SEMINARIO DE ANÁLISIS COMPLEJO
(COMPLEX ANALYSIS SEMINAR)

Navier-Stokes on manifolds

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Viernes, 11 de mayo de 2018 a las 10:30
(Friday, May 11, 2018 at 10:30)

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Resumen / Abstract:

Navier-Stokes equations on $\mathbb{R}^n$ are typically written as follows:

$$u_t + u\nabla u - \mu\Delta u + \nabla p = f$$
$$\nabla \cdot u = 0$$

where u is the velocity field and p is the pressure. To formulate this on a Riemannian manifold M one can write $\mathsf{grad}(p) = \sharp\nabla p = g^{ij} p_{;j}$ and $\mathsf{div}(u) = \mathsf{tr}(\nabla u) = u^i_{;i}$ where ∇ is now the covariant derivative. The nonlinear term is $\nabla_u u = u^k_{;i} u^i$. [a]

The diffusion term is more problematic because there are various ways to define Laplacian for vector fields. I will discuss some of the issues which arise and argue why a certain operator is physically most reasonable. This choice shows that Killing vector fields play an essential role in the analysis of the problem. Killing fields are the inifinitesimal generators of volume preserving maps, in other words they satisfy the condition $\mathsf{div}(u) = 0$. Killing fields are also solutions of the Euler equations, i.e. the Navier Stokes equations without viscosity.

It turns out that the full curvature tensor does not appear in the equations, only its contraction, known as the Ricci tensor. In the two dimensional case this reduces to the Gaussian curvature. I will discuss some properties of the solutions when M is compact and without boundary. From the numerical point of view the linearized version of the system is also interesting, and I will discuss that case also.

The main motivation for studying flows on the manifolds comes from atmospheric models. The two dimensional flow on the sphere is too simplistic for real applications; however, since the atmosphere is a relatively thin layer, this flow is anyway an important part of the model.

[a] Einstein summation convention is used.