Isoperimetric inequalities for polygons: Available methods and open problems.

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Topics to discuss:

(a) Isoperimetric Inequalities.

(b) Symmetrization.

(c) Polygons.

Important sources:

- George Pólya and Gabor Szegö, *Isoperimetric inequalities in mathematical physics.* 1951.
- Catherine Bandle, *Isoperimetric Inequalities and Applications* 1980.
- Yurii Burago and Viktor Zalgaller, *Geometric inequalities.* 1980.
- Vladimir Dubinin, *Condenser Capacity and Symmetrization in Geometric Function Theory.* 2014.
- Al Baernstein II, *Symmetrization in analysis.* 2019.

Classical Isoperimetric Inequality

The isoperimetric theorem states:

Theorem A. *Among all planar regions* Ω *with a given perimeter* $L(\Omega)$*, the circle encloses the greatest area:*

$$\frac{A(\Omega)}{L(\Omega)^2} \leq \frac{1}{4\pi}.$$

Queen Dido Story: Birth of Isoperimetry

This remarkable theorem even has a literary history dating back some twenty-one centuries to Virgil's Aeneid and the saga of Queen Dido. Apparently, the good Queen had more than her fair share of entrepreneurial skill and mathematical ability—as well as misfortune of epic proportion. Her legend recounts, among other tragedies, the murder of her father by her brother, who then directed his intentions toward her. She was obliged to assemble her valuables and flee her native city of Tyria in ancient Phoenicia. In due course, her ship landed in North Africa, where she made the following offer to a local chieftain. In return for her fortune, she would be ceded as much land as she could isolate with the skin of an ox. The proposition must have seemed too good to refuse. It was agreed to, and a large ox was sacrificed for its hide. Queen Dido broke it down into extremely thin strips of leather, which she tied together to construct a giant semicircle that, when combined with the natural boundary imposed by the sea, turned out to encompass far more area than anyone might have imagined. And upon this land, the city of Carthage was born.

Jacob Steiner Story: Invention of Symmetrization

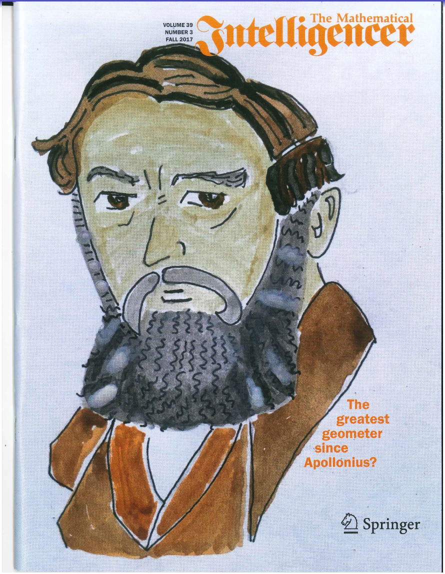
The first geometric transformation bearing the name *symmetrization* was introduced by Jacob Steiner in 1836.

Figure: Jakob Steiner (1796–1863)

"Calculating replaces thinking while geometry stimulates it".

Jakob Steiner, a self made Swiss farmer's son and contemporary of Gauss was the foremost "synthetic geometer." He hated the use of algebra and analysis and distrusted figures. He proposed several arguments to prove that the circle is the largest figure with given boundary length. Besides symmetrization, his four-hinge method has great intuitive appeal, but is limited to two dimensions. He published several proofs trying to avoid analysis and the calculus of variations.

Inventor of Symmetrization



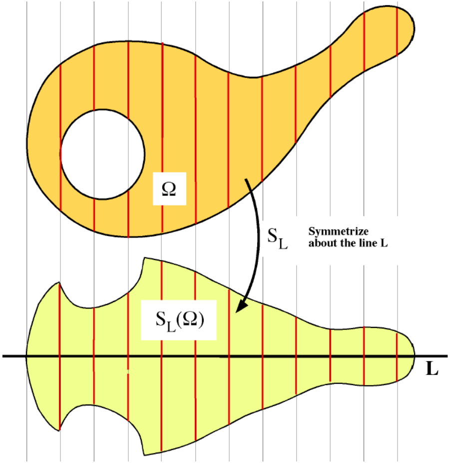
Steiner Symmetrization

Let C be a closed contour on $\mathbb{R}^2$ enclosing a domain D and let $m_D(x)$ denote the Lebesgue measure of the intersection of D with the vertical line $v_x = \{(x, y) \in \mathbb{R}^2 : -\infty < y < \infty\}$. Then Steiner's symmetrization of D with respect to the x-axis is defined by

$$D^* = \{(x, y) \in \mathbb{R}^2 : |y| < (1/2)m_D(x)\}.$$

This implies, in particular, that D^* is symmetric with respect to the x-axis and convex in the y-direction. Let $C^* = \partial D^*$ be the boundary of D^*.

Steiner Symmetrization Visual



Schwarz and Pólya Symmetrizations

(2)

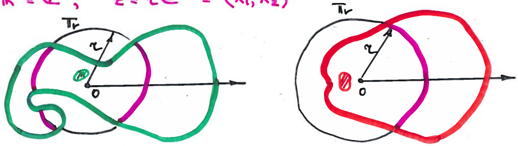
(2) Schwarz symmetrization : of $\Omega \subset \mathbb{R}^n$ w.r.t. a point $x_0 \in \mathbb{R}^n$:

$$\Omega^{\star} = B_r(x^0) := \{ x \in \mathbb{R}^n : |x - x_0| < r \}$$

s.t. $|\Omega| = |B_r(x^0)|$.

(3) Circular symmetrization (Polya - sym.) of $\Omega \subset \overline{\mathbb{R}^2}$ w.r.t. a ray $l_0 = \{ (x_1, 0) : x_1 \geq 0 \}$

$\mathbb{R}^2 = \mathbb{C}$, $z = re^{i\theta} = (x_1, x_2)$



Results obtained via Symmetrization

Steiner used his symmetrization to show that

(a) $\text{area}\, D = \text{area}\, D^*$, (b) $\text{length}\, C^* \leq \text{length}\, C$,

which implies the classical isoperimetric inequality

$$\frac{\text{area}\, D}{(\text{length}\, C)^2} \leq \frac{1}{4\pi}.$$

The ingenious idea of Steiner was exploited over the years by many mathematicians, who proved numerous, so-called, *isoperimetric inequalities* for several important geometrical and physical quantities: s.t. transfinite diameter $d(\bar{D})$ that is equal to the logarithmic capacity $\text{cap}\,(\bar{D})$, for the inner radius $r(D, a)$, for the torsional rigidity $P(D)$, and for the principal frequency $\lambda(D)$:

(c) $d(\overline{D^*}) \leq d(\overline{D})$, (d) $r(D^*, a^*) \geq r(D, a) \quad a \in D$,

(e) $P(D^*) \geq P(D)$, (f) $\lambda(D^*) \leq \lambda(D)$.

Conformal radius and logarithmic capacity

We recall that the conformal radius $R(\Omega, z_0)$ of a simply connected domain $\Omega \subset \mathbb{C}$ with respect to its point z_0 can be defined as

$$R(\Omega, z_0) = \lim_{z \to z_0} \exp(g_\Omega(z, z_0) + \log|z - z_0|).$$

$$\frac{1}{2\pi} \log R(\Omega, z_0) = m(\Omega, z_0) = \lim_{\varepsilon \to 0} \left(\left(\int_{\Omega \setminus D_\varepsilon(z_0)} |\nabla u(z)|^2 \, dA \right)^{-1} + \log \varepsilon$$

Similarly, the logarithmic capacity $\operatorname{cap} E$ of a compact set E with connected complementary set $\Omega(E) = \overline{\mathbb{C}} \setminus E$ can be defined as

$$\operatorname{cap} E = \lim_{z \to \infty} \exp(g_{\Omega(E)}(z, \infty) - \log|z|).$$

Torsional rigidity and Principal frequency

The torsional rigidity of a cylindrical beam quantifies its ability to resist to twisting. In case when a cross-section is Ω, the torsional rigidity $P = P(\Omega)$ can be defined by the following maximization problem:

$$P := \sup_{u \in C_0^1(\overline{\Omega})} \frac{4\left(\int_\Omega u(z)\, dA\right)^2}{\int_\Omega |\nabla u(z)|^2\ dA},$$

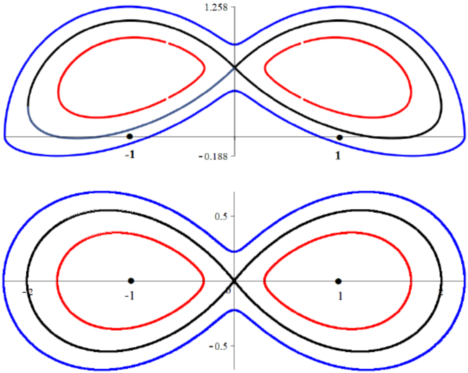
where $C_0^1(\overline{\Omega})$ – the set of differentiable functions on Ω that vanish on its boundary $\partial\Omega$.

To define the principal frequency $\Lambda = \Lambda(\Omega)$ of Ω, we consider Ω as a uniformly stretched elastic membrane of a drum fixed along the boundary $\partial\Omega$. Then Λ is the lowest bass tone of this drum defined as

$$\Lambda^2 := \inf_{u \in C_0^1(\overline{\Omega})} \frac{\int_\Omega |\nabla u(z)|^2\ dA}{\int_\Omega (u(z))^2\, dA}.$$

Symmetrizing functions $u \geq 0$ with compact support on Ω.

The c-supersets $\{z : u^*(z) \geq c\}$ of the symmetrized function $u^*(z)$ are defined to be the symmetrization of the c-supersets $\{z : u(z) \geq c\}$ of the original function $u(z)$.



Symmetrization and the Dirichlet Integral.

A key result of the Pólya-Szegö symmetrization theory is that symmetrization type transformations do not increase the Dirichlet integral.

Theorem (Pólya and Szegö, 1951)

Suppose that $u \geq 0$ with compact support in Ω has continuous partial derivatives in Ω. Then

$$\int_\Omega |\nabla u_i^*(z)|^2 \, dA \leq \int_\Omega |\nabla u(z)|^2 \, dA.$$

Thus, the Dirichlet integral decreases under symmetrization.

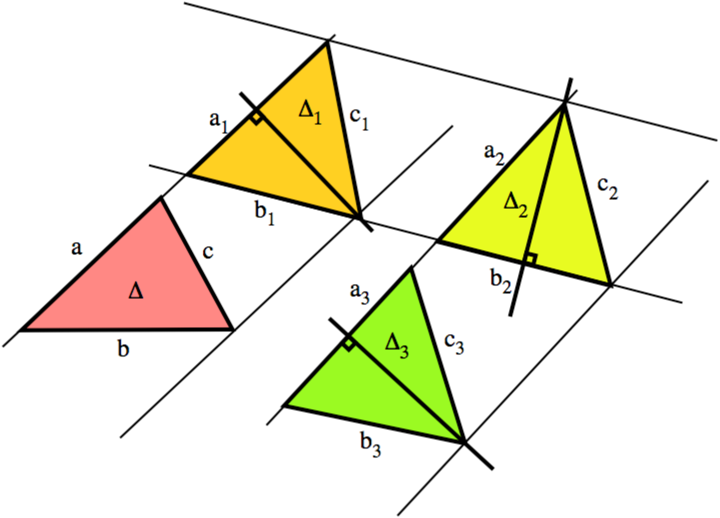
Pólya-Szegö problems on polygons

G. Pólya and G. Szegö used Steiner symmetrization to prove the following.

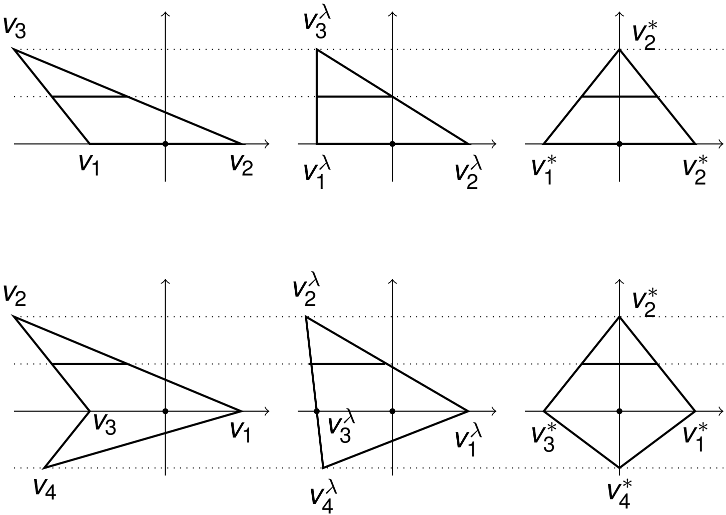
Theorem (Pólya and Szegö, 1951)

The regular triangle and square have the maximal conformal radius $R(\Omega) = \max_{z_0 \in \Omega} R(\Omega, z_0)$ and torsional rigidity $P(\Omega)$ and the minimal logarithmic capacity $\operatorname{cap} E$ (that is $=$ transfinite diameter $d(E)$), the principal frequency $\lambda(\Omega)$, the polar moment of inertia with respect its center of gravity $I(\Omega)$, and the electrostatic capacity $C(\Omega)$ among all triangles and quadrilaterals of given area.

Symmetrizing triangles.

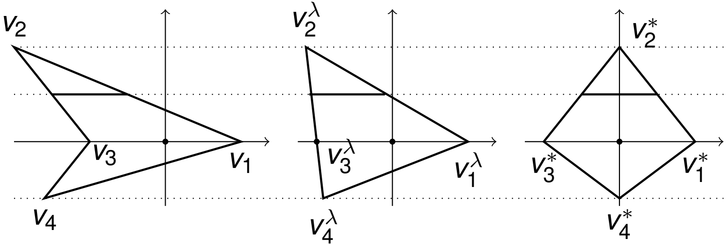


Symmetrizing triangles and quadrilaterals.



Pólya-Szegö problems on polygons

The proof based on symmetrization fails for polygons with $n \geq 5$ sides since Steiner symmetrization increases number of sides of a polygon in general.



G. Pölya and G. Szegö wrote the following:

Problem

"... to prove (or disprove) the analogous theorems for regular polygons with more than four sides is a challenging task".

Pólya-Szegö problems on polygons

G. Pólya and G. Szegö mentioned the following functionals:

1. Conformal radius $R(\Omega, z_0)$.
2. Torsional rigidity $P(\Omega)$.
3. Logarithmic capacity cap E.
4. Principal frequency $\lambda(\Omega)$.
5. Polar moment of inertia $I(\Omega)$.
6. Electrostatic capacity $C(\Omega)$.

For two of these functionals, conformal radius and logarithmic capacity, their problem is solved.

Triangulation and conformal radius

How it was solved for the conformal radius?
Let D_n denote a polygon with $n \geq 3$ sides; D_n^* - regular n-gon.
If it is not possible to calculate characteristics of polygons, then it is a good idea to find a way to estimate those using simpler configurations, say triangles. Thus, we are looking for appropriate triangulations of polygons.

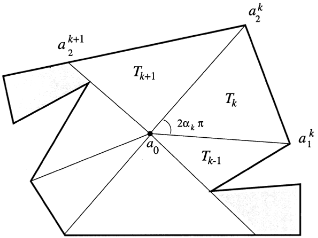
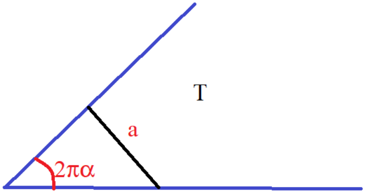


Figure: Triangulation of a polygon

Reduced modulus of a triangle.

Definition: Reduced module of T is defined as the limit:

$$m(T; a_0|a_1, a_2) = \lim_{\varepsilon \to 0} \left(\operatorname{mod}(T(\varepsilon)) + \frac{1}{2\pi\alpha} \log \varepsilon \right).$$



$$m(T; \infty|a_1, a_2) = \frac{1}{2\pi\alpha} \log \frac{2^{4\alpha+1} \alpha B(\beta_1, \beta_2) \sin \pi\beta_2}{a \sin 2\pi\alpha}.$$

Here $\beta_1\pi$ and $\beta_2\pi$ - angles of the triangle.

Conformal radius

Step (a) Finding an upper bound using triangulation.

Lemma (Solynin 88)

Let $T_1, \ldots, T_m$ be a triangulation of D_n centered at $a_0 \in D_n$ and let T_k has an angle $2\alpha_k\pi$ at its vertex at a_0. Then

$$m(D_n, a_0) = \frac{1}{2\pi}\log R(D_n, a_0) \leq \sum_{k=1}^{m} \alpha_k^2 m(T_k|0_0, a_1^k, a_2^k).$$

Equality occurs if and only if the triangulation is done along images of radial segment under a conformal mapping from $\mathbb{D}$ onto D_n.

Conformal radius

Step (b) Calculate characteristics for a triangle.

Lemma (Solynin 1988)

Let T be a triangle with angles $\alpha\pi$ and $\beta\pi$ at its vertices $a_0 = 0$, $a_1 = a > 0$. Then

$$m(T|a_0, a_1, a_2) = \frac{1}{\pi}\log 4 + \frac{1}{\alpha\pi}\log\frac{a}{\alpha B(\alpha,\beta)},$$

where $B(\alpha,\beta)$ is Euler's beta-function.

Step (c) Show that an isosceles triangle is extremal.

Lemma (Solynin 1988)

Let T be a triangle with angles $2\alpha\pi$ and $\beta\pi$ at its vertices $a_0 = 0$, $a_1 = a > 0$, where $a = a(S)$ is such that area$(T) = S$ - fixed. Then

$$m(T|a_0, a_1, a_2) < \frac{1}{2\pi}\log 4 + \frac{1}{\alpha}\log\frac{\sqrt{S\cot(\alpha\pi)}}{\alpha B(1/2, 1/2+\alpha)}.$$

Conformal radius

Step (d) Maximize a special weighted sum for characteristics of triangles.

Theorem (Solynin 1988)

Let D_n be a Euclidean polygon having $n \geq 3$ sides. Let $R(D_n) = \max_{z \in D_n} R(D_n, z)$ be the maximal conformal radius of D_n. Then

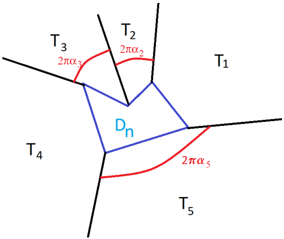
$$\frac{R^2(D_n)}{\text{area}(D_n)} \leq \frac{2^{4/n}}{\pi} \frac{\Gamma(1-\frac{1}{n})\Gamma(\frac{1}{2}+\frac{1}{n})}{\Gamma(1+\frac{1}{n})\Gamma(\frac{1}{2}-\frac{1}{n})}$$

with the sign of equality only for the regular Euclidean n-gons.

Logarithmic capacity

If we try to apply our triangulation approach to the problems on the logarithmic capacity, torsional rigidity, or principal eigenvalue, it fails. Let us see which exactly part fails for the logarithmic capacity.

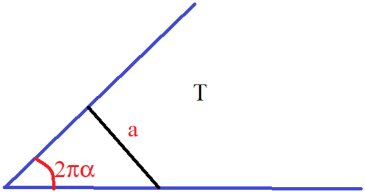
(a) Partitioning of the exterior of D_n works.



$$m(D_n, \infty) \leq \sum_{k=1}^{m} \alpha_k^2 m(T_k|\infty, a_1^k, a_2^k).$$

Logarithmic capacity

(b) Reduced module of T is computable:

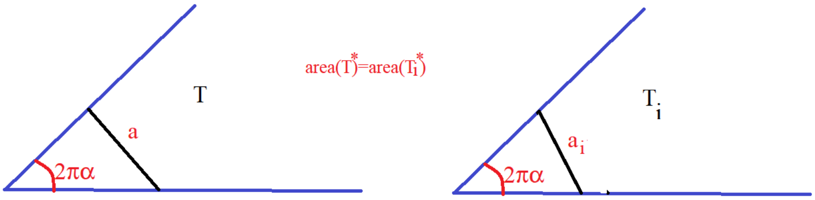


$$m(T|\infty|a_1, a_2) = \frac{1}{2\pi\alpha} \log \frac{2^{4\alpha+1}\alpha B(\beta_1, \beta_2)\ \sin \pi\beta_2}{a \sin 2\pi\alpha}.$$

Here $\beta_1\pi$ and $\beta_2\pi$ - angles of the triangle.

Logarithmic capacity

(c) Isosceles triangle has the largest reduced module for a fixed angle and area:



$$m(T|\infty|a_1, a_2) \leq m(T_i|\infty, a_1, a_2).$$

Logarithmic capacity

(d) Maximization of the weighted sum of reduced moduli?

$$m(\Omega(\overline{D}_n),\infty) \leq \sum_{k=1}^{m} \alpha_k^2 m(T_k^i|\infty|a_1^k,a_2^k) \longrightarrow \max,$$

$$m(T_k^i|\infty|a_1^k,a_2^k) = \frac{1}{2\pi\alpha_k}\log\frac{\pi^{1/2}4^{\alpha_k}\Gamma(1/2+\alpha_k)}{(\sigma_k\tan\pi\alpha_k)^{1/2}\Gamma(\alpha_k)}$$

Does not work!

$$\sup_{\alpha_1,\dots,\alpha_m}\sum_{k=1}^{m}\alpha_k^2 m(T_k^i|\infty|a_1^k,a_2^k) = \infty.$$

Too many choices for triangles!

Logarithmic capacity

Is there a smaller set of triangles such that the maximization problem on this set gives the desired result?

Proportional systems? What they are? - area$(T_k)/\alpha_k = constant$. Simple examples:

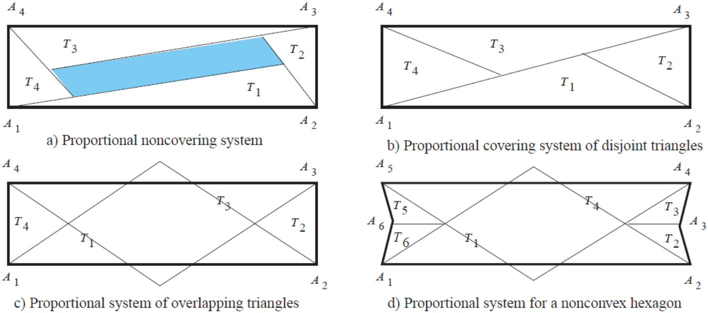


Figure: Proportional systems of triangles

Logarithmic capacity

Do proportional systems exist for any polygon D_n? Is there one which covers D_n? - Those were questions, I asked Victor Abramovich Zalgaller.

Let α_k denote the angle of T_k - now it is a complementary triangle. An admissible system $\{T_k\}_{k=1}^n$ is called *proportional* if the quotient $\alpha_k/\text{area}(T_k)$ does not depend on $k = 1, \dots, n$.

Theorem (Solynin and Zalgaller 2004)

For every n-gon D_n there is at least one proportional system $\{T_k\}_{k=1}^n$ that covers D_n, i.e.

$$\cup_{k=1}^n \overline{T}_k \supset D_n.$$

Logarithmic capacity

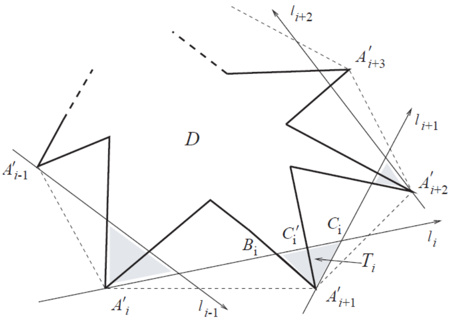
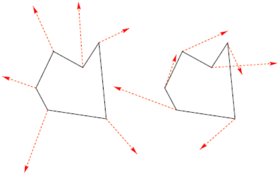


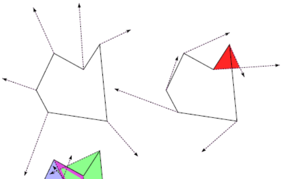
Figure 3. Regular proportional system for small θ

For the main parameter θ, we choose the inclination of $l_{1,1}$: $\theta = \varphi_{1,1}$. Let θ^* be the angle formed by the sides $[A'_1, A'_2]$ and $[A'_1, A'_{\hat{n}}]$ of the convex hull $\hat{D}$, then $0 < \theta < \theta^*$.

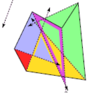
A. Siegel's Theorem, 2003: Covering by triangles.



Theorem B. Let P be a simple polygon that is not necessarily convex. Let the vertices of P be in, counterclockwise order, $v_1, v_2, \dots, v_n$. Let, for $i = 1, 2, \dots, n$ the ray r_i emanate from v_i and form an the angle θ_i with a horizontal ray emanating to the right from v_i, and suppose that $\theta_1 < \theta_2 < \dots < \theta_n < \theta_1 + 2\pi$.



Whenever the rays from two consecutive vertices intersect, let them induce the triangular region defined by the two vertices and the intersection point.



Then there is a fixed α such that if all of the assigned angles are increased by α, the triangular regions induced by the redirected rays cover the interior of P.

Logarithmic capacity

Theorem (Solynin and Zalgaller 2004)

For any polygon D_n having a given number of sides $n \geq 3$,

$$\frac{cap^2\,(\overline{D}_n)}{\text{Area}\, D_n} \geq \frac{cap^2\,(\overline{D}_n^*)}{\text{Area}\, D_n^*} = \frac{n\tan(\pi/n)\Gamma^2(1+1/n)}{\pi 2^{4/n}\Gamma^2(1/2+1/n)}$$

with the sign of equality only for the regular n-gons.

Solynin, Alexander Yu.; Zalgaller, Victor A. *An isoperimetric inequality for logarithmic capacity of polygons.* Ann. of Math. (2) 159 (2004), no. 1, 277–303.

Bogdan Grechuk selected our theorem for his book *Theorems of the 21st century.* Vol. I. Springer, Cham, 2019. xvi+446 pp. He suggested the following interpretation for a general public:

How to find a quiet place to build a house?

Assume that there is a short street where you can build a house for your families. Each family would like to live quietly, as far away from others as possible. Where you should build the houses to maximize “average happiness” for everyone?

The reason for this is interpretation of the log capacity as transfinite diameter:

$$\operatorname{cap}(E) \approx \max \left(\prod_{i=1}^{k-1} \prod_{j=i+1}^{k} |A_j - A_i| \right)^{\frac{2}{k(k-1)}} .$$

Interpretation for city government:

The log capacity = the maximal cost to build a network between large number of households in a city if the cost is proportional to the average distance between households measured on the logarithmic scale.

Request to the city government: Save taxpayers money! Do not overpay contractors!

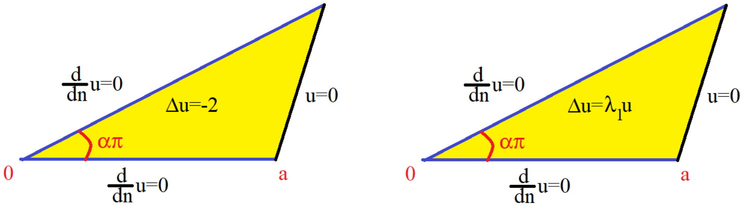
Sincerely Yours, Mathematician.

Torsional rigidity and Principal frequency

Problem remains widely open:

(a) Appropriate proportional triangulation is not know.

(b) Explicit formulas for triangles (with mixed boundary conditions) are not know.

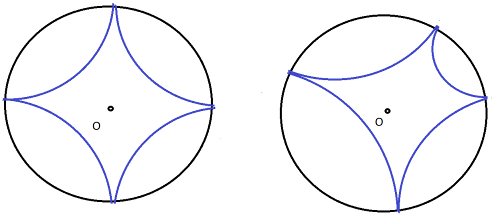


(c) Extremal property of the isosceles triangles is not known.

(d) As a result - there is no function to maximize/minimize.

Hyperbolic polygons

Joseph Hersch conjectured that the regular hyperbolic n-gon has the maximal conformal radius among all hyperbolic n-gons with vertices on the unit circle. These polygons are fundamental domains of automorphic functions. Reiner Kühnau proved this for triangles.



Hyperbolic polygons

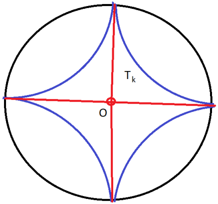
We say that a circle C intersecting the absolute $\mathbb{T}$ has the β-property, $0 \leq \beta \leq 1/2$, if the angle between C and the radius hitting the point of intersection of C and $\mathbb{T}$ is at most $\beta\pi$. A circular n-gon $D_n \subset U$ is called β-circular if its sides lie on circles having the β-property. Obviously, the 0-circular n-gons are precisely the hyperbolic n-gons. We denote by $D_n(\beta)$ the regular circular n-gon with vertices at the points $\tau_k = \exp(2\pi i(k-1)/n)$, $k = 1, \dots, n$, and with angles equal to $2\beta\pi$, $0 \leq \beta \leq 1$. Let $R(D, z_0)$ be the conformal radius of a simply connected domain D with respect to a point $z_0 \in D$, $R(D) = \max_{z \in D} R(D, z)$.

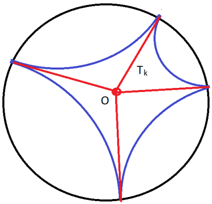
Theorem 1. *Let D_n be a β-circular n-gon, $n \geq 3$, $0 \leq \beta \leq 1/2$. Then*

$$R(D_n) \leq 2^{\frac{4}{n}} \frac{\Gamma(1-\frac{2}{n})\Gamma(\frac{1}{2}+\frac{1}{n})\Gamma(\frac{1}{2}+\beta+\frac{1}{n})}{\Gamma(1+\frac{2}{n})\Gamma(\frac{1}{2}-\frac{1}{n})\Gamma(\frac{1}{2}+\beta-\frac{1}{n})}. \qquad (1)$$

Here $\Gamma(z)$ is Euler's gamma function.

Equality in (1) holds only in the case $D_n = e^{i\theta} D_n(\beta)$ with θ real.





Hyperbolic polygons

The most advanced generalization.

Theorem (R. Barnard, P. Hadjicostas, A. Solynin, 2005)

Let $D_n \ni z_0$ be a hyperbolic polygon with $n \geq 3$ sides and h-area A, $0 < A \leq \sigma_n$, and let $\beta = 1/2 - 1/n - 2A/\pi n$. Then

$$R_h^2(D_n, z_0) \leq R_h^2(D_n(A), 0) = \frac{\Gamma^2\left(1 - \frac{1}{n}\right)\Gamma\left(\frac{1}{2} + \frac{1}{n} + \beta\right)\Gamma\left(\frac{1}{2} + \frac{1}{n} - \beta\right)}{\Gamma^2\left(1 + \frac{1}{n}\right)\Gamma\left(\frac{1}{2} - \frac{1}{n} + \beta\right)\Gamma\left(\frac{1}{2} - \frac{1}{n} - \beta\right)},$$

where Γ denotes the Euler gamma function, with the sign of equality only for the regular hyperbolic n-gons centered at z_0.

In other words, this theorem asserts that the regular hyperbolic polygon has the maximal conformal h-radius among all hyperbolic polygons with a fixed number of sides and prescribed hyperbolic area.

Identities and Inequalities for Transcendental Functions

Recent work: Md. Shafiul Alam and Toshiyuki Sugawa, "Geometric deduction of the solutions to modular equations", arXiv, 12 May, 2021.

For given integers $p \geq 2$, Ramanujan, an Indian mathematical genius, considered the equation

$$\frac{{}_2F_1(a, 1-a; 1; 1-\beta)}{{}_2F_1(a, 1-a; 1; \beta)} = p\frac{{}_2F_1(a, 1-a; 1; 1-\alpha)}{{}_2F_1(a, 1-a; 1; \alpha)}, \tag{1.1}$$

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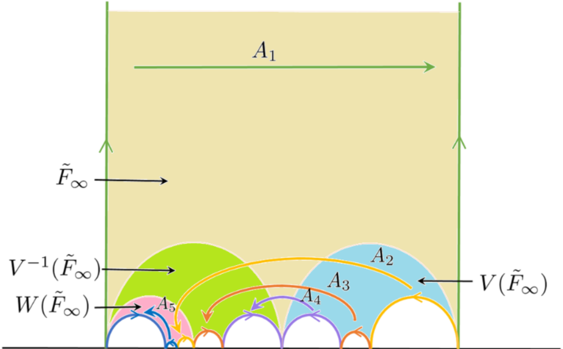
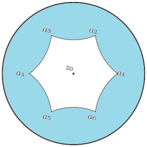


Hyperbolic capacity

Analog of Pólya-Szegö problem for the hyperbolic capacity. Hyperbolic polygonal condenser - $(D_n^h, \overline{\mathbb{C}} \setminus \mathbb{D})$, where D_n^h is a hyperbolic n-gon.

Problem

Prove that the regular hyperbolic polygonal condenser $(D_n^{h*}, \overline{\mathbb{C}} \setminus \mathbb{D})$ has the minimal hyperbolic capacity among all hyperbolic polygonal condensers having n sides and a prescribed hyperbolic area of D_n^h.



Regular 6-gon centered at $z_0 = 0$

Some related questions were discussed in: M.M.S. Nasser, O. Rainio and M. Vuorinen, *Condenser capacity and hyperbolic diameter.* Preprint 2020.

A.A. Gonchar's Problem: Invention of Dissymmetrization

Let $0 < r < 1$, $n \geq 2$, $\Theta = \{\theta_k\}_{k=1}^n$, $0 = \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$, $\Theta^* = \{e^{2\pi i(k-1)/n}\}_{k=1}^n$. Consider compact sets $E_\Theta = \cup_{k=1}^n E_k$, where $E_k = e^{i\theta_k}[r, 1]$.

Problem: Prove that $\omega(0, E_\Theta, \mathbb{D} \setminus E_\Theta) \leq \omega(0, E_\Theta^*, \mathbb{D} \setminus E_\Theta^*)$.

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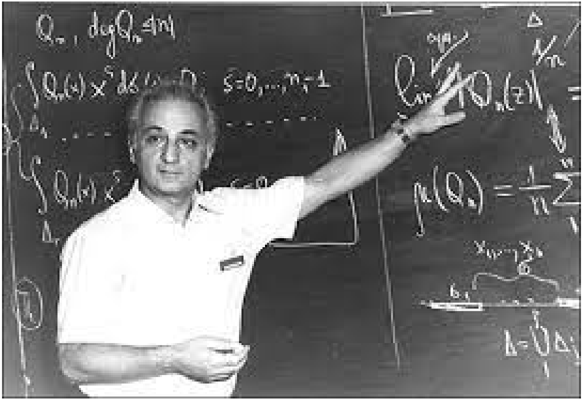

A.A. Gonchar's Problem

Theorem (Dubinin1984)

If E_Θ is not a rotation of E_{Θ^*}, then

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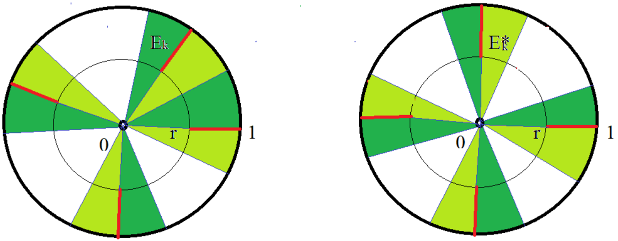
Proof:



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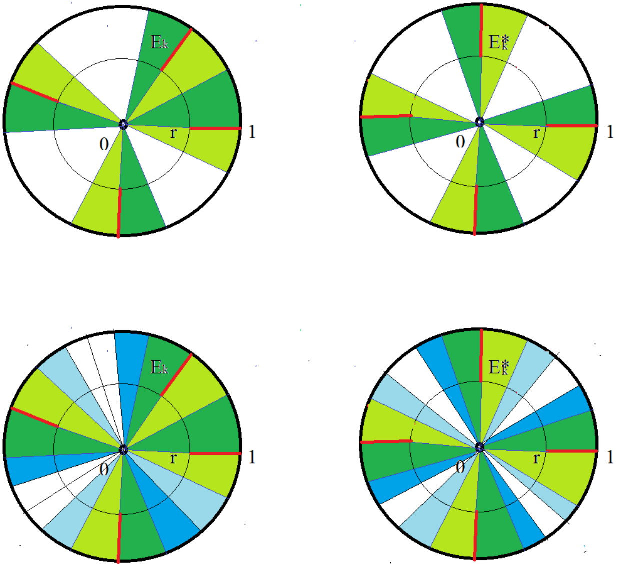
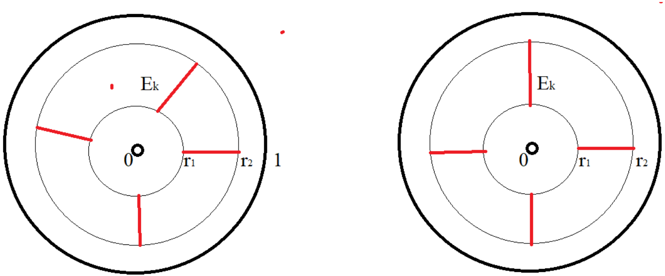


Generalized Gonchar's Problem.

Open Problem: Al Baernstein II considered a generalization of Gonchar's problem for the union of rotations of an arbitrary compact set $E \subset (0, 1]$.



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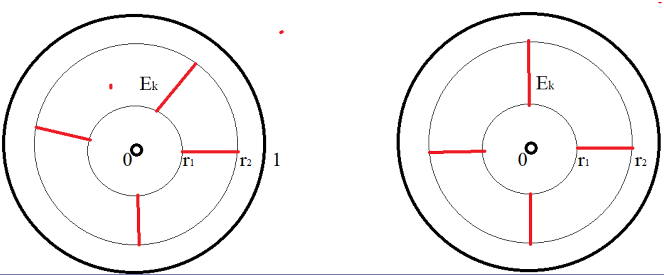

Theorem (Solynin1998)

Given $0 < r_1 < r_2 < 1$, *let* $E_\Theta = \cup_{k=1}^{n} E_k$, *where* $E_k = e^{i\theta_k}[r_1, r_2]$. *Then*

$$\omega(0, E_\Theta, \mathbb{D} \setminus E_\Theta) < \omega(0, E_\Theta^*, \mathbb{D} \setminus E_\Theta^*).$$

Quotients of Hypergeometric Functions.

$$S(v,t) = \frac{\theta_2(\frac{1}{2}v \mid i\pi t)}{\theta_2(0 \mid i\pi t)}\,.$$

We now prove a result on the monotonicity of $S(v,t)$ which is required for what follows but, in our view, is also of interest for the theory of elliptic functions. It is possible that experts already know of this property, but we could not find references to it in the literature.

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CONVEXITY OF QUOTIENTS OF THETA FUNCTIONS

ATUL DIXIT, ARINDAM ROY AND ALEXANDRU ZAHARESCU

ABSTRACT. For fixed u and v such that $0 \le u < v < 1/2$, the monotonicity of the quotients of Jacobi theta functions, namely, $\theta_j(u|i\pi t)/\theta_j(v|i\pi t)$, $j = 1, 2, 3, 4$, on $0 < t < \infty$ has been established in the previous works of A.Yu. Solynin, K. Schiefermayr, and Solynin and the first author. In the present paper, we show that the quotients $\theta_2(u|i\pi t)/\theta_2(v|i\pi t)$ and $\theta_3(u|i\pi t)/\theta_3(v|i\pi t)$ are convex on $0 < t < \infty$.

Paying back to Approximation Theory.

Surprisingly my estimate of the harmonic measure appeared to be useful in Approximation Theory. Recent paper by (Igor Pritsker, House of algebraic integers symmetric about the unit circle, 2021)

The subject of algebraic integers located near (or on) the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is classical. Kronecker [8] proved that if an algebraic integer and all of its conjugates are located in the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$, then it is either a root of unity or zero. For an algebraic integer $\alpha = \alpha_1$, with the complete set of conjugates $\{\alpha_k\}_{k=1}^n$, the house of this algebraic integer is defined by

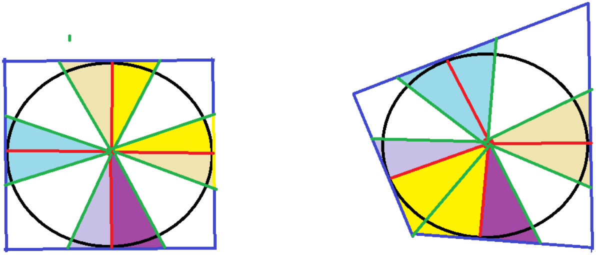
$$\overline{|\alpha|} := \max_{1<k<n} |\alpha_k|.$$

This brings us to celebrated Lehmer's conjecture [9], see also [1], [7], [17], [18] for more details and references. Lehmer observed from computations that the smallest Mahler measure of a non-zero and non-cyclotomic algebraic integer seems to be coming from the largest (in absolute value sense) root α_L of

Theorem 1. *If α is a reciprocal algebraic integer of degree n, with complete set of conjugates $\{\alpha_k\}_{k=1}^n \bigcap \mathbb{T} = \emptyset$, then*

$$\overline{|\alpha|} \ge (1+\sqrt{2})^{\frac{1}{2n}} > 1 + \frac{\log(1+\sqrt{2})}{2n}. \tag{1.5}$$

Dissymmetrization of polygons.



Theorem 1. *Let D_m be an m-gon circumscribed about the circle $\mathbb{T}_r$; with $m \leq n$. Then*

$$A_n \leq R(D_m, 0)/r \leq 4/\pi, \tag{1.1}$$

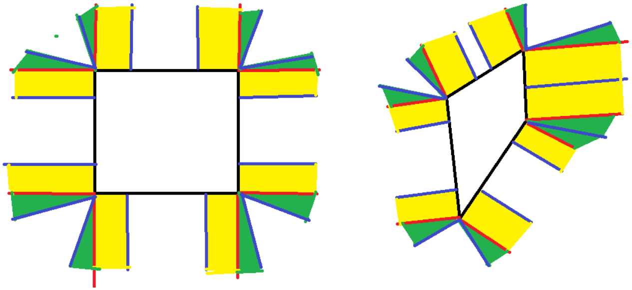
where A_n is defined in (8); *moreover,*

$$P(D_m)/r^4 \geq P(D_n^*), \tag{1.2}$$

$$r\Lambda(D_m) \leq \Lambda(D_n^*). \tag{1.3}$$

Equality in the left inequality of (1.1) *and in* (1.2) *and* (1.3) *is attained in the case $m = n$ and only for a regular n-gon. Equality in the right inequality in* (1.1) *is attained only in the case when D_m degenerates into the band $e^{i\alpha}$, where $\Pi = \{z: |\mathrm{Im}\, z| < r\}$ and $\alpha \in \mathbb{R}$.*

Dissymmetrization of the exterior of polygon.



2.2. Let us prove a theorem about the existence of an (n, r)-dissymmetrization of the set $\widehat{D}_{n,r}$ onto the exterior of a convex m-gon with $m \leq n$. This theorem is fundamental for the applications that follow.

Theorem 4. *Let D_m be a convex m-gon, and $\widehat{D}_m = \overline{\mathbb{C}} \setminus D_m$. If $m \leq n$ and $L(\partial D_m) \leq L(\partial D^*_{n,r}) = 2nr\sin(\pi/n)$, then there is an n-symmetric partition $\{P_{k,s}, R_{p,q}\}$ of the set $\widehat{D}_{n,r}$ and an (n, r)-dissymmetrization $\{\lambda_{k,s}, \mu_{p',q'}\}$ such that*

$$\mathrm{Dis}_{n,r}\widehat{D}_{n,r} = \widehat{D}_m.$$

Dissymmetrization of the exterior of polygon.

Theorem 7. *Suppose that the m-gon D_m is inscribed in the circle $\mathbb{T}_r$ and $m \le n$. Then the following assertions are true.*

1) *The inequality*

$$d(\overline{D}_m) \le B_n r \tag{3.1}$$

holds. Equality is attained in (3.1) only when $m = n$ and $D_n = e^{i\alpha} D^*_{n,r}$, $\alpha \in \mathbb{R}$.

2) *If in addition $\overline{D}_m \ni a$, where a is a fixed point of the disk U_r, then*

$$d(\overline{D}_m) \ge \tfrac{1}{2}\sqrt{r^2 - |a|^2}. \tag{3.2}$$

Equality is attained in (3.2) only in the case when the m-gon $\overline{D}_m$ degenerates into a chord of the disk U_r that has a as its midpoint.

Of all the m-gons

Theorem 10. *Let Γ_m be a closed m-link polygonal line with $m \le n$. Then*

$$d(\Gamma_m)/L(\Gamma_m) \le B_n(2n\sin(\pi/n))^{-1}. \tag{3.8}$$

Equality holds in (3.8) only when $m = n$ and Γ_n bounds a regular n-gon.

Hersch's amplification coefficient.

Let D_n be a polygon on a Riemann surface $\mathcal{R}$ with vertices a_j, $j = 1, \ldots, n$, and let $\zeta = f(z)$ be a Riemann mapping function of D_n, $A_j = f(a_j)$.

We say that D_n admits the reflection $D_{n,j}$ with respect to its side $a_j a_{j+1}$ if the inverse Riemann mapping $f^{-1}(\zeta)$ can be continued conformally into the domain $U' = \overline{\boldsymbol{C}} \setminus \overline{U}$ (the extended mapping might not be univalent) across the arc $A_j A_{j+1}$ of $\boldsymbol{T}$ so that $f^{-1}(U') = D_{n,j}$. If D_n admits reflection with respect to each of its sides, then $\widehat{D}_n$ will denote the full n-sides reflection, i.e. a Riemann surface obtained from D_n by "sticking" along each side $a_j a_{j+1}$ the reflected $D_{n,j}$. Following J. Hersch [6], the quantity

$$\kappa(D_n, z_0) = \frac{R(\widehat{D}_n, z_0)}{R(D_n, z_0)}$$

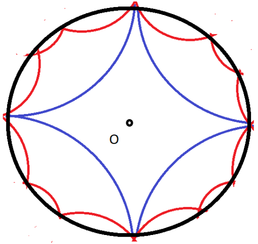
will be called the "*amplification coefficient*" of the conformal radius of D_n at the point $z_0 \in D_n$.

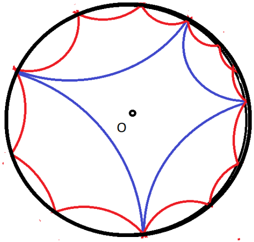
Hersch's amplification coefficient.

Let

$$\kappa(D_n) = \inf_{z \in D_n} \kappa(D_n, z). \tag{1.5}$$

By an "*amplification center*" we mean a point $z_0 \in D_n$ such that $\kappa(D_n) = \kappa(D_n, z_0)$. Clearly, $\kappa(D_n, z_0)$ and $\kappa(D_n)$ are conformal invariants.





Hersch's amplification coefficient.

If D_n is a hyperbolic n-gon, then $\widehat{D}_n$ is also a hyperbolic polygon having $n(n-1)$ sides. Due to this fact J. Hersch [6] was able to consider further reflections of appeared polygons. When studying the conformal mapping onto such polygons he posed the problem of finding of the minimal value of the amplification coefficient among all the hyperbolic n-gons having vertices on $\boldsymbol{T}$. The following theorem solves this problem.

THEOREM 2 *Let D_n be an arbitrary n-gon, $n \geq 2$, admitting the full n-sides reflection $\widehat{D}_n$, $z_0 \in D_n$. Then*

$$\kappa(D_n, z_0) \geq \frac{n+1}{n-1}. \tag{1.6}$$

Equality in (1.6) is attained only in the case when D_n is a regular n-gon centered at z_0.

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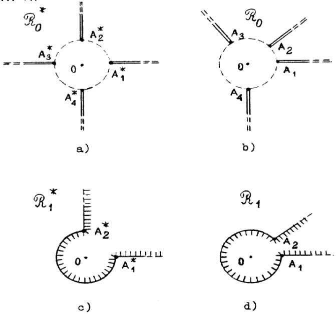


Dissymmetrization of curvilinear polygons

In Solynin, Alexander Yu.; Zalgaller, Victor A. *The inradius, the first eigenvalue, and the torsional rigidity of curvilinear polygons.* Bull. Lond. Math. Soc. 42 (2010), no. 5, 765–783., we proved several inequalities for λ_1, P, and ρ in the case, when Ω is a curvilinear polygon with n sides, each of which is a smooth arc of curvature $\leq \kappa$. Here λ_1 – the first eigenvalue of the Dirichlet Laplacian, P – the torsional rigidity, and ρ – the inradius of a planar domain Ω. Our main proofs rely on the method of dissymmetrization and on a special geometrical “containment theorem” for curvilinear polygons.

For $n \geq 3$ and $\kappa \in \mathbb{R}$, let $\mathcal{D}(n, \kappa)$ be the class of simply connected domains Ω such that $\partial\Omega$ consists of at most n smooth arcs, each of which has piecewise continuous curvature $\leq \kappa$.

Let $D(n, \kappa)$ be the regular circular n-gon circumscribed about the unit circle.

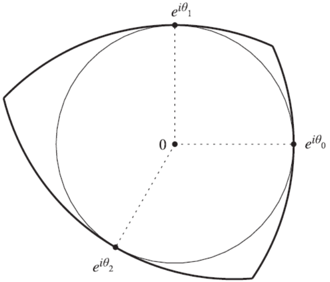


FIGURE 1. *Circular polygon for the partition* $\Theta_3 = \{0, \pi/2, 4\pi/3\}$ *with* $\kappa = 0.5$.

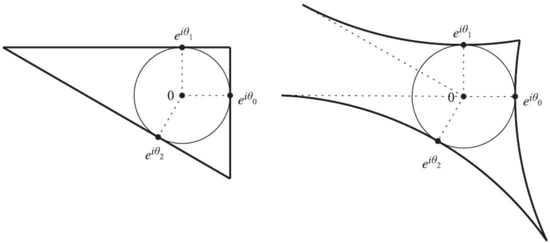


FIGURE 2. *Circular polygons for the partition* $\Theta_3 = \{0, \pi/2, 4\pi/3\}$ *with* $\kappa = 0$ *and* $\kappa = -0.15$.

Theorem (Solynin and Zalgaller 2010)

Let D be a curvilinear polygon bounded by $n \geq 3$ smooth arcs having piecewise continuous curvature not exceeding κ. If $\kappa_1 = \kappa\rho(D_n) \leq 1$, then $1 - \csc(\pi/n) \leq \kappa_1$ and

$$\lambda_1(D)\rho^2(D) \leq \lambda_1(D(n, \kappa_1)),$$

$$P(D)\rho^{-4}(D) \geq P(D(n, \kappa_1)).$$

Equality occurs if and only if D coincides with $D(n, \kappa)$ up to a linear transformation of $\mathbb{C}$.

Theorem (Solynin and Zalgaller 2010)

Let $D \in \mathcal{D}(n, \kappa)$ be a curvilinear polygon and let $\rho = \rho(D)$ be the inradius of D. If $\kappa \leq 1/\rho$, then $1 - \csc(\pi/n) \leq \kappa\rho(D)$ and there is a circular polygon D_n circumscribed about some circle C of radius ρ such that ∂D_n consists of n circular arcs, each of curvature κ, such that $D_n \subset D$.

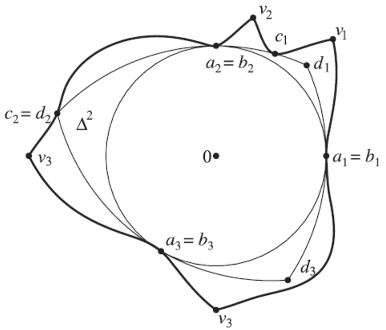


FIGURE 5. *Construction of a circular polygon D_n: case of positive curvature.*

Lemma (Solynin and Zalgaller 2010)

Let $0 < r \leq 1$. Let $C = C_0 \cup C_1$, where $C_1 = \{z = re^{i\theta} : \theta_1 \leq \theta \leq 2\pi\}$, $0 < \theta_1 \leq \pi$, and C_0 is a Jordan arc in $\mathbb{C} \setminus \mathbb{D}_r$ joining the points $z_0 = r$ and $z_1 = re^{i\theta_1}$. Let Ω be the domain bounded by C. Assume that C is a smooth curve having piecewise smooth curvature $\leq \kappa$, for some $0 < \kappa < 1$.

Then Ω contains an open disk of radius greater than 1.

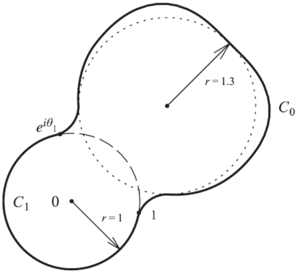
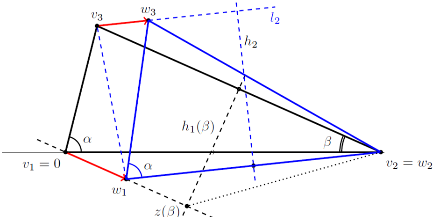


FIGURE 4. *Proof of Lemma 2: inscribing a bigger circle.*

One application of continuous symmetrization

M. Fleeman and B. Simanek conjectured in their recent paper (*Torsional rigidity and Bergman analytic content of simply connected regions.* Comput. Methods Funct. Theory **19** (2019), no. 1, 37–63) that the isosceles right triangle has the maximal torsional rigidity among all right triangles with fixed area. In my 2020 paper, I proved the following.

Theorem 6.5. *Let α, $0 < \alpha < \pi$, be fixed. Then the torsional rigidity $P(T(\alpha, \beta))$ and the maximal conformal radius $R(T(\alpha, \beta))$ of the triangles $T(\alpha, \beta)$ are strictly increasing functions of β and the logarithmic capacity $\mathrm{cap}\,(T(\alpha, \beta))$ and the principal frequency $\Lambda(T(\alpha, \beta))$ are strictly decreasing functions of β on the interval $0 < \beta \le (\pi - \alpha)/2$.*



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Thank You !