Interpolating sequences for pairs of spaces

Georgios Tsikalas

Washington University in St. Louis/
Vanderbilt University

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Definition

A sequence $\{z_n\}$ in $\mathbb{D}$ is interpolating for H^∞ if for every sequence $\{w_n\} \in \ell^\infty$, there exists $f \in H^\infty$ such that

$$f(z_n) = w_n, \quad \forall n.$$

Write $\{z_n\}$ satisfies (IS).

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(WS) is weakly separated if there exists $\delta > 0$ such that,

$$d(z_n, z_m) := \left| \frac{z_n - z_m}{1 - \overline{z_n} z_m} \right| > \delta, \quad \text{for all } m \neq n,$$

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(CM) satisfies the Carleson measure condition if there exists $M > 0$ such that

$$\sum_j (1 - |z_j|^2) |f(z_j)|^2 \leq M \int_{\partial \mathbb{D}} |f|^2 dm, \quad \forall f \in \mathbb{C}[z],$$

i.e. $\mu = \sum_j (1 - |z_j|^2) \delta_{z_j}$ is a Carleson measure on $\mathbb{D}$.

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Theorem (Carleson, 1958)

(IS) $\Leftrightarrow$ (WS) + (CM).

Reproducing kernel Hilbert spaces

A RKHS $\mathcal{H}_k$ on a set X is a Hilbert space of functions $f : X \to \mathbb{C}$ such that point evaluations are continuous.
Thus, $\forall w \in X$ there exists $k_w \in \mathcal{H}_k$ such that

$$f(w) = \langle f, k_w \rangle_{\mathcal{H}_k}, \quad \forall f \in \mathcal{H}_k.$$

The function $k : X \times X \to \mathbb{C}$ defined as $k(z,w) := k_w(z)$ is the reproducing kernel of $\mathcal{H}_k$.

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$$\mathrm{Mult}(\mathcal{H}_k) = \{\phi : X \to \mathbb{C} \mid \phi \cdot f \in \mathcal{H}_k \text{ for all } f \in \mathcal{H}_k\}.$$

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Example

Let $H^2 = \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}) : ||f||^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$.
Then, $k(z, w) = \frac{1}{1 - z\overline{w}}$ and $\mathrm{Mult}(H^2) = H^\infty$ with equality of norms.

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Lemma (Shapiro–Shields)

A sequence $\{z_n\}$ in $\mathbb{D}$ is interpolating for H^∞ if and only if the operator

$$f \mapsto \left\{ f(z_n)\sqrt{1-|z_n|^2} \right\}_n = \left\{ \frac{f(z_n)}{||k_{z_n}||} \right\}_n$$

maps H^2 onto ℓ^2.

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Key property

H^2 and $\mathcal{D}$ are complete Pick spaces.

Nevanlinna-Pick Interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, z_2, \ldots, z_n \in \mathbb{D}$ and $w_1, w_2, \ldots, w_n \in \mathbb{C}$. There exists $\phi \in \text{Mult}(H^2) = H^\infty$ with

$$\phi(z_i) = w_i \text{ for } 1 \leq i \leq n \quad \text{and} \quad ||\phi||_{\text{Mult}(H^2)} \leq 1$$

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if and only if the matrix

$$\left[\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right]_{i,j=1}^n = \left[(1 - w_i\overline{w_j})k(z_i, z_j)\right]_{i,j=1}^n$$

is positive semi-definite.

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is positive semi-definite. Recall that $k(z, w) = (1 - z\overline{w})^{-1}$ is the reproducing kernel of H^2.

Complete Pick spaces

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- $\mathcal{H}_k$ is called a Pick space if this condition is also sufficient.
- $\mathcal{H}_k$ is called a complete Pick space if the analogue of this condition for matrix-valued functions is sufficient.

Examples

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- The Bergman space $A^2 = \mathrm{Hol}(\mathbb{D}) \cap L^2(\mathbb{D})$ is not a Pick space.
- The Dirichlet space $\mathcal{D}$ is a complete Pick space (Agler).
- The Drury-Arveson space H^2_d is the RKHS on $\mathbb{B}_d$, the open unit ball in $\mathbb{C}^d$, with kernel

$$k(z,w) = \frac{1}{1 - \langle z, w\rangle} = \frac{1}{1 - \sum_{i=1}^d z_i \overline{w}_i}.$$

H^2_d is a complete Pick space and is also universal among all such spaces (McCullough–Quiggin, Agler–McCarthy).

A distance function for RKHS's

Let $\mathcal{H}_k$ be a RKHS on a set X with kernel k. Also, let $\hat{k}_x := \dfrac{k_x}{||k_x||}$ denote the *normalized* kernel function at x.

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Example

If $\mathcal{H}_k = H^2$, then

$$d_k(z,w) = \left| \frac{z-w}{1-\overline{z}w} \right|$$

is the pseudohyperbolic metric on $\mathbb{D}$.

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i.e. $\mu := \sum_j \frac{1}{k(z_j,z_j)}\delta_{z_j}$ is a Carleson measure for $\mathcal{H}_k$.

Old and new developments

Lemma

In every RKHS $\mathcal{H}_k$, (IS) $\Rightarrow$ (WS) $+$ (CM).

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- Bøe, 2005 holds in those spaces on the unit ball $\mathbb{B}_d$ with kernel

$$k(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\alpha}, \quad \text{where } \alpha \in (0, 1);$$

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Theorem (Aleman–Hartz–McCarthy–Richter, 2017)

In every complete Pick space, (IS) $\Leftrightarrow$ (WS) + (CM).

Grammians

Let $\mathcal{H}_k$ be a RKHS on X with kernel k, let $\{z_n\} \subset X$. Recall that $\hat{k}_z = k_z/||k_z||$ and define the Grammian

$$G[\{z_n\}] = \left[\left\langle \hat{k}_{z_i}, \hat{k}_{z_j} \right\rangle\right]_{i,j}.$$

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Theorem (Marshall-Sundberg, '94)

If $\mathcal{H}_k$ is a complete Pick space, then

$\boxed{\text{(IS)}} \iff \boxed{G[\{z_n\}] : \ell^2 \to \ell^2 \ \text{ bounded and bounded below}}$

Two proofs of the A.-H.-M.-R. characterization

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- Original proof used the solution to the Kadison-Singer problem by Marcus, Spielman and Srivastava (2013).
- New proof uses the column-row property:

Theorem (Hartz, 2020)

Assume $\mathcal{H}_k$ is a complete Pick space and $\{\phi_n\} \subset \text{Mult}(\mathcal{H}_k)$. Then,

$$\left|\left| \begin{bmatrix} M_{\phi_1} & M_{\phi_2} & \cdots \end{bmatrix} \right|\right| \leq \left|\left| \begin{bmatrix} M_{\phi_1} \\ M_{\phi_2} \\ \vdots \end{bmatrix} \right|\right|.$$

Pairs of spaces

Let $\mathcal{H}_k$, $\mathcal{H}_\ell$ be two RKHSs on a set X with kernels k, ℓ, resp. Define

$$\text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell) := \{\phi : X \to \mathbb{C} \mid \phi \cdot f \in \mathcal{H}_\ell, \, \forall f \in \mathcal{H}_k\}.$$

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Example

- $\mathcal{H}_k = H^2 =$ Hardy space
- $\mathcal{H}_\ell = A^2 =$ Bergman space on $\mathbb{D}$

$$\boxed{H^2 \subset A^2} \ \Rightarrow \ \boxed{\mathrm{Mult}(H^2) \subset \mathrm{Mult}(H^2, A^2)}.$$

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Actually, we even have

$$\boxed{H^2 \subset \mathrm{Mult}(H^2, A^2)}.$$

Interpolating sequences for pairs of kernels

Let $\mathcal{H}_k$ and $\mathcal{H}_\ell$ be two RKHSs on X with kernels k and ℓ.

Observation

If $\phi \in \text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$, then $|\phi(z)| \leq ||\phi||_{\text{Mult}} \dfrac{||\ell_z||}{||k_z||}$, for all $z \in X$.

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Definition

A sequence $\{z_n\}$ in X is interpolating for $\text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ (write (IS)) if for all $\{w_n\} \in \ell^\infty$, there exists $\phi \in \text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ such that

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Lemma

$$\boxed{\text{(IS)} \quad \text{wrt} \quad \text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)} \Rightarrow \boxed{\text{(CM) for } k} + \boxed{\text{(WS) by } \ell}$$

Complete Pick factors

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Question (Aleman–Hartz–McCarthy–Richter, 2017)

Suppose that k is a complete Pick factor of ℓ. Is it true that

$$\boxed{\text{(IS) wrt } \operatorname{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)} \Leftrightarrow \boxed{\text{(CM) for } k} + \boxed{\text{(WS) by } \ell} \ ?$$

Beyond weak separation

Definition

Let ℓ be a kernel on X and $\{z_n\} \subset X$. Given $m \geq 2$, we say that $\{z_n\}$ is *m*-weakly separated by ℓ (write (m-WS)) if there exists $\delta_m > 0$ such that for every m-point subset $\{\mu_1, \ldots, \mu_m\} \subset \{z_n\}$ we have

$$d = \operatorname{dist}\left(\hat{\ell}_{\mu_1},\ \operatorname{span}\left\{\hat{\ell}_{\mu_2}, \ldots, \hat{\ell}_{\mu_m}\right\}\right) \geq \delta_m.$$

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Let $X = \{1, 2, 3\}$, $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $v_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$.

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Then, $\{1, 2, 3\}$ will be (2-WS) but not (3-WS) by ℓ.

The characterization

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Let ℓ be a kernel on X and $\{z_n\} \subset X$. Given $m \geq 2$, we say that $\{z_n\}$ is *m-weakly separated by ℓ* (write *(m-WS)*) if there exists $\delta_m > 0$ such that for every m-point subset $\{\mu_1, \dots, \mu_m\} \subset \{z_n\}$ we have

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Theorem (T., 2022)

$$\boxed{\text{(IS)}} \Leftrightarrow \boxed{\text{(CM) for } k} + \boxed{(m\text{-WS}) \text{ by } \ell,\ \ \forall m \geq 2}$$

Moreover, the separation condition cannot, in general, be relaxed.

The automatic separation property

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This is equivalent to:
for any fixed $m \geq 2$, a kernel $\hat{\ell}_z$ can be "close" to the span of m other kernels $\hat{\ell}_{w_1}, \hat{\ell}_{w_2}, \dots, \hat{\ell}_{w_m}$ if and only if it is "close" to one of them.

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Then, ℓ does not have the automatic separation property.

The missing link

Let $\mathcal{H}_k, \mathcal{H}_\ell$ be two RKHSs on X such that k is a complete Pick factor of ℓ. Recall: (IS)=interpolating for $\mathrm{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$.

Question (Aleman–Hartz–McCarthy–Richter, 2017)

Is it true that $\boxed{\text{(IS)}} \Leftrightarrow \boxed{\text{(CM) for } k} + \boxed{\text{(WS) by } \ell}$?

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For "regular" kernels, the answer is yes IFF ℓ has the automatic separation property.

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- kernels of Bergman spaces on $\mathbb{D}$ with exponentially decaying weights (Borichev–Dhuez–Kellay);
- kernels of weighted Bargmann-Fock spaces on $\mathbb{C}^n$, e.g. $\ell(z, w) = e^{\alpha \cdot z\overline{w}}$ (Massaneda–Thomas).

A counterexample

Theorem (T., 2022)

Assume, in addition, that k, ℓ are "regular" kernels. TFAE:

- $\boxed{\text{(IS) wrt } \operatorname{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)} \Leftrightarrow \boxed{\text{(CM) for } k} + \boxed{\text{(WS) by } \ell}$
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Example

Let ρ be the kernel corresp. to the Bergman space on $\mathbb{D}$ with weight $e^{-\frac{1}{1-|z|^2}}$.For $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{D}^2$, define

$$\ell(z, w) = \frac{\rho(z_1, w_1) + \rho(z_2, w_2)}{(1 - z_1\overline{w}_1)(1 - z_2\overline{w}_2)}.$$

ℓ is "regular", but doesn't have the automatic sep. property.

A "metric" description of (m-WS)

Let ℓ be a kernel on X and assume that $\{z_n\} \subset X$ is (WS). Then, given $m \geq 3$, $\{z_n\}$ will be (m-WS) if and only if there exists $\delta > 0$ (depending on m) such that

$$d_\ell(z, w; \mu_1, \mu_2, \dots, \mu_{m-2}) > \delta,$$

for all $z \neq w$ and for any $m-2$ point subset $\{\mu_1, \dots, \mu_{m-2}\}$ of $\{z_n\}$ that does not contain either z or w, where $d_\ell(\cdot, \cdot\, ; \mu_1, \mu_2, \dots, \mu_{m-2})$ is the metric associated with the subspace of $\mathcal{H}_\ell$ given by

$$\{f \in \mathcal{H}_\ell \;:\; f(\mu_1) = \cdots = f(\mu_{m-2}) = 0\}.$$