MEROMORPHIC FUNCTIONS WITHOUT MULTIPLE VALUES IN THE PUNCTURED PLANE

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Madrid 2015

Introduction

The study of meromorphic functions without multiple values in the complex plane $\mathbb{C}$ was initiated by F.Nevanlinna [2].
We shall try to extend his results to the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
The results of F.Nevanlinna are obtained in the frame of the Value Distribution Theory of meromorphic functions due to R.Nevanlinna.
We shall make use of the Value Distribution Theory for meromorphic functions in annuli developed by K.H.Khrystiyanin and A.A.Kondratyuk [4], [5] and R.Korhonen [1]

F.Nevanlinna's results on meromorphic functions without multiple values

We summarize some results of F.Nevanlinna on meromorphic functions without multiple values in the complex plane $\mathbb{C}$

Theorem (F.Nevanlinna). *If $f(z)$ is a meromorphic function of finite order ρ* in $\mathbb{C}$ *without multiple values, then*

1) *$f(z)$ is of regular growth*
2) *ρ is a multiple of* $1/2$
3) *The sum of the deficiencies of $f(z)$ is* 2
4) *Each deficiency is a multiple of* $1/\rho$
5) Every deficient value is asymptotic

Two Theorems on Differential Equations in C*

The research of F.Nevanlinna is based very strongly on some results differential equations [2] on the holomorphic solutions of complex linear differential equations.
Next, we present two results on differential equations in the context of $\mathbb{C}^*$.
To do this we recall the definition of the Schwarzian derivative $\{f(z), z\}$ of a meromorphic function $f(z)$

$$\{f(z), z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

Two Theorems on Differential Equations in C*

Theorem. *Let* $p(z)$ *be an holomorphic function in* $\mathbb{C}^*$. *If* $f(z)$ *is a locally injective meromorphic solution of the third order non-linear differential equation*

$$\frac{1}{2}\{f(z), z\} = p(z) \ , \tag{1}$$

in $\mathbb{C}^*$, *then there exist two linearly independent solutions* $u_1(z), u_2(z)$ *of the second order linear differential equation*

$$u'' + p(z)\, u = 0 \ , \tag{2}$$

such that

$$f(z) = \frac{u_1(z)}{u_2(z)} \ , \tag{3}$$

and this representation is valid all over $\mathbb{C}^*$. *This representation is unique if we choose* $u_2(z_0)$ *for a fixed* $z_0 \in \mathbb{C}^*$.

Two Theorems on Differential Equations in C*

Theorem. (Continuation)
Conversely, given two linearly independent local solutions $u_1(z), u_2(z)$ *of the equation* (2) *in a simply connected neighbourhood of* z_0 , *say* $D(z_0, c)$, *then* (3) *yields a local solution in* $D(z_0, c)$ of (1) *and by analytic continuation we obtain a solution* $f(z)$ *of* (1) *in* $\mathbb{C}^*$, *which in general will be multivalued, in such a way that there exists a Möbius transformation* T, *such that given two local branches* $f(z), f^*(z)$ *in an arbitrary disc* $D(\alpha, \epsilon) \subset \mathbb{C}^*$, *there exists an integer* $k \in \mathbb{Z}$ *such that*

$$f^*(z) = T^k \circ f(z) \ .$$

Two Theorems on Differential Equations in C*

Theorem (E.Hille). *Given the second order complex differential equation*

$$u'' + p(z)\, u = 0 \tag{1}$$

where $p(z)$ *is an holomorphic function in* $R_0 < |z| < \infty$ *such that*

$$p(z) = a^2 z^m [1 + O(1)] \ , \ a \in \mathbb{C} \ , \ m \ \text{ a positive integer} \tag{2}$$

then making the change of variables

$$\zeta = \phi(z) = \int_{z_0}^{z} (p(t))^{\frac{1}{2}}\, dt \ , \quad |z_0| > R_0 \ , \quad U(\zeta) = p^{\frac{1}{4}}(z)\, u(z)$$

we get to the equation

$$U''(\zeta) + (1 - h(\zeta))\, U(\zeta) = 0$$

where $h(\zeta) = O\left(\zeta^{-2}\right)$ *and in these conditions equation* (2) *has two linearly independent solutions* $U_1(\zeta), U_2(\zeta)$ *such that for large* ζ

$$U_1(\zeta) \sim e^{-i\zeta} \ , \ U_2(\zeta) \sim e^{i\zeta} \ \text{in} \quad |\arg \zeta| < \pi - \delta$$

Two theorems on Differential Equations in C*

We conclude putting these two theorems together that if in equation (1) we have $p(z) = \frac{1}{2}\{f(z), z\}$ then

$$f(z) = \frac{U_1(\zeta)}{U_2(\zeta)} \sim e^{2i\zeta} \text{ in } \quad |\arg\zeta| < \pi - \delta$$

Condition (2) is satisfied for finite order meromorphic functions without multiple values.
In fact.

$$f(z) = c_0 + c_p(z - z_0)^p + c_{p+1}(z - z_0)^{p+1} \;\; , \; c_p \neq 0 \; , \; p \gtrless 0$$

then

$$\{f(z), z\} = \frac{1 - p^2}{2}(z - z_0)^{-2} - \frac{1 - p^2}{pc_p} c_{p+1}(z - z_0)^{-1} + ..$$

and by the logarithmic derivative Lemma in Nevanlinna Theory

$$T(r, \{f(z), z\}) = O(\log r)$$

that is $\{f(z), z\}$ is a polynomial.

Nevanlinna Theory on Annuli

We recall the basic facts of the Value Distribution Theory or Nevanlinna Theory on Annuli.
We shall pay special attention to the particular case of the punctured plane $\mathbb{C}^*$.
We shall follow K.H.Krystiyanin and A.A.Kondratyuk [4][5].

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Let $f(z)$ be a meromorphic function in the annulus

$$A = A(R_0) = \left\{ z \;\middle|\; \frac{1}{R_0} < |z| < R_0 \right\} \text{ where } 1 < R_0 \leq \infty \; .$$

For a finite value $a \in \mathbb{C}$ we define the proximity function

$$m_0\left(r, \frac{1}{f-a}\right) = m\left(r, \frac{1}{f-a}\right) + m\left(\frac{1}{r}, \frac{1}{f-a}\right) \quad ,$$

where $m\left(r, \frac{1}{f-a}\right), m\left(\frac{1}{r}, \frac{1}{f-a}\right)$ denote the classical proximity functions, and for $a = \infty$ we set analogously

$$m_0(r, \infty) = m_0(r, f) = m(r, f) + m\left(\frac{1}{r}, f\right)$$

Nevanlinna Theory on Annuli. The Counting Function.

We define

$$N_1\left(r, \frac{1}{f-a}\right) = \int_{1/r}^{1} \frac{n_1\left(t, \frac{1}{f-a}\right)}{t} dt \ ,$$

$$N_2\left(r, \frac{1}{f-a}\right) = \int_{1}^{r} \frac{n_2\left(t, \frac{1}{f-a}\right)}{t} dt \ ,$$

for $1 < r < R_0$, where $n_1\left(t, \frac{1}{f-a}\right)$ is the simple counting function of poles of $\frac{1}{f(z)-a}$ in $\{z \mid \ t < |z| \leq 1\}$ and $n_2\left(t, \frac{1}{f-a}\right)$ is the simple counting function of the poles of this function in $\{z \mid \ 1 < |z| \leq t\}$ and

$$N_0\left(r, f\right) = N_1\left(r, \frac{1}{f-a}\right) + N_2\left(r, \frac{1}{f-a}\right) \ .$$

In an analogous way we define $N_0\left(r, f\right) = N_0\left(r, \infty\right)$.

Nevanlinna Theory in Annuli. The Characteristic Function

We call

$$T_0(r,f) = m_0(r,f) + N_0(r,f) \ , \ 1 < r < R_0 \ ,$$

the characteristic function of $f(z)$ and yields an indicator of the global growth of the function at zero and infinity

First Main Theorem for Meromorphic Functions in the Punctured Plane

Let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}^*$ then

$$T_0\left(r, \frac{1}{f-a}\right) = T_0(r, f) + O(1) \ , \ \text{as } r \to \infty \ ,$$

for every fixed $a \in \mathbb{C}$

Second Main Theorem for Meromorphic Functions in the Punctured Plane

Let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}^*$ and let $a_1, a_2, .., a_p$ be $p \geq 2$ distinct finite complex numbers, then

$$m_0(r,f) + \sum_{\nu=1}^{p} m_0\left(r, \frac{1}{f - a_\nu}\right) \leq 2T_0(r,f) - N_0^1(r,f) + S(r,f) \ ,$$

(Fundamental Inequality)

where $N_0^1(r,f)$ is the ramification index given by

$$N_0^1(r,f) = N_0\left(r, \frac{1}{f'}\right) + 2N_0(r,f) - N_0(r,f') \ ,$$

and $S(r,f)$ plays the role of an error term, it satisfies the relation

$$S(r,f) = o(T(r,f)) \ ,$$

as $r \to \infty$ outside an exceptional set of finite measure.

Asymptotic behaviour in the punctured plane

Let $f(z)$ be a meromorphic function in $\mathbb{C}^*$, and let $z_0 \in \mathbb{C}^*$ and

$$f(z) = c_0 + c_p (z - z_0)^p + c_{p+1} (z - z_0)^{p+1} + \dots \quad (c_p \neq 0) \ ,$$

be the power series expansion of $f(z)$ near z_0 , then as in the plane case we shall make use of th Schwarzian derivative $\{f(z), z\}$ of $f(z)$ in $\mathbb{C}^*$

$$\{f(z), z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \ ,$$

and obtain

$$\{f(z), z\} = \frac{1-p^2}{2}(z - z_0)^{-2} - \frac{1-p^2}{pc_p} c_{p+1} (z - z_0)^{-1} + \dots$$

near z_0. this is also true near a pole, that is for $p < 0$. We deduce that if $f(z)$ has no multiple values, that is $p = 1$, then the Schwarzian derivative is holomorphic in $\mathbb{C}^*$.

Asymptotic behaviour in the punctured plane

Further, if $f(z)$ is of finite order, we obtain by the logarithmic derivative lemma

$$T_0(r,\{f(z),z\}) = m_0(r,\{f(z),z\})$$

$$= m_0\left(r,\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)\right) \leq m_0\left(r,\frac{f'''}{f'}\right)+m_0\left(r,\frac{f''}{f'}\right)+O(1) = O(\log$$

that is we deduce in this case that $\{f(z),z\}$ is a rational function $R(z)$ in $\mathbb{C}$ with only possible poles at $z=0,\infty$.
By the above Theorem of Hille $f(z)$ must be the quotient of two linearly independent solutions of the linear equation

$$w''+\frac{1}{2}R(z)w=0.$$

The method of integration of this equation is also due E.Hille [2].

Asymptotic behaviour in the punctured plane

If $R(z)$ is a rational function with only possible poles at $z = 0, \infty$ we shall have on one hand

$$R(z) = a^2 z^m \left[1 + O(1)\right] \ , \ a \in \mathbb{C} \ , \ m \in \mathbb{Z} \ ,$$

as $z \to \infty$.

On the other hand, we might also consider the meromorphic function

$$g(\lambda) = f\left(\frac{1}{z}\right) \ , \ z, \lambda \in \mathbb{C}^* \ ,$$

which does not have multiple values either and is also of finite order, so that we can proceed with $g(\lambda)$ as we have done with $f(z)$ to conclude that $g(\lambda)$ is the quotient of two linearly independent solutions of

$$w'' + T(\lambda)\, w = 0 \ ,$$

where $T(\lambda)$ is the Schwarzian derivative $\{g(\lambda), \lambda\}$ of $g(\lambda)$, which is also a rational function and also satisfies as $\lambda \to \infty$

$$T(\lambda) = b^2 \lambda^n \left[1 + O(1)\right] \ , \ b \in \mathbb{C} \ , \ n \in \mathbb{Z}$$

Asymptotic behaviour in the punctured plane

We recall that making the change of variable

$$\zeta = \phi(z) = \int_{z_0}^{z} R(t)^{\frac{1}{2}}\, dt ,$$

we obtain

$$f(z) \sim e^{2i\zeta},$$

and similarly setting

$$\xi = \psi(\lambda) = \int_{\lambda_0}^{\lambda} T(\lambda)^{\frac{1}{2}}\, d\lambda ,$$

we obtain

$$f\left(\frac{1}{z}\right) = g(\lambda) \sim e^{2i\lambda}$$

Asymptotic behaviour in the punctured plane

F.Nevanlinna shows in the that there are $m+2$ sectors A_k where asymptotic relations of the type

$$\zeta = \phi(z) \sim Bz^{\frac{m+2}{2}}$$

the image of these sectors in the $\zeta-$plane for $|z| < r_0$ will cover $|\arg \zeta| < \pi - \delta$, $|\zeta| > R_0$ and the solution of the above second order differential equation is given by

$$f(z) \sim e^{2i\zeta}$$

From these two relationships and an application of the argument principle it is derived that the number of solutions $n_k(t, f)$ of $f(z) = K$ in one of these sectors satisfies

$$n_k(t, f) = At^{m+2/2}$$

and as a consequence

$$N_k(t, f) = A_1 t^{m+2/2}$$

for every K except a finite number of exceptional values

Asymptotic behaviour in the punctured plane

We can apply the same arguments in $\mathbb{C}^*$ to both equations the corresponding to $f(z)$ for $|z| > R_0$ and the corresponding to $g(\lambda) = f\left(\frac{1}{z}\right)$ for $|\lambda| > R_0$ that is for $|z| < 1/R_0$
We conclude that except for a finite number of values K,it holds

$$N_2(r, K) \sim A_\infty r^{\frac{m+2}{2}} \quad ,$$

where A_∞ is a positive constant and $m \in \mathbb{Z}$, and on other hand except for a finite number of values, we have

$$m(r, K) = o\left(r^{\frac{m+2}{2}}\right),$$

so that from the First Main Theorem for meromorphic functions on annuli we obtain

$$T_2(r, f) = m(r, K) + N_2(r, K) + O(\log r)$$

$$= A_\infty r^{\frac{m+2}{2}}(1 + o(1)) \sim A_\infty r^{\frac{m+2}{2}}$$

Asymptotic behaviour in the punctured plane

Similarly, considering the function $g(\lambda)$ we can conclude from that except for a finite number of values of K, it holds

$$N_1(r, K) \sim T_1(r, f) \sim A_0 r^{\frac{m+2}{2}} \ ,$$

and

$$m\left(\frac{1}{r}, K\right) = o\left(r^{\frac{n+2}{2}}\right) \ ,$$

and again from the First Main Theorem for meromorphic functions on annuli we obtain

$$T_1(r, f) = m\left(\frac{1}{r}, K\right) + N_1(r, K) + O(\log r)$$

$$A_0 r^{\frac{n+2}{2}} (1 + o(1)) \sim A_0 r^{\frac{n+2}{2}}$$

Finally we obtain for $T_0(r, f)$ the relation

$$T_0(r, f) = T_1(r, f) + T_2(r, f) = A_o r^{\frac{n+2}{2}} + A_\infty r^{\frac{m+2}{2}} \ ,$$

that is

$$T_0(r, f) = Cr^{\frac{l+2}{2}} + O(\log r) \quad \text{where} \quad l = \max\{m, n\}$$

Deficiencies in the punctured plane

Given $a \in \widehat{\mathbb{C}}$ we define the deficiency of $f(z)$ with respect to a by

$$\delta_0(a, f) = \liminf_{r\to\infty} \frac{m_0(r, f, a)}{T_0(r, f)} \ ,$$

where

$$m_0(r, f) = m(r, f, a) + m\left(\frac{1}{r}, f, a\right) \ .$$

We shall say that a value $a \in \widehat{\mathbb{C}}$, is a deficient value for $f(z)$ when $\delta_0(a, f) > 0$.
To relate the deficiencies and deficient values of $f(z)$ to the behaviour of $f(z)$ at zero and infinity, we introduce the numbers

$$\delta_1(a, f) = \liminf_{r\to\infty} \frac{m\left(\frac{1}{r}, f, a\right)}{T_1(r, f)} \ \ , \ \ \delta_2(a, f) = \liminf_{r\to\infty} \frac{m(r, f, a)}{T_2(r, f)} \ ,$$

where

$$T_1(r, f) = m\left(\frac{1}{r}, f\right) + N_1(r, f) \quad \text{and} \quad T_2(r, f) = m(r, f) + N_2(r, f)$$

Deficiencies in the punctured plane

F.Nevanlinna, making use of the ideas of E.Hille, proved for meromorphic functions $f(z)$ in the plane satisfying

$$p(z) = \frac{1}{2}\{f(z), z\} = Az^m [1 + O(1)] \ , A \in \mathbb{C} \ , \ m \in \mathbb{Z} \qquad (1)$$

for $R_0 < |z| < \infty$, the existence of $m+2$ sectors A_k where the function $f(z)$ has respective asymptotic values a_k and these asymptotic values are deficient values with

$$\delta(a_k, f) = \liminf_{r\to\infty} \frac{m(r, f, a_k)}{T_2(r, f)} = \frac{2}{m+2} \ ,$$

so that

$$\sum_k \delta(a_k, f) = 2$$

Deficiencies in the punctured plane

In the case of meromorphic functions $f(z)$ in $\mathbb{C}^*$ such that the Schwarzian derivative satisfies the relation (1) and the Schwarzian derivative of the associated function

$$g(\lambda) = f\left(\frac{1}{z}\right) \ , \ z, \lambda \in \mathbb{C}^* \ ,$$

satisfies an analogous relation

$$t(\lambda) = \frac{1}{2}\{g(\lambda), \lambda\} = B\lambda^n [1 + O(1)] \ , \ B \in \mathbb{C} \ , \ n \in \mathbb{Z} \ ,$$

there are $m+2$ values a_k and $n+2$ of asymptotic values a_l^λ of $f(z)$ and $g(\lambda)$ respectively such that

$$\begin{aligned} \delta_2(a_k, f) &= \liminf_{r\to\infty} \frac{m(r, f, a_k)}{T_2(r, f)} = \frac{2}{m+2} \ , \\ \delta_1\left(a_l^\lambda, f\right) &= \liminf_{r\to\infty} \frac{m\left(\frac{1}{r}, f, a_{\lambda k}\right)}{T_2(r, f)} = \frac{2}{n+2} \end{aligned}$$

so that

$$\sum_k \delta_2(a_k, f) = 2 \ , \ \sum_l \delta_1\left(a_l^\lambda, f\right) = 2$$

Deficiencies in the punctured plane

The following situations can be considered

i) $T_2(r,f) >> T_1(r,f)$, that is $T_1(r,f) = o(T_2(r,f))$

In this case if we assume for a value $a \in \widehat{\mathbb{C}}$ that

$$\delta_2(a,f) = \liminf_{r\to\infty} \frac{m(r,f,a)}{T_2(r,f)} > 0 \ ,$$

then we can also conclude

$$\delta_0(a,f) = \liminf_{r\to\infty} \frac{m(r,f,a) + m\left(\frac{1}{r},f,a\right)}{T_1(r,f) + T_2(r,f)}$$

$$\geq \liminf_{r\to\infty} \frac{m(r,f,a)}{T_1(r,f) + T_2(r,f)} \ ,$$

that is $\delta_0(a,f) \geq \delta_2(a,f) > 0$ so that a is a deficient value for $f(z)$ and $\sum \delta_0(a,f) = 2$.

ii) $T_1(r,f) >> T_2(r,f)$

Similarly in this case from the asumption $\delta_1(a,f) > 0$, we conclude $\delta_0(a,f) \geq \delta_1(a,f) > 0$ so that a is also a deficient value for $f(z)$ and $\sum \delta_0(a,f) = 2$.

Deficiencies of meromorphic functions in the punctured plane

Finally we consider the case

iii) $T_1(r,f) \sim T_2(r,f)$

In this case we have

$$T_1(r,f) \sim A_0 r^{\frac{n+2}{2}} \quad \text{and} \quad T_2(r,f) \sim A_\infty r^{\frac{n+2}{2}}$$

so that

$$T_0(r,f) = T_1(r,f) + T_2(r,f) = (A_0 + A_\infty) r^{\frac{n+2}{2}}$$

and let us consider again the partial deficiencies

$$\delta_1(a,f) = \liminf_{r\to\infty} \frac{m\left(\frac{1}{r}, f, a\right)}{T_1(r,f)} \quad , \quad \delta_2(a,f) = \liminf_{r\to\infty} \frac{m(r,f,a)}{T_2(r,f)}$$

then we get

$$\delta_0(a,f) = \liminf_{r\to\infty} \frac{m(r,f,a) + m\left(\frac{1}{r}, f, a\right)}{T_1(r,f) + T_2(r,f)} = \liminf_{r\to\infty} \frac{m(r,f,a) + m\left(\frac{1}{r}, f, a\right)}{(A_0 + A_\infty) r^{\frac{n+2}{2}}}$$

Deficiencies of meromorphic functions in thepunctured plane

We obtain

$$\delta_0(a,f) \geq \frac{A_0}{A_0+A_\infty}\liminf_{r\to\infty}\frac{m(r,f,a)}{A_0 r^{\frac{n+2}{2}}} + \frac{A_\infty}{A_0+A_\infty}\liminf_{r\to\infty}\frac{m\left(\frac{1}{r},f,a\right)}{A_\infty r^{\frac{n+2}{2}}}$$

$$= \frac{A_0}{A_0+A_\infty}\liminf_{r\to\infty}\frac{m(r,f,a)}{T_1(r,f)} + \frac{A_\infty}{A_0+A_\infty}\liminf_{r\to\infty}\frac{m\left(\frac{1}{r},f,a\right)}{T_2(r,f)}$$

$$= \frac{A_0}{A_0+A_\infty}\delta_1(a,f) + \frac{A_\infty}{A_0+A_\infty}\delta_2(a,f) \quad ,$$

so that

$$\sum_a \delta_0(a,f) \geq \frac{A_0}{A_0+A_\infty}\sum_a \delta_1(a,f) + \frac{A_\infty}{A_0+A_\infty}\sum_a \delta_2(a,f) = 2.$$

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