second countable LCA group G. For any $\lambda \in \Lambda$, define $T_\lambda\phi(g) = \phi(g-\lambda)$ on $\phi \in L^2(G)$ and $M_\lambda h(\chi) = e_\lambda(\chi)h(\chi)$ on $h \in L^2(\widehat{G})$. Let us denote with $Q_{\Lambda^\perp}$ a cross section for the annihilator lattice $\Lambda^\perp$. Thus, by Plancherel theorem and Weil's formula (3) we have*

$$\begin{aligned}
\langle \phi, T_\lambda\psi\rangle_{L^2(G)} = \langle \mathcal{F}_G(\phi), M_\lambda\mathcal{F}_G(\psi)\rangle_{L^2(\widehat{G})} &= \int_{\widehat{G}} \mathcal{F}_G(\phi)(\chi)\overline{\mathcal{F}_G(\psi)(\chi)}e_{-\lambda}(\chi)d\chi \\
&= |Q_{\Lambda^\perp}| \int_{\widehat{G}/\Lambda^\perp} \sum_{\tilde{\lambda}\in\Lambda^\perp} \mathcal{F}_G(\phi)(\dot{\chi}+\tilde{\lambda})\overline{\mathcal{F}_G(\psi)(\dot{\chi}+\tilde{\lambda})}e_{-\lambda}(\dot{\chi}+\tilde{\lambda})d\dot{\chi} \\
&= |Q_{\Lambda^\perp}| \int_{\widehat{G}/\Lambda^\perp} \sum_{\tilde{\lambda}\in\Lambda^\perp} \mathcal{F}_G(\phi)(\dot{\chi}+\tilde{\lambda})\overline{\mathcal{F}_G(\psi)(\dot{\chi}+\tilde{\lambda})}e_{-\lambda}(\dot{\chi})d\dot{\chi}.
\end{aligned}$$

Since $\mathcal{F}_G(\phi)\overline{\mathcal{F}_G(\psi)} \in L^1(\widehat{G})$, we have that

$$[\phi,\psi]_T(\dot{\chi}) := |Q_{\Lambda^\perp}| \sum_{\tilde{\lambda}\in\Lambda^\perp} \mathcal{F}_G(\phi)(\dot{\chi}+\tilde{\lambda})\overline{\mathcal{F}_G(\psi)(\dot{\chi}+\tilde{\lambda})} \qquad a.e. \quad \dot{\chi}\in\widehat{G}/\Lambda^\perp$$

defines a sesquilinear map $[\cdot,\cdot]_T : L^2(G)\times L^2(G) \to L^1(\widehat{G}/\Lambda^\perp)$, so T is a dual integrable representation of Λ on $\mathcal{H} = L^2(G)$.

A relevant application of dual integrable representations is the possibility to characterize bases of unitary orbits in terms of their associated bracket maps. The following result has been proved in [6, Proposition 5.1].

Theorem 2.3. *Let G be a countable abelian group, let π be a dual integrable representation of G on a Hilbert space $\mathcal{H}$, and let $\phi \in \mathcal{H}$. The system $\{\pi(g)\phi : g \in G\}$ is an orthonormal basis for its closed linear span if and only if $[\phi, \phi]_\pi(\chi) = 1$ for almost every $\chi \in \widehat{G}$.*

As a consequence of the preceding theorem, and that $|Q_\Lambda||Q_{\Lambda^\perp}| = 1$ (see e.g. [5, Lemma 6.2.3]), we have the following result.

Corollary 2.4. *Let T and Λ be as in Example 2.2, and let $\phi \in L^2(G)$. Then the system of translates $\{T_\lambda \phi : \lambda \in \Lambda\}$ is an orthonormal system in $L^2(G)$ if and only if*

$$\sum_{\tilde{\lambda}\in\Lambda^\perp} |\mathcal{F}_G(\phi)(\chi + \tilde{\lambda})|^2 = |Q_\Lambda| \quad a.e. \;\; \chi \in \widehat{G}\,.$$

3 Proof of Theorem 1.3

In this section we shall prove Theorem 1.3 and its corollaries.

Proof of Theorem 1.3.

1) $\Rightarrow$ 4) It is well-known ([20]) that, for any cross section Q_Λ, the exponential set $E_{\Lambda^\perp}$ is an orthogonal basis for $L^2(Q_\Lambda)$. Thus, for all $f \in L^2(Q_\Lambda)$,

$$\sum_{\tilde{\lambda}\in\Lambda^\perp} |\langle f, \frac{1}{\sqrt{|Q_\Lambda|}} e_{\tilde{\lambda}}\rangle_{L^2(Q_\Lambda)}|^2 = \|f\|^2_{L^2(Q_\Lambda)} \tag{4}$$

by the Plancherel theorem. Since condition (1) says that Ω is contained in some cross section Q_Λ, then the previous identity still holds for all $f \in L^2(\Omega)$. Hence $E_{\Lambda^\perp}$ is a tight frame for $L^2(\Omega)$ with constant $|Q_\Lambda|$.

4) $\Rightarrow$ 1) Suppose, by contradiction, that Ω is not a subtiling set. Then we claim that for all cross section Q_Λ there exist $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_2 \neq 0$, such that

$$|(Q_\Lambda + \lambda_1) \cap \Omega \cap (\Omega + \lambda_2)| > 0. \tag{5}$$

If this is true, then let $\Omega_1 = (Q_\Lambda + \lambda_1) \cap \Omega \cap (\Omega + \lambda_2)$, and $\Omega_2 = \Omega_1 - \lambda_2$. Both are subsets of Ω with positive measure and, since $\lambda_2 \neq 0$, they are disjoint because $\Omega_1 \subset Q_\Lambda + \lambda_1$ and $\Omega_2 \subset Q_\lambda + \lambda_1 - \lambda_2$. Therefore, the function

$$f = \mathbf{1}_{\Omega_1} - \mathbf{1}_{\Omega_2}$$

is nonzero and belongs to $L^2(\Omega)$. Then, for all $\tilde{\lambda} \in \Lambda^\perp$ we have

$$\langle f, e_{\tilde{\lambda}}\rangle_{L^2(\Omega)} = \int_{\Omega_+} e_{\tilde{\lambda}}(g)dg - \int_{\Omega_-} e_{\tilde{\lambda}}(g)dg = \int_{\Omega_+} \big(e_{\tilde{\lambda}}(g) - e_{\tilde{\lambda}}(g - \lambda^*)\big)dg = 0.$$

This implies that the system $E_{\Lambda^\perp}$ can not be a frame for $L^2(\Omega)$.

In order to prove (5), let us proceed by contradiction and suppose that for all $\lambda \in \Lambda$ and all $\lambda^* \in \Lambda$, $\lambda^* \neq 0$ we have

$$|(Q_\Lambda + \lambda) \cap \Omega \cap (\Omega + \lambda^*)| = 0.$$

Now take $\lambda' \in \Lambda$, $\lambda' \neq 0$. By definition of cross section, we have

$$\Omega \cap (\Omega + \lambda') = \bigsqcup_{\lambda \in \Lambda} (Q_\Lambda + \lambda) \cap \Omega \cap (\Omega + \lambda')$$

which implies that $|\Omega \cap (\Omega + \lambda')| = 0$. Hence, Ω would be a subtiling set of G by Λ, which is a contradiction.
1) $\Rightarrow$ 2) Since (4) holds, we can obtain 2) by choosing $f = \overline{e_\chi}\,\mathbf{1}_\Omega$.

2) $\Rightarrow$ 3) This follows as an application of Corollary 2.4.

3) $\Rightarrow$ 1) By orthogonality, we have that for all $\lambda \in \Lambda$, $\lambda \neq 0$

$$0 = \langle \mathbf{1}_\Omega, \mathbf{1}_\Omega(\cdot - \lambda)\rangle_{L^2(G)} = |\Omega \cap (\Omega + \lambda)|$$

so Ω is sub-tiling. ☐

Proof of Corollary 1.4. If (Ω, Λ) is a tiling pair then it is well-known that $E_{\Lambda^\perp}$ is an orthogonal basis for $L^2(\Omega)$. To prove the converse, assume by contradiction that Ω is not tiling. Then one of the following cases holds

i. Ω is a strictly sub-tiling set, i.e. there exists a cross section Q_Λ such that $\Omega \subset Q_\Lambda$ and $|Q_\Lambda \setminus \Omega| > 0$.

ii. Ω is not a sub-tiling set, so that (5) holds.

For case i., observe that the assumption of $E_{\Lambda^\perp}$ being an orthogonal basis for $L^2(\Omega)$ implies

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f, \frac{1}{\sqrt{|\Omega|}} e_{\tilde{\lambda}}\rangle_{L^2(\Omega)}|^2 = \|f\|^2_{L^2(\Omega)} \quad \forall\, f \in L^2(\Omega).$$

On the other hand, since $E_{\Lambda^\perp}$ is an orthogonal basis for $L^2(Q_\Lambda)$, we have

$$\sum_{\tilde{\lambda} \in \Lambda^\perp} |\langle f\mathbf{1}_\Omega, \frac{1}{\sqrt{|Q_\Lambda|}} e_{\tilde{\lambda}}\rangle_{L^2(Q_\Lambda)}|^2 = \|f\mathbf{1}_\Omega\|^2_{L^2(Q_\Lambda)} \quad \forall\, f \in L^2(\Omega)$$

so that $|\Omega| = |Q_\Lambda|$, which contradicts i.
For case ii., with Theorem 1.3 we already proved that $E_{\Lambda^\perp}$ can not even be a frame. ☐

Proof of Corollary 1.5. Assume 4) holds, i.e. that $E_{\Lambda^\perp}$ is a tight frame for $L^2(\Omega)$ with constant $|Q_\Lambda|$. Then

$$\sum_{\tilde{\lambda}\in\Lambda^\perp} |\langle f, M_{\tilde{\lambda}}\psi\rangle_{L^2(\Omega)}|^2 = \sum_{\tilde{\lambda}\in\Lambda^\perp} |\langle f\overline{\psi}, e_{\tilde{\lambda}}\rangle_{L^2(\Omega)}|^2 = |Q_\Lambda|\, \|f\overline{\psi}\|^2_{L^2(\Omega)} \quad \forall\ f \in L^2(\Omega)\,.$$

Since $A\|f\|^2_{L^2(\Omega)} \leq \|f\overline{\psi}\|^2_{L^2(\Omega)} \leq B\|f\|^2_{L^2(\Omega)}$, this proves 5). Conversely, assume 5) holds. Then, since $A > 0$, for all $f \in L^2(\Omega)$ we can write

$$\sum_{\tilde{\lambda}\in\Lambda^\perp} |\langle f, e_{\tilde{\lambda}}\rangle_{L^2(\Omega)}|^2 = \sum_{\tilde{\lambda}\in\Lambda^\perp} |\langle f/\overline{\psi}, M_{\tilde{\lambda}}\psi\rangle_{L^2(\Omega)}|^2\,,$$

so that, by the hypotheses on ψ, we get

$$\frac{A}{B}|Q_\Lambda|\, \|f\|^2_{L^2(\Omega)} \leq \sum_{\tilde{\lambda}\in\Lambda^\perp} |\langle f, e_{\tilde{\lambda}}\rangle_{L^2(\Omega)}|^2 \leq \frac{B}{A}|Q_\Lambda|\, \|f\|^2_{L^2(\Omega)} \quad \forall\ f \in L^2(\Omega).$$

Thus $E_{\Lambda^\perp}$ is a frame, hence proving 4). Observe that, by Theorem 1.3, this implies that $E_{\Lambda^\perp}$ is a tight frame with constant $|Q_\Lambda|$, hence improving the inequalities above. ☐

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