On Blaschke-oscillatory and Blaschke-critical differential equations

by

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In the celebrated 1949 paper due to Nehari, necessary and sufficient conditions are given for a locally univalent meromorphic function to be univalent in the unit disc $\mathbb{D}$. The proof involves a second order differential equation of the form

$$f'' + A(z)f = 0, \tag{1}$$

where $A(z)$ is analytic in $\mathbb{D}$. As an immediate consequence of the proof, it follows that if $|A(z)| \leq 1/(1-|z|^2)^2$ for every $z \in \mathbb{D}$, then (1) is disconjugate (any non-trivial solution of (1) has at most one zero in $\mathbb{D}$).

Since 1949 a number of papers provide with different types of growth conditions for the coefficient $A(z)$ such that the solutions of (1) have at most finitely many zeros in $\mathbb{D}$. If there exists at least one solution with infinitely many zeros in $\mathbb{D}$, then (1) is oscillatory. If the zeros z_n still satisfy the classical Blaschke condition

$$\sum_n (1-|z_n|) < \infty,$$

then (1) is called Blaschke-oscillatory (=BO).

We give necessary and sufficient conditions for BO-equations, and consider prescribed zero sequences of solutions. The BO-concept extends to differential equations of arbitrary order also. All questions regarding the zeros of solutions can be rephrased for the critical points of solutions. This gives rise to Blaschke-critical (=BC) equations. There are examples showing that the concepts BO and BC are not the same.

Some classical tools and closely related topics that are often related to the non-oscillation case: the Schwarzian derivative, properties of univalent functions, Green's identity, conformal mappings, and a certain Hardy-Littlewood inequality. The BO-case also makes use of interpolation theory, various growth estimates for logarithmic derivatives of Blaschke products, Bank-Laine functions and recently updated Wiman-Valiron theory.