Recent results on operator theory in weighted Bergman spaces

Jari Taskinen, University of Helsinki

Presentation at Universidad Autónoma de Madrid, Jan. 11th, 2024

January 11, 2024

Toeplitz operators on polygonal domains

We first consider reflexive Bergman spaces $A^p(\Omega)$ on polygonal domains Ω of the complex plane. With some restrictions to the angles of the boundary of Ω, we show that the boundedness of the Toeplitz operator $T_g : A^p(\Omega) \to A^p(\Omega)$ with a positive symbol g is equivalent to the boundedness of the Berezin transform of g, or to g times the area measure being a Carleson measure. The main technical tool is a weighted Forelli-Rudin-type estimate.

Based on:

JT: Berezin transform and Toeplitz operators on polygonal domains, Compl. Variables Elliptic Eq. 67, 3 (2022), 773–787

Bergman kernel, Berezin transform, Toeplitz operator

• $L^p(\Omega)$, $1 < p < \infty$, is the Lebesgue space with respect to the (real) area measure dA on a domain $\Omega \subset \mathbb{C}$, normalized by $\int_\Omega dA = 1$.
• Notation:

$$\|f\|_{p,\Omega}^p = \int_\Omega |f|^p dA, \qquad \langle f, g \rangle = \int_\Omega f \overline{g} dA.$$

• Bergman space $A^p(\Omega) \subset L^p(\Omega)$, closed subspace of analytic functions.

• Bergman projection P_Ω is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$ = integral operator with the Bergman kernel $K_\Omega : \Omega \times \Omega \to \mathbb{C}$,

$$P_\Omega f(z) = \int_\Omega K_\Omega(z, w) f(w) dA(w).$$

Example. In the case $\Omega = \mathbb{D}$ (open unit disc), we have

$$K_{\mathbb{D}}(z, w) = \frac{1}{(1 - z\overline{w})^2} \quad \text{and normalized kernel } k_{\mathbb{D}}(z, w) := \frac{1 - |z|^2}{(1 - z\overline{w})^2}$$

If $g \in L^1(\Omega)$, we define the Toeplitz operator T_g on $A^p(\Omega)$ by

$$T_g f(z) = \int_\Omega K_\Omega(z,w)g(w)f(w)dA(w)$$

if the integral converges for all $f \in A^p(\Omega)$. The question of characterizing the boundedness of $T_g : A^p(\Omega) \to A^p(\Omega)$ in terms of the symbol g is an open problem even in the case $p = 2$ and $\Omega = \mathbb{D}$.

• A finite, positive measure μ on Ω is a Carleson measure for $A^p(\Omega)$, if

$$\int_\Omega |f|^p d\mu \le C \int_\Omega |f|^p dA \quad \forall\ f \in A^p(\Omega).$$

• If $\Omega = \mathbb{D}$ and $g \in L^1(\mathbb{D})$ we define the Berezin transform $B_g : \mathbb{D} \to \mathbb{C}$ by

$$B_g(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^2 g(w)}{|1-z\overline{w}|^4} dA(w)$$

Bergman kernel, Berezin transform, Toeplitz operator

The following is a classical result in the theory of Toeplitz operators in Bergman spaces; McDonald-Sundberg (1979), Hastings (1975), Luecking (1983,1985).

Theorem

Let $g \in L^1(\mathbb{D})$ be a non-negative function. The following statements are equivalent.
(i) The Toeplitz operator $T_g : A^p(\mathbb{D}) \to A^p(\mathbb{D})$ is bounded.
(ii) The Berezin transform B_g is a bounded function on $\mathbb{D}$.
(iii) The measure gdA is Carleson for the space $A^2(\mathbb{D})$.

There exist a number of generalizations, for example for Bergman spaces with regular (radial) weights in Peláez, Rättyä, Sierra (2016).
In the next we will consider this result in the case Ω is a polygon (so that its boundary is no more smooth).

Simply connected domains

- Next: $\Omega \subset \mathbb{C}$ is a simply connected domain, e.g. polygon
- Riemann conformal map $\varphi : \Omega \to \mathbb{D}$ with $\psi = \varphi^{-1} : \mathbb{D} \to \Omega$
- The Bergman kernel is given by

$$K_\Omega(z, w) := \frac{\varphi'(z)\overline{\varphi'(w)}}{(1 - \varphi(z)\overline{\varphi(w)})^2}$$

- Change of integration variables with the Jacobian $|\varphi'|^2$:

$$\int_\Omega f|\varphi'|^2 dA = \int_{\mathbb{D}} f \circ \psi \, dA \quad \text{for all } f \in L^\infty(\Omega).$$

- Isometric operators

$$I_\psi : A^2(\Omega) \to A^2(\mathbb{D}), \quad I_\psi f(z) = f \circ \psi(z)\psi'(z) \quad \text{with } I_\psi^{-1} = I_\varphi,$$
$$I_{\psi,p} : A^p(\Omega) \to L^p(\mathbb{D}), \quad I_\psi f(z) = f \circ \psi(z)|\psi'(z)|^{2/p}$$

Normalized Bergman kernel is defined by

$$k_{\Omega,z}(w) := \frac{\varphi'(w)\overline{\varphi'(z)}d_\Omega(z)}{(1-\varphi(w)\overline{\varphi(z)})^2} = K_\Omega(w,z)d_\Omega(z), \quad d_\Omega(z) := \frac{1-|\varphi(z)|^2}{|\varphi'(z)|}.$$

It follows from the Koebe distortion theorem that d_Ω is proportional to the boundary distance: $\exists$ constant $C > 0$ such that

$$\frac{1}{C}d_\Omega(z) \le \mathrm{dist}(z, \partial\Omega) \le Cd_\Omega(z) \quad \text{for all } z \in \Omega$$

Moreover, we get

$$\|k_{\Omega,z}\|_{2,\Omega}^2 = d_\Omega(z)^2 \int_\Omega \frac{|\varphi'(w)|^2|\varphi'(z)|^2}{|1-\varphi(w)\overline{\varphi(z)}|^4} dA(w),$$

and since the quantities

$$\int_\Omega \frac{|\varphi'(w)|^2}{|1-\varphi(w)\overline{\varphi(z)}|^4} dA(w) \quad \text{and} \quad \frac{1}{(1-|\varphi(z)|^2)^2}$$

are proportional to each other by the Forelli-Rudin estimates, we obtain

$$c < \|k_{\Omega,z}\|_{2,\Omega} < C$$

for some constants $0 < c < C$.

Berezin transform in simply connected domains

The Berezin transform $B_g : \Omega \to \mathbb{C}$ of g and T_g is defined by

$$B_g(z) = \langle T_g k_{\Omega,z}, k_{\Omega,z} \rangle = d_\Omega(z)^2 \int_\Omega \frac{|\varphi'(z)|^2 g(w) |\varphi'(w)|^2}{|1 - \varphi(z)\overline{\varphi(w)}|^4} dA(w),$$

assuming the integral converges for all $z \in \Omega$.

Polygonal domains

- Now assume $\Omega \subset \mathbb{C}$ is a bounded polygon with $N \geq 3$ corners.
- Corners at the points $w_k \in \partial\Omega$, $k = 1, \ldots, N$.

Denote $z_k = \varphi(w_k) \in \partial\mathbb{D}$

- The boundary $\partial\Omega$ has a corner with an angle $\pi\alpha_k$ at w_k
- We have $0 < \alpha_k < 2$, $\alpha_k \neq 1$.
  Outward corner: $0 < \alpha_k < 1$. Inward: $1 < \alpha_k < 2$.

Schwartz-Christoffel formula yields the Riemann map $\psi : \mathbb{D} \to \Omega$,

$$\psi(z) = A \int_0^z \prod_{k=1}^{N} (1 - w\overline{z_k})^{\alpha_k - 1} dw + B \ , \quad z \in \mathbb{D},$$

where $A \neq 0$ and B are constants. This implies

$$\psi'(z) = A \prod_{k=1}^{N} (1 - z\overline{z_k})^{\alpha_k - 1} \text{ for } z \in \mathbb{D},$$

$$\varphi'(z) = A^{-1} \prod_{k=1}^{N} (1 - \varphi(z)\overline{\varphi(w_k)})^{1 - \alpha_k} \text{ for } z \in \Omega.$$

In particular, $\psi' : \mathbb{D} \to \mathbb{C}$ is never bounded on $\mathbb{D}$.

Main result

Theorem

Let Ω be a polygon with the Riemann map $\psi : \mathbb{D} \to \Omega$, and let p and the numbers α_k satisfy condition

$$p > |2 - p| \max_{k=1,\ldots,N} \alpha_k. \tag{1}$$

Also, let $g \in L^1(\Omega)$ be such that $g(z) \geq 0$ for almost all $z \in \Omega$. The following statements are equivalent.
(i) The Toeplitz operator $T_g : A^p(\Omega) \to A^p(\Omega)$ is bounded.
(ii) The Berezin transform B_g is a bounded function on Ω.
(iii) The measure gdA is Carleson for the space $A^2(\Omega)$.

One of the complications is that the Bergman projection is not bounded for all polygons and $p \neq 2$. For the convex ones (with $0 < \alpha_k < 1$) there is no problem, however.

Lemma

Let Ω be a polygon with the Riemann map ψ such that

$$p > |2-p| \max_{k=1,\ldots,N} \alpha_k, \tag{1}$$

then the Bergman projection P_Ω is a bounded operator from $L^p(\Omega)$ onto $A^p(\Omega)$. Consequently, the dual space of $A^p(\Omega)$ can be canonically identified with the space $A^q(\Omega)$, where $1/p + 1/q = 1$, by using the dual pairing $\langle f, g\rangle$.

This goes back at least until Békollé, Canadian J.Math. (1986).

Generalized Forelli-Rudin estimates

The proof uses the following two conditions,

$$|\psi'|^{2-p} \in L^1(\mathbb{D}) \tag{2}$$

and "generalized Forelli-Rudin estimate"

$$C(1-|z|)^{-2p+2}|\psi'(z)|^{2-p} \le \int_{\mathbb{D}} \frac{1}{|1-z\overline{w}|^{2p}}|\psi'(w)|^{2-p}dA(w)$$

$$\le C'(1-|z|)^{-2p+2}|\psi'(z)|^{2-p}\ \forall\, z \in \mathbb{D}, \qquad C' > C > 0 \text{ constants.} \tag{3}$$

Condition (2) guarantees that the integral in (3) converges for all $z \in \mathbb{D}$, and it is known to hold for the Riemann map of any simply connected planar domain Ω with at least two boundary points, if $4/3 < p < 3.422$.

• Brennan conjecture: (2) holds for all simply connected domains, if and only if $4/3 < p < 4$. The conjecture remains open.

The main technical step of the proof consists of showing that (1) $\Rightarrow$ (3).

Bounded Toeplitz $\Rightarrow$ bounded Berezin

Lemma

If $\varphi : \Omega \to \mathbb{D}$ satisfies (3), then

$$c_p\Big(\frac{1-|\varphi(z)|}{|\varphi'(z)|}\Big)^{2/p-1} \le \|k_{\Omega,z}\|_{\Omega,p} \le C_p\Big(\frac{1-|\varphi(z)|}{|\varphi'(z)|}\Big)^{2/p-1}$$

for all $z \in \mathbb{D}$.

We assume (3) and that the Toeplitz-operator T_g is bounded and show that the Berezin transform B_g is a bounded function. Let $1/p + 1/q = 1$, and $M_z = \|k_{\Omega,z}\|_{\Omega,p}\,\|k_{\Omega,z}\|_{\Omega,q}$. Since (3) holds, we can use the above lemma for $\|k_{\Omega,z}\|_{\Omega,p}$ to $M_z \cong 1$.
For all $z \in \mathbb{D}$ we have, since $T_g : A^p(\Omega) \to A^p(\Omega)$ is assumed bounded

$$B_f(z) = \langle T_g k_{\Omega,z}, k_{\Omega,z}\rangle \cong M_z^{-1}\langle T_g k_{\Omega,z}, k_{\Omega,z}\rangle$$

$$= \Big\langle T_g\Big(\frac{k_{\Omega,z}}{\|k_{\Omega,z}\|_{\Omega,p}}\Big), \frac{k_{\Omega,z}}{\|k_{\Omega,z}\|_{\Omega,q}}\Big\rangle \le C.$$

Solid hulls and cores: introduction

It is often impossible to describe a non-Hilbert Banach space of analytic functions on the disc or the plane in terms of the Taylor coefficients. The next best thing is to find the solid hull of the given space. This means, rougly, finding the strongest growth condition that the coefficients of the functions in the given space have to satisfy. In the next, we characterize the solid hulls of a large class of Bergman spaces H_v^∞ and A_v^p, $1 < p < \infty$ (on the complex plane $\mathbb{C}$ or the open unit disc $\mathbb{D}$).

The methods consist of new combinations of those in G.Bennet, D.Stegenga, R.Timoney, Illinois J. Math 25 (1981), which use S.Kislyakov, Trudy Math.Inst.Steklov (1981), and W.Lusky, Studia Math. 175 (2006).

References

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Weighted Bergman spaces.

Let $\Omega = \mathbb{D}$ or $\mathbb{C}$. A *weight* v is a continuous, radial function $v : \Omega \to]0, \infty[$, which is non-increasing w.r.t. r and satisfies $v(r) \to 0$, as $r \to 1$ (for $\mathbb{D}$), or $r^m v(r) \to 0$, as $r \to \infty$, for each $m \in \mathbb{N}$ (for $\mathbb{C}$).
Weighted Bergman spaces (with $1 < p < \infty$, dA is the area measure)

$$A_v^p(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic } : \|f\|_{p,v}^p := \int_{\mathbb{D}} v(z)|f(z)|^p dA < \infty\},$$
$$H_v^\infty(\Omega) = \{f : \Omega \to \mathbb{C} \text{ analytic } : \|f\|_v := \sup_{z \in \Omega} v(z)|f(z)| < \infty\}$$

Solid hulls and cores: Definitions.

A vector space of complex sequences A is *solid* if $a = (a_m)_{m=0}^{\infty} \in A$ and $|b_m| \le |a_m|$ for each m implies $b = (b_m)_{m=0}^{\infty} \in A$. The **solid hull of** A is

$$S(A) := \big\{ (b_m)_{m=0}^{\infty} \, : \, \exists (a_m)_{m=0}^{\infty} \in A \text{ such that } |b_m| \le |a_m| \ \forall m \in \mathbb{N} \big\}.$$

Thus, $S(A)$ is the smallest solid space containing A. The **solid core** $s(A)$ of the space A is the largest solid space contained in A:

$$s(A) := \big\{ (b_m)_{m=0}^{\infty} \, : \, (b_m a_m)_{m=0}^{\infty} \in A \text{ for all } (a_m)_{m=0}^{\infty} \in \ell^{\infty} \big\}.$$

The solid hull and core of an analytic function space on $\mathbb{D}$ or $\mathbb{C}$ are defined by applying the previous definition to the sequences of Taylor coefficients.

Main general result on solid hulls for H_v^∞

Let us fix $v : \Omega \to \mathbb{R}^+$. Given $m > 0$, we denote by r_m the global maximum point of $r^m v(r)$. Notice that $r_m \to 1$ (if $\Omega = \mathbb{D}$) and $r_m \to \infty$ (if $\Omega = \mathbb{C}$) as $m \to \infty$. We assume that the weight v satisfies the technical

condition (b),

which gives an increasing sequence $(m_n)_{n=1}^\infty$ of positive numbers for v. (Similar but weaker than the weight condition (B) introduced by W.Lusky and used in several of his works.)

Thus, the numbers r_{m_n}, $n = 1, 2, \ldots$, form an increasing sequence of radii tending to the boundary of the domain.

Theorem

Assume v satisfies condition (b). Then, the solid hull of $H_v^\infty(\Omega)$ is

$$\Big\{ f = \sum_{m=0}^\infty b_m z^m \ : \ \sup_n v(r_{m_n}) \Big(\sum_{m=m_n+1}^{m_{n+1}} |b_m|^2 r_{m_n}^{2m} \Big)^{1/2} < \infty \Big\}. \qquad [**]$$

Remark. There is some freedom in the choice of the numbers m_n and r_{m_n}.

Examples.

Examples. 1°. The Bergman space $A^2(\mathbb{D})$ and the Hardy space $H^2(\mathbb{D})$ are solid, but $A^p(\mathbb{D})$ and $H^p(\mathbb{D})$ are not, if $1 \le p < \infty$, $p \neq 2$. The space $H^\infty(\mathbb{D})$ is not solid.

2°. For $H^p(\mathbb{D})$, $2 \le p < \infty$, the solid hull is ℓ^2. Bennet, Stegenga, Timoney (1981): $\Omega = \mathbb{D}$, v is doubling with respect to the boundary distance. Then,

$$S(H_v^\infty(\mathbb{D})) = \Big\{ (b_m)_{m=0}^\infty \, : \, \Big(\sum_{m \le n} |b_m|^2 \Big)^{1/2} \le Cv(1 - 1/n) \ \forall \, n \in \mathbb{N} \Big\}.$$

3°. The case $p = \infty$: for $v(r) = (1-r)^a$, $a \ge 0$, the solid hull and core of $H_v^\infty(\mathbb{D})$ are (see the monograph by Jevtic, Vukotic, Arsenovic, 2016)

$$S\big(H_v^\infty(\mathbb{D})\big) = \Big\{ (b_m)_{m=0}^\infty \, : \, \sup_{n \in \mathbb{N}} \sum_{m=n^2+1}^{(n+1)^2} |b_m|^2 (m+1)^{-2a} < \infty \Big\},$$

$$s\big(H_v^\infty(\mathbb{D})\big) = \Big\{ (b_m)_{m=0}^\infty \, : \, \sup_{n \in \mathbb{N}} \sum_{m=n^2+1}^{(n+1)^2} |b_m| (m+1)^{-a} < \infty \Big\}.$$

Examples.

Examples (our results).

• $\Omega = \mathbb{D}$, $v(r) = \exp(-1/(1-r))$, can be generalized for $v(r) = \exp(-a/(1-r)^b)$ with $a, b > 0$, but the expressions get messy:

$$S\big(H_v^\infty(\mathbb{D})\big) = \Big\{(b_m)_{m=0}^\infty \ : \ \sup_{n\in\mathbb{N}} e^{-2n^2} \sum_{m=n^4+1}^{(n+1)^4} |b_m|^2 \Big(1 - \frac{1}{n^2}\Big)^{2m} < \infty \Big\},$$

$$s\big(H_v^\infty(\mathbb{D})\big) = \Big\{(b_m)_{m=0}^\infty \ : \ \sup_{n\in\mathbb{N}} e^{-n^2} \sum_{m=n^4+1}^{(n+1)^4} |b_m| \Big(1 - \frac{1}{n^2}\Big)^{m} < \infty \Big\}.$$

• $\Omega = \mathbb{C}$, $v(r) = \exp(-ar^p)$ on $\mathbb{C}$ with $a, p > 0$ constants. Then,

$$S\big(H_v^\infty(\mathbb{C})\big) = \Big\{(b_m)_{m=0}^\infty \ : \ \sup_{n\in\mathbb{N}} \sum_{m=pn^2+1}^{p(n+1)^2} |b_m|^2 e^{-2n^2} n^{4m/p} (ap)^{-m/p} < \infty \Big\}.$$

Examples.

- Let $\Omega = \mathbb{D}$, $v(r) = \exp(-1/(1-r))$, $2 < p < \infty$. We have

$$S(A_v^p) = \Big\{ (b_m)_{m=0}^{\infty} \ : \ \sum_{n=1}^{\infty} e^{-n^2} \Big(\frac{1}{n^3}\Big) \Big(\sum_{m=n^4/p+1}^{(n+1)^4/p} |b_m|^2 \Big(1 - \frac{1}{n^2}\Big)^{2m} \Big)^{p/2} < \infty \Big\}.$$

- Let $\Omega = \mathbb{D}$, $v(r) = \exp(-1/(1-r))$, $1 < p < 2$. Then,

$$s(A_v^p) = \Big\{ (b_m)_{m=0}^{\infty} \ : \ \sum_{n=1}^{\infty} e^{-n^2} \Big(\frac{1}{n^3}\Big) \Big(\sum_{m=n^4/p+1}^{(n+1)^4/p} |b_m|^2 \Big(1 - \frac{1}{n^2}\Big)^{2m} \Big)^{p/2} < \infty \Big\}.$$

Weighted Bergman spaces on the disc $\mathbb{D}$.

Recall the notation for weighted Bergman spaces:

$$A_v^p = A_v^p(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic } : \|f\|_{p,v}^p := \int_{\mathbb{D}} v|f|^p dA < \infty\},$$
$$H_v^\infty = H_v^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic } : \|f\|_v := \sup_{z\in\Omega} v(z)|f(z)| < \infty\}$$

Bergman projection P_v is the orthogonal projection from L_v^2 onto A_v^2; integral kernel denoted by $K_v : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$. Its boundedness with respect to other L^p-norms is an interesting question.

• Dostanic (2004,2007): Let $\alpha, \beta > 0$ and $v(r) = \exp(\alpha/(1-r)^\beta)$. Then, P_v is never bounded on L_v^p if $p \neq 2$!

• W.Lusky, JT, Bounded holomorphic projections for exponentially decreasing weights, J. Function Spaces Appl. 6 (2008): there are bounded projection operators $L_v^p \to A_v^p$ for a large class of weights satisfying condition (B), which includes the exponential weights but also much more general rapidly decreasing weights.

On the boundedness of the Bergman projection

The following result and its generalizations have been proved by Constantin and Pelaéz (2015), Arroussi (2016), He, Lv, Schuster (2019).

Theorem (Constatin, Pelaéz, Arroussi, He, Lv, Schuster)

Let $0 < \alpha, \tilde{\alpha}, \beta, \tilde{\beta} < \infty$ *and* $1 \le p < \infty$. *Put*

$$v_\ell(r) = \exp\left(-\frac{\alpha}{(1-r^\ell)^\beta}\right) \quad \textit{and} \quad w_\ell(r) = \exp\left(-\frac{\tilde{\alpha}}{(1-r^\ell)^{\tilde{\beta}}}\right).$$

- *If* $\beta = \tilde{\beta}$, $\tilde{\alpha} = 2\alpha/p$ *then* $P_{w_\ell} : L^p_{v_\ell} \to A^p_{v_\ell}$ *is bounded* $\forall \ell > 0$.
- *If* $\beta = \tilde{\beta}$, $\tilde{\alpha} = 2\alpha$ *then* $P_{w_\ell} : L^\infty_{v_\ell} \to H^\infty_{v_\ell}$ *is bounded* $\forall \ell > 0$.

The paper J.Bonet,W.Lusky,JT: Unbounded Bergman projections on weighted spaces with respect to exponential weights IEOT 93 (2021) contains the following

- If $\beta = \tilde{\beta}$, $\tilde{\alpha} \neq 2\alpha/p$ then P_{w_ℓ} is unbounded on $(L^p_{v_\ell}, \|\cdot\|_{p,v_\ell})$ $\forall \ell > 0$.
- If $\beta = \tilde{\beta}$, $\tilde{\alpha} \neq 2\alpha$ then P_{w_ℓ} is unbounded on $(L^\infty_{v_\ell}, \|\cdot\|_{\infty,v_\ell})$ $\forall \ell > 0$.
- If $\tilde{\beta} \neq \beta$ then $P_{\tilde{w}_\ell}$ is unbounded on $(L^p_{v_\ell}, \|\cdot\|_{p,v_\ell})$ for all $\ell > 0$.

We consider Toeplitz operators in $H_v^\infty(\mathbb{D})$ for weights v with condition (B).

• Given $a \in L^1 = L^1(\mathbb{D})$ (symbol), Toeplitz operator T_a on H_v^∞ is defined as

$$T_a = P_v M_a = \int_{\mathbb{D}} K_v(z,w)a(w)f(w)dA(w)$$

with the pointwise multiplier $M_a : f \mapsto af$.

Toeplitz operators with harmonic symbols, $\Omega = \mathbb{D}$

In the case of reflexive Bergman spaces with no weight or with standard weights, any bounded symbol a induces a bounded Toeplitz operator $T_a = P_v M_a$. This follows directly from the boundedness of the Bergman projection P_v in these cases. In J.Bonet,W.Lusky,JT: On boundedness and compactness of Toeplitz operators in weighted H1-spaces, JFA(2020), we observe that

Theorem

There exists a bounded harmonic function $a : \mathbb{D} \to \mathbb{C}$ such that the Toeplitz operator T_a is not bounded $H_v^\infty \to H_v^\infty$ for any weight v.

In the case a is analytic, T_a is the pointwise multiplier with a, and T_a is bounded, if and only if a is a bounded function (belongs to H^∞).

Corollary

The Bergman projection P_v is never (for any weight) bounded $L_v^\infty \to L_v^\infty$.

Let us consider on the unit circle $\partial\mathbb{D}$ the function which is equal to 1 on $\partial\mathbb{D} \cap \{\mathrm{Re} z > 0\}$ and equal to 0 elsewhere. The function f of the above theorem is simply defined as the harmonic extension of this function.

Toeplitz operators with radial symbols, $\Omega = \mathbb{D}$

Let $a(z) = a(|z|)$ for $z \in \mathbb{D}$. One expects that vanishing of the symbol on the boundary of the disc may imply the boundedness of T_a in H_v^∞ (cf. the previous theorem). We prove in J.Bonet,W.Lusky,JT, J.Math.Anal.Appl. (2021):

Theorem

Assume the radial weight v is normal or exponential. Let $a \in L^1$ be such that the restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$ with

$$\limsup_{r\to 1} a'(r) < \infty \qquad \text{or} \qquad \liminf_{r\to 1} a'(r) > -\infty. \quad (*)$$

Then

$$\limsup_{r\to 1} |a(r)\log(1-r)| < \infty \ \Rightarrow \ T_a \text{ is bounded } H_v^\infty \to H_v^\infty$$

$$\limsup_{r\to 1} |a(r)\log(1-r)| = 0 \ \Rightarrow \ T_a \text{ is compact } H_v^\infty \to H_v^\infty.$$

The theorem actually holds for all weights satisfying condition (B) (next slide) and, for some $\varepsilon > 0$,

$$\sup_{n=1,2,\dots} \frac{\int_0^1 r^{n-n^\epsilon} v(r)dr}{\int_0^1 r^n v(r)dr} < \infty. \quad (**)$$

We say that a radial weight v satisfies condition (B), if

$$\forall b_1 > 1 \ \exists b_2 > 1 \ \exists c > 0 \ \forall m, n > 0$$
$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \ \text{ and } \ m, n, |m-n| \ge c \ \Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_2.$$

Here, $r_n \in (0,1)$ is the maximum point of the function $r \mapsto v(r)r^n$ for any $n > 0$,

Corollary

Let v satisfy (B) and $(**)$. Then, $T_a : H_v^\infty \to H_v^\infty$ is bounded for every (radial) symbol, which is continuously differentiable on $[0,1]$.

The above results are consequences of a general characterization of the boundedness of Toeplitz operators $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ in the case the weight v satisfies condition (B). The characterization uses the classical de la Vallée-Poussin operators (details skipped here). See J.Bonet, W.Lusky, JT (JFA 2020).

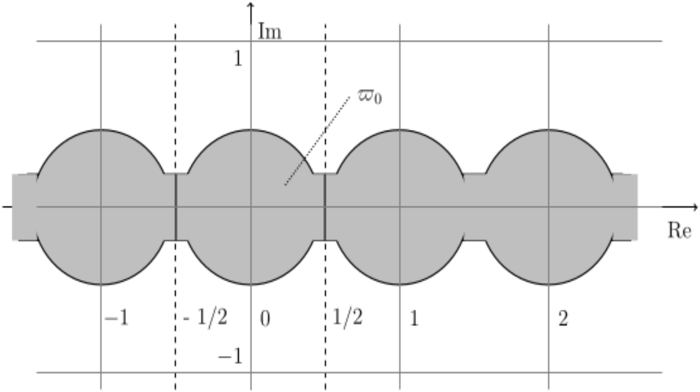
Periodic domain and cell

• Periodic cell $\varpi \subset]-\frac{1}{2},\frac{1}{2}[\times]-M,M[\subset \mathbb{R}^2 \cong \mathbb{C}$ for some $M>0$, see picture below. Translates of ϖ are $\varpi_m = \varpi + m$, where $m \in \mathbb{Z} \subset \mathbb{C}$,
• Periodic domain Π is the interior of the set

$$\bigcup_{m\in\mathbb{Z}} \mathrm{cl}(\varpi_m).$$

• Some geometric assumptions: ϖ and Π are Lipschitz domains such that the boundaries $\partial\varpi$ and $\partial\Pi$ are in addition piecewise smooth. Excludes cusps both in ϖ and Π. Consequently, $\partial\varpi$ is a Jordan curve, polynomials form a dense subspace of the Bergman space $A^2(\varpi)$.

Periodic domain and cell



Floquet transform in $L^2(\Pi)$

The definition of the Floquet transform reads for $f \in L^2(\Pi)$ as

$$\mathsf{F}f(z,\eta) = \widehat{f}(z,\eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-i\eta m} f(z+m), \quad z \in \varpi,\ \eta \in [-\pi,\pi],$$

$$\mathsf{F} : L^2(\Pi) \to L^2\big(-\pi,\pi; L^2(\varpi)\big) \ .$$

Here, $L^2(-\pi,\pi; L^2(\varpi))$ is the vector valued L^2-space (or Bochner space) on $[-\pi,\pi]$ of functions $g = g(z,\eta)$ with values $g(\cdot,\eta)$ in $L^2(\varpi)$, with norm

$$\|g\|^2 = \int_{-\pi}^{\pi} \|g(\cdot,\eta)\|^2_{L^2(\varpi)} d\eta$$

The series converges in $L^2(-\pi,\pi; L^2(\varpi))$, thus pointwise for a.e. η, z etc.

Theorem

F *is a unitary map from* $L^2(\Pi)$ *onto* $L^2(-\pi,\pi; L^2(\varpi))$ *with inverse*

$$\mathsf{F}^{-1}g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i[\mathrm{Re} z]\eta} g(z - [\mathrm{Re}\, z], \eta) d\eta, \quad z \in \Pi.$$

Floquet transform in $A^2(\Pi)$

The Floquet transform is simply defined in $A^2(\Pi) \subset L^2(\Pi)$ as the restriction. Then, the question is about its range.

Theorem

Floquet transform F *maps* $A^2(\Pi)$ *onto* $L^2(-\pi,\pi; A^2_\eta(\varpi))$. *Its inverse* $\mathsf{F}^{-1} : L^2(-\pi,\pi; A^2_\eta(\varpi)) \to A^2(\Pi)$ *is given by the formula (1).*

• For $\eta \in [-\pi,\pi]$, we denote by $A^2_{\eta,\mathrm{ext}}(\varpi)$ the subspace of $A^2(\varpi)$ of such f which can be extended as analytic functions to a neighborhood in Π of $\mathrm{cl}\,(\varpi) \cap \Pi$ and satisfy the boundary condition

$$f(\frac{1}{2} + iy) = e^{i\eta} f(-\frac{1}{2} + iy) \text{ for all } a < y < b.$$

• We define the space $A^2_\eta(\varpi)$ as the closure of $A^2_{\eta,\mathrm{ext}}(\varpi)$ in $A^2(\varpi)$.

Projections in Π and in ϖ

• We denote by $P_\eta : L^2(\varpi) \to A^2_\eta(\varpi)$ the orthogonal projection with kernel $K_\eta : \varpi \times \varpi \to \mathbb{C}$,

$$P_\eta f(z) = \int_\varpi K_\eta(z,w) f(w) dA(w).$$

Theorem

The map $\mathcal{P}f(z,\eta) = \big(P_\eta f(\cdot,\eta)\big)$ is the orthogonal projection from $L^2(-\pi,\pi; L^2(\varpi))$ onto $L^2(-\pi,\pi; A^2_\eta(\varpi))$. The Bergman projection $P_\Pi : L^2(\Pi) \to A^2(\Pi)$ equals $P_\Pi = \mathsf{F}^{-1}\mathcal{P}\mathsf{F}$.

From now on we consider Toeplitz operators $T_a : A^2(\Pi) \to A^2(\Pi)$ with periodic symbols $a \in L^\infty(\Pi)$: we assume

$$a(z) = a(z+1) \qquad \text{for a.e. } z \in \Pi.$$

• We define for all $\eta \in [-\pi, \pi]$ the bounded, Toeplitz-type operator $T_{a,\eta} : A^2_\eta(\varpi) \to A^2_\eta(\varpi)$,

$$T_{a,\eta} f = P_\eta(a|_\varpi f)$$

• In the Bochner space, $\mathcal{T}_a : L^2(-\pi, \pi; A^2_\eta(\varpi)) \to L^2(-\pi, \pi; A^2_\eta(\varpi))$,

$$\mathcal{T}_a : f(\cdot, \eta) \mapsto T_{a,\eta} f(\cdot, \eta),$$

The following is an immediate consequence of the definitions.

Lemma

$T_a f = \mathrm{F}^{-1} \mathcal{T}_a \mathrm{F} f$ *for all* $f \in A^2(\Pi)$.

Spectra

We denote the spectrum of $T_{a,\eta}$ in the space $A^2_\eta(\varpi)$ by $\sigma(T_{a,\eta})$.

Theorem

The essential spectrum of the Toeplitz-operator $T_a : A^2(\Pi) \to A^2(\Pi)$ *can be described by the formula*

$$\sigma_{\rm ess}(T_a) = \bigcup_{\eta \in [-\pi,\pi]} \sigma(T_{a,\eta}).$$

Moreover, there holds $\sigma(T_a) = \sigma_{\rm ess}(T_a)$.

(An analogous formula is classical in spectral problems for periodic elliptic operators which are in particular *unbounded operators in Sobolev-type Hilbert spaces*; S.A.Nazarov, P.Kuchment and many others.)

Application: in some periodic domain with thin ligaments, one can construct examples of Toeplitz operators with essential spectra containing any finite number of disjoint components.

FINALE

Thank you for your attention!