THE CENTERED HARDY-LITTLEWOOD MAXIMAL OPERATOR IN HIGH DIMENSIONS

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Here, now.

DEFINITION

Hardy-Littlewood maximal operator:

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

- Variants:
- Uncentered
- Over different balls
- Over more general sets
- Using other measures
- $M\nu$, ν a measure

Useful because:

• Mf is larger than $|f|$ but not much larger

• Mf is more regular (lower semi-continuous) than $|f|$ (for f measurable).

Mf IS LARGER THAN $|f|$ BUT NOT MUCH LARGER.

Consider Lebesgue measure on $\mathbb{R}^d$.

- $|f| \leq Mf$ a.e.
- M satisfies the strong type (p, p) inequality: For $1 < p \leq \infty$,

$$\|Mf\|_p \leq C_p \|f\|_p.$$

- $\|Mf\|_\infty \leq \|f\|_\infty$, so $\|Mf\|_\infty = \|f\|_\infty$

- M is unbounded on L_1

- Example: Take $f = \mathbf{1}_{[0,1]}$

- Hence $\lim_{p \downarrow 1} C_p = \infty$

• Boundedness properties depend on the operator.

• Example: Stein's spherical maximal operator is bounded for $p > \frac{d}{d-1}$, $d \geq 3$ (Stein, 1976)); and for $d = 2$ (Bourgain, 1986).

GENERAL QUESTION: HOW DO THE CONSTANTS c_1, C_p BEHAVE?

• For $p = 1$, M satisfies the weak type $(1, 1)$ inequality

$$\sup_{\alpha>0} \alpha|\{Mf > \alpha\}| \leq c_1\|f\|_1.$$

• Dimension $d = 1$, Lebesgue measure. Centered maximal operator M, best $c_1 \approx 1.56$ (the constant below).

$$|\{x\colon Mf(x) > \alpha\}| \leq \frac{11+\sqrt{61}}{12\alpha}\|f\|_1$$

Proof. Discretization (Melas, TAMS 2002, Ann. Math. 2003).

Dimension $d = 1$, Lebesgue measure, Strong type (p, p) inequality, $1 < p < \infty$:

$$\|Mf\|_p \leq C_p \|f\|_p.$$

- Centered maximal operator M_c, best C_p unknown.

• Optimal bounds are hard to find. In general, we are happy if we find “good" bounds.

BEHAVIOR OF THE CONSTANTS $c_{1,d}$, $C_{p,d}$ AS THE DIMENSION $d \to \infty$

- Uncentered Hardy-Littlewood maximal operator:
- Euclidean balls, Lebesgue measure in $\mathbb{R}^d$
- $C_{p,d}$, $c_{1,d}$ grow exponentially in d.

STRONG TYPE BOUNDS, CENTERED MAXIMAL OPERATOR, LEBESGUE MEASURE IN $\mathbb{R}^d$:

• Euclidean balls, fixed $p > 1$, $\sup_{d \geq 1} C_{p,d} < \infty$, (Stein, 1982).

Proof. Use the boundedness of Stein's spherical maximal function.

• Arbitrary balls, $p \geq 2$, $\sup_{d \geq 1} C_{p,d} < \infty$ (Bourgain, 1986)

Proof. Put the ball in isotropic position. Use Fourier Analysis.

• Arbitrary balls, fixed $p > 3/2$, $\sup_{d \geq 1} C_{p,d} < \infty$ (Bourgain, 1986, Carbery, 1986).

Proof. Fourier Analysis.

• Recall: ℓ_q balls ($1 \leq q < \infty$) given by the norm

$$\|x\|_q := \left(|x_1|^q + |x_2|^q + \cdots + |x_d|^q\right)^{1/q}$$

• ℓ_q balls ($1 \leq q < \infty$) fixed $p > 1$, $\sup_{d \geq 1} C_{p,d} < \infty$, (D. Müller, 1990).

• B_d ball with unit volume. Müller proved: If minimal sections (through the origin) and maximal projections (over hyperplanes passing through the origin) are bounded independently of d, then $\sup_{d \geq 1} C_{p,d} < \infty$.

Proof. Fourier Analysis.

- Left open: ℓ_∞ balls (cubes with sides parallel to the axes), $1 < p \leq 3/2$.

- Volume of minimal sections: 1.

- Volume of maximal projections: $\sqrt{d}$.

• ℓ_∞ balls (cubes), uniform bounds also exist for fixed $1 < p \leq 3/2$:

$$\sup_{d \geq 1} C_{p,d} < \infty.$$

(Bourgain, Math ArXiv, December 2012).

• Summing up: For all $p > 1$ and all ℓ_q balls ($1 \leq q \leq \infty$), there exist strong bounds uniform in the dimension.

• For all $p > 3/2$ and all balls, there exist strong bounds uniform in the dimension.

WEAK TYPE $(1,1)$ BOUNDS, CENTERED MAXIMAL OPERATOR, LEBESGUE MEASURE IN $\mathbb{R}^d$:

• Vitali Covering Theorem, $c_{1,d} \leq 3^d$, or $c_{1,d} \leq 2^d$.

• Arbitrary balls, $c_{1,d} \leq O(d \log d)$, covering lemma of "Vitali type" (Stein and Strömberg, 1983).

• Euclidean balls, $c_{1,d} \leq O(d)$, maximal ergodic theorem for the heat semigroup. (Stein and Strömberg, 1983).

• Question: For Euclidean balls, is $\sup_{d \geq 1} c_{1,d} < \infty$? (Stein and Strömberg, 1983).

• Answer not known.

• My guess: NO.

• Reason: For ℓ_∞ balls (cubes), uniform bounds do NOT exist: $\lim_{d \to \infty} c_{1,d} = \infty$ (JMA, 2011).

Proof. Discretization + calculus + normal approximation to the binomial distribution.

• Rate of divergence: $c_{1,d} \geq \Theta(\log^{1-o(1)} d)$ (G. Aubrun, 2009, shortly after my paper).

Proof. Discretization (same example) + stochastic process (the Brownian bridge).

GUESSES, INTUITIONS

• Maximal functions associated to cubes and to euclidean balls (or more general balls) should behave roughly in the same way.

• So for Lebesgue measure on $\mathbb{R}^d$, $p > 1$, and arbitrary balls, we expect uniform bounds for $C_{p,d}$, but not for $c_{1,d}$.

• Good bounds seem to depend on "balls of the same radius have similar measures", rather than on doubling.

NON-UNIFORM BOUNDS, CENTERED MAXIMAL OPERATOR, EUCLIDEAN BALLS, MEASURES IN $\mathbb{R}^d$:

• Finite measures defined via radial, radially decreasing bounded densities (example, standard gaussian):

• Exponential increase of $c_{1,d}$ with d (JMA, 2007)

Proof. Discretization (1 delta). Use that balls centred far away from the origin have much smaller measure than balls of the same radius centred at the origin.

• Refinements of the same idea yield exponential increase for small values of $p > 1$

• For finite measures defined via radial, radially decreasing bounded densities (Criado, 2010, Ph. D. Thesis 2012 under Prof. F. Soria)

• For more general (radial, radially decreasing) measures, including some doubling measures (J. Pérez Lázaro-JMA, 2011) and some high values of p in the latter case.

• For the standard gaussian measure and all $1 < p < \infty$, $C_{p,d}$ increases exponentially with d (Criado-Sjögren, 2012)

Some doubling measures:

• $t \in (0, 1)$, $d\mu_{t,d} := \|x\|_2^{-td} dx$ on $\mathbb{R}^d$.

• $\mu_{t,d}$ is doubling.

• For all $t \in (1/2, 1)$ and all $1 < p < \infty$, $C_{p,d}$ increases exponentially with d (J. Pérez Lázaro-JMA, A. Criado, Ph. D. Thesis, independently)

Proof. Adaptation of the Criado-Sjögren argument for the gaussian measure.

BEYOND $\mathbb{R}^d$:

• Volume in d-dimensional hyperbolic space, geodesic balls

• Volume is not doubling. In fact, hyperbolic space admits no doubling measures

• Balls of equal radius have equal volume

• $c_{1,d} \leq O(d \log d)$ (Li-Lohoué, 2012)

Proof. Semigroup methods

• $p > 1$, $\sup_{d \geq 1} C_{p,d} < \infty$ (Li, personal communication)

Proof. Semigroup methods

Centered Hardy-Littlewood maximal operator, Metric measure spaces

• Metric measure space: Separable metric space with a Radon measure.

• Naor and Tao (2010): the Stein-Strömberg $c_{1,d} \leq O(d \log d)$ bound holds for metric measure spaces that satisfy the "Strong Microdoubling Condition" (ex., Ahlfors-David regular spaces).

• Strong d-Microdoubling with constant K: For all x, all $r > 0$ and all $y \in B(x, r)$,

$$\mu B\left(y, \left(1 + \frac{1}{d}\right) r\right) \leq K \mu B(x, r).$$

• Idea: Use ultrametric spaces, random martingales, Doob's maximal inequality.

• A second argument: Use E. Lindenstrauss "Random Vitali Covering Theorem" (2001).

THANKS FOR LISTENING!