Spectral properties of the Cesàro operator acting on various spaces of analytic functions

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Outline

The birth of the Cesàro operator

The Cesàro operator $\mathcal{C}$ was first considered on sequence spaces as the mapping that assigns to a sequence $\boldsymbol{x} = x_0, x_1, \ldots$ the sequence of *Cesàro means*:

$$(\mathcal{C}\boldsymbol{x})_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k.$$

Hardy (1920): $\mathcal{C}$ is bounded on l^2. Let $x \in l^2$ and consider the sequences:

$$s_n = \sum_{k=0}^{n} x_k \quad \text{and} \quad r_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2}.$$

Note that $s_n^2 - s_{n-1}^2 = 2x_1x_n + \ldots 2x_{n-1}x_n + x_n^2 \leq 2x_ns_n$ and that

$$r_n \leq \frac{1}{(n+1)^2} + \int_{n+1}^{\infty} \frac{dx}{x^2} = \frac{1}{(n+1)^2} + \frac{1}{n+1} \leq \frac{2}{n+1}.$$

This yields

$$\sum_{n=0}^{N} \left(\frac{s_n}{n+1}\right)^2 = \sum_{n=0}^{N} s_n^2(r_n - r_{n+1}) = \sum_{n=0}^{N}(s_n^2 - s_{n-1}^2)r_n - s_N^2 r_{N+1} \leq$$

$$\leq 4\sum_{n=0}^{N} x_n \frac{s_n}{n+1} \leq 4\left(\sum_{n=0}^{N} x_n^2\right)^{1/2}\left(\sum_{n=0}^{N}\left(\frac{s_n}{n+1}\right)^2\right)^{1/2}$$

for all natural numbers N. Hence, $\|\mathcal{C}\boldsymbol{x}\| \leq 4\|\boldsymbol{x}\|$.

Remark: The constant 4 is not sharp. It has been proven later that $\mathcal{C}$ is bounded on l^p, $p > 1$ with norm $\frac{p}{p-1}$.

Hardy's inequality:

$$\sum_{n=0}^{\infty}\left(\frac{s_n}{n+1}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=0}^{\infty} x_n^p.$$

(see Hardy, Littlewood, Polya: *Inequalities* (1934))

Spectrum of $\mathcal{C}$ on l^2

(1965) *Brown, Halmos, Shields:* determined the spectrum of $\mathcal{C}$ on l^2 and proved that $\mathcal{C}$ is hyponormal on l^2.

Theorem (Brown, Halmos, Shields)

i). $\|I - \mathcal{C}\| = 1$ *and* $\|\mathcal{C}\| = 2$. *(I is the identity operator).*

ii). *For* $f \in l^2$ *with* $\|f\| = 1$, $\|(I - \mathcal{C})f\| < 1$ *and* $\|(I - \mathcal{C}^*)f\| < 1$.

iii). *The point spectrum* $\sigma_p(\mathcal{C})$ *of* $\mathcal{C}$ *is empty, the point spectrum of* $\mathcal{C}^*$ *is given by* $\sigma_p(\mathcal{C}^*) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$

iv). *The spectrum of* $\mathcal{C}$ *is given by* $\sigma(\mathcal{C}) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.

v). $\mathcal{C}$ *is hyponormal (*$\mathcal{C}^*\mathcal{C} - \mathcal{C}\mathcal{C}^*$ *is positive.)*

Outline of the proof:
The authors consider the matrices k and k^* associated to $\mathcal{C}$ and $\mathcal{C}^*$:

$$k = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ \vdots & & & \ddots \end{pmatrix}, \quad k^* = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \\ 0 & 0 & \frac{1}{3} & \\ \vdots & & & \ddots \end{pmatrix} \text{ and } kk^* = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ \vdots & & & \ddots \end{pmatrix}.$$

and use the fact that $\mathcal{C} + \mathcal{C}^* - \mathcal{C}\mathcal{C}^*$ is a diagonal operator D with matrix

$$d = k + k^* - kk^* = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & \frac{1}{3} & \\ \vdots & & & \ddots \end{pmatrix}.$$

to prove the first part of (i) and the inequality $\|\mathcal{C}\| \leq 2$.
$(I - D = I - \mathcal{C} - \mathcal{C}^* + \mathcal{C}\mathcal{C}^* = (I - \mathcal{C})(I - \mathcal{C}^*))$.

The sequences $f_\alpha(n) = \frac{1}{(n+1)^\alpha}$ $(\alpha > 1/2)$ are then used to show that $\|\mathcal{C}^* f_\alpha\| \to 2\|f_\alpha\|$ as $\alpha \to 1/2$ and hence to conclude that the norm inequality above cannot be improved.

The assertions (iii) are obtain by fairly straightforward computations (solving the equations $\mathcal{C}f = \lambda f$ and $\mathcal{C}^* f = \lambda f$).
To prove the fact that the point spectrum of $\mathcal{C}$ is empty, take $f = (f_0, f_1, \dots) \in l^2$ and note that $\mathcal{C}f = \lambda f$ implies

$$f_0 = \lambda f_0 \text{ and } f_n = \lambda((n+1)f_n - nf_{n-1}), \text{for } n \geq 1.$$

This gives $(\lambda(n+1)-1)f_n = \lambda n f_{n-1}$. If m is the smallest integer such that $f_m \neq 0$ then $\lambda = 1/(m+1)$ and thus $\lambda \in (0,1]$. For $n \geq 1$ this gives

$$|f_n| = \left| \frac{\lambda n}{\lambda n - (1-\lambda)} f_{n-1} \right| \geq |f_{n-1}|,$$

which is impossible for a non-zero sequence in l^2.

The equation $\mathcal{C}^* f = \lambda f$ (where $(\mathcal{C}^* f)_n = \sum_{i=n}^{\infty} \frac{f_i}{i+1}$) gives $f_n = \lambda(n+1)(f_n - f_{n+1})$ which implies

$$f_{n+1} = \left(1 - \frac{1}{\lambda(n+1)}\right) f_n = \prod_{j=1}^{n+1} (1 - \frac{1}{\lambda j}) f_0.$$

This implies that 0 cannot be an eigenvalue and all existing eigenvalues must be simple. The authors then show that for λ such that $|1-\lambda| < 1$ ($\Re(1/\lambda) > 1/2$) a corresponding sequence f above is in l^2 and also that λ with $|1-\lambda| = 1$ cannot be an eigenvalue of $\mathcal{C}^*$.

Since $\|I - \mathcal{C}\| = 1$, the spectrum of $I - \mathcal{C}$ is included in the closed unit disc, and hence, the spectrum of $\mathcal{C}$ is included in the closed disc $\{z : |1-z| \leq 1\}$. On the other hand, the spectrum of $I - \mathcal{C}^*$ includes the open unit disc and hence the same is true for the spectrum of $I - \mathcal{C}$, which proves the assertion about $\sigma(\mathcal{C})$.

Hyponormality

To prove the hyponormality, the authors use the fact that both matrices k^*k and kk^* are so called L- shaped, i.e. of the form

$$k = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ \alpha_1 & \alpha_1 & \alpha_2 & \\ \alpha_2 & \alpha_2 & \alpha_2 & \\ \vdots & & & \ddots \end{pmatrix},$$

with $\alpha_n = \sum_{j=n}^{\infty} \frac{1}{(j+1)^2}$ in the case of kk^* and $\alpha_n = \frac{1}{n+1}$ for k^*k. Since $k^*k - kk^*$ is also L-shaped, proving hyponormality reduces to establishing when such a matrix is positive. We have that

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_2 & \alpha_2 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & \alpha_n & \alpha_n & \dots & \alpha_n \end{vmatrix} = \begin{vmatrix} \alpha_0-\alpha_1 & \alpha_1-\alpha_2 & \alpha_2-\alpha_3 & \dots & \alpha_n \\ 0 & \alpha_1-\alpha_2 & \alpha_2-\alpha_3 & \dots & \alpha_n \\ 0 & 0 & \alpha_2-\alpha_3 & \dots & \alpha_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \alpha_n \end{vmatrix} = (\alpha_0-\alpha_1)\dots(\alpha_{n-1}-\alpha_n)\alpha_n$$

is positive for all n if the sequence $\{\alpha_n\}$ is positive and decreasing, which is the case for the sequence defining the L-shaped $k^*k - kk^*$.

Definition

Let $H(\mathbb{D})$ be the space of analytic functions on the unit disc $\mathbb{D}$ endowed with the topology of uniform convergence on compacts. For $f \in H(\mathbb{D})$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$ we define the Cesàro operator applied to f as

$$\mathcal{C}f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

An equivalent representation of $\mathcal{C}$ as an integral operator is given by

$$\mathcal{C}f(z) = \frac{1}{z} \int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta.$$

Indeed, for f as above (with Taylor coefficients in l^2) we have

$$\mathcal{C}f(z) = \frac{1}{z} \int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta = \frac{1}{z} \int_0^z \left(\sum_{k=0}^{\infty} a_k \zeta^k \right) \left(\sum_{i=0}^{\infty} \zeta^i \right) d\zeta$$

$$= \frac{1}{z} \int_0^z \sum_{n=0}^{\infty} (a_0 + a_1 + \ldots a_n) \zeta^n d\zeta = \sum_{n=0}^{\infty} \frac{a_0 + a_1 + \ldots a_n}{n+1} z^n.$$

The Hardy spaces

The Hardy space H^2 is the space of analytic functions on $\mathbb{D}$ with Taylor coefficients in l^2. For $0 < p < \infty$ the Hardy spaces H^p are defined as the spaces of analytic functions f such that

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right)^{1/p} < \infty$$

while and H^∞ is the space of all bounded analytic functions on $\mathbb{D}$, i.e.

$$H^\infty = \{ f \in H(\mathbb{D}) : \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty \}.$$

These are Banach spaces for $1 \leq p \leq \infty$ and for $p = 2$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n \in H^2$ we have

$$\|f\|_2^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} = \sup_{0<r<1} \int_0^{2\pi} f(re^{it}) \overline{f(re^{it})} \frac{dt}{2\pi} = \sup_{0<r<1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} |a_n|^2.$$

The Hardy spaces

A function $f \in H^p$ has non-tangential limits

$$f(e^{it}) := \lim_{r \to 1} f(re^{it})$$

for almost every $t \in [0, 2\pi]$ and the resulting boundary function $f(e^{it})$ is p-integrable on the unit circle.

The norm of $f \in H^p$ is then given by

$$\|f\|_p^p = \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} = \lim_{r \to 1} \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} = \int_0^{2\pi} |f(e^{it})|^p \frac{dt}{2\pi}.$$

The Cesàro operator on H^2

(1920) *Hardy*: $\mathcal{C}$ is bounded on H^2.

(1965) *Brown, Halmos, Shields:* The spectrum of $\mathcal{C}$ on H^2 is given by $\sigma(\mathcal{C}) = \{z \in \mathbb{D} : |z - 1| \leq 1\}$; $\mathcal{C}$ is hyponormal on H^2.

More remarkable properties of $\mathcal{C}$ on H^2:

(1971) *Kriete, Trutt:* $\mathcal{C}$ on H^2 is subnormal. Using duality, they proved that $\mathcal{C}$ has a normal extension by showing that $I - \mathcal{C}$ is unitarily equivalent to the operator of multiplication by the identity function on a certain Hilbert space $\mathcal{H}$ identified as the closure of analytic polynomials in a space of the form $L^2(\mathbb{D}, \mu)$.

(1984) *Cowen:* alternative proof to Kriete-Trutt Theorem using the fact that $\mathcal{C}$ is the resolvent at 0 of the infinitesimal generator of a certain composition semigroup.

The Cesàro operator on H^p

(1987) *Siskakis:*

- used composition semigroups to determine the norm and spectrum of $\mathcal{C}$ acting on $H^p, p \geq 2$ and to give estimates for these when $1 < p < 2$ namely:

Theorem

i.) $\|\mathcal{C}_p\|_p = p, \;\; \sigma(\mathcal{C}_p) = \{z : |z - \frac{p}{2}| \leq \frac{p}{2}\}$, *for* $p \geq 2$

ii.) $p \leq \|\mathcal{C}_p\|_p \leq 2, \;\; \sigma(\mathcal{C}_p) \supset \{z : |z - \frac{p}{2}| \leq \frac{p}{2}\}$, *for* $1 < p < 2$

where $\mathcal{C}_p$ *denotes the restriction* $\mathcal{C}|_{H^p}$.

(1990) *Siskakis:* - proved that $\mathcal{C}$ is bounded on H^1 (using a result by Hardy-Littlewood)

(1992) *Miao*: $\mathcal{C}$ is bounded on H^p, $p \in (0, 1)$.

The Bergman spaces

The Bergman spaces L_a^p are the unit disc counterparts of the Hardy spaces.
For $0 < p < \infty$

$$L_a^p = \{f \in H(\mathbb{D}), \|f\|_{L_a^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\},$$

where $dA(z) = \frac{1}{\pi} dxdy$ denotes the normalized area measure on $\mathbb{D}$.

(1996) *Siskakis:* - extended the composition semigroup method to determine the norm and spectrum of $\mathcal{C}$ acting on the Bergman spaces $L_a^p, p \geq 4$ and to give estimates for these for $1 < p < 4$.

Theorem

i.) $\|\mathcal{C}_p\|_{L_a^p} = \frac{p}{2}, \ \ \sigma(\mathcal{C}_p) = \{z : |z - \frac{p}{4}| \leq \frac{p}{4}\}$, *for* $p \geq 4$

ii.) $\frac{p}{2} \leq \|\mathcal{C}_p\|_{L_a^p} \leq 2, \ \ \sigma(\mathcal{C}_p) \supset \{z : |z - \frac{p}{4}| \leq \frac{p}{4}\}$, *for* $1 \leq p < 4$

where $\mathcal{C}_p$ *denotes the restriction* $\mathcal{C}|_{L_a^p}$.

(The Cesàro operator is bounded on weighted Bergman spaces $L_a^{p,\alpha}$ (more general work by Andersen) and on weighted Dirichlet spaces $D^{2,\alpha}$, $\alpha \in (0,1)$ (Galanopoulos))

Composition semigroups - definitions

Let X be a Banach space of analytic functions on $\mathbb{D}$ and $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Consider the corresponding composition operator C_ϕ defined by $C_\phi(f) = f \circ \phi$ and the operator powers C_ϕ^n, $n \geq 0$ induced by the discrete iterates ϕ_n of ϕ given by $\phi_n = \phi \circ \phi \circ \cdots \circ \phi$ n-times.

We say that ϕ has fractional iterates if there is a family $\{\phi_t : t \geq 0\}$ of analytic self-maps of the unit disc such that

i.) $\phi_1 = \phi$

ii.) $\phi_{t+s} = \phi_t \circ \phi_s, \ \forall t, s \geq 0$

iii.) $\phi_0(z) = z, \ \forall z \in \mathbb{D}$

iv.) $\phi_t(z)$ is jointly continuous in (t, z) on $[0, \infty) \times \mathbb{D}$

Fractional powers of C_ϕ are defined by setting $T_t(f) = f \circ \phi_t$, $t \geq 0$. The corresponding family $\{T_t : t \geq 0\}$ satisfies

i.) $T_0 = I$, the identity operator on X,

ii.) $T_{t+s} = T_t \circ T_s, \ \forall t, s \geq 0.$

and for T_t, $t \geq 0$ bounded on X, the family is called *one-parameter semigroup* of bounded operators on X.

More basic facts about semigroups of operators

Moreover, the family $T_t,\ t \geq 0$ is called *strongly continuous* if $\lim_{t\to 0^+} \|T_t(x) - x\| = 0$ for every $x \in X$ and *uniformly continuous* if the (stronger) property $\lim_{t\to 0^+} \|T_t - I\| = 0$ holds.

Function theoretic properties of ϕ and $\{\phi_t\}$ are closely connected to operator properties of $\{T_t\}$ (and questions about C_ϕ can be translated into questions about $\{T_t\}$, as for example questions about spectrum). The information is related to the *infinitesimal generator* of the semigroup.

The *infinitesimal generator* of a strongly continuous semigroup is the (unbounded) operator A defined by

$$A = \lim_{t\to 0} \frac{T_t(x) - x}{t} = \frac{\partial T_t(x)}{\partial t}\Big|_{t=0},$$

whose domain $\mathcal{D}(A)$ consists of $x \in X$ such that the limit above exists. $\mathcal{D}(A)$ is a dense linear subspace of X that coincides with X if and only if A is bounded (that is whenever $\{T_t\}$ is uniformly continuous).

The densely defined operators that are infinitesimal generators of strongly continuous semigroups of contractions are characterized by the classical Hille-Yosida theorem, the characterization is given by a resolvent estimate.

More basic facts about semigroups of operators

The resolvent set $\rho(A)$ of the infinitesimal generator is given by

$$\rho(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \text{ has a bounded inverse}\}.$$

For $\lambda \in \rho(A)$, the corresponding resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$. The spectrum of A is $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

The infinitesimal generator A of a strongly continuous semigroup is always closed (its graph is closed on $X \times X$) and hence $\sigma(A)$ is closed.

The *type* or *growth bound* of a strongly continuous semigroup $\{T_t\}$ is defined as $\omega = \lim_{t\to\infty} \dfrac{\log \|T_t\|}{t}$. One can show that ω is either finite or $-\infty$ and that for all $w > \omega$, $\{T_t\}$ satisfy estimates of the type $\|T_t\| \leq Me^{wt}$ where M is a constant depending only on w.
The spectral radius of T_t is $r(T_t) = e^{\omega t}$ and for λ such that $Re\lambda > \omega$ then $R(\lambda, A)$ is bounded and the following holds:

$$R(\lambda, A)(x) = \int_0^\infty e^{-\lambda t} T_t(x)\, dt, \ x \in X.$$

More basic facts about semigroups of operators

By the Hille-Yosida-Phillips theorem: if $\|T_t\| \leq e^{\omega t}$ for each t then for all real $\lambda > \omega$

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega}$$

for each $\lambda > \omega$.

Intuitively, operator semigroups are the analogue of the exponential function. For uniformly continuous semigroups we have $T_t = e^{tA}$.

By the spectral theorem for semigroups we have $e^{t\sigma(A)} \subseteq \sigma(T_t),\ t \geq 0$. Equality holds (up to the point 0) for special classes of semigroups like uniformly continuous semigroups. Nevertheless, the point spectra satisfy $e^{t\sigma_p(A)} = \sigma_p(T_t) \setminus \{0\},\ t \geq 0$.

Composition semigroups

The following statements hold for composition semigroups $\{T_t\}$ on both H^p and L^p_a, $1 \leq p < \infty$ (Berkson and Porta, Siskakis):

- $\{T_t\}$ is strongly continuous.
- The infinitesimal generator A of $\{T_t\}$ is given by

$$A(f)(z) = \frac{\partial f(\phi_t(z))}{\partial t}\bigg|_{t=0} = G(z)f'(z)$$

  where $G(z)$ is the corresponding generator of $\{\phi_t\}$, i.e

$$G(z) = \lim_{t \to 0} \frac{\partial \phi_t(z)}{\partial t}$$

To determine the point spectrum of A solve the equation $Af = \lambda f$ for f and λ. This corresponds to solving the differential equation $G(z)f'(z) = \lambda f(z)$.

Weighted composition semigroups

Let X be a Banach space of analytic functions and $\{\phi_t\}$ be a semigroup of analytic self-maps of $\mathbb{D}$. For a suitable analytic map w on $\mathbb{D}$

$$S_t(f)(z) = \frac{w(\phi_t(z))}{w(z)} f(\phi_t(z)),\ f \in X,$$

are bounded operators on X and the family $\{S_t : t \geq 0\}$ is a (weighted) operator semigroup.

Strong continuity of weighted semigroups is more delicate than in the unweighted case and depends on the weight. Nevertheless, spectra of generators of weighted composition semigroups can be treated in a similar manner as in the unweighted case.

The Cesàro operator and its adjoint in H^2, the averaging operator $\mathcal{A}$ that has integral representation $\mathcal{A}(f)(z) = \frac{1}{z-1}\int_1^z f(\zeta)d\zeta$ are both related to certain weighted composition semigroups given by one-parameter semigroups of analytic functions in $\mathbb{D}$.

The semigroup related to $\mathcal{C}$

Let $\varphi(z) = \frac{z}{1-z}$, $z \in \mathbb{D}$ and $\forall t \geq 0$ let

$$\varphi_t(z) = \varphi^{-1}(e^{-t}\varphi(z)) = \frac{e^{-t}z}{(e^{-t}-1)z+1}, \quad z \in \mathbb{D}$$

The family $\{\varphi_t(z) : t \geq 0\}$ is indeed a one-parameter semigroup of analytic functions of $\mathbb{D}$, i.e. the conditions below are satisfied:

i.) $\forall t \geq 0, \ \varphi(\mathbb{D}) \subset \mathbb{D}$

ii.) $\varphi_{t+s} = \varphi_t \circ \varphi_s, \ \forall t, s \geq 0$

iii.) $\varphi_0(z) = z, \ \forall z \in \mathbb{D}$

iv.) $\varphi_t(z)$ is continuous in (t, z) on $[0, \infty) \times \mathbb{D}$

Consider now the weighted composition operators S_t defined by

$$S_t(f)(z) = \frac{\varphi_t(z)}{z} f \circ \varphi_t(z).$$

The Hardy space case

The infinitesimal generator of $\{S_t\}$ is given by

$$\Delta(f)(z) = -z(1-z)f'(z) - (1-z)f(z) = -(1-z)(zf(z))'.$$

and we have that $\mathcal{C} = R(0, \Delta)$. For the results about the norm and spectrum of $\mathcal{C}$ on H^p stated above, Siskakis determined the spectrum of Δ:

$$\sigma(\Delta) = \{\lambda \in \mathbb{C} : Re\lambda < -1/p\}.$$

By transferring to Hardy spaces in the half plane the author obtained that $\|S_t\|_{H^p} = e^{-t/p}$ for $p \geq 2$. The corresponding growth bound is then $\omega = -1/p$ and the Hille-Yosida-Phillips theorem gives $\|\mathcal{C}\|_{H^p} = p$.

The Bergman space case

In the Bergman space case, Siskakis obtained growth estimates using an identity for φ_t together with a suitable change of variables in the integrals involved.

The identity used is $\left(\frac{\varphi_t(z)}{z}\right)^2 = e^{-t}\varphi_t'(z)$ and it gives for $p \geq 4$, $t \geq 0$

$$\begin{aligned}
\|S_t(f)\|_{La^p}^p &= \int_{\mathbb{D}} \left|\frac{\varphi_t(z)}{z}\right|^p |f(\varphi_t(z))|^p dA(z) \\
&= e^{-2t} \int_{\mathbb{D}} \left|\frac{\varphi_t(z)}{z}\right|^{p-4} |f(\varphi_t(z))|^p |\varphi_t'(z)|^2 dA(z) \\
&\leq e^{-2t} \int_{\mathbb{D}} |f(\varphi_t(z))|^p |\varphi_t'(z)|^2 dA(z) \\
&= e^{-2t} \int_{\varphi_t(\mathbb{D})} |f(w)|^p dA(w) \leq e^{-2t} \|f\|_{L_a^p}^p.
\end{aligned}$$

For $1 \leq p < 4$ the author uses another version of the semigroup, namely:

$$T_t(f)(z) = \left(\frac{\varphi_t(z)}{z}\right)^{4/p} f(\varphi_t(z)),\ t \geq 0$$

and shows that $\|T_t\|_{L_a^p} \leq e^{-2t/p}$.

Subdecomposability

(1996) *Miller, Miller, Smith:* used this composition semigroup to prove that $\mathcal{C}$ is subdecomposable on $H^p, 1 < p < \infty$, i.e. it has a decomposable extension.

(Recall that an operator T is said to be decomposable on a complex Banach space X provided that for each open cover $\{U, V\}$ of $\mathbb{C}$, there exist closed $T-$invariant subspaces Y and Z s.t. $X = Y + Z, \sigma(T|_Y) \subset U, \sigma(T|_Z) \subset V$.)

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This method breaks down at certain values of p!

The Weighted Bergman Space Case

(2003) *Dahlner:* determined the spectrum of $\mathcal{C}$ acting on standard weighted Bergman spaces $L_a^{p,\alpha}$, $\alpha > -1$, $1 < p < \infty$ using Hardy-type inequalities.

$$(L_a^{p,\alpha} = \{f \in H(\mathbb{D}), \|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty\})$$

Theorem

If $1 < p < \infty$ *and* $\alpha \geq 1$ *then*

i.) $\sigma_p(\mathcal{C}|_{L_a^{p,\alpha}}) = \{\frac{1}{n}, n \in \mathbb{Z}^+, n < \frac{2+\alpha}{p}\}$

ii.) $\sigma(\mathcal{C}|_{L_a^{p,\alpha}}) = \{z : |z - \frac{p}{2(2+\alpha)}| \leq \frac{p}{2(2+\alpha)}\}$,

iii.) $\sigma_l(\mathcal{C}|_{L_a^{p,\alpha}}) = \{z : |z - \frac{p}{2(2+\alpha)}| = \frac{p}{2(2+\alpha)}\}$, *and* $ind(\lambda - \mathcal{C}) = -1, \forall \lambda \in \{z : |z - \frac{p}{2(2+\alpha)}| < \frac{p}{2(2+\alpha)}\}$

The Weighted Bergman Space Case

(2003) *Dahlner:* determined the spectrum of $\mathcal{C}$ acting on standard weighted Bergman spaces $L_a^{p,\alpha}$, $\alpha > -1$, $1 < p < \infty$ using Hardy-type inequalities.

$$(L_a^{p,\alpha} = \{f \in H(\mathbb{D}), \|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty\})$$

Theorem

If $1 < p < \infty$ *and* $\alpha \geq 1$ *then*

i.) $\sigma_p(\mathcal{C}|_{L_a^{p,\alpha}}) = \{\frac{1}{n}, n \in \mathbb{Z}^+, n < \frac{2+\alpha}{p}\}$

ii.) $\sigma(\mathcal{C}|_{L_a^{p,\alpha}}) = \{z : |z - \frac{p}{2(2+\alpha)}| \leq \frac{p}{2(2+\alpha)}\}$,

iii.) $\sigma_l(\mathcal{C}|_{L_a^{p,\alpha}}) = \{z : |z - \frac{p}{2(2+\alpha)}| = \frac{p}{2(2+\alpha)}\}$, *and* $ind(\lambda - \mathcal{C}) = -1, \forall \lambda \in \{z : |z - \frac{p}{2(2+\alpha)}| < \frac{p}{2(2+\alpha)}\}$

Using this result: $\mathcal{C}$ is subdecomposable on these $L_a^{p,\alpha}$.

Bishop's Property (Beta)

- (1984) *Eschmeier, Putinar*: A bounded operator T on a reflexive Banach space X is decomposable if and only if both T and its adjoint T^* have the Bishop property (β).

Definition

An operator T is said to have the Bishop property (β) if for every open subset Ω of $\mathbb{C}$ and every sequence of X-valued analytic functions on Ω such that

$$\sup_{K\in\Omega} \lim_{n\to\infty} \sup_{\lambda\in K} \|(\lambda I - T)f_n(\lambda)\|_X = 0$$

with K compact subset of Ω, implies

$$\sup_{K\in\Omega} \lim_{n\to\infty} \sup_{\lambda\in K} \|f_n(\lambda)\|_X = 0.$$

the property (β) if and only if it is subdecomposable.

The result of Dahlner

Theorem

(Dahlner) Let X be a Banach space and let T be a bounded linear operator on X with the following properties:
(i) $\sigma(T) = \bar{\mathbb{D}} \cup F$ where F is a finite subset of $\mathbb{C} \setminus \bar{\mathbb{D}}$.
(ii) There is a finite subset $E \in \partial\mathbb{D}$ and numbers $A, B, C \geq 0$ such that for all $\lambda \in \mathbb{C} \setminus (\partial\mathbb{D} \cup F)$ with $|\lambda| < 2\|T\|$ and all $f \in X$ we have

$$\|f\|_X \leq c_{A,B,C}(\lambda)\|(\lambda I - T)f\|_X,$$

where

$$c_{A,B,C}(\lambda) = A(dist(\lambda, \partial\mathbb{D} \cup F))^{-B} e^{C\sum_{\zeta \in E} |Re 1/(\lambda-\zeta)|(1-\log|\lambda-\zeta|)}.$$

Then T has Bishop's property (β).

The reflexive case remained open both for Hardy and Bergman spaces.

Basic facts

For $\mathcal{C}$ acting on $H(\mathbb{D})$, the following hold:

- $\sigma_p(\mathcal{C}) = \{\frac{1}{n}, n \in \mathbb{Z}^+\}$ (eigenfunctions $z^{n-1}(1-z)^{-n}$) and $Ran(\frac{1}{n}I - \mathcal{C}), n \in \mathbb{Z}^+$ does not contain the function $z \to z^{n-1}$.

Choosing in the integral representation of $\mathcal{C}$ the integration path $\Gamma(t) = \varphi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}$, $t \geq 0$, we obtain

$$\mathcal{C}f(z) = \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}d\zeta = -\frac{1}{z}\int_0^\infty \frac{(f\circ\varphi_t)(z)}{1-\varphi_t(z)}\frac{\partial\varphi_t(z)}{\partial t}dt$$

$$= \int_0^\infty \frac{\varphi_t(z)}{z} f\circ\varphi_t(z)dt = \int_0^\infty S_t(f)(z)dt.$$

Here we used that $\varphi(\varphi_t(z)) = e^{-t}\varphi(z)$ and

$$\frac{\partial\varphi_t(z)}{\partial t} = -e^{-t}\varphi(z)(1-\varphi_t(z))^2 = -\frac{-e^{-t}z(1-z)}{((e^{-t}-1)z+1)^2}$$

which gives

$$\frac{1}{1-\varphi_t(z)}\frac{\partial\varphi_t(z)}{\partial t} = -e^{-t}\varphi(z)(1-\varphi_t(z)) = -\varphi_t(z).$$

Solving the resolvent equation $(\lambda I - \mathcal{C})f = h$, $\lambda \in \mathbb{C} \setminus \{0\}, f, h \in H(\mathbb{D})$ one obtains

$$f(z) = R_\lambda h(z) = \frac{1}{\lambda z}\left(\frac{z}{1-z}\right)^{\frac{1}{\lambda}} \left(\left[\left(\frac{\zeta}{1-\zeta}\right)^{-\frac{1}{\lambda}} \zeta h(\zeta) \right]_0^z + \frac{1}{\lambda} \int_0^z \left(\frac{\zeta}{1-\zeta}\right)^{-\frac{1}{\lambda}-1} \frac{\zeta h(\zeta)}{(1-\zeta)^2} d\zeta \right)$$

If the multiplicity of the zero of h at the origin, $m_0(h)$ is s.t. $m_0(h) > Re\frac{1}{\lambda} - 1$ then with $\varphi(z) = \frac{z}{1-z}$ we can write, formally,

$$f(z) = R_\lambda h(z) = \frac{h(z)}{\lambda} + \frac{1}{\lambda^2} z^{\frac{1}{\lambda}-1}(1-z)^{-\frac{1}{\lambda}} \int_0^z \zeta^{-\frac{1}{\lambda}}(1-\zeta)^{\frac{1}{\lambda}-1} h(\zeta) d\zeta \qquad (1)$$

Solving for Polynomials

In general, for $Re\frac{1}{\lambda} > 1$ and $m_0(h) < Re\frac{1}{\lambda} - 1$, let m be the least positive integer such that $m > Re\frac{1}{\lambda} - 1$ and write $h = h_1 + h_2$ where $m_0(h_2) = m$ and h_1 polynomial of degree (at most) $m - 1$. Since a solution to $(\lambda I - \mathcal{C})f = h_2$ is given by $R_\lambda h_2$ it suffices by linearity to solve the resolvent equation for the polynomial h_1.

The construction of an inverse or left-inverse of $(\lambda I - \mathcal{C})$ in some space X of analytic functions on $\mathbb{D}$ essentially amounts to two steps: to verify the boundedness of the operator R_λ given by acting on the subspace $Z_m(X)$ which consists of all functions h with $m_0(h) = m > Re\frac{1}{\lambda} - 1$, and to solve the resolvent equation when necessary, for polynomials of degree less than m.