Foreword

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Wavelet analysis can be defined as an alternative to the classical windowed Fourier analysis. In the latter case the goal is to measure the local frequency content of a signal, while in the wavelet case one is comparing several magnifications of this signal, with distinct resolutions. The building blocks of a windowed Fourier analysis are sines and cosines (waves) multiplied by a sliding window. They are usually referred to as time-frequency atoms. In a wavelet analysis, the window is already oscillating and is called a mother wavelet. This mother wavelet is no longer multiplied by sines or cosines. Instead it is translated and dilated by arbitrary translations and dilations. That is the way the mother wavelet generates the other wavelets which are the building blocks of a wavelet analysis. These dilations are precisely the magnifications we alluded to, and the building blocks are called time-scale atoms.

Fourier analysis, windowed Fourier analysis, and wavelet analysis are based on an identical recipe. In the three cases, the analysis of a function amounts to computing all the correlations between this function and the time-frequency or time-scale atoms which are being used. The synthesis is obtained exactly as if these building blocks were an orthonormal basis.

A common wisdom among numerical analysts and image processing people is that the inverse of a scale is a frequency: small scales correspond to large frequencies and large scales to small frequencies. Moreover, very distinct scales should provide independent (i.e., non-redundant) information. Wavelet analysis could be defined as an attempt to give a very precise meaning to this folk belief.

Wavelets were implicit in mathematics, physics, signal or image processing, and numerical analysis long before they were given the status of a unified scientific field.

In pure mathematics, three algorithms have been created to overcome some drawbacks of standard Fourier series expansions. These difficulties appear when one is facing the problem of measuring the size or the smoothness of a function. For example, the simplest norms, based on quadratic estimates, can easily be extracted from Fourier coefficients. But as soon as L^p or H^p estimates are addressed, Fourier coefficients do not answer the problem, while the algorithms that do answer it involve the **Haar basis** (1909), the **Franklin orthonormal system** (1927), or the **Littlewood-Paley theory** (1930); these have, in the past, proven to be the correct tools.

Later, **Calderón's reproducing identity** (1960) and **atomic decompositions** (1972) were widely used in other functional settings (Hardy spaces, for example). Both the Littlewood-Paley theory and atomic decompositions play a key role in a branch of operator theory created by Calderón, Zygmund, and their school which is known as the Calderón-Zygmund Theory. Just before wavelets became popular, J.O. Strömberg used this precise tool for solving a celebrated problem in the geometry of Banach spaces: the existence of a specific unconditional basis for the Hardy space $H^1(\mathbb{R})$.

In signal or image processing a similar and parallel evolution started from the standard windowed Fourier analysis and culminated in some discrete versions of Calderón's reproducing identity. Indeed, D. Gabor (1946) introduced **time-frequency atoms** in speech signal processing; Croisier, Esteban, and Galand developed **subband coding** in signal processing (1975); and only a little later Burt and Adelson described **pyramidal algorithms** in image processing (1982). D. Marr was convinced that both human vision and computer vision were based on similar algorithms which should be, in some sense, independent of the "wires" used in their realizations. These specific algorithms involve the **zero-crossings** of the wavelet transform of a two-dimensional signal (1982). In numerical analysis, wavelets are related to spline approximation. Before wavelets became fashionable, V. Rokhlin created the so-called **multipole algorithms**: refinement schemes that play a key role in computer graphics.

Finally, let us turn to mathematical physics. **Coherent states** are fundamental in quantum mechanics. **Renormalization in quantum field theory** is needed for extracting finite numbers from divergent integrals. It is based on some variants of Littlewood-Paley techniques which were mainly developed by K. Wilson, K. Gawedzki and A. Kupiainen, J. Glimm and A. Jaffe, G. Battle and P. Federbush.

Therefore wavelets were implicit in several scientific fields but nobody knew that, for instance, Littlewood-Paley theory and the Burt & Adelson pyramidal algorithms were telling the same story. The great unification was a shock, and many people still do not accept it. This unification was a fairy tale come true, which explains why the subject became immediately popular. The great unification meant a scientific status incorporating the heuristics and the wisdom of the distinct fields where protowavelets were already used. This unification was made possible through the efforts of several people. Let me especially mention Alex Grossmann and Stephane Mallat.

I have a vivid and nostalgic memory of many discussions with Antoni Zygmund. He used to test me on whatever problem he was dreaming about. He silently waited for my answer. Then he listened with a smile to my often stupid comments. Finally he often tried to correct my erroneous viewpoints. This happened when R.R. Coifman and G. Weiss and their collaborators launched the so-called **atomic decompositions** program. Zygmund asked my opinion about what Guido Weiss was doing. Zygmund immediately recognized the relevance of this endeavour, while it took me a slightly longer time.

But it is hard to believe that Zygmund would have guessed that atomic decompositions are also relevant in signal processing. He would have been surprised to learn that the celebrated composer and conductor Pierre Boulez and his collaborators decided to find a compact atomic decomposition for an aria by Mozart interpreted by Rita Streich. P. Boulez and his collaborators were indeed using (time-frequency) waveforms instead of (time-scale) wavelets.

We now come to the present book. It is not just one more book about wavelets. This unique book is distinct, since it is co-authored by one of the pioneers of atomic decompositions. Who else is more appropriate to talk about wavelets? Indeed atomic decompositions are at the heart of signal and image processing.

The careful writing of the authors, Eugenio Hernández and Guido Weiss, is well known and this book reflects their desire to make this subject most accessible. It will be applauded by all lovers of the precise, powerful, and elegant mathematics which Guido Weiss and his school have promoted.

This book contains many new and impressive results. Nowadays, there is a tendency to derive wavelets from the multiresolution analysis construction. By this method one cannot address basic issues like the ones that are discussed in this book and are, indeed, crucial. For example, the Fourier localization of a wavelet is discussed in full detail. This has been neglected by other authors. I hope the reader will enjoy this remarkable contribution as much as I did, and I thank the authors for letting me read the manuscript.