Preface

Introduction

Wavelets were introduced relatively recently, in the beginning of the 1980s. They attracted considerable interest from the mathematical community and from members of many diverse disciplines in which wavelets had promising applications. A consequence of this interest is the appearance of several books on this subject and a large volume of research articles. In order to explain why we have written this book, describe where it might play a useful role in this field and to whom it is addressed, we find it necessary to state what we mean by the word "wavelet" and mention some of its properties. Let us do this for wavelets defined on the real line $\mathbb{R}$.

The real line is endowed with two basic algebraic operations, addition and multiplication. From these two operations we obtain two families of operators acting on functions defined on $\mathbb{R}$: the **translations** and the **dilations**. More precisely, translation by $h \in \mathbb{R}$ is the operator τ_h that maps a function f into the function whose value at $x \in \mathbb{R}$ is $(\tau_h f)(x) = f(x-h)$. The dilation ρ_r, $r > 0$, is defined by the equality $(\rho_r f)(x) = f(rx)$. Many of the important linear operators acting on functions defined on $\mathbb{R}$ have simple relations with these two families. For example, differentiation commutes with the translations. More generally, in the setting of tempered distributions, the class of convolution operators are characterized by this property of commuting with translations (differentiation is obtained by convolving with the distribution that is the derivative of the "Dirac-delta function"). Similar observations can be made about the family of dilations. A most important operator acting on functions (or, more generally, on tempered distributions) is the **Fourier Transform**, which maps f into $\hat{f}$, where

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx.$$

It is well known that convolution operators are converted, via the Fourier transform, into multiplication operators. This is a consequence of the formula $(f * g)^\wedge = \hat{f}\,\hat{g}$. In particular, $(\tau_h f)^\wedge(\xi) = e^{-ih\xi}\hat{f}(x)$; that is, translation by h corresponds to multiplication by the exponential $e^{-ih\xi}$. All these properties are particularly natural if we consider them in the context of $L^2(\mathbb{R})$: the Fourier transform can then be expressed in terms of a unitary operator, and this allows one to study many convolution operators in terms of particularly simple multiplier operators.

In view of these observations, it is only natural to look for bases of $L^2(\mathbb{R})$ having properties that reflect the importance of translations, dilations and the Fourier transform. For example, in the analogous periodic case, the "trigonometric" system, $\{\frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z}\}$, is an orthonormal basis for $L^2(0, 2\pi)$ that simultaneously diagonalizes all the bounded operators on this space that commute with translations. This property makes this system a most important basis for $L^2(0, 2\pi)$ and is of fundamental importance to the study of Fourier series. The various wavelets provide us with orthonormal bases for $L^2(\mathbb{R})$ that are particularly natural when dealing with the analysis that involves the action of translations, dilations and the Fourier transform (that is, Harmonic Analysis). We see that this is most plausible from their definition: a function $\psi \in L^2(\mathbb{R})$ is an **orthonormal wavelet** provided the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) \qquad \text{for all} \;\; j, k \in \mathbb{Z}.$$

That is, this system is generated from one function, ψ, by translating it by the integers and applying the dyadic dilations ρ_r, where $r = 2^j$, to these translates. The multiplication by the factor $2^{j/2}$ is forced upon us if we require each member of this system to have L^2-norm equal to one; moreover, it renders the action of the Fourier transform on this system particularly simple: if $\gamma = \hat{\psi}$, then the Fourier transform of $\psi_{j,k}$ is

$$(\psi_{j,k})^\wedge(\xi) = e^{-i2^{-j}k\xi}2^{-\frac{j}{2}}\gamma(2^{-j}\xi).$$

That is, we still have dyadic dilations and the translations are converted into "modulations" (which is a term that means multiplication by exponentials).

The philosophy of the book

The purpose of this book is to show how such wavelets can be constructed, illustrate why they provide us with a particularly powerful tool in mathematical analysis, and indicate how they can be used in applications. The title of the book reflects our hope that it can be read by those who are familiar with the Fourier transform and its basic properties; we feel that this amount of knowledge suffices for the understanding of the material presented. Let us explain in more detail what we mean by this. We shall show that wavelets can be applied to a large variety of mathematical subjects. For example, they can be used to characterize several function spaces: the Lebesgue, Hardy, Sobolev, Besov and Triebel-Lizorkin spaces are some of these. The Lebesgue spaces are easily defined, and some of their basic properties are not hard to explain. This is not the case for all these spaces. For example, the Hardy spaces have many different, but equivalent, definitions. Originally they were introduced as spaces of holomorphic functions in the domain in the complex plane that lies above the x-axis. About twenty-five years ago it was discovered that they can be identified as functions (really, distributions) on $\mathbb{R}$ having an appropriate maximal function. A few years later their "atomic" characterization was discovered. This approach involves certain "building blocks" called atoms, which are particularly simple functions, that can be used to express the general element of the Hardy space. It would carry us way beyond the scope of this book if, before discussing these spaces, we were to present all the material that is necessary to establish the equivalence of these various versions. It is not difficult, however, to present clear statements of those properties that are most relevant to the use of wavelets; when we do this, we do give appropriate references. In this sense this book is not "self-contained," but this does not mean that more is demanded from the reader in order to appreciate the roles that are played by wavelets in these applications.

Wavelets can be defined on other domains. For example, we can introduce a natural extension of the definition of the function $\psi_{j,k}$ by considering ψ to be defined in $\mathbb{R}^n$, n-dimensional Euclidean space, by letting $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ be an n-tuple of integers and replacing $2^{j/2}$ by $2^{nj/2}$ (so that $\|\psi_{j,k}\|_2 = \|\psi\|_2$). The situation, in this case, is more complicated: if one makes certain natural assumptions, it can be shown that one cannot obtain an orthonormal basis of $L^2(\mathbb{R}^n)$ by such a construction; in fact, $2^n - 1$ such generating functions are needed if one wants to obtain such a basis. Other domains can be considered where the roles played by the translations and dilations need to be played by different actions on the domain. We decided in this "first course on wavelets" not to present the theory of wavelets in these more complicated settings and to concentrate on the one-dimensional case. We felt that a good understanding of the one-dimensional theory provides a good background for its extensions to other domains.

Let us make a few comments about some of the other books on wavelets. Perhaps the two most important treatises on the subject are Y. Meyer's three-volume set ([Me1], [Me2], and [CM1] – the third one is co-authored with R. Coifman) and I. Daubechies "Ten lectures on wavelets" ([Da1]).

Both are excellent presentations, and we recommend them with enthusiasm. They are more advanced than this book and cover much more material. Since they were written, however, the theory has advanced considerably (partly due to their contributions). Some of the original constructions have been simplified and extended. We hope that this book can serve as an introduction to these two treatises. The book by C. Chui ([Chu]) should also be mentioned. It is a good complement to the ones by Daubechies and Meyer (as, we hope, is ours). We cite it often, particularly when we discuss spline wavelets.

Description of the book

It is, perhaps, useful to describe this book in more detail and give some advice about how to read it. The first four chapters, together with Chapter 7, make up a "natural" inter-related group. They are devoted to the construction of wavelets. We feel that Chapter 7 is the most important one in the book. There are two simple equations that completely characterize all orthonormal wavelets. They are

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(2^j\xi)\right|^2 = 1 \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}, \tag{1}$$

and for every odd integer m,

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi+2m\pi))} = 0 \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}. \tag{2}$$

More precisely, $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if ψ satisfies (1) and (2), provided $\|\psi\|_2 = 1$. The proof of this is elementary but it is not simple, and we present it in the seventh chapter. These equations are known and have been used by many investigators working with wavelets. The proof of this characterization in full generality, however, did not appear in the published literature until recently. It can be found in a paper by G. Gripenberg ([Gri1]), the Ph.D. thesis of one of our students, X. Wang ([Wan]), and will appear in an expository article we wrote with him ([HWW3]). It has been one of our goals to study the properties of wavelets by examining their Fourier transforms. One of the principal features of this book, in fact, is the important role played by the Fourier transform.

The first four chapters are devoted to different ways of constructing wavelets. Chapter 1 deals with the local sine and cosine bases that were discovered by R. Coifman and Y. Meyer. We show how they lead us to bases for $L^2(\mathbb{R})$ that have the important features described in the beginning of this introduction; that is, they enjoy particularly simple relations with the basic operators: translations, dilations and the Fourier transform. We use these bases to construct the wavelets of Lemarié and Meyer, the first class of orthonormal wavelets that were introduced and that includes ones such that they and their Fourier transform are smooth.

In the second chapter we develop a general method that was introduced by Mallat and Meyer for constructing wavelets: the **multiresolution analysis** (MRA). We apply this method to obtain the compactly supported wavelets introduced by Daubechies. The third chapter is devoted to the "band-limited" wavelets (the ones having compactly supported Fourier transforms). We show that the elements of this class have some surprising properties; for example, their Fourier transforms vanish in a neighborhood of the origin. Perhaps one of the best reasons for studying this class separately is that the basic equations (1) and (2) are particularly easy to study. Among other things, the series involved have only a finite number of non-zero terms and we do not need to worry about their convergence. This allows us to pave the way for the technically more difficult analysis involved in the seventh chapter. The fourth chapter introduces the reader to the "spline wavelets." This class appears to be particularly important in the various applications of wavelet theory to signal and image analyses. We also explain in this fourth chapter how one can construct periodic wavelets.

By the end of the first four chapters we have enough examples and have obtained sufficiently many properties of wavelets to introduce the reader to some of the uses of wavelets and their connection to other parts of Analysis. We therefore interrupt our program of characterizing all wavelets in terms of their Fourier transform and show how they provide us with tools for the study of the important scales of function spaces we mentioned above. In addition to providing us with orthonormal bases for the Hilbert space $L^2(\mathbb{R})$, some wavelets give us natural bases for these other topological linear spaces as well. Let us illustrate this with the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, of all those measurable functions f such that

$$\|f\|_p = \Big(\int_{-\infty}^{\infty} |f(x)|^p \, dx\Big)^{\frac{1}{p}} < \infty. \tag{3}$$

When $p = 2$ the finiteness of this norm $\|f\|_2$ is equivalent to the finiteness of the norm

$$\|\mathfrak{c}\|_2 = \Big(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |c_{j,k}|^2\Big)^{\frac{1}{2}}$$

of the coefficient sequence $\mathfrak{c} = \{c_{j,k}\} = \big\{\langle f, \psi_{j,k}\rangle\big\}$ that provides us with the representation

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\, \psi_{j,k}\, .$$

Thus, $L^2(\mathbb{R})$ can be represented as the space $\ell^2(\mathbb{Z}\times\mathbb{Z})$ of all sequences $\mathfrak{c}$ such that $\|\mathfrak{c}\|_2 < \infty$. Appropriate wavelet bases provide us with the characterization of $L^p(\mathbb{R})$ in terms of a sequence space for the other indices $p \in (1,\infty)$. It can be shown that f belongs to $L^p(\mathbb{R})$ if and only if

$$\|\mathfrak{c}\|^{(p)} = \big\|\{c_{j,k}\}\big\|^{(p)} = \Big(\int_{-\infty}^{\infty} \Big\{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^j |c_{j,k}|^2 \chi_{j,k}(x)\Big\}^{\frac{p}{2}} dx\Big)^{\frac{1}{p}} < \infty,$$

where $\chi_{j,k}$ is the characteristic function of the interval $[2^{-j}k, 2^{-j}(k+1)]$ and $\mathfrak{c}$ is the sequence of coefficients of f associated with $\{\psi_{j,k}\}$. Observe that the finiteness of $\|\mathfrak{c}\|^{(p)}$ is a condition on the size (or absolute value) of the coefficients $c_{j,k}$. This provides us with the ability to study $L^p(\mathbb{R})$ in terms of a corresponding sequence space in a way that is analogous to the reduction of properties of $L^2(\mathbb{R})$ to properties of $\ell^2(\mathbb{Z}\times\mathbb{Z})$. Note that $\|\mathfrak{c}\|^{(2)} = \|\mathfrak{c}\|_2$. Thus, an operator that is diagonalized by the basis $\{\psi_{j,k}\}$ can be analyzed in terms of its proper values, as is the case in Hilbert space theory. Many important operators are "essentially" diagonalized by wavelet bases. It is this circle of ideas that is presented in Chapter 5 and Chapter 6. More specifically, we present a brief treatment of bases in Banach spaces, with an emphasis on the notion of unconditionality, in Chapter 5. In Chapter 6 we give the characterizations described above. This treatment uses properties of **Calderón-Zygmund operators**; consequently, we have an opportunity to see how wavelets are associated with the study of these important operators.

In Chapter 7 we resume the study of wavelets in $L^2(\mathbb{R})$. We not only give a characterization of all wavelets, as described above, but we also characterize all wavelets that arise from an MRA and the basic functions (the scaling functions and low-pass filters) involved in this method. These characterizations allow us to construct several other classes of wavelets as well.

Though most of the bases discussed in the first seven chapters are orthonormal, we do mention some other types of bases. In Chapter 8 we present a more thorough treatment of systems that are more general, with particular attention to **frames** and their importance to wavelets. We pay special attention to the way they can be used to analyze and reconstruct functions; we also extend the Balian-Low theorem to frames.

The last chapter is devoted to certain topics that are important and relevant to the applications of the theory of wavelets. We indicate how the mathematical theory is transformed when it is applied to "discrete" signals. We develop the Discrete Fourier (and Cosine) Transform in what is, probably, a manner that is different from the usual treatment but has some features that are adaptable to programming for computers. We also describe the decomposition and reconstruction algorithms for wavelets and we end the chapter with a treatment of "wavelet packets."

One of our colleagues, M.V. Wickerhauser, has recently written a book, "Adapted wavelet analysis from theory to software" ([Wi2]), that treats the subject we just mentioned, and many more applications, in great detail. We believe that his book will prove to be most useful. We found no need, therefore, to go further than we did in this direction. As we stated about the books by Daubechies and Meyer, we hope that our book makes a good companion to, and complements, the book by Wickerhauser.

Some advice to the reader

The background we assume of the reader is a "good undergraduate" preparation in mathematics. We do use the language of measure theory; for example, we talk about "measurable functions." One should not be discouraged if he/she only knows the ordinary Riemann integral. Substituting the Lebesgue integral for the latter will, in general, not affect the meaning or the validity of most statements. Some notions in elementary functional analysis are used; again, the results or statements involving these notions can almost always be understood by ignoring unfamiliar language.

It is our hope that graduate students in mathematics, the sciences and engineering can profit from our presentation. We advise the reader not to be discouraged by the **few** somewhat technical notions we introduce at times (distributions, maximal functions, vector-valued inequalities, etc). If it's "too much," just skip it at first; there is enough material that can be understood with the background mentioned in the previous paragraph. It is our experience that even those whose main interest is in the applications can profit by learning about the theory we present.

In each section we number results consecutively; that is, we do **not** form separate lists of theorems, propositions, corollaries, lemmas, formulae and inequalities. These items are listed as ordered pairs n.m, where "n" denotes the section (in the chapter) and "m" the m^{th} item so numbered in the section. If we need to refer to something in another chapter, we mention the chapter and the relevant ordered pair. The sections in each chapter are also assigned an ordered pair, n.m; in this case, “n” denotes the number of the chapter and “m” the m$^{\rm th}$ section.

We do not present a list of exercises at the end of each chapter. In many cases we leave certain calculations to be worked out by the reader. This is particularly true of the comments made in the last section of each chapter, which is labelled “Notes and references.”

We also feel that we should state quite clearly that, though the bibliography we include is quite large, it is far from a list that comes anywhere close to exhausting what has been published in the theory of wavelets during the relatively short period of its existence. We have tried to give proper credits; however, since some of the material we discuss is quite new, we realize that it is very likely we omitted some references that should have been included.

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