Variable Exponent Bergman Spaces

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$$1 < p^- \leq p^+ < \infty$$

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For a complex-valued measurable function $\varphi : \Omega \to \mathbb{C}$ we define the *modular* $\rho_{p(\cdot)}$ by

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and the *Luxemburg-Nakano norm* by

$$\|\varphi\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left(\frac{\varphi}{\lambda} \right) \leqslant 1 \right\}.$$

Variable Exponent Lebesgue Spaces

Definition

Let $p \in \mathscr{P}(\Omega)$. The *variable Lebesgue space* $L^{p(\cdot)}(\Omega)$ is introduced as the set of all complex-valued measurable functions $\varphi : \Omega \to \mathbb{C}$ for which the modular is finite, i.e. $\rho_{p(\cdot)}(\varphi) < \infty$. Equipped with the Luxemburg-Nakano norm this is a Banach space.

Variable Exponent Lebesgue Spaces

Proposition (Hölder's inequality)

Let $p \in \mathscr{P}(\Omega)$, $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{p'(\cdot)}(\Omega)$, *where* $1/p'(x) + 1/p(x) = 1$. *Then* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \leqslant 2\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)}.$$

Variable Exponent Lebesgue Spaces

Proposition

Let $p \in \mathscr{P}(\Omega)$*, then the dual space to* $L^{p(\cdot)}(\Omega)$ *is* $L^{p'(\cdot)}(\Omega)$*(up to an isomorphism), where* $1/p'(x) + 1/p(x) = 1$.

Variable Exponent Lebesgue Spaces

$$\|\!|f|\!\|_{L^{p(\cdot)}(\Omega)} := \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \leqslant 1} \int_\Omega f(x)g(x)\,\mathrm{d}x,$$

$$\|\!|f|\!\|_{L^{p(\cdot)}(\Omega)} \sim \|f\|_{L^{p(\cdot)}(\Omega)}\,.$$

Variable Exponent Lebesgue Spaces

Definition

A function $p : \Omega \to \mathbb{R}$ is said to be *locally* log-*Hölder continuous* on Ω if there exists a positive constant C such that

$$|p(x) - p(y)| \leqslant \frac{C}{\log\left(\mathrm{e} + 1/|x-y|\right)},$$

for all $x, y \in \Omega$ We denote by $\mathscr{P}^{\log}(\Omega)$ the set of all locally log-Hölder continuous functions in Ω for which $1 < p_- \leqslant p_+ < \infty$.

Variable Exponent Lebesgue Spaces

Definition

Given a function $f \in L^1_{\mathrm{loc}}(\Omega)$, the *Hardy-Littlewood maximal function* of f, denoted by Mf, is defined for any $x \in \mathbb{R}^n$ by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy.$$

Variable Exponent Lebesgue Spaces

Proposition

Let $p \in \mathscr{P}^{\log}(\Omega)$. *Then the Hardy-Littlewood maximal function is bounded in* $L^{p(\cdot)}(\Omega)$,

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leqslant \|f\|_{L^{p(\cdot)}(\Omega)}.$$

Rubio de Francia Extrapolation

We recall that by $\mathcal{F}$ we denote a family of pairs of non-negative, measurable functions and by A_1 we denote the Muckenhoupt A_1 weight.

Proposition

Given Ω, suppose that for some $p_0 \geq 1$ the family $\mathcal{F}$ is such that for all $w \in A_1$,

$$\int_\Omega F(x)^{p_0} w(x)\,dx \leq C_0 \int_\Omega G(x)^{p_0} w(x)\,dx, \quad (F, G) \in \mathcal{F}.$$

Given $p \in \mathscr{P}(\Omega)$, if $p_0 \leq p_- \leq p_+ < \infty$ and the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\Omega)$, then

$$\|F\|_{L^{p(\cdot)}(\Omega)} \leq C_{p(\cdot)} \|G\|_{L^{p(\cdot)}(\Omega)}.$$

Bergman Spaces

Let $\mathbb{D}$ denote the open unit disk in the complex plane and $\mathrm{d}A$ the normalized Lebesgue measure on $\mathbb{D}$. For a given $1 \leqslant p < \infty$ define the *Bergman space* $A^p(\mathbb{D})$ as the space of all analytic functions on $\mathbb{D}$ that satisfies:

$$\|f\|_{A^p}^p := \int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}A(z) < \infty.$$

Bergman Spaces

For every $z \in \mathbb{D}$, the *evaluation functional* $\gamma_z : A^p(\mathbb{D}) \to \mathbb{C}$ defined as

$$\gamma_z(f) := f(z)$$

is bounded. Therefore convergence in $A^p(\mathbb{D})$ implies uniform convergence on compact subsets of $\mathbb{D}$ and consequently L^p-limits of sequences in A^p are analytic. Hence $A^p(\mathbb{D})$ is a closed subspace of $L^p(\mathbb{D})$ (and hence a Banach space).

Bergman Spaces

If $p = 2$, then A^2 is a Hilbert space.

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, \mathrm{d}A(z)$$

Bergman Spaces

$A^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$, there exists a bounded projection operator $P : L^2(\mathbb{D}) \to A^2(\mathbb{D})$ (called the *Bergman projection*).

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w}z)^2} \, dA(w).$$

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$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w}z)^2} \, dA(w).$$

$P : L^p(\mathbb{D}) \to A^p(\mathbb{D})$ is also bounded for $1 < p < \infty$.

Variable Exponent Bergman Spaces

Definition

Given a measurable function $p \in \mathscr{P}(\mathbb{D})$ we define the *variable exponent Bergman space* $A^{p(\cdot)}(\mathbb{D})$ as the space of all analytic functions on $\mathbb{D}$ that belong to the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{D})$ with respect to the area measure dA on the unit disk $\mathbb{D}$, i.e.

$$A^{p(\cdot)}(\mathbb{D}) = \left\{ f \text{ is analytic and } \int_{\mathbb{D}} |f(z)|^{p(z)}\, dA(z) < \infty \right\}.$$

Variable Exponent Bergman Spaces

For every $z \in \mathbb{D}$, the *evaluation functional* $\gamma_z : A^{p(\cdot)}(\mathbb{D}) \to \mathbb{C}$ defined as

$$\gamma_z(f) := f(z)$$

is bounded. Therefore convergence in $A^{p(\cdot)}(\mathbb{D})$ implies uniform convergence on compact subsets of $\mathbb{D}$ and consequently $L^{p(\cdot)}$-limits of sequences in $A^{p(\cdot)}$ are analytic. Hence $A^{p(\cdot)}(\mathbb{D})$ is a closed subspace of $L^{p(\cdot)}(\mathbb{D})$ (and hence a Banach space).

Variable Exponent Bergman Spaces

Lemma

Let $p \in \mathscr{P}(\mathbb{D})$*, then for every function* $f \in L^{p(\cdot)}(\mathbb{D})$ *we have*

$$\|f\|_{L^{p(\cdot)}(\mathbb{D})} \leqslant \|f\|_{L^1(\mathbb{D})}^{1/p^+} \|f\|_{L^\infty(\mathbb{D})}^{1-1/p^+}.$$

Theorem

Let $p \in \mathscr{P}(\mathbb{D})$*. Then for every* $a \in \mathbb{D}$ *the evaluation functional* γ_a *is bounded. Moreover,*

$$\frac{1}{(1-|a|)^{2/p^+}} \lesssim \|\gamma_a\| \lesssim \frac{1}{(1-|a|)^{2/p^-}}.$$

Approximation

For $z \in \mathbb{C}$, we define the following radial test function:

$$\eta(z) := \begin{cases} C \exp\left(\frac{1}{|z|^2-1}\right), & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geqslant 1, \end{cases}$$

where $C > 0$ is the normalizing constant in the sense $\int_{\mathbb{C}} \eta(z)\, \mathrm{d}A(z) = 1$.

Approximation

Let $r \in [1/2, 1)$ and define

$$\eta_r(z) := \frac{4r^2}{(1-r)^2}\eta\left(\frac{2rz}{1-r}\right).$$

Notice that η_r is a C^∞ function supported on the set $\frac{1-r}{2r}\overline{\mathbb{D}}$ and $\int_{\mathbb{C}} \eta_r(z)\, \mathrm{d}A(z) = 1$. Notice also that for $z \in \mathbb{D}$ the function $w \mapsto \eta_r(z-w)$ is supported on the closed ball $\overline{B\left(z, \frac{1-r}{2r}\right)}$.

Approximation

Definition (Mollified dilation)

Given a function $f \in A^{p(\cdot)}(\mathbb{D})$, we will define its *mollified dilation* $f_r : \frac{1+r}{2r}\mathbb{D} \to \mathbb{C}$ as

$$f_r(z) := \int_{\mathbb{D}} f(rw)\eta_r(z-w)\,\mathrm{d}A(w). \tag{1}$$

where $\rho\mathbb{D}$ stands for the complex disk with radius ρ.

Approximation

Lemma

The function f_r is analytic on $\frac{1+r}{2r}\mathbb{D}$.

Approximation

Lemma

We have that $f_r(z) \to f(z)$ *as* $r \to 1^-$.

Approximation

Lemma

For every $z \in \mathbb{D}$ and $1/2 \leqslant r < 1$, the mollified dilation is pointwise dominated by the Hardy-Littlewood maximal operator, namely

$$|f_r(z)| \lesssim Mf(z).$$

Approximation

Theorem

Let $p \in \mathscr{P}^{\log}(\mathbb{D})$ *and let* $f \in A^{p(\cdot)}(\mathbb{D})$*. Then for* $1/2 \leqslant r < 1$, $f_r \in A^{p(\cdot)}(\mathbb{D})$ *and*

$$\sup_{1/2 \leqslant r < 1} \|f_r\|_{A^{p(\cdot)}(\mathbb{D})} \lesssim \|f\|_{A^{p(\cdot)}(\mathbb{D})}.$$

Moreover, $\|f_r - f\|_{A^{p(\cdot)}(\mathbb{D})} \to 0$ *as* $r \to 1^-$.

Approximation

Corollary

Let $p \in \mathscr{P}^{\log}(\mathbb{D})$. *Then the set of analytic polynomials is dense in* $A^{p(\cdot)}(\mathbb{D})$.

Bergman Projection

Definition

Given a positive function w on $\mathbb{D}$, and $1 < p < \infty$, the weighted Lebesgue space $L^p(w)$ consists of every function $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{L^p(w)}^p := \int_{\mathbb{D}} |f(z)|^p w(z)\, \mathrm{d}A(z) < \infty.$$

Bergman Projection

Definition

We say that w belongs to the *Békollé-Bonami class* B_p if there exists a constant $C > 0$ such that for every circular segment $I \subset \partial\mathbb{D}$,

$$\left(\int_{S(I)} w \, \mathrm{d}A\right)\left(\int_{S(I)} w^{-p'/p} \, \mathrm{d}A\right)^{p'/p} \leq CA(S(I))^p$$

where $1/p + 1/p' = 1$ and $S(I)$ denotes the *Carleson square*:

$$S(I) = \left\{ re^{it} : e^{it} \in I, \ 1 - \frac{|I|}{2\pi} \leq r < 1 \right\}.$$

Bergman Projection

Proposition (Békollé-Bonami)

Let $w \in L^1(\mathbb{D})$. A necessary and sufficient condition for the Bergman projection to be bounded on $L^p(w)$ is that w belongs to the class B_p.

Bergman Projection

Theorem

Let $p \in \mathscr{P}^{\log}(\mathbb{D})$. Then the Bergman projection is bounded from $L^{p(\cdot)}(\mathbb{D})$ onto $A^{p(\cdot)}(\mathbb{D})$.