Schwarzian derivative of harmonic mappings

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Let f be a locally univalent (analytic) function.

The classical Schwarzian derivative

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = P(f)_z - \frac{1}{2}P(f)^2\,,$$

$$P(f) = \frac{f''}{f'} = \frac{\partial}{\partial z}\left(\log|f'|^2\right)\,.$$

Classical notation (Cayley, 1880):

$$S(f)(z) = \{f, z\}\,.$$

Hermann Amandus Schwarz, 1843–1921

Define an operator S with the property that $S(T \circ f) = S(f)$ for all linear fractional transformations

$$T(z) = \frac{az+b}{cz+d}\,, \quad ad - bc \neq 0\,,$$

and all locally univalent functions f.

Let $g = T \circ f$. Then,

$$g = \frac{af+b}{cf+d}$$

or

$$c(fg) + dg - af = b\,.$$

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Taking derivatives, we get

$$\left\{ \begin{array}{c} c(fg)' + dg' - af' = 0 \\ c(fg)'' + dg'' - af'' = 0 \\ c(fg)''' + dg''' - af''' = 0 \end{array} \right. .$$

Hence,

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \equiv \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 .$$

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A function f has Schwarzian $Sf = 2p$ if and only if $f = u_1/u_2$ where u_1 and u_2 are independent solutions of

$$u'' + pu = 0.$$

$$Sf = Sg \iff g = T \circ f\,.$$

Chain Rule

$$S(f \circ g)(z) = Sf(g(z)) \cdot g'(z)^2 + Sg(z)\,.$$

Let ϕ be an automorphism of $\mathbb{D}$ with $\phi(0) = z$. Then,

$$S(f \circ \phi)(0) = S(f)(z)(1 - |z|^2)^2 \,.$$

The Schwarzian norm

$$\|S(f)\| = \sup_{z \in \mathbb{D}} |S(f)(z)|(1 - |z|^2)^2 \,.$$

Krauss, 1932

If f is univalent in the unit disk, then $\|S(f)\| \leq 6\,.$

Nehari, 1949

If $\|S(f)\| \leq 2\,$, then f is univalent.

The pre-Schwarzian norm

$$\|P(f)\| = \sup_{z\in\mathbb{D}} |P(f)(z)|(1-|z|^2)\,.$$

If f is univalent in the unit disk, then $\|P(f)\| \leq 6\,.$

If $\|P(f)\| \leq 1\,,$ then f is univalent.

Use $\mathcal{S}$ to denote the family of all analytic functions f in the unit disk which are univalent in $\mathbb{D}$ and satisfy

$$f(0) = f'(0) - 1 = 0\,.$$

The function $f = u + iv : \mathbb{D} \to \mathbb{C}$ is harmonic if

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\frac{\partial^2 f}{\partial \bar{z}\partial z} \equiv 0, \quad z = x + iy\,.$$

The canonical decomposition of f equals $h + \overline{g}$, where h, $g \in \mathcal{H}(\mathbb{D})$, and $g(0) = 0$.

Lewy, 1936

The harmonic mapping $f = h + \overline{g}$ is locally univalent if and only if

$$J_f = |h'|^2 - |g'|^2 \neq 0\,.$$

We call f orientation-preserving if $J_f > 0 \equiv h$ is locally univalent and $|\omega_f| = |g'/h'| < 1$.

Use S_H to denote the family of all orientation-preserving harmonic functions $f = h + \overline{g}$ that are univalent in $\mathbb{D}$ with the normalizations

$$h(0) = g(0) = h'(0) - 1 = 0\,.$$

S_H is a normal family, but it is not compact:

$$f_n(z) = z + \frac{n}{n+1}\overline{z}\,.$$

Theorem

The family

$$S_H^0 = \{f \in S_H \colon g'(0) = 0\}$$

is normal and compact.

Take $f \in S_H$ with $|\omega_f(0)| = |g'(0)| = |a| < 1$. Consider

$$\varphi(w) = \frac{w - \overline{aw}}{1 - |a|^2}.$$

Then,

$$f_0 = \varphi \circ f \in S_H^0.$$

Also, given $f_0 \in S_H^0$ and $a \in \mathbb{D}$, the function

$$f = f_0 + \overline{af_0}$$

belongs to S_H and satisfies $g'(0) = a$.

The harmonic Koebe function

$$K(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)} \in S_H^0 .$$

$K = h + \overline{g}$, where h and g satisfies

$$\begin{cases} h - g = k, \quad k(z) = \frac{z}{(1-z)^2} \\ g'/h' = Id \end{cases}, \quad h(0) = g(0) = 0 .$$

Clunie and Sheil-Small, 1984

Let $f = h + \overline{g}$ be locally univalent. Then, f is univalent and *convex in the horizontal direction* (CHD) if and only if the analytic function $h - g$ is univalent and CHD.

An orientation-preserving harmonic mapping $f = h + \overline{g}$ can be lifted (locally) to a regular surface given by isothermal parameters if and only if $\omega_f = q^2$ for some analytic function q (with $|q| < 1$). The minimal surface has the *Weierstrass-Enneper representation*

$$u = \operatorname{Re}\left\{\int_{z_0}^{z} h'(1+q^2)d\zeta\right\}, \quad v = \operatorname{Im}\left\{\int_{z_0}^{z} h'(1-q^2)d\zeta\right\},$$

and

$$w = 2\operatorname{Im}\left\{\int_{z_0}^{z} h'q\,d\zeta\right\}.$$

The metric of the surface has the form $ds = \rho|dz|$ where $\rho = |h'| + |g'| > 0$.

$$\mathbb{S}f = 2(\sigma_{zz} - \sigma_z^2)\,,$$

where $\sigma = \log(|h'| + |g'|)$.

In terms of the canonical representation $f = h + \overline{g}$ and the dilatation $\omega = q^2$,

Chuaqui, Duren, and Osgood, 2003

$$\mathbb{S}(f) = S(h) + \frac{2\overline{q}}{1+|q|^2}\left(q'' - q'\frac{h''}{h'}\right) - 4\left(\frac{q'\overline{q}}{1+|q|^2}\right)^2 .$$

Properties

- $\mathbb{S}(f \circ \phi) = \mathbb{S}(f)(\phi(z))\phi'(z)^2 + S(\phi)(z)$.
- $\mathbb{S}(f)$ is analytic if and only if $f = h + a\overline{h}$ with $|a| < 1$ and $S(h) = \mathbb{S}(f)$.
- $\|\mathbb{S}(f)\| < \infty$ if and only if $\|S(h)\| < \infty$ if and only if f is uniformly locally univalent.
- There exists a constant M such that

$$\|\mathbb{S}(f)\| = \sup_{z\in\mathbb{D}} |\mathbb{S}(f)(z)|(1-|z|^2)^2 \leq M$$

for all univalent harmonic mappings f with $\omega_f = q^2$.

Disadvantage

If f is an orientation-preserving harmonic mapping with $\omega_f = q^2$, $a \in \mathbb{D}$, and $L(z) = z + a\bar{z}$, then $F = L \circ f$ has dilatation

$$\omega_F = \frac{q^2 + \bar{a}}{1 + aq^2} .$$

Definition

Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping with dilatation $\omega = g'/h'$. We define

$$S_f = S(h) + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\omega'\,\overline{\omega}}{1-|\omega|^2}\right)^2 .$$

$$S_f = S(h) + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\,\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\omega'\,\overline{\omega}}{1-|\omega|^2}\right)^2 ????????$$

Approximating by Möbius transformations

Let f be LU in $\mathbb{D}$. Consider

$$T_f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0\,,$$

with $T_f(0) = f(0)$, $T_f'(0) = f'(0)$, and $T_f''(0) = f''(0)$.

$$F_f(z) = (T_f^{-1}\circ f)(z) = \sum_{n=0}^{\infty} s_n \frac{z^n}{n!} = z + \frac{1}{3!} S(f)(0)\, z^3 + \ldots\,,$$

$$S(f)(w) = S(F_f)(w)\,.$$

Let $f = h + \overline{g}$ be sense-preserving in $\mathbb{D}$. Consider

$$M_f = T + \alpha\overline{T}, \quad T(z) = \frac{az+b}{cz+d}, |\alpha| < 1\,,$$

with $M_f(0) = f(0)$,

$$\frac{\partial M_f}{\partial z}(0) = \frac{\partial f}{\partial z}(0) = h'(0), \quad \frac{\partial M_f}{\partial \overline{z}}(0) = \frac{\partial f}{\partial \overline{z}}(0) = \overline{g'(0)}\,,$$

and

$$\frac{\partial^2 M_f}{\partial z^2}(0) = \frac{\partial^2 f}{\partial z^2}(0) = h''(0)\,.$$

$$F_f(z) = (M_f^{-1} \circ f)(z)\,.$$

$$S_f = S(h) + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\,\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\omega'\,\overline{\omega}}{1-|\omega|^2}\right)^2 .$$

$$\begin{aligned}
F_f(z) &= z - \frac{1}{2!}\left(\frac{\overline{\omega(0)}\omega'(0)}{1-|\omega(0)|^2}\right) z^2 + \frac{1}{2!}\left(\frac{\overline{h'(0)\omega'(0)}}{h'(0)(1-|\omega(0)|^2)}\right)\overline{z}^2 \\
&+ \frac{1}{3!}\left(Sh(0) + \frac{\overline{\omega(0)}}{1-|\omega(0)|^2}\left(\omega'(0)\frac{h''}{h'}(0) - \omega''(0)\right)\right) z^3 \\
&- \frac{1}{3!}\left(\frac{h''(0)\overline{h'(0)\omega'(0)}}{(h'(0))^2(1-|\omega(0)|^2)}\right) z\overline{z}^2 \\
&- \frac{1}{3!}\left(\frac{1}{1-|\omega(0)|^2}\left(\overline{\omega''(0) + 2\omega'(0)\frac{h''}{h'}(0)}\right)\right)\overline{z}^3 + \cdots
\end{aligned}$$

If ϕ is analytic and locally univalent,

$$S(\phi) = \rho_{zz} - \frac{1}{2}\rho_z^2 = P(\phi)_z - \frac{1}{2}P(\phi)^2\,,$$

where $\rho = \log(J_f)$ (and $P(\phi) = \frac{\partial}{\partial z}(\log J_f)$).
The same formula holds for locally univalent harmonic functions:

$$S_f = \sigma_{zz} - \frac{1}{2}\sigma_z^2 = (P_f)_z - \frac{1}{2}(P_f)^2\,,$$

where $\sigma = \log(J_f)$ and

$$P_f = \frac{\partial}{\partial z}(\log J_f) = \frac{h''}{h'} - \frac{\overline{\omega}\,\omega'}{1-|\omega|^2}\,.$$

Properties

- $S_{(f\circ\phi)}(z) = S_f(\phi(z))\phi'(z)^2 + S(\phi)(z)$.
- S_f is analytic if and only if $f = h + a\overline{h}$ with $|a| < 1$ and $S(h) = S_f$.
- $\|\mathbb{S}(f)\| < \infty$ if and only if $\|S(h)\| < \infty$ if and only if f is uniformly locally univalent.
- $$\sup_{f\in S_H} \|S_f\| < \infty \,.$$
- Take $L(z) = az + b\overline{z}$ with $|a| \neq |b|$. Then, $S_{(L\circ f)} \equiv S_f$.

Properties

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- S_f is analytic if and only if $f = h + a\overline{h}$ with $|a| < 1$ and $S(h) = S_f$.
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- Take $L(z) = az + b\bar{z}$ with $|a| \neq |b|$. Then, $S_{(L\circ f)} \equiv S_f$.

Properties

- $P_{(f\circ\phi)}(z) = P_f(\phi(z))\phi'(z) + P(\phi)(z)$.
- P_f is analytic if and only if $f = h + a\overline{h}$ with $|a| < 1$ and $P(h) = P_f$.
- $$\sup_{f\in S_H} \|P_f\| < \infty\,.$$
- Take $L(z) = az + b\overline{z}$ with $|a| \neq |b|$. Then, $P_{(L\circ f)} \equiv P_f$.

Theorem

If $f = h + \overline{g}$ is a sense-preserving harmonic mapping in the unit disk with second complex dilatation ω and

$$|P_f(z)|(1-|z|^2) + \frac{|\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2} \leq 1$$

for all $|z| < 1$, then f is univalent in $\mathbb{D}$. The constant 1 is sharp.

The condition in the previous theorem implies that $h + ag$ is univalent for all $|a| \leq 1$.

The Jacobian of a function f of complex values equals $J_f = |f_z|^2 - |f_{\overline{z}}|^2$. For sense-preserving mappings $w = f(z)$, we have

$$(|f_z| - |f_{\overline{z}}|)\,|dz| \leq |dw| \leq (|f_z| + |f_{\overline{z}}|)\,|dz|\,.$$

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}\,.$$

A sense-preserving homeomorphism f is K-quasiconformal if $f \in W^{1,2}_{loc}$ and $D_f \leq K$
(or $|\mu_f| = |f_{\overline{z}}/f_z| = |\omega_f| \leq k = (K-1)/(K+1) < 1$).

Becker and Ahlfors, 1972; 1974

If

$$\sup_{z\in\mathbb{D}} |P\phi(z)|\,(1-|z|^2) \leq k < 1\,,$$

then ϕ has a K-quasiconformal extension to the whole complex plane $\mathbb{C}$, where $K=(1+k)/(1-k)$.
An explicit quasiconformal extension is defined by

$$\Phi(z) = \begin{cases} \widetilde{\phi}(z), & |z|\leq 1 \\ \phi\left(\frac{1}{\bar{z}}\right) + u\left(\frac{1}{\bar{z}}\right), & |z|>1 \end{cases},$$

where, for $z\in\mathbb{D}\setminus\{0\}$, $u(z)=\phi'(z)(1-|z|^2)/\bar{z}$.

Theorem 1

Assume that

$$|P_f(z)|(1-|z|^2)+|\omega^*(z)|\leq k<1, \quad z\in\mathbb{D}.$$

Then, f has a continuous and injective extension $\widetilde{f}$ to $\overline{\mathbb{D}}$ and

$$F(z)=\begin{cases}\widetilde{f}(z), & |z|\leq 1\\ f\left(\frac{1}{\overline{z}}\right)+U\left(\frac{1}{\overline{z}}\right), & |z|>1\end{cases}$$

is a homeomorphic extension of f to the whole complex plane onto itself. The function U equals

$$U(z)=\frac{h'(z)(1-|z|^2)}{\overline{z}}+\frac{\overline{g'(z)}(1-|z|^2)}{z}, \quad z\in\mathbb{D}\setminus\{0\}.$$

Theorem 2

If, in addition, $\|w\|_\infty < 1$, then $\widetilde{f}(\partial\mathbb{D})$ is a quasicircle and f can be extended to a quasiconformal map in $\mathbb{C}$. Indeed, the function F is an explicit K-quasiconformal extension of f whenever

$$k < \frac{1-\|w\|_\infty}{1+\|w\|_\infty}\,.$$

The constant K equals

$$K = \frac{(1+k)+(1-k)\|w\|_\infty}{(1-k)-(1+k)\|w\|_\infty}\,.$$

Let $\mathcal{F}$ be a family of locally univalent holomorphic functions f in the unit disk normalized by the conditions $f(0) = 1 - f'(0) = 0$. If $\mathcal{F}$ closed under the *Koebe transform* defined by

$$F_\zeta(z) = \frac{f\left(\dfrac{\zeta + z}{1 + \bar{\zeta} z}\right) - f(\zeta)}{(1 - |\zeta|^2) f'(\zeta)}, \quad \zeta \in \mathbb{D},$$

we call $\mathcal{F}$ a linear invariant family. The order of $\mathcal{F}$ is

$$\alpha(\mathcal{F}) = \sup_{f \in \mathcal{F}} |a_2(f)| = \frac{1}{2} \sup_{f \in \mathcal{F}} |f''(0)| .$$

Pommerenke, 1964

$$\mathcal{F}_\lambda = \{f : \mathbb{D} \to \mathbb{C} \colon f' \neq 0 \text{ in } \mathbb{D},\ f(0) = 0,\ f'(0) = 1, \|Sf\| \leq \lambda\}$$

is

$$\alpha(\mathcal{F}_\lambda) = \sqrt{1 + \frac{\lambda}{2}}\,.$$

Let $\mathcal{F}$ be a family of sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$, normalized with $h(0) = g(0) = 0, h'(0) = 1$. The family is said to be affine and linearly invariant (AL family) if it closed under the two operations:

$$K_\zeta(f)(z) = \frac{f\left(\dfrac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)h'(\zeta)}, \tag{1}$$

and

$$A_\varepsilon(f)(z) = \frac{f(z) + \varepsilon\overline{f(z)}}{1+\varepsilon g'(0)}. \tag{2}$$

The order of the family, given by

$$\alpha(\mathcal{F}) = \sup_{f\in\mathcal{F}} |a_2(h)| = \frac{1}{2}\sup_{f\in\mathcal{F}} |h''(0)|.$$

The order of the family $\mathcal{F}_\lambda$ satisfies

$$\alpha(\mathcal{F}_\lambda) \leq \sqrt{\frac{\lambda}{2} + 1 + \frac{1}{2}\sup_{f\in\mathcal{F}_\lambda^0} |g''(0)|^2} + \frac{1}{2}\sup_{f\in\mathcal{F}_\lambda} |g'(0)|$$
$$< \sqrt{\frac{\lambda}{2} + \frac{3}{2}} + \frac{1}{2}.$$

Moreover,

$$S = \frac{1}{2}\sup_{f\in\mathcal{F}_\lambda^0} |h''(0)| \leq \sqrt{\frac{\lambda}{2} + \frac{3}{2}}.$$