The Hilbert matrix operator on spaces of analytic functions

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1. The Hilbert matrix.

The (one-sided) Hilbert matrix is the one-sided infinite matrix

$$H = \left(\frac{1}{i+j+1} \right)_{i,j=0}^{\infty}$$

$$= \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdot \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdot \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdot \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Observe that H is symmetric. Its (i, j) entry is

$$\frac{1}{i+j+1} = \lambda_{i+j}$$

where (λ_k) is the moment sequence

$$\lambda_k = \int_0^1 x^k \, dx, \quad k = 0, 1, \cdots$$

of the Lebesgue measure on $[0, 1]$.

2. Some History.

Hilbert's double series theorem (∼1900):

If $\sum_{k=0}^{\infty} a_k^2 < \infty$ *then* $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_i a_j}{i+j+1} < \infty$.

The proof was published by H. Weyl (1908). This was generalized by Hardy and Riesz (1925) as follows:

If $\sum_{k=0}^{\infty} |a_k|^p < \infty$ *and* $\sum_{k=0}^{\infty} |b_k|^q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{|a_i||b_j|}{i+j+1} \le \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{1/p} \left(\sum_{k=0}^{\infty} |b_k|^q \right)^{1/q}.$$

Equivalently, by duality,

$$\left(\sum_{i=0}^{\infty} \left| \sum_{j=0}^{\infty} \frac{a_j}{i+j+1} \right|^p \right)^{1/p} \le \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{1/p},$$

with the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ best possible.

Thus on the space l^p of p-summable sequences $a = (a_k)$ with norm,

$$\|a\|_p = \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{1/p},$$

the operator

$$\mathcal{H} : l^p \to l^p, \quad (a_k) \longrightarrow (A_k),$$

where

$$A_k = \sum_{j=0}^{\infty} \frac{a_j}{k+j+1},$$

is bounded and

$$\|\mathcal{H}\|_{l^p \to l^p} = \frac{\pi}{\sin(\frac{\pi}{p})}, \qquad 1 < p < \infty.$$

The matrix of $\mathcal{H}$ with respect to the usual Schauder basis $\{e_j\}$,

$$e_j = (0, \cdots, 0, 1, 0, \cdots), \quad (1 \text{ in } j\text{th place}),$$

of l^p is clearly the Hilbert matrix:

$$(A_0, A_1, A_2, ...) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & . \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . \\ . & . & . & . \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \end{pmatrix}.$$

3. The Hilbert matrix on analytic functions.

Let $\mathbb{D} = \{z : |z| < 1\}$, the unit disc,

$\mathcal{A}(\mathbb{D})$ = space of all analytic functions on $\mathbb{D}$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}(\mathbb{D})$, and let the Hilbert matrix H multiply the Taylor coefficients (a_n). We obtain formally the power series

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

This series is not always defined: If

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

then $A_n = \sum\limits_{k=0}^{\infty} \frac{1}{n+k+1} = \infty$ for each n. Thus $\mathcal{H}$ cannot be defined on the whole space $\mathcal{A}(\mathbb{D})$. On the other hand if $f(z)$ is a polynomial then

$$A_n = \sum_{k=0}^{N} \frac{a_k}{n+k+1}, \quad N = \text{degree of } f,$$

so the power series $\mathcal{H}(f)(z)$ is well defined and converges for $|z| < 1$, thus $\mathcal{H}(f)$ is analytic on $\mathbb{D}$.

Integral representation. If f is a polynomial then

$$\mathcal{H}(f)(z) = \int_0^1 f(t) \frac{1}{1 - tz} \, dt.$$

Indeed in this case

$$\sum_{k=0}^{N} \frac{a_k}{n + k + 1} = \int_0^1 t^n f(t) \, dt,$$

and

$$\begin{aligned} \mathcal{H}(f)(z) &= \sum_{n=0}^{\infty} \left(\int_0^1 t^n f(t) \, dt \right) z^n \\ &= \int_0^1 f(t) \sum_{n=0}^{\infty} t^n z^n \, dt \\ &= \int_0^1 f(t) \frac{1}{1 - tz} \, dt. \end{aligned}$$

Some other representations will be given later.

Restricting the domain. We need to restrict the domain of $\mathcal{H}$ to linear subspaces $X \subset \mathcal{A}(\mathbb{D})$ on which it is well defined.

Two such classes of spaces are the Hardy spaces H^p and the Bergman spaces A^p. We will concentrate on these two cases.

4. The Hilbert matrix on Hardy spaces.

Hardy spaces. Let $f : \mathbb{D} \to \mathbb{C}$ analytic. For $0 < p \leq \infty$ we say that $f \in H^p$ if

$$\|f\|_{H^p} = \sup_{r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty,$$

or

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty, \quad \text{(for } p = \infty).$$

We use $\| \, \|_p$ instead of $\| \, \|_{H^p}$. Each H^p is a linear space. For $1 \leq p \leq \infty$, the function $\| \, \|_p$ is a complete norm, making H^p a Banach space. For $p = 2$ the norm of $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^2$ turns out to be

$$\begin{aligned} \|f\|_2^2 &= \sup_r \int_0^{2\pi} |f(re^{i\theta})|^2 \, \frac{d\theta}{2\pi} \\ &= \sup_r \int_0^{2\pi} f(re^{i\theta}) \overline{f(re^{i\theta})} \, \frac{d\theta}{2\pi} \\ &= \sup_r \sum_{k=0}^{\infty} |a_k|^2 r^{2k} \\ &= \sum_{k=0}^{\infty} |a_k|^2 \end{aligned}$$

and this identifies H^2 with l^2. In particular H^2 is a Hilbert space, with inner product

$$\langle f, g\rangle = \lim_{r\to 1}\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\frac{d\theta}{2\pi}$$
$$= \sum_{k=0}^{\infty} a_k\overline{b_k}, \quad (f \sim \sum a_k z^k, g \sim \sum b_k z^k)$$

Properties of H^p:

• It is easy to see that if $1 < p < q < \infty$ then

$$H^1 \supset H^p \supset H^q \supset H^\infty,$$

with strict containment in each case.

• If $f \in H^p$ then the limit

$$f^*(e^{i\theta}) =: \lim_{r\to 1} f(re^{i\theta})$$

exists for almost all $\theta \in [0, 2\pi]$, the resulting function $f^*(e^{i\theta})$ is p-integrable on the circle $\mathbb{T}$, and

$$\|f^*\|_{L^p(\mathbb{T})} = \|f\|_{H^p}.$$

• If $f, g \in H^p$ and $f^*(e^{i\theta}) = g^*(e^{i\theta})$ on a set of positive measure on $\mathbb{T}$ then $f \equiv g$. (a form of the identity principle).

• We have

$$\{f^* : f \in H^p\} = \overline{\left\{p(e^{i\theta}) : p \text{ polynomial}\right\}},$$

closure in $L^p(\mathbb{T})$. Thus H^p can be identified, isometrically, with this closed subspace of $L^p(\mathbb{T})$.

Further, for $1 \le p \le \infty$, an $f \in L^p(\mathbb{T})$ is the boundary function of some H^p function if and only if the negative part of the Fourier series of f is zero.

• The Riesz Projection $P_+ : L^p(\mathbb{T}) \to H^p$ is

$$P_+(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)}{1 - e^{-i\theta}z} d\theta,$$

or, in terms of Fourier series,

$$P_+ : \sum_{-\infty}^{\infty} \widehat{f}(n)e^{in\theta} \longrightarrow \sum_{n=0}^{\infty} \widehat{f}(n)e^{in\theta}.$$

P_+ is a bounded operator for $1 < p < \infty$ (more on this later).

• If $f \in H^p$ then $|f(z)| \le \frac{C_p\|f\|_p}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}.$

• Hardy's inequality: If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$ then

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \|f\|_{H^1}.$$

• Fejér-Riesz inequality: If $f \in H^p$ then

$$\int_{-1}^{1} |f(t)|^p dt \leq \frac{1}{2} \|f\|_{H^p}^p, \qquad (0 < p < \infty).$$

• If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic then the composition operator $C_\varphi(f) = f \circ \varphi$ is bounded on H^p and

$$\|C_\varphi\|_{H^p \to H^p} \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/p}$$

• We will also need the following:

1. $\frac{1}{1-z} \in H^p$ for $p < 1$, but $f \notin H^1$. This implies that for $r \in \mathbb{R}$,

$$f_r(z) = \frac{1}{(1-z)^r} \in H^p$$

if and only if $r < \frac{1}{p}$, and $\lim_{r \to \frac{1}{p}} \|f_r\|_p = \infty$.

2. For $s \in \mathbb{R}$ the function

$$g_s(z) = \frac{1}{(1-z)(\frac{1}{z}\log\frac{1}{1-z})^s}$$

is in H^1 if and only if $s > 1$.

Hardy spaces of the half-plane.

Let $\Pi = \{z : \mathrm{Re}(z) > 0\}$, the right half-plane.

For $0 < p < \infty$, $H^p(\Pi)$ contains all analytic $f : \Pi \to \mathbb{C}$ such that

$$\|f\|^p_{H^p(\Pi)} = \sup_{0<x<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy < \infty.$$

$H^p(\Pi)$ are Banach spaces for $1 \le p < \infty$. We will only need the fact that $H^p(\Pi)$ are isometrically isomorphic to H^p. Indeed let

$$\mu : \mathbb{D} \to \Pi, \quad \mu(z) = \frac{1+z}{1-z},$$

the conformal map and let

$$V : H^p(\Pi) \to H^p,$$

be the linear map

$$V(f)(z) = (4\pi)^{1/p}(1-z)^{-2/p} f(\mu(z)).$$

It can be checked that V is 1-1, onto, $\|V(f)\|_{H^p} = \|f\|_{H^p(\Pi)}$ for each $f \in H^p(\Pi)$, and has inverse

$$V^{-1} : H^p \to H^p(\Pi),$$

$$V^{-1}(g)(z) = \pi^{-1/p}(1+z)^{-2/p} g(\mu^{-1}(z)).$$

The Hilbert matrix on H^p.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$. Then

$$\left|\sum_{n=0}^{\infty} \frac{a_n}{n+k+1}\right| \le \sum_{n=0}^{\infty} \frac{|a_n|}{n+k+1} \le \sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le \pi \|f\|_{H^1},$$

by Hardy's inequality. Thus the series

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n,$$

is well defined and converges on $\mathbb{D}$.

In other words the matrix H transforms each $f \in H^1$ (and therefore each $f \in H^p$ for $p > 1$) into a function $\mathcal{H}(f) \in \mathcal{A}(\mathbb{D})$.

But $\mathcal{H}(H^1) \not\subseteq H^1$. To see this let $1 < s \le 2$ and consider the test functions

$$f_s(z) = \frac{1}{(1-z)(\frac{1}{z} \log \frac{1}{1-z})^s} \in H^1.$$

Assuming that

$$\mathcal{H}(f_s)(z) = \sum_{n=0}^{\infty} \left(\int_0^1 t^n f_s(t) dt \right) z^n \ \in H^1$$

we can apply Hardy's inequality to obtain

$$\sum_{n=0}^{\infty} \frac{\int_0^1 t^n f_s(t)dt}{n+1} dt < \infty.$$

But

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^1 t^n f_s(t) dt &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{t^n}{n+1} \right) f_s(t) dt \\
&= \int_0^1 (\frac{1}{t} \log \frac{1}{1-t}) f_s(t) dt \\
&= \int_0^1 \frac{1}{(1-t)(\frac{1}{t} \log \frac{1}{1-t})^{s-1}} dt \\
&= \infty,
\end{aligned}$$

because $0 < s - 1 \leq 1$, a contradiction. Thus $\mathcal{H}(H^1) \not\subseteq H^1$.

Remark. In a recent paper of B. Lanucha, M. Nowak and M. Pavlovic it was proved:

If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$ with $a_n \geq 0$ then

$$\mathcal{H}(f) \in H^1 \ \Leftrightarrow \ \sum_{n=0}^{\infty} \frac{a_n \log(n+1)}{n+1} < \infty.$$

We also see that $\mathcal{H}(H^\infty) \not\subseteq H^\infty$. In fact for $f \equiv 1$ we find

$$\mathcal{H}(1)(z) = \frac{1}{z} \log \frac{1}{1-z} \notin H^\infty.$$

On the other hand since $H^2 \simeq l^2$,

$$\mathcal{H} : H^2 \to H^2,$$

is bounded and $\|\mathcal{H}\|_{H^2 \to H^2} = \pi$.

For other $p > 1$, observe that the integral representation

$$\mathcal{H}(f)(z) = \int_0^1 f(t) \frac{1}{1-tz}\, dt$$

is valid for all $f \in H^1$. This is verified by using the Fejér-Riesz inequality and the absolute convergence of Taylor series to justify the exchange of sums and integrals.

We can view this as an "improper line integral". The change of the path of integration

$$t = t(s) = \frac{s}{(s-1)z+1}, \quad 0 \le s \le 1,$$

gives

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{(s-1)z+1} f\left(\frac{s}{(s-1)z+1}\right) ds.$$

This says that

$$\mathcal{H} = \int_0^1 T_s \, ds, \quad T_s(f) = w_s f \circ \varphi_s,$$

i.e. $\mathcal{H}$ is an average of weighted composition operators with

$$w_s(z) = \frac{1}{(s-1)z+1}, \quad \varphi_s(z) = \frac{s}{(s-1)z+1}.$$

Observe that φ_s maps $\mathbb{D}$ into $\mathbb{D}$.

We would like to estimate $\|T_s\|_{H^p \to H^p}$.

To do this use the isometry $V : H^p(\Pi) \to H^p$ and equivalently estimate $\|\tilde{T}_s\|_{H^p(\Pi) \to H^p(\Pi)}$ where

$$\tilde{T}_s = V^{-1} T_s V : H^p(\Pi) \to H^p(\Pi).$$

For $f \in H^p(\Pi)$ we find,

$$\tilde{T}_s(f)(z) = (1-s)^{-\frac{2}{p}} \left(s + \frac{2}{z-1}\right)^{\frac{2}{p}-1} f(\psi_s(z))$$

where $\psi_s(z) = \frac{s}{1-s} z + \frac{1}{1-s}$ maps Π into Π.

Then integrate for the norm in $H^p(\Pi)$ to obtain the desired estimate for $\|\tilde{T}_s\|_{H^p(\Pi)\to H^p(\Pi)} = \|T_s\|_{H^p\to H^p}$ as in the following Lemma.

Lemma. Suppose $0 < s < 1$. We have:
(i) If $2 \le p < \infty$ then

$$\|T_s\|_{H^p\to H^p} \le s^{\frac{1}{p}-1}(1-s)^{-\frac{1}{p}}.$$

(ii) If $1 < p < 2$ then

$$\|T_s(f)\|_{H^p} \le s^{\frac{1}{p}-1}(1-s)^{-\frac{1}{p}}\|f\|_{H^p}.$$

for all $f \in H^p$ with $f(0) = 0$.

We then have for $2 \le p < \infty$,

$$\begin{aligned}
\|\mathcal{H}\|_{H^p\to H^p} &\le \int_0^1 \|T_s\|_{H^p\to H^p}\, ds \\
&\le \int_0^1 s^{\frac{1}{p}-1}(1-s)^{-\frac{1}{p}}\, ds \\
&= B(\frac{1}{p}, 1-\frac{1}{p}) \\
&= \Gamma(\frac{1}{p})\Gamma(1-\frac{1}{p}) \\
&= \frac{\pi}{\sin(\frac{\pi}{p})}
\end{aligned}$$

where we have used the standard identities for the classical Beta and Gamma functions B and Γ. A variation of this argument gives an analogous result for $1 < p < 2$, so we have,

Theorem [Diamantopoulos, S. , Studia Math. (2000)]
(i) If $2 \leq p < \infty$ then

$$\|\mathcal{H}\|_{H^p \to H^p} \leq \frac{\pi}{\sin(\frac{\pi}{p})}.$$

(ii) If $1 < p < 2$ then

$$\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \|f\|_{H^p}$$

for each $f \in H^p$ with $f(0) = 0$.

$\mathcal{H}$ as a Hankel operator.

Consider the Riesz projection

$$P_+ : L^p(\mathbb{T}) \longrightarrow H^p, \quad f \sim \sum_{-\infty}^{\infty} a_n e^{in\theta} \longrightarrow \sum_{n=0}^{\infty} a_n z^n,$$

and recall that P_+ is a bounded operator when $1 < p < \infty$.

Consider also the multiplication operator

$$M_\phi : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T}), \quad f \longrightarrow \phi f,$$

where the function

$$\phi(t) = i(\pi - t)e^{-it} \in L^\infty(\mathbb{T}),$$

has $\|\phi\|_{L^\infty(\mathbb{T})} = \pi$ and has Fourier coefficients

$$\widehat{\phi}(n) = \frac{1}{n+1}, \quad n \geq 0.$$

Let J be the isometric "conjugation" (or "flip") operator

$$J : H^p \longrightarrow L^p(\mathbb{T}), \quad f(e^{it}) \longrightarrow f(e^{-it})$$

A calculation gives

$$\mathcal{H} = P_+ \circ M_\phi \circ J, \qquad \begin{array}{ccc} L^p(\mathbb{T}) & \xrightarrow{M_\phi} & L^p(\mathbb{T}) \\ \uparrow J & & \downarrow P_+ \\ H^p & \xrightarrow{\mathcal{H}} & H^p \end{array}$$

It follows immediately that $\mathcal{H} : H^p \to H^p$ is bounded for $1 < p < \infty$ and

$$\begin{aligned}\|\mathcal{H}\|_{H^p\to H^p} &\leq \|P_+\|_{L^p(\mathbb{T})\to H^p}\|M_\phi\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \\ &= \|\phi\|_\infty\|P_+\|_{L^p(\mathbb{T})\to H^p} \\ &= \pi\|P_+\|_{L^p(\mathbb{T})\to H^p}.\end{aligned}$$

Thus if we knew the norm of P_+ we would have an estimate for $\|\mathcal{H}\|_{H^p\to H^p}$.

The norm of the Riesz projection.

I. Gohberg and N. Krupnik (1968) proved:

$$\|P_+\|_{L^p(\mathbb{T})\to H^p} \ \geq \ \frac{1}{\sin(\frac{\pi}{p})}, \quad 1 < p < \infty,$$

and conjectured that equality holds for the norm.

B. Hollenbeck and I. Verbitsky (J. Funct. Anal. 2000) proved the conjecture. Thus

$$\|P_+\|_{L^p(\mathbb{T})\to H^p} \ = \ \frac{1}{\sin(\frac{\pi}{p})}, \quad 1 < p < \infty.$$

It follows that for $\mathcal{H} = P_+ M_\phi J$ we have the uniform upper estimate,

$$\|\mathcal{H}\|_{H^p\to H^p} \ \leq \ \frac{\pi}{\sin(\frac{\pi}{p})}, \quad 1 < p < \infty.$$

Lower estimate of $\|\mathcal{H}\|_{H^p\to H^p}$.

Theorem [M. Dostanić, M. Jevtić and D. Vukotić (J. Funct. Anal. 2008)]. Let $1 < p < \infty$, then

$$\|\mathcal{H}\|_{H^p\to H^p} \geq \frac{\pi}{\sin(\frac{\pi}{p})}.$$

Sketch of proof:

Step 1. Fix $\varepsilon \in (0,1)$, let $\gamma \in (\varepsilon, 1)$ and consider the functions

$$f_\gamma(z) = \frac{1}{(1-z)^{\gamma/p}}.$$

Then $f_\gamma \in H^p$ and

$$\|f_\gamma\|_{H^p} \to \infty, \qquad \text{as } \gamma \to 1.$$

We can write

$$\begin{aligned}
\mathcal{H}(f_\gamma)(z) &= \int_0^1 (1-t)^{-\gamma/p} \frac{1}{1-tz}\, dt \\
&= \int_0^1 \frac{x^{-\gamma/p}}{1-z+xz}\, dx, \quad (\text{set } 1-t = x) \\
&= \int_0^\infty \frac{x^{-\gamma/p}}{1-z+xz}\, dx - \int_1^\infty \frac{x^{-\gamma/p}}{1-z+xz}\, dx \\
&= g_\gamma(z) - R_\gamma(z)
\end{aligned}$$

Step 2. The function $z^{1-\gamma/p}g_\gamma(z)$ is analytic on

$$\mathbb{C}\setminus[-\infty,0]\cup[1,\infty], \quad (z\neq\frac{1}{1-x}).$$

If $0<z<1$ we have

$$\begin{aligned} z^{1-\gamma/p}g_\gamma(z) &= \frac{z^{1-\gamma/p}}{1-z}\int_0^\infty \frac{x^{-\gamma/p}}{1+\frac{z}{1-z}x}\,dx \\ &= (1-z)^{-\gamma/p}\int_0^\infty \frac{u^{-\gamma/p}}{1+u}\,du, \\ &\text{(set } u=\tfrac{z}{1-z}x\text{)} \\ &= \Gamma(\gamma/p)\Gamma(1-\gamma/p)(1-z)^{-\gamma/p} \\ &= \frac{\pi}{\sin(\frac{\gamma\pi}{p})}(1-z)^{-\gamma/p}, \end{aligned}$$

and by the identity principle, for $z\in\mathbb{D}\setminus(-1,0]$,

$$z^{1-\gamma/p}g_\gamma(z)=\frac{\pi}{\sin(\frac{\gamma\pi}{p})}(1-z)^{-\gamma/p}.$$

It follows that

$$\begin{aligned} \|g_\gamma(z)\|_{L^p(\mathbb{T})} &= \|z^{1-\gamma/p}g_\gamma(z)\|_{L^p(\mathbb{T})} \\ &= \frac{\pi}{\sin(\frac{\gamma\pi}{p})}\|f_\gamma\|_{H^p}. \end{aligned}$$

Next $R_\gamma(z)$ is analytic on $\mathbb{C}\setminus[-\infty,0]$, thus in particular well defined a.e. on $\mathbb{T}$, and we have

$$\begin{aligned}\|\mathcal{H}\|_{H^p\to H^p}\cdot\|f_\gamma\|_{H^p} &\ge \|\mathcal{H}(f_\gamma)\|_{H^p}\\ &\ge \left|\|g_\gamma\|_{L^p(\mathbb{T})}-\|R_\gamma\|_{L^p(\mathbb{T})}\right|\\ &= \left|\frac{\pi}{\sin(\frac{\gamma\pi}{p})}\|f_\gamma\|_{H^p}-\|R_\gamma\|_{L^p(\mathbb{T})}\right|.\end{aligned}$$

Thus

$$\|\mathcal{H}\|_{H^p\to H^p}\ge\left|\frac{\pi}{\sin(\frac{\gamma\pi}{p})}-\frac{\|R_\gamma\|_{L^p(\mathbb{T})}}{\|f_\gamma\|_{H^p}}\right|$$

Step 3. Show that

$$\|R_\gamma\|_{L^p(\mathbb{T})}\le C=C(\varepsilon)$$

where C does not depend on $\gamma\in(\varepsilon,1)$.

Step 4. Let $\gamma\to 1$ to obtain

$$\|\mathcal{H}\|_{H^p\to H^p}\ge\frac{\pi}{\sin(\frac{\gamma\pi}{p})}.$$

As a conclusion

$$\|\mathcal{H}\|_{H^p\to H^p}\;=\;\frac{\pi}{\sin(\frac{\pi}{p})},\qquad 1<p<\infty.$$

5. $\mathcal{H}$ on Bergman spaces.

Bergman spaces. Let $f : \mathbb{D} \to \mathbb{C}$ analytic. For $0 < p < \infty$ we say that $f \in A^p$ if

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dx dy$ denotes the normalized area measure on $\mathbb{D}$.

For $1 \leq p < \infty$ these are Banach spaces, and for $p = 2$,

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^2 \, dA(z) &= \int_{\mathbb{D}} f(z)\overline{f(z)} \, dA(z) \\ &= \int_0^1 \int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{k=0}^{\infty} \overline{a}_k r^k e^{-ik\theta} \frac{1}{\pi} r d\theta dr \\ &= 2 \int_0^1 \sum_{n=0}^{\infty} |a_n|^2 r^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.\end{aligned}$$

A^2 is Hilbert space with inner product

$$\begin{aligned}\langle f, g \rangle &= \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z) \\ &= \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}, \quad (f \sim \sum a_k z^k, g \sim \sum b_k z^k)\end{aligned}$$

Properties of A^p:

• If $1 < p < q < \infty$ then

$$A^1 \supset A^p \supset A^q \supset H^\infty,$$

with strict containment in each case. In addition for each p we have

$$H^p \subset A^{2p}.$$

• In contrast to Hardy spaces, functions $f \in A^p$ need not have boundary values on $\mathbb{T}$.

• If $2 < p < \infty$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A^p$ then

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq C_p \|f\|_{A^p}.$$

(substitute of Hardy's inequality)

• If $f \in A^p$ then

$$|f(z)| \leq \frac{C_p \|f\|_{A^p}}{(1-|z|)^{2/p}}, \qquad z \in \mathbb{D}.$$

As a consequence, when $f \in A^p$ with $p > 2$,

$$\int_0^1 |f(t)|\, dt \leq C'_p \|f\|_{A^p}$$

• If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic then the composition operator $C_\varphi(f) = f \circ \varphi$ is bounded on A^p and

$$\|C_\varphi\|_{H^p \to H^p} \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{2/p}$$

• For $r \in \mathbb{R}$,

$$f_r(z) = \frac{1}{(1-z)^r} \in A^p$$

if and only if $r < \frac{2}{p}$, and $\lim_{r \to \frac{2}{p}} \|f_r\|_p = \infty$.

$\mathcal{H}$ on Bergman spaces.

On A^2, $\mathcal{H}$ is not defined. Indeed

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{\log(n+1)} z^n \in A^2$$

since

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)\log^2(n+1)} < \infty,$$

while

$$\mathcal{H}(f)(0) = \sum_{n=0}^{\infty} \frac{1}{(n+1)\log(n+1)} \quad \text{is divergent.}$$

If $2 < p < \infty$ the substitute of Hardy's inequality for $f \in A^p$ implies that the series

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n$$

is a well defined analytic function on $\mathbb{D}$.

The growth estimate for A^p implies that if $f \in A^p$ with $p > 2$ then the integral

$$\int_0^1 f(t) \frac{1}{1-tz}\, dt$$

is absolutely convergent for $z \in \mathbb{D}$ and defines an analytic functions on $\mathbb{D}$. A further calculation gives

$$\mathcal{H}(f)(z) = \int_0^1 f(t) \frac{1}{1-tz}\, dt,$$

the integral representation for Bergman functions.

We can change the variable again to obtain

$$\mathcal{H} = \int_0^1 T_s\, ds, \quad T_s(f) = w_s f \circ \varphi_s,$$

where w_s and ϕ_s are the same as for Hardy spaces.

We want to estimate $\|T_s\|_{A^p}$. For this we need the following identities

1. $w_s(z)^2 = \frac{1}{s(1-s)}\phi_t'(z)$,
2. $w_s(\phi_s^{-1}(z)) = \frac{z}{s}$,

which are valid for all $0 \leq s < 1, z \in \mathbb{D}$. Thus

$$\begin{aligned}
\|T_s(f)\|_{A^p}^p &= \int_{\mathbb{D}} |w_s(z)|^p |f(\phi_s(z))|^p \, dA(z) \\
&= \int_{\mathbb{D}} |w_s(z)|^{p-4} |w_s(z)|^4 |f(\phi_s(z))|^p \, dA(z) \\
&= \frac{1}{s^2(1-s)^2} \int_{\mathbb{D}} |w_s(z)|^{p-4} |f(\phi_s(z))|^p |\phi_s'(z)|^2 \, dA(z) \\
&= \frac{1}{s^2(1-s)^2} \int_{\phi_s(\mathbb{D})} |w_s(\phi_s^{-1}(u))|^{p-4} |f(u)|^p \, dA(u) \\
&= \frac{1}{s^{p-2}(1-s)^2} \int_{\phi_s(\mathbb{D})} |u|^{p-4} |f(u)|^p \, dA(u) \\
&\quad \text{(assumimg } p \geq 4\text{)} \\
&\leq \frac{1}{s^{p-2}(1-s)^2} \int_{\mathbb{D}} |f(u)|^p \, dA(u) \\
&= \frac{1}{s^{p-2}(1-s)^2} \|f\|_{A^p}^p.
\end{aligned}$$

For $2 < p < 4$ a little more work gives a similar result with an additional constant C in the last step. This gives the following

Lemma For $0 < s < 1$ we have:
(i) If $4 \le p < \infty$ then $\|T_s\|_{A^p} \le s^{\frac{2}{p}-1}(1-s)^{-\frac{2}{p}}$.
(ii) If $2 < p < 4$ then $\|T_s\|_{A^p} \le C s^{\frac{2}{p}-1}(1-s)^{-\frac{2}{p}}$
for some constant $C = C_p$.

The usual calculation then gives

$$\begin{aligned}\|\mathcal{H}\|_{A^p} &\le \int_0^1 t^{2/p-1}(1-t)^{-2/p}\,dt \\ &= \Gamma(\frac{2}{p})\Gamma(1-\frac{2}{p}) \\ &= \frac{\pi}{\sin(\frac{2\pi}{p})},\end{aligned}$$

so we have the following.

Theorem [Diamantopoulos, Illinois J. Math. 2004]

1. $\|\mathcal{H}\|_{A^p} \le \frac{\pi}{\sin(\frac{2\pi}{p})}$ for $4 \le p < \infty$.

2. $\|\mathcal{H}\|_{A^p} \le C_p \frac{\pi}{\sin(\frac{2\pi}{p})}$ for $2 < p < 4$.

Remark. $C_p \to \infty$ as $p \to 2$. This was improved to a uniform C valid for all $2 < p < 4$ by M. Dostanić, M. Jevtić and D. Vukotić.

Lower bound for $\|\mathcal{H}\|_{A^p}$.

Theorem [D-J-V]. Let $2 < p < \infty$, then

$$\|\mathcal{H}\|_{A^p \to A^p} \geq \frac{\pi}{\sin(\frac{2\pi}{p})}.$$

Sketch of proof:

Consider the test functions

$$f_\gamma(z) = \frac{1}{(1-z)^{\gamma/p}}, \quad \gamma < 2 < p,$$

then $f_\gamma \in A^p$. A calculation shows that for the weighted composition operators T_s we have

$$T_s(f_\gamma)(z) = \frac{((s-1)z+1)^{\gamma/p-1}}{(1-s)^{\gamma/p}} f_\gamma(z),$$

thus

$$\mathcal{H}(f_\gamma)(z) = \left(\int_0^1 \frac{((s-1)z+1)^{\gamma/p-1}}{(1-s)^{\gamma/p}}\, ds\right) f_\gamma(z)$$

(change of variable $u = 1 - s$)

$$= \left(\int_0^1 \frac{(1-uz)^{\gamma/p-1}}{u^{\gamma/p}}\, du\right) f_\gamma(z)$$

$$= \phi_\gamma(z) f_\gamma(z)$$

We check that the function

$$\phi_\gamma(z) = \int_0^1 \frac{(1-uz)^{\gamma/p-1}}{u^{\gamma/p}}\,du$$

is in fact defined and continuous on $|z| \le 1$ for all $\gamma \le 2$. Since $\gamma/p - 1 < 0$ and

$$|1-uz| \ge 1-u|z| \ge 1-u, \qquad z \in \overline{\mathbb{D}},$$

we have

$$\begin{aligned} |\phi_\gamma(z)| \le |\phi_\gamma(1)| \\ &= \int_0^1 \frac{(1-u)^{\gamma/p-1}}{u^{\gamma/p}}\,du \\ &= \frac{\pi}{\sin(\frac{\gamma\pi}{p})}. \end{aligned}$$

Now let $g_\gamma(z) = \frac{f_\gamma(z)}{\|f_\gamma\|_{A^p}}$ and consider the family

$$\{|g_\gamma(z)|^p : 0 \le \gamma \le 2, z \in \mathbb{D}\}.$$

We can check that it satisfies all properties of an approximate identity:

1. $|g_\gamma(z)|^p \ge 0$
2. $\int_{\mathbb{D}} |g_\gamma(z)|^p \, dA(z) = 1$ for each γ
3. $|g_\gamma(z)|^p$ tends uniformly to 0, as $\gamma \to 2$, on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$.

Further the function $\phi_\gamma(z) : [0,2] \times \overline{\mathbb{D}} \to \mathbb{C}$ is uniformly continuous and uniformly bounded in any set of the form

$$\overline{\{z \in \mathbb{D} : |z-1| < \varepsilon\}} \times [0,2].$$

In addition

$$\begin{aligned}\int_{\mathbb{D}} |\mathcal{H}(g_\gamma)(z)|^p dA(z) - |\phi_2(1)|^p \\ &= \int_{\mathbb{D}} (|\phi_\gamma(z)|^p - |\phi_2(1)|^p) dA(z) \\ &\to 0, \quad \text{as } \gamma \to 2\end{aligned}$$

and it follows that

$$\lim_{\gamma \to 2} \|\phi_\gamma g_\gamma\|_{A^p} = \|\phi_2\|_\infty = \phi_2(1) = \frac{\pi}{\sin(\frac{2\pi}{p})}.$$

Since $\|\mathcal{H}\|_{A^p \to A^p} \geq \|\mathcal{H}(g_\gamma)\|_{A^p} = \|\phi_\gamma g_\gamma\|_{A^p}$ for each $\gamma < 2$ we conclude

$$\|\mathcal{H}\|_{A^p \to A^p} \geq \frac{\pi}{\sin(\frac{2\pi}{p})}.$$

As a consequence,

$$\|\mathcal{H}\|_{A^p \to A^p} = \frac{\pi}{\sin(\frac{2\pi}{p})}, \quad 4 \leq p < \infty.$$

The value $\|\mathcal{H}\|_{A^p \to A^p}$ is not known for $2 < p < 4$.

6. Some other representations of $\mathcal{H}$.

a. As an area integral. ([D-J-V]) If $2 < p < \infty$ and $f \in A^p$ then

$$\mathcal{H}(f)(z) = \int_{\mathbb{D}} \frac{f(\overline{w})}{(1-w)(1-\overline{w}z)} dA(w).$$

Indeed

$$\int_{\mathbb{D}} w^m \overline{w}^n dA(w) = \begin{cases} \frac{1}{n+1}, & \text{when } m = n, \\ 0 & \text{when } m \neq n \end{cases}$$

so

$$\int_{\mathbb{D}} \frac{1}{1-w} \overline{w}^n dA(w) = \sum_{m=0}^{\infty} \int_{\mathbb{D}} w^m \overline{w}^n dA(w) = \frac{1}{n+1}.$$

If $f(z) = \sum_{n=0}^{N} a_n z^n$ is a polynomial then

$$\begin{aligned} \sum_{k=0}^{N} \frac{a_k}{n+k+1} &= \sum_{k=0}^{N} a_k \int_{\mathbb{D}} \frac{1}{1-w} \overline{w}^{n+k} dA(w) \\ &= \int_{\mathbb{D}} \frac{1}{1-w} \left(\sum_{k=0}^{N} a_k \overline{w}^k \right) \overline{w}^n dA(w) \\ &= \int_{\mathbb{D}} \frac{f(\overline{w})}{1-w} \overline{w}^n dA(w), \end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(f)(z) &= \sum_{n=1}^{\infty}\left(\int_{\mathbb{D}}\frac{f(\overline{w})}{1-w}\overline{w}^n dA(w)\right)z^n\\ &= \int_{\mathbb{D}}\frac{f(\overline{w})}{1-w}\left(\sum_{n=0}^{\infty}(\overline{w}z)^n\right)dA(w)\\ &= \int_{\mathbb{D}}\frac{f(\overline{w})}{(1-w)(1-\overline{w}z)}dA(w).\end{aligned}$$

The validity of this for all functions in A^p is verified by appealing to the uniform convergence of the Taylor series of f.

b. As average of composition operators.

We can change the path in the integral

$$\mathcal{H}(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}\,dt$$

to $t = t(s) = \phi_s(z)$ where $\{\phi_s(z) : 0 \leq s \leq 1\}$ is a family of functions with $\phi_s : \mathbb{D} \to \mathbb{D}$ analytic for each $0 \leq s < 1$, $\phi_0(z) = 0$ and $\phi_1(z) = 1$ for each $z \in \mathbb{D}$, and $\frac{\partial \phi_s(z)}{\partial s}$ exists and is a bounded analytic function of z for each s, to obtain

$$\mathcal{H}(f)(z) = \int_0^1 \frac{\frac{\partial \phi_s(z)}{\partial s}}{1-\phi_s(z)z}f(\phi_s(z))\,dt.$$

Families of this form can be created as follows:

Let $h : \mathbb{D} \to \mathbb{C}$ be a starlike univalent function (i.e. $f(0) = 0$ and if $w \in h(\mathbb{D})$ then $[0, w] \subset h(\mathbb{D})$), and put

$$\psi_s(z) = h^{-1}(sh(z)), \quad 0 \le s \le 1.$$

Then $\psi_s : \mathbb{D} \to \mathbb{D}$ are analytic, $\psi_s(0) = 0$ and we set

$$\phi_s(z) = \frac{\psi_s(z)}{z},$$

which are also analytic self-maps of $\mathbb{D}$ by Schwarz's Lemma. The path

$$t(s) = \phi_s(z), \quad 0 \le s \le 1,$$

joins 0 to 1, and we evaluate the integral along this path to find

$$\mathcal{H}(f)(z) = \frac{h(z)}{z} \int_0^1 \frac{1}{(1 - z\phi_s(z))h'(z\phi_s(z))} f(\phi_s(z))\, ds$$

or

$$\mathcal{H}(f)(z) = \frac{h(z)}{z} \int_0^1 w(z\phi_s(z)) f(\phi_s(z))\, ds$$

where $w(z) = \frac{1}{(1-z)h'(z)}$.

The previous representation $\mathcal{H} = \int_0^1 T_s\, ds$ can be recovered in this way with the starlike function $h(z) = \frac{z}{1-z}$.

As a further example, let

$$h(z) = \log \frac{1}{1-z},$$

then we find $\psi_s(z) = 1 - (1-z)^s$, and $w(z) \equiv 1$. Thus

$$\mathcal{H}(f)(z) = \frac{1}{z} \log \frac{1}{1-z} \int_0^1 f\left(\frac{1-(1-z)^s}{z}\right) ds.$$

Second part

7. The point spectrum of $\mathcal{H}$ on H^p.

Let F_n be the nth finite section of the Hilbert matrix

$$F_n = \left(\frac{1}{i+j+1}\right)_{i,j=0}^{n-1},$$

then

$$\det(F_n) = \frac{((n-1)!!)^4}{(2n-1)!!},$$

where $n!! = \prod_{k=1}^{n} k!$. For example

$$\det(F_3) = \frac{1}{2160},$$
$$\det(F_6) = \frac{1}{186313420339200000},$$
$$\det(F_9) \sim \frac{1}{10^{42}}.$$

In particular the inverse of F_n has very large integer entries and the computation of its eigenvalues is a very sensitive problem.

In Numerical Analysis, F_n are typical examples of "ill-conditioned" matrices, difficult to use in numerical computation.

Coming back to the infinite Hilbert matrix H,

Various papers, $\sim$1950-1960, discuss H and more general versions of it like

$$H_\lambda = \left(\frac{1}{i+j+\lambda}\right), \quad \lambda \in \mathbb{C}.$$

These works concentrate mostly on the spectrum on l^2 and in particular on "latent roots".

A latent root is a phony eigenvalue in the sense that the corresponding eigenvector is not necessarily in the space under consideration. For H_λ, this will be a complex number c for which there is a nonzero sequence (x_n), not necessarily in l^2, such that

$$\sum_{k=0}^{\infty} \frac{x_k}{n+k+\lambda} = c x_n, \quad n = 0, 1, \cdots$$

In these works one can find results connecting the eigenvectors to hypergeometric functions.

Sample results of this kind are:

Theorem [Hill, J. London Math. Soc. (1960)] If $x_n = x_n(\lambda, \mu)$ is defined by

$$x_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\Gamma(k+\mu)\Gamma(k+1-\mu)}{k!\Gamma(k+\lambda)}$$

where $0 < \text{Re}(\mu) \leq \frac{1}{2}$. Then

$$\sum_{k=0}^{\infty} \frac{x_k}{n+k+\lambda} = \frac{\pi}{\sin(\pi\mu)} x_n, \quad n = 0, 1, 2, \cdots$$

i.e $\frac{\pi}{\sin(\pi\mu)}$ is a latent root of H_λ.

Theorem. [W.Magnus, Amer, J. Math., (1950] The spectrum of $H = H_1$ on l^2 (and thus on H^2) is the interval $[0, \pi]$, and there are no eigenvalues.

A more complete study of the point spectrum on Hardy spaces (and on some other spaces defined by the growth of f) was done recently by A. Aleman, A. Montes and A. Sarafoleanu [Constr. Approx., to appear]. They consider matrices

$$H_\lambda, \quad \lambda \in \mathbb{C} \setminus \mathbb{Z},$$

and prove the representation

$$H_\lambda(f)(z) = \int_0^1 f(t)\frac{t^{\lambda-1}}{1-tz}\,dt, \qquad \text{if } \mathrm{Re}(\lambda) > 0,$$
$$= \frac{1}{\kappa}\int_\gamma f(t)\frac{t^{\lambda-1}}{1-tz}\,dt, \quad \text{if } \mathrm{Re}(\lambda) \le 0,$$

where $\kappa = e^{2\pi i\lambda} - 1$ and γ is appropriate closed curve such as the boundary of a Stolz angle

$$\{z \in \mathbb{D} : |1-z| \le \sigma(1-|z|)\}, \quad \sigma > 1.$$

The key observation in this work is that H_λ "almost commutes" with two specific second order linear differential operators with polynomial coefficients. More precisely if

$$D_{1,\lambda}(f)(z) = (z^2-1)f'(z) + \lambda z f(z)$$

and

$$D_{2,\lambda}(f)(z) =$$
$$= z(z-1)^2 f''(z) + (z-1)[(\lambda+2)z-\lambda]f'(z) + \lambda z f(z)$$

then

$$(H_\lambda D_{1,\lambda} - D_{1,\lambda}H_\lambda)(f) = \frac{1}{\kappa}(\lambda-1)\int_\gamma f(t)t^{\lambda-2}\,dt,$$

and

$$H_\lambda D_{2,\lambda} = D_{2,\lambda}H_\lambda.$$

Next, the eigenvalue equation

$$D_{1,\lambda}(f) = \nu f,$$

has analytic solutions which are constant multiples of

$$f_\lambda(z) = (1 - z^2)^{-\lambda/2} \left(\frac{1+z}{1-z}\right)^{-\nu/2}.$$

Similarly the eigenvalue equation

$$D_{2,\lambda}(g) = \nu g,$$

has solutions which are constant multiples of

$$g_a(z) = (1-z)^a \, {}_2F_1(a+1, a+\lambda; \lambda; z),$$

where a is a root (any of the two) of the quadratic

$$x^2 + x + \lambda - \nu = 0,$$

and

$${}_2F_1(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{1!\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} z^2 + \cdots$$

is the classical hypergeometric function.

Observe that these functions g_a make sense for each $a \in \mathbb{C}$ and that for each $a \in \mathbb{C}$ we can find

ν such that a is root of the quadratic. In view of this, consider all a in the strip

$$\{z : -1 < \text{Re}(z) \leq -\frac{1}{2}\}.$$

If a' is the other root of the quadratic then since $a+a' = -1$ we will have $-\frac{1}{2} \leq \text{Re}(a')$. There are three cases to consider

1. If $a = -\frac{1}{2} = a'$ then $g_a \in H^p$ for all $p < 2$.

2. If $a \neq -\frac{1}{2}$ and λ is such that $-(a'+\lambda) \notin \mathbb{N} \cup \{0\}$ then

$$g_a(z) \sim \frac{d}{(1-z)^a}, \;\; d \neq 0,$$

as $z \to 1$ along the radius. In this case $g_a \in H^p$ if and only if $\text{Re}(a) < \frac{1}{p}$.

3. If $a \neq -\frac{1}{2}$ and λ is such that $-(a' + \lambda) = n \in \mathbb{N} \cup \{0\}$ then

$$\text{Re}(\lambda) \leq \frac{1}{2}, \quad \text{and } g_a(z) = (1-z)^{a'} Q(z),$$

with Q a polynomial of degree n. In this case $g_a \in H^p$ if and only if $1+\text{Re}(a) < \frac{1}{p}$.

Further for all values of a in the strip, $H_\lambda(g_a)$ is defined and satisfies

$$H_\lambda(g_a) = -\frac{\pi}{\sin(\pi a)} g_a.$$

This last assertion is seen as follows:

By the commutation of H_λ with $D_{2,\lambda}$ we have

$$D_{2,\lambda} H_\lambda(g_a) = H_\lambda D_{2,\lambda}(g_a)$$

or

$$D_{2,\lambda}(H_\lambda(g_a)) = H_\lambda(\nu g_a) = \nu H_\lambda(g_a).$$

This says that $H_\lambda(g_a)$ is an eigenfunction of $D_{2,\lambda}$, so there is a ξ such that

$$H_\lambda(g_a) = \xi g_a,$$

and the value of ξ is found by observing that $g_a(0) = 1$, thus

$$\begin{aligned} \xi &= H_\lambda(g_a)(0) \\ &= \frac{1}{\kappa} \int_\gamma g_a(t) t^{\lambda - 1}\, dt \\ &= \cdots \\ &= -\frac{\pi}{\sin(\pi a)}. \end{aligned}$$

Thus using $D_{2,\lambda}$, and for spaces H^p with $1 < p < 2$, we have detected a large set of eigenvalues of the form

$$\left\{-\frac{\pi}{\sin(\pi a)} : -\frac{1}{p} < \mathrm{Re}(a) \leq -\frac{1}{2}\right\}$$

with corresponding eigenfunctions

$$g_a(z) = (1-z)^a \, {}_2F_1(a+1, a+\lambda; \lambda; z)$$

(except when possibly $a = -\lambda, -\lambda - 1$). These kind of eigenvalues however disappear for H^p with $p \geq 2$.

A similar analysis for $D_{1,\lambda}$ detects eigenfunctions of the form

$$f_n(z) = \frac{1}{(1-z)^\lambda} \left(\frac{1+z}{1-z}\right)^n, \ 0 \leq n \leq N$$

where N is the largest integer for which

$$(1-z)^{-\lambda-N} \in H^p.$$

The corresponding eigenvalues are $(-1)^n \frac{\pi}{\sin(\pi\lambda)}$. In all cases the eigenspaces are finite dimensional.

The results as written in the article do not cover the case $\lambda = 1$ (but A. Aleman says that he can do that case too, by a variation of the arguments involved).

8. Generalized Hilbert operators

a. Changing the Lebesgue measure.

We have seen that $H = (h_{i,j})$ where

$$h_{i,j} = \frac{1}{i+j+1} = \lambda_{i+j}$$

with (λ_k) is the moment sequence

$$\lambda_k = \int_0^1 x^k \, dx, \quad k = 0, 1, \cdots$$

of the Lebesgue measure on $[0,1]$.

For a general finite positive Borel measure μ on $[0,1)$ consider the moments

$$\mu_k = \int_{[0,1)} x^k \, d\mu(x), \quad k = 0, 1, \cdots$$

and the resulting generalized Hilbert matrix

$$H_\mu = \left(h_{i,j}\right)_{i,j=0}^{\infty}$$

where $h_{i,j} = \mu_{i+j}$, that is

$$H_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \cdot \\ \mu_1 & \mu_2 & \mu_3 & \cdot \\ \mu_2 & \mu_3 & \mu_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

For $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}(\mathbb{D})$ we have, by analogy, the formal transformation

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n.$$

If μ satisfies

$$(*) \quad \mu((t,1)) = \mathrm{O}((1-t)), \quad \text{all } t \text{ near } 1,$$

then

$$\mu_n = \mathrm{O}(1/(n+1)).$$

Using Hardy's inequality then it follows that the power series $\mathcal{H}_\mu(f)$ is analytic on $\mathbb{D}$ for every $f \in H^1$ and

$$\mathcal{H}_\mu(f)(z) = \int_{[0,1)} f(t) \frac{1}{1-tz} \, d\mu(t).$$

Condition $(*)$ is known to be equivalent to that

$$\mathcal{H}_\mu : H^2 \to H^2$$

is bounded.

Recall that the Hilbert matrix $\mathcal{H}$ is not bounded on H^1. For general μ we have

Theorem[P. Galanopoulos and J.A. Pelaez, Studia Math. 2010] Suppose μ satisfies $(*)$, then

1. $\mathcal{H}_\mu : H^1 \to H^1$ is bounded if and only if
$$\mu((t,1)) \log \frac{1}{1-t} = \mathrm{O}((1-t)), \quad \text{all } t \text{ near } 1,$$
2. $\mathcal{H}_\mu : H^1 \to H^1$ is compact if and only if
$$\mu((t,1)) \log \frac{1}{1-t} = \mathrm{o}((1-t)).$$

Further, [G-P] proved that $\mathcal{H}_\mu$ is Hilbert-Schmidt on H^2 if and only if

$$(HS) \quad \int_{[0,1)} \frac{\mu([t,1])}{(1-t)^2}\, d\mu(t) < \infty$$

(Hilbert-Schmidt means: $\sum_{k=0}^{\infty} \|\mathcal{H}_\mu(e_k)\|_{H^2}^2 < \infty$ for any orthonormal basis (e_k))

On the Bergman space A^2,

Theorem[G-P] If μ satisfies (HS) then for each $f \in A^2$ the power series $\mathcal{H}_\mu(f)$ represents an analytic function on $\mathbb{D}$ and

$$\mathcal{H}_\mu(f)(z) = \int_{[0,1)} f(t) \frac{1}{1-tz}\, d\mu(t).$$

But for the boundedness of $\mathcal{H}_\mu$ on A^2 something more is required. In fact [G-P] proved that if (HS) holds then $\mathcal{H}_\mu : A^2 \to A^2$ is bounded if and only if

$$\int_{\mathbb{D}} |f(z)|^2 |\mu'(z)|^2 dA(z) \leq C \int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

for all f for which the right hand-side integral is finite, where $\mu(z)$ is the function

$$\mu(z) = \sum_{n=0}^{\infty} \mu_n z^n.$$

To appreciate this, compare with the following relevant result

Theorem [G-P] For each $\beta \in [0,1)$ there is a μ such that

$$\int_{[0,1)} \frac{\mu([t,1))}{(1-t)^2} \left(\log \frac{1}{1-t} \right)^{\beta} d\mu(t) < \infty,$$

but $\mathcal{H}_\mu$ is not bounded on A^2.

b. Changing the kernel function.

Observe that

$$\mathcal{H}(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}\,dt$$
$$= \int_0^1 f(t)g'(tz)\,dt$$

where $g(z) = \log(\frac{1}{1-z})$. We may consider the more general transformation

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz)\,dt,$$

where g is any analytic function on $\mathbb{D}$.

By Fejer-Riesz's inequality the integral converges for each $f \in H^1$.

The question arises to describe the symbols g for which

$$\mathcal{H}_g : H^p \to H^p$$

is a bounded operator (or compact or ...)

Preliminary remarks

• If g is a polynomial of degree n then $\mathcal{H}_g(f)$ is also a polynomial of degree $n-1$. Thus $\mathcal{H}_g$ is a finite rank operator when g is a polynomial.

• Since

$$\mathcal{H}_{\lambda g_1+\mu g_2} = \lambda \mathcal{H}_{g_1} + \mu \mathcal{H}_{g_2}, \quad \lambda, \mu \in \mathbb{C},$$

the set

$$V = V_p =: \{g : \mathcal{H}_g : H^p \to H^p \text{ is bounded }\}$$

is a linear space of analytic functions and contains the polynomials.

• We can define a norm on V,

$$\|g\|_V =: \|\mathcal{H}_g\|_{H^p \to H^p}, \quad g \in V,$$

to make $(V, \| \ \|_V)$ a normed space.

• If we set

$$V_0 = V_{p,0} =: \text{closure of polynomials in V}$$

then $\mathcal{H}_g : H^p \to H^p$ is a compact operator, for each $g \in V_0$.

• If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $g(z)=\sum_{n=0}^{\infty} b_n z^n$ then

$$\begin{aligned}\mathcal{H}_g(f)(z) &= \sum_{n=0}^{\infty}\left((n+1)b_{n+1}\int_0^1 t^n f(t)dt\right)z^n\\ &= \sum_{n=0}^{\infty}\left((n+1)b_{n+1}\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)z^n.\end{aligned}$$

In particular if the series of g has gaps, then the series of $\mathcal{H}_g(f)$ has the same gaps.

• We can view $\mathcal{H}_g$ as a product of the classical Hilbert operator $\mathcal{H}$ and a coefficient multiplication operator. For an analytic $h(z)\sim\sum\lambda_n z^n$ let

$$\Lambda_h(f)(z)=\sum_{n=0}^{\infty}\lambda_n a_n z^n, \quad f\sim\sum a_n z^n.$$

the coefficient multiplier operator. Then we have

$$\mathcal{H}_g=\Lambda_{g'}\circ\mathcal{H}.$$

In particular if g is a function such that

$$\Lambda_{g'}: H^p\to H^p$$

is a bounded multiplier, then $\mathcal{H}_g : H^p\to H^p$ is bounded. In other words the space V of those g

for which $\mathcal{H}_g$ is bounded contains all antiderivatives of functions that act as coefficient multipliers of H^p.

It turns out that we can almost completely describe all g which give bounded $\mathcal{H}_g$ in terms of mean Lipschitz spaces.

Mean Lipschitz spaces.

Let $1 \leq p < \infty$, $0 < \alpha \leq 1$. The integral modulus of continuity of order p for an integrable function f on $\mathbb{T}$ is defined to be

$$\omega_p(t) = \sup_{0<h\leq t} \left(\int_0^{2\pi} |g(e^{i(\theta+h)}) - g(e^{i\theta})|^p d\theta \right)^{1/p}$$

$\Lambda(p, \alpha)$ contains all f for which

$$\omega_p(t) = O(t^\alpha), \quad \text{as } t \to 0.$$

If either p or α is kept fixed and the other is let to increase then $\Lambda(p, \alpha)$ decreases in size.

If $\alpha > \frac{1}{p}$ then $\Lambda(p,\alpha)$ consists entirely of continuous functions. The borderline space $\Lambda\left(p,\frac{1}{p}\right)$ contains unbounded functions, in fact

$$\log \frac{1}{1-z} \in \Lambda\left(p, \frac{1}{p}\right), \quad \text{all } p > 1,$$

and these spaces increase with p but they stay always inside $BMOA$:

$$\Lambda\left(q, \frac{1}{q}\right) \subset \Lambda\left(p, \frac{1}{p}\right) \subset BMOA, \quad 1 \leq q < p < \infty,$$

These will be the spaces of interest to us.

Equivalent characterizations:

By taking Poisson integrals of functions on the boundary we may assume that we are working with analytic functions on $\mathbb{D}$ when dealing with the above spaces.

Suppose $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on $\mathbb{D}$. We write as usual

$$M_p(r,g) = \left(\int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta\right)^{1/p}$$

and

$$\Delta_n g(z) = \sum_{2^n \leq k \leq 2^{n+1}-1} a_k z^k.$$

the n-th dyadic section of g. Then

Lemma. The following are equivalent:

1. $g \in \Lambda(p, \alpha)$,
2. $M_p(r, g') = \mathrm{O}\left((1-r)^{\alpha-1}\right)$ as $r \to 1$,
3. $\|\Delta_n g\|_{H^p} = \mathrm{O}\left(2^{-n\alpha}\right)$ as $n \to \infty$,
4. $\|\Delta_n g'\|_{H^p} = \mathrm{O}\left(2^{n(1-\alpha)}\right)$ as $n \to \infty$.

The following theorems describes the boundedness of $\mathcal{H}_g$ on H^p and on A^p.

Theorem 1. [P. Galanopoulos D. Girela, J.A. Pelaez, and A. S. (2011), preprint]

Suppose g is analytic on $\mathbb{D}$.

1. If $1 < p \leq 2$ then $\mathcal{H}_g : H^p \to H^p$ is bounded if and only if $g \in \Lambda\left(p, \frac{1}{p}\right)$.

2. If $2 < p < \infty$ and $\mathcal{H}_g : H^p \to H^p$ is bounded then $g \in \Lambda\left(p, \frac{1}{p}\right)$.

3. If $2 < p < \infty$ and $g \in \Lambda\left(q, \frac{1}{q}\right)$ for some $q < p$ then $\mathcal{H}_g : H^p \to H^p$ is is bounded.

Theorem 2.[G-G-P-S]

Suppose $2 < p < \infty$. Then $\mathcal{H}_g : A^p \to A^p$ is bounded if and only if $g \in \Lambda\left(p, \frac{1}{p}\right)$.

Sketch of proof of:

$\mathcal{H}_g : H^p \to H^p$ bounded $\Rightarrow g \in \Lambda\left(p, \frac{1}{p}\right)$,

Write $(f*g)(z) = \sum_{n=0}^{\infty} \widehat{f}(n)\widehat{g}(n)z^n$, the Hadamard product of two power series f and g.

We are going to use the following lemma from the general theory of "smooth Cesaro means".

Lemma Let Φ be a C^∞ function with compact support in $(0, \infty)$, write

$$A_\Phi = \|\Phi\|_\infty + \|\Phi''\|_\infty$$

and let

$$W_m^\Phi(z) = \sum_{k\in\mathbb{N}} \Phi(k/m) z^k,$$

Then for each $p \in (0, \infty)$ there is a C_p such that

$$\|W_m^\Phi * f\|_{H^p} \le C_p A_\Phi \|f\|_{H^p}, \;\; f \in H^p$$

Next we construct the C^∞ function for our purpose. Fix $1 < p < \infty$. Let

$$a_N = 1 - \frac{1}{N}, \quad N = 2, 3, \cdots$$

and let ϕ_N be defined by

$$\phi_N(s) = \int_0^1 \frac{t^{sN}}{(1-a_N t)^2} dt, \ \ (\phi_N \in C^\infty).$$

Find a C^∞ function $\Phi_N : \mathbb{R} \to \mathbb{C}$ such that:
(i) $\mathrm{supp}(\Phi_N) \subset [\frac{1}{2}, 4]$,
(ii) In the smaller interval $[1, 2]$,

$$\Phi_N(s) = \frac{N^{2-\frac{1}{p}}}{\phi_N(s)}, \quad 1 \leq s \leq 2.$$

We can estimate that

$$A_{\Phi_N} = \|\Phi_N\|_\infty + \|\Phi_N''\|_\infty \leq C N^{1-\frac{1}{p}}.$$

Put

$$W_N(z) = \sum_{k \in \mathbb{N}} \Phi_N(k/N) z^k,$$

a polynomial. Let also

$$f_N(z) = \frac{N^{\frac{1}{p}-2}}{(1-a_N z)^2}, \quad N = 2, 3, \cdots$$

Then $f_N \in H^p$ and

$$\sup_N \|f_N\|_{H^p} = C < \infty,$$

therefore by hypothesis,

$$\sup_N \|\mathcal{H}_g(f_N)\|_{H^p} = C' < \infty.$$

Applying the Lemma to $\mathcal{H}_g(f_N) \in H^p$ we find

$$\begin{aligned}\|W_N * \mathcal{H}_g(f_N)\|_{H^p} &\le C A_{\Phi_N} \|\mathcal{H}_g(f_N)\|_{H^p} \\ &\le C' N^{1-\frac{1}{p}}.\end{aligned}$$

On the other hand, if $g(z) = \sum_{n=0}^{\infty} b_k z^k$ we find

$$\begin{aligned} W_N * \mathcal{H}_g(f_N)(z) & \\ &= \sum_{\frac{N}{2} \le k \le 4N} \left(\Phi_N(\frac{k}{N})(k+1) b_{k+1} \int_0^1 t^k f_N(t) dt \right) z^k \\ &= \sum_{\frac{N}{2} \le k \le N-1} \cdots + \sum_{N \le k \le 2N-1} \cdots + \sum_{2N \le k \le 4N} \cdots \\ &= F_1^N(z) + F_2^N(z) + F_3^N(z), \end{aligned}$$

and we compute, for $N \le k \le 2N$,

$$
\begin{aligned}
\Phi_N(k/N) &= \frac{N^{2-\frac{1}{p}}}{\phi_N(k/N)} \\
&= \frac{N^{2-\frac{1}{p}}}{\int_0^1 \frac{t^{\frac{k}{N}N}}{(1-a_N t)^2}\,dt} \\
&= \frac{1}{\int_0^1 t^k \frac{N^{\frac{1}{p}-2}}{(1-a_N t)^2}\,dt} \\
&= \frac{1}{\int_0^1 t^k f_N(t)\,dt},
\end{aligned}
$$

therefore,

$$
F_2^N(z) = \sum_{N\le k\le 2N-1} (k+1) b_{k+1} z^k .
$$

Thus

$$
\Delta_n g'(z) = \sum_{k=2^n}^{2^{n+1}-1} (k+1) b_{k+1} z^k = F_2^{2^n}(z)
$$

Now, from M. Riesz's projection theorem

$$
\|F_2^N\|_{H^p} \le C \|W_n \star \mathcal{H}_g(f_N)\|_{H^p} \le C' N^{1-\frac{1}{p}}
$$

Therefore,

$$\|\Delta_n g'\|_{H^p} = \|F_2^{2^n}\|_{H^p} \leq C2^{n(1-\frac{1}{p})}$$

and it follows that $g \in \Lambda\left(p, \frac{1}{p}\right)$, Q.E.D.

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