An extremal problem for H^p

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Based on joint work with Ole Fredrik Brevig and Sigrid Grepstad.

The Hardy space H^p

For $0 < p < \infty$ the Hardy space of analytic functions on the unit disc $H^p(\mathbb{D})$ consists of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p}^p = \lim_{r\to 1^-} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p \frac{\mathrm{d}\theta}{2\pi} < \infty.$$

H^∞ is the space of bounded analytic functions in $\mathbb{D}$, endowed with the norm

$$\|f\|_{H^\infty} = \sup_{|z|<1} |f(z)|.$$

The Hardy space H^p

- H^p is a Banach space for $1 \leq p \leq \infty$.
- H^p is a Quasi-Banach space for $0 < p < 1$.
- H^p is strictly convex for $1 < p < \infty$.
- If $f \in H^p$ for some $0 < p \leq \infty$, then

$$f^*(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})$$

  exists for almost every θ and $f^* \in L^p(\mathbb{T})$.
- It follows that $\|f\|_{H^p(\mathbb{D})} = \|f^*\|_{L^p(\mathbb{T})}$.

An extremal problem

$$\Psi_k(p) = \sup\left\{ \mathrm{Re}\frac{f^{(k)}(0)}{k!} : \|f\|_{H^p} = 1 \right\}$$

$\Psi_p(k) = 1$ for $1 \leq p \leq \infty$.

$$|a_k| = \left| \frac{1}{2\pi} \lim_{r \to 1^-} \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} \, \mathrm{d}\theta \right| \leq \|f\|_{H^1} \leq \|f\|_{H^p}$$

- For $1 < p \leq \infty$ we have the unique extremal function $f(z) = z^k$.
- For $p = 1$ the extremals are functions of the form $f(z) = A\prod_{j=1}^k (z - \alpha_j)(1 - \overline{\alpha_j} z)$.

The extremal problem

We want to study the extremal problem

$$\Phi_k(p,t) = \sup\left\{ \mathrm{Re}\,\frac{f^{(k)}(0)}{k!} : \|f\|_{H^p} \leq 1 \text{ and } f(0) = t \right\},$$

for $k \in \mathbb{N}$, $0 \leq t \leq 1$ and $0 < p \leq \infty$.

- There always exists at least one function attaining the supremum
- The norm of the extremal function will always be 1

Example

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$\Phi_k(2,t) = \sqrt{1-t^2}$ and the unique corresponding extremal function is $f(z) = t + \sqrt{1-t^2}z^k$.

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Proof

It follows from Parseval's identity that

$$\|f\|_{H^2}^2 = \sum_{n\geq 0} \left| \frac{f^{(n)}(0)}{n!} \right|^2 .$$

Thus the extremal function $f = \sum_{n\geq 0} c_n z^n$ must be such that $c_0 = t$ and $c_k = \sqrt{1-t^2}$, with all other coefficients equal to 0.

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$\Phi_1(\infty, t) = 1 - t^2$ and the unique corresponding extremal function is $f(z) = (t + z)/(1 + tz)$.

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Proof

This follows from Schwarz-Pick inequality. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then

$$|f'(w)| \leq \frac{1 - |f(w)|^2}{1 - |w|^2}.$$

Beneteau and Korenblum solved the case $k = 1$ and $1 \leq p \leq \infty$ by an interpolating argument.

Structure of extremals

Theorem (Macintyre–Rogosinski, Havinson and Kabaila)

If $f \in H^p$ is extremal for $\Phi_k(p,t)$, then there are complex numbers $|\lambda_j| \leq 1$ for $j = 1, \ldots, k$ and a constant C such that

$$f(z) = C \prod_{j=1}^{l} \frac{\lambda_j - z}{1 - \overline{\lambda_j} z} \prod_{j=1}^{k} (1 - \overline{\lambda_j} z)^{2/p},$$

for some $0 \leq l \leq k$, and the strict inequality $|\lambda_j| < 1$ holds for $0 < j \leq l$.

Structure of extremals for k=1

From the structure theorem we know that there are essentially two candidates for extremal functions for $k = 1$, depending on whether $l = 0$ or $l = 1$.

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Observe that

$$f_0(0) = \frac{1}{(1 + \beta^2)^{1/p}}, \quad \text{and} \quad f_1(0) = \frac{\alpha}{(1 + \alpha^2)^{1/p}}.$$

Solving $\Phi_1(p,t)$ idea

We use the structure theorem to find an extremal function for $\Phi_1(p,t)$.
The idea is:

1. We define the functions $\beta \mapsto t(\beta) = (1+\beta^2)^{-1/p}$ and $\alpha \mapsto t(\alpha) = \alpha(1+\alpha^2)^{-1/p}$ for $0 \leq \beta \leq 1$ and $0 \leq \alpha < 1$.
2. Check which t-values $f_0(0)$ and $f_1(0)$ can obtain, and which α and β obtain these values.

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2. Check which t-values $f_0(0)$ and $f_1(0)$ can obtain, and which α and β obtain these values.

3(a). If there is only one candidate for each t we have found the unique extremal function, and can calculate $\Phi_1(p,t)$.

3(b). If there are several candidates for some t we need to compare the values $f'(0)$.

Possible t values for $f_0(0)$ and $f_1(0)$ for $p = 2$

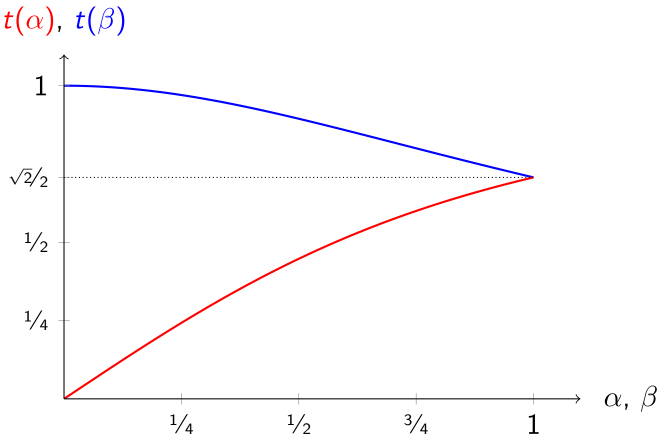

Figure: Plot of the curves $\alpha \mapsto t(\alpha) = \alpha(1+\alpha^2)^{-1/2}$ and $\beta \mapsto t(\beta) = (1+\beta^2)^{-1/2}$

The analysis in the case $k = 1$ and $1 \leq p \leq \infty$

Let $1 \leq p \leq \infty$. Observe that

$$f_0(0) = 1/(1+\beta^2)^{1/p} \in [2^{-1/p}, 1],$$

is strictly decreasing in $0 \leq \beta \leq 1$ and that

$$f_1(0) = \alpha/(1+\alpha^2)^{1/p} \in [0, 2^{-1/p}),$$

is strictly increasing in $0 \leq \alpha < 1$.

Consequence

There is exactly one candidate for the extremal function for each t, solving the problem $\Phi_1(p,t)$ for $1 \leq p \leq \infty$.

$\Phi_1(p, t)$ for $1 \le p \le \infty$

Theorem (Beneteau–Korenblum)

Fix $1 \le p \le \infty$. Then the function $t \mapsto \Phi_1(p, t)$ is decreasing and takes the values $[0, 1]$. For $0 \le t < 2^{-1/p}$ we have a unique extremal function f_1, and for $2^{-1/p} \le t \le 1$ we have a unique extremal function f_0.

For $0 \le t < 2^{-1/p}$ we define α implicitly by $t = \alpha(1 + \alpha^2)^{-1/p}$ and the unique extremal function is

$$f_1(z) = \frac{\alpha + z}{1 + \alpha z} \frac{(1 + \alpha z)^{2/p}}{(1 + \alpha^2)^{1/p}}.$$

For $2^{-1/p} \le t \le 1$, we define β implicitly by $t = (1 + \beta^2)^{-1/p}$ and the unique extremal function is

$$f_0(z) = \frac{(1 + \beta z)^{2/p}}{(1 + \beta^2)^{1/p}}.$$

The analysis in the case $k = 1$ and $0 < p < 1$.

Fix $0 < p < 1$. We want to find extremal candidates for each $0 \leq t \leq 1$, so we must study the functions

$$t(\alpha) = \frac{\alpha}{(1+\alpha^2)^{1/p}}, \quad 0 \leq \alpha < 1,$$

and

$$t(\beta) = \frac{1}{(1+\beta^2)^{1/p}}, \quad 0 \leq \beta \leq 1.$$

Fact

There is a number $2^{-1/p} \leq c_p \leq 1/2$ such that $t(\alpha)$ takes the values $[0, c_p)$ an $t(\beta)$ takes the values $[2^{-1/p}, 1]$. The function $t(\alpha)$ first increases, and then decreases, whereas $\beta(t)$ is strictly decreasing.

Possible t values for $f_0(0)$ and $f_1(0)$ for $p = 1/2$

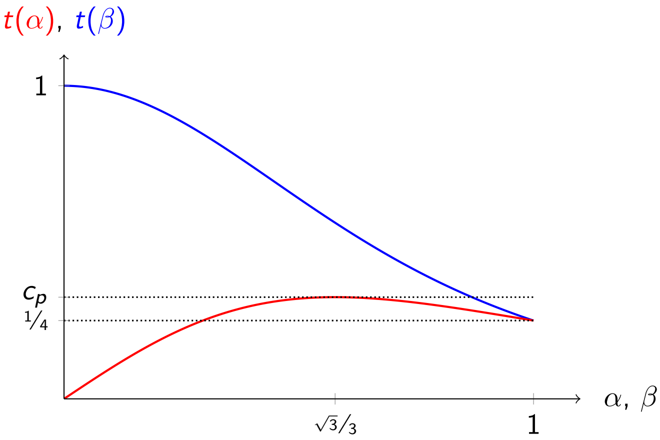

Figure: Plot of the curves $\alpha \mapsto t(\alpha) = \alpha(1+\alpha^2)^{-2}$ and $\beta \mapsto t(\beta) = (1+\beta^2)^{-2}$

The analysis in the case $k = 1$ and $0 < p < 1$

There are 3 cases to consider:

(a) For $0 < t < t^{-1/p}$ there is only one candidate; f_1.

(b) For $2^{-1/p} \leq t < c_p$ there are three candidates; $f_1^{\alpha_1}$, $f_1^{\alpha_2}$ and f_0.

(c) For $c_p \leq t \leq 1$ there is only one candidate; f_0.

Consequence (Connelly)

This immediately gives the unique extremal function and the value $\Phi_1(p, t)$ in the cases (a) and (c).

The case (b)

For $2^{-1/p} \leq t < c_p$ there are three candidates; $f_1^{\alpha_1}$, $f_1^{\alpha_2}$ and f_0. We compare the candidates and find that

Proposition

There is a point $t_p \in (2^{-1/p}, c_p)$ such that $f_1^{\alpha_1}$ is the unique extremal function for $0 \leq t < t_p$ and f_0 is the unique extremal function for $t_p < t \leq 1$. Both f_0 and f_1 are extremal functions for $\Phi_1(p, t_p)$.

Plot of the curve $p \mapsto t_p$

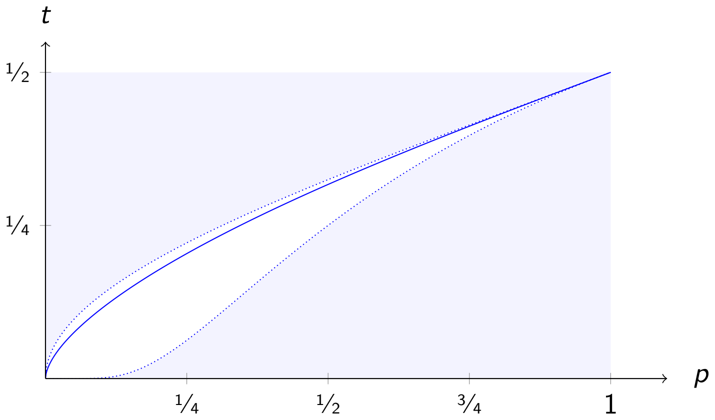

Figure: Plot of the curve $p \mapsto t_p$. Points (p, t) above and below the curve correspond to the cases where f_0 and f_1 is the extremal, respectively. The estimates $2^{-1/p} < t_p < c_p$ are represented by dotted curves.

The extremal for $0 < p < 1$ and $k = 1$.

If $0 \leq t \leq t_p$, let α denote the unique real number in the interval $0 \leq \alpha < \sqrt{p/(2-p)}$ such that $t = \alpha(1+\alpha^2)^{-1/p}$. Then

$$\Phi_1(p,t) = \frac{1}{(1+\alpha^2)^{1/p}} \left(1 + \left(\frac{2}{p} - 1\right)\alpha^2\right),$$

and an extremal is

$$f(z) = \frac{\alpha + z}{1 + \alpha z} \frac{(1+\alpha z)^{2/p}}{(1+\alpha^2)^{1/p}}.$$

The extremal for $0 < p < 1$ and $k = 1$.

If $t_p \leq t \leq 1$, let β denote the unique real number in the interval $0 \leq \beta \leq 1$ such that $t = (1+\beta^2)^{-1/p}$. Then

$$\Phi_1(p,t) = \frac{1}{(1+\beta^2)^{1/p}} \frac{2\beta}{p},$$

and an extremal is

$$f(z) = \frac{(1+\beta z)^{2/p}}{(1+\beta^2)^{1/p}}.$$

$0 < p < 1$ and $k = 1$

Theorem (Brevig–Grepstad–I.)

Fix $0 < p < 1$. The function $t \mapsto \Phi_1(p,t)$ is increasing from $\Phi_1(p,0) = 1$ to

$$\Phi_1(p,(1-p/2)^{1/p}) = (1-p/2)^{1/p}\frac{2}{\sqrt{p(2-p)}}$$

and then decreasing to $\Phi_1(p,1) = 0$. There exist a number $0 \le t_p \le 1$ such that $\Phi_1(p,t_p)$ has exactly 2 extremal functions. For all other $0 \le t \le 1$ the extremal function is unique.

$\Phi_1(p, t)$ for some p.

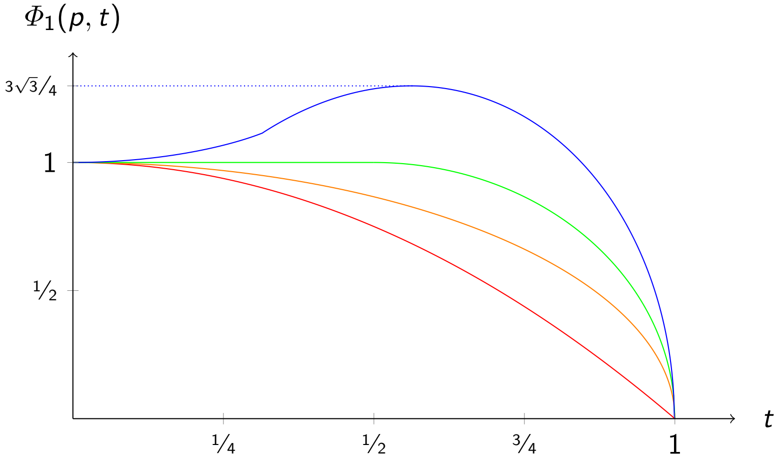

Figure: Plot of the curves $t \mapsto \Phi_1(p, t)$ for $p = 1/2$, $p = 1$, $p = 2$ and $p = \infty$.

The Wiener transform

Let $\omega_k = \exp(2\pi i/k)$. Then for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we define the Wiener transform as

$$W_k f(z) = \frac{1}{k} \sum_{j=0}^{k-1} f(\omega_k^j z) = \sum_{n=0}^{\infty} a_{kn} z^{kn}$$

F. Wieners trick for $k \geq 2$ and $1 \leq p \leq \infty$.

Consider $1 \leq p \leq \infty$ and $k \geq 2$. Let f be extremal for $\Phi_k(p, t)$. Then $W_k f$ is also extremal for $\Phi_k(p, t)$. This gives:

Theorem (Beneteau–Korenblum)

Let $k \geq 2$ be an integer. For every $1 \leq p \leq \infty$ and every $0 \leq t \leq 1$,

$$\Phi_k(p, t) = \Phi_1(p, t).$$

Let f_1 be the extremal function for $\Phi_1(p, t)$, then $f_k(z) = f_1(z^k)$ is an extremal function for $\Phi_k(p, t)$

The case $k \geq 2$ and $0 < p < 1$

F. Wieners trick only gives

$$\Phi_1(p,t) \leq \Phi_k(p,t) \leq k^{1/p-1}\Phi_1(p,t)$$

Question

Let $0 < p < 1$. Is it true that the extremal for $\Phi_k(p,1)$ has at most one zero in $\mathbb{D}$?

Question

Fix $k \geq 2$ and $0 < p < 1$. Is there some t_0 such that $\Phi_k(p,t) = \Phi_1(p,t)$ holds for every $t_0 \leq t \leq 1$?

Thanks

Thank you for the attention!