Counterexample of normability in Hardy and Bergman spaces with $0 < p < 1$

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Preliminary facts. Hardy spaces

Definition

Given $0 < p < \infty$, for a function f analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the integral means of order p are defined by

$$M_p(r, f) := \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p}, \quad 0 \leq r < 1.$$

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For $p = \infty$, we define

$$M_\infty(r, f) := \max_{|z|=r} |f(z)|, \quad 0 \leq r < 1.$$

Definition

Given $0 < p < \infty$, the Hardy space H^p is the set

$$H^p := \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^p} := \sup_{0 \le r < 1} M_p(r, f) < \infty \right\}.$$

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Also, H^∞ is the set of bounded analytic functions in $\mathbb{D}$, and

$$\|f\|_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z)|$$

Theorem (Hardy)

Let $0 < p < \infty$. If $f \in H^p$, then $M_p(r, f)$ is an increasing function of $r \in (0, 1)$. In particular,

$$\|f\|_{H^p} = \lim_{r \to 1^-} M_p(r, f)$$

Theorem (Fatou, 1906)

Let $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. *If* $0 < p \leq \infty$ *and* $f \in H^p$. *Given* $e^{i\theta} \in \mathbb{T}$, *we define*

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$$f^*(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta}).$$

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Therefore, $H^p \subset L^p(\mathbb{T})$. Actually, H^p is a closed subspace of $L^p(\mathbb{T})$.

Bergman spaces

Definition

Given $0 < p < \infty$, the Bergman space A^p is the set

$$A^p =: \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty \right\},$$

where

$$dA(z) = \frac{dx\, dy}{\pi} = \frac{r\, dr\, d\theta}{\pi}, \quad z = x + iy = re^{i\theta},$$

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is the normalized Lebesgue area measure on $\mathbb{D}$.

In other words, A^p is the set of analytic functions in $\mathbb{D}$ that belong to the space $L^p(\mathbb{D})$. Moreover, A^p is a closed subspace of $L^p(\mathbb{D})$.

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- However, $\|\cdot\|_{L^p(X)}$ in general does not define a norm in $L^p(X)$ when $0 < p < 1$, since it fails to satisfy the triangle inequality.

Natural question

Is $(H^p, \|\cdot\|_{H^p})$ a normed space when $0 < p < 1$? And $(A^p, \|\cdot\|_{A^p})$?

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- There are many known monographs or texts that treat Hardy spaces or Bergman spaces that mention this fact. However, none of them provide proof.
- In the 1950´s, Livingston proved, by an indirect method, that $\| \cdot \|_{H^p}$ is not a norm in H^p when $0 < p < 1$. Moreover, what Livingston demonstrates is that , in this case, it is not possible to define an equivalent norm with the metric.

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Aim of this work

Our purpose is to fill this gap in the literature by giving specific examples of two functions, in both H^p and A^p spaces with $0 < p < 1$, that do not satisfy the triangle inequality.

The Hardy space case. Indirect proof by Livingston.

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The space H^p, $0 < p < 1$ is not normable.

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Definition (Topological vector space)

Given a vector space X and a topology T on X, we say that X is a topological vector space if

1. every point $x \in X$ is a closed set, and
2. the vector space operations are continuous with respect to T.

Sketch of the proof

(1) H^p, $0 < p < 1$, is a linear topological space vector space that is, in fact, metrizable, with the metric given by $d(f,g) := \|f - g\|_{H^p}^p$. If $\| \cdot \|_{H^p}$ were a norm, it would give us the same topology.

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(2) Theorem (Kolmogorov, 1934) A topological vector space X has an equivalent norm topology if and only if X contains a bounded open convex set.

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(2) Theorem (Kolmogorov, 1934) A topological vector space X has an equivalent norm topology if and only if X contains a bounded open convex set.

(3) Livingston´s proof is based on proving that the unit ball $B = \{x \in H^p : \|x\| < 1\}$ contains no convex neighborhood of the origin.

(4) In order to do that, it is supposed that B contains a convex neighborhood of the origin V. Since V is open, there exists $\epsilon > 0$ such that $B_\epsilon = \{x \in H^p : \|x\| < \epsilon\} \subset V$. Then, it can be shown that there exists a finite sequence of points $x_1, \cdots, x_n \in B_\epsilon$ and a convex combination $\sum_{k=1}^n a_k x_k$ of these points so that $\sum_{k=1}^n a_k x_k \notin B$, and hence $\sum_{k=1}^n a_k x_k \notin V$.

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(5) To demonstrate this, Livingston constructs a collection of k continuous functions on $\mathbb{T}$ with special properties. This enables him to find, using the Weierstrass-Fejér theorem, a particular collection of k trigonometric polynomials. From these, a family of k polynomials is built that satisfy (4).

The Hardy space case. Direct proof.

Given $0 < p < 1$, we want to find two functions $f, g \in H^p$ such that

$$\|f + g\|_{H^p} > \|f\|_{H^p} + \|g\|_{H^p}$$

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Actually, we will find two functions, f and g, such that $\|f\|_{H^p} = \|g\|_{H^p} = r$, yet their midpoint $(f + g)/2$ is not in the disc $D = \{\varphi \in H^p : \|\varphi\|_{H^p} \leq r\}$.

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$\|\cdot\|_{H^p}$ is invariant under rotations. In particular, if $g(z) = f(-z)$, then $\|g\|_{H^p} = \|f\|_{H^p}$.

Lemma (Boundedness of the composition with the function $z \mapsto z^2$)

Let $f \in H^p$ and $h(z) = f(z^2)$, for $z \in \mathbb{D}$. Then, $h \in H^p$ and $\|h\|_{H^p} = \|f\|_{H^p}$.

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Proof.

By the change of variable $t = 2\theta$ and by periodicity,

$$\|h\|_{H^p}^p = \int_0^{2\pi} |f^*(e^{2i\theta})|^p \frac{d\theta}{2\pi} = \frac{1}{2}\int_0^{4\pi} |f^*(e^{it})|^p \frac{dt}{2\pi}$$
$$= \int_0^{2\pi} |f^*(e^{it})|^p \frac{dt}{2\pi} = \|f\|_{H^p}^p$$

□

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In this case, by the previous lemma, we have that

$$\left\| \frac{1}{1-z^2} \right\|_{H^p}^p = \left\| \frac{1}{1-z} \right\|_{H^p}^p .$$

The counterexample in the Hardy Space case

Let $0 < p < 1$. Then the functions f and g, defined respectively by

$$f(z) = \frac{1+z}{1-z}, \quad g(z) = -\frac{1-z}{1+z},$$

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This means that

$$\|f+g\|_{H^p} > \|f\|_{H^p} + \|g\|_{H^p} = 2\,\|f\|_{H^p}.$$

Proof.

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Now, applying the lemma, and using the simple fact that $|1 + z| < 2$ for all $z \in \mathbb{T} \setminus \{1\}$, we have that

$$\begin{aligned}\|f + g\|_{H^p} &= \left\|\frac{4}{1 - z^2}\right\|_{H^p} = 4\left\|\frac{1}{1 - z}\right\|_{H^p} > 2\left\|\frac{1 + z}{1 - z}\right\|_{H^p} \\ &= 2\|f\|_{H^p} = \|f\|_{H^p} + \|g\|_{H^p}\end{aligned}$$

□

The Bergman space case

Some basic facts that we need:

- As in the Hardy space case, $\|\cdot\|_{A^p}$ is invariant under rotations. In particular, if $g(z) = f(-z)$, then $\|g\|_{A^p} = \|f\|_{A^p}$.

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- In the special case $f = \sum_{n=0}^{\infty} a_n z^n \in A^2$, we can compute the norm of f in terms of the Taylor coefficients:

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- Given $\alpha > 0$,

$$h(z) = \frac{1}{(1-z)^\alpha} \in A^p \Leftrightarrow p\alpha < 2.$$

Lemma (Boundedness of the composition with the function $z \mapsto z^2$)

If $h \in A^p$, then

$$\int_{\mathbb{D}} |h(z)|^p dA(z) = 2 \int_{\mathbb{D}} \left|h\left(z^2\right)\right|^p |z|^2 dA(z).$$

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Proof.

By the change of variable $2\theta = \varphi$ and periodicity,followed by another change of variable $r^2 = \rho$

$$\begin{aligned} \int_{\mathbb{D}} \left| h\left(z^2\right) \right|^p |z|^2 dA(z) &= \int_0^1 2r^3 \int_0^{2\pi} \left| h(r^2 e^{2i\theta}) \right|^p \frac{d\theta}{2\pi} dr \\ &= \int_0^1 2r^3 \int_0^{2\pi} \left| h(r^2 e^{i\varphi}) \right|^p \frac{d\theta}{2\pi} dr \\ &= \int_0^1 \rho M_p^p(\rho, h) d\rho. \end{aligned}$$

Theorem

Let $1/2 \le p < 1$ and let $\epsilon \le 1$ and $(1-p)/p \le \epsilon < 2(1-p)/p$. Then the functions f and g, given by

$$f(z) = \frac{(1+z)^{2-\epsilon}}{(1-z)^{2+\epsilon}}, \quad g(z) = -f(-z) = -\frac{(1-z)^{2-\epsilon}}{(1+z)^{2+\epsilon}},$$

fail to satisfy the triangle inequality for $\| \cdot \|_{A^p}$.

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Proof

As we know, $\|f\|_{A^p} = \|g\|_{A^p}$. We oberve that

$$f(z) + g(z) = \frac{(1+z)^4 - (1-z)^4}{(1-z^2)^{2+\epsilon}} = \frac{8z(1+z^2)}{(1-z^2)^{2+\epsilon}}.$$

We need to prove that $\|f+g\|_{A^p} > \|f\|_{A^p} + \|g\|_{A^p}$, which is the same as $\|f+g\|_{A^p}^p > 2^p \|f\|_{A^p}^p$.

Proof.

By the computation, and applying the lemma, this is equivalent to

$$2^{3p}\int_{\mathbb{D}}\frac{|z|^p|1+z^2|^p}{|1-z^2|^{(2+\epsilon)p}}dA(z) > 2^p\int_{\mathbb{D}}\frac{|1+z|^{(2-\epsilon)p}}{|1-z|^{(2+\epsilon)p}}dA(z)$$

$$= 2^{p+1}\int_{\mathbb{D}}\frac{|z|^2|1+z^2|^{p(2-\epsilon)}}{|1-z^2|^{(2+\epsilon)p}}dA(z).$$

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Rewriting this, we have

$$\int_{\mathbb{D}}\frac{|z|^p|1+z^2|^p(2^{2p-1}-|z|^{2-p}|1+z^2|^{p(1-\epsilon)})}{|1-z^2|^{(2+\epsilon)p}}dA(z) > 0.$$

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By the election of ϵ and the restrictions on p, we have $(2p-1)-(1-\epsilon)p = p+\epsilon p - 1 \geq 0$ and $(1-\epsilon)p \geq 0$, so that

$$2^{2p-1}-|z|^{2-p}|1+z^2|^{(1-\epsilon)p} > 2^{2p-1}-2^{(1-\epsilon)p} \geq 0$$

for all $z \in \mathbb{D}$, which finishes the proof. □

Theorem

Let $0 < p < 1/2$ *and define*

$$f(z) = (1+z)^{4/p}, \quad g(z) = -f(-z) = -(1-z)^{4/p},$$

choosing the appropriate branch of the complex logarithm so that, say, $\log 1 = 0$*. Then, the functions* f *and* g *both belong to* A^p *but fail to satisfy the triangle inequality for* $\|\cdot\|_{A^p}$*.*

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In order to prove this result, we need the following elementary lemma:

Lemma

If $a, b > 0$ *and* $q > 1$*, then* $|a^q - b^q| \geq |a-b|^q$*.*

Proof

As in the previous examples, $\|f\|_{A^p} = \|g\|_{A^p}$. We can compute this value using the formula for the A^2 functions based on the Taylor coefficients:

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |1 + 2z + z^2|^2 dA(z) = 1 + 2 + \frac{1}{3} = \frac{10}{3}.$$

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Next, using the previous lemma, integrating in polar coordinates and using Fubini, we obtain:

$$\begin{aligned}\|f + g\|_{A^p}^p &= \int_{\mathbb{D}} |(1+z)^{4/p} - (1-z)^{4/p}|^p dA(z) \\ &\geq \int_{\mathbb{D}} \left| |1+z|^{4/p} - |1-z|^{4/p} \right|^p dA(z) \\ &\geq \int_{\mathbb{D}} \left| |1+z|^4 - |1-z|^4 \right| dA(z)\end{aligned}$$

Proof.

$$= \int_{\mathbb{D}} |(1+|z|^2+2\mathrm{Re} z)^2 - (1+|z|^2-2\mathrm{Re} z)^2| dA(z)$$

$$= 8\int_{\mathbb{D}} (1+|z|^2)|\mathrm{Re} z| dA(z)$$

$$= \frac{8}{\pi}\int_0^1 r^2(1+r^2)dr \cdot 2\int_{-\pi/2}^{\pi/2} \cos(\theta)d\theta$$

$$= \frac{2^8}{15\pi} > 2^p\frac{10}{3} = (\|f\|_{A^p} + \|g\|_{A^p})^p$$

provided that $p < 1/2$. Actually, the inequality holds for a larger range of values of p ($0 < p < \log_2(128/25\pi) \approx 0,705$). □

Thank you for your attention!