Operator Theory from Holomorphic Semigroups

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Contents

1. Preliminaries from operator theory

Let $\mathbf{X}$ be a Banach space.
A family $(T_t)_{t\geq 0}$ of bounded operators on $\mathbf{X}$ is an *operator semigroup* if

- $T_tT_s = T_{t+s}$, $t, s \geq 0$,
- $T_0 = I$, the identity operator.

And $(T_t)_{t\in\mathbb{R}}$ is an *operator group* if the above are satisfied for $t, s \in \mathbb{R}$.

Elementary examples: ($\mathbb{R}$ real line, $\mathbb{D}$ unit disc in the complex plane $\mathbb{C}$, $\mathbb{T}$ the unit circle)

- The group of translates

$$T_t(f)(x) = f(x+t), \quad t \in \mathbb{R}$$

on $\mathbf{X} = L^p(\mathbb{R})$, $p \geq 1$, or on $\mathbf{X} = C(\mathbb{R})$, space of continuous functions.

- The group of rotations

$$T_t(f)(z) = f(e^{it}z), \quad t \in \mathbb{R}$$

on $L^p(\mathbb{T})$, or on $L^p(\mathbb{D}, dA(z))$, or on Hardy spaces $H^p(\mathbb{D})$, $p \geq 1$.

• Semigroup of dilations

$$T_t(f)(z) = f(e^{-t}z), \quad t \geq 0,$$

on appropriate space $\mathbf{X}$ of analytic functions, or of integrable functions on $\mathbb{D}$.

• Multiplication semigroups

$$T_t(f)(x) = e^{tm(x)} f(x)$$

on appropriate spaces of integrable or continuous functions, where $m(x)$ is an appropriate function.

• A composition semigroup

$$T_t(f)(z) = f(e^{-t}z + 1 - e^{-t}),\ t \geq 0$$

on a Hardy space H^p or even on $L^p(0,1)$.

• (less elementary)

$$T_t(f)(x) = \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{f(x-s)}{s^2 + t^2}\, ds,\ t \geq 0$$

on $L^p(\mathbb{R})$.

A semigroup (T_t) is called:

- *uniformly continuous* if

$$\lim_{t \to 0} \|T_t - I\| = 0.$$

- *strongly continuous* (or a c_0-*semigroup*) if

$$\lim_{t \to 0} \|T_t(x) - x\| = 0 \text{ for each } x \in \mathbf{X}.$$

Clearly, uniformly continuous $\Rightarrow$ strongly continuous, converse not true.

The *infinitesimal generator* of a (T_t) is the (unbounded in general) operator Γ defined for those $x \in \mathbf{X}$ for which the limit

$$\Gamma(x) = \lim_{t \to 0} \frac{T_t(x) - x}{t} = \left. \frac{\partial T_t(x)}{\partial t} \right|_{t=0},$$

exists, the limit taken in $\| \, \|_{\mathbf{X}}$.

$D(\Gamma) = \{x \in \mathbf{X} : \text{ the limit exits}\}$, the domain of Γ. It is a dense linear subspace of $\mathbf{X}$, and

$$\begin{aligned} D(\Gamma) = \mathbf{X} &\Leftrightarrow (T_t) \text{ is uniformly continuous,} \\ &\Leftrightarrow \Gamma : \mathbf{X} \to \mathbf{X} \text{ bounded operator.} \end{aligned}$$

In that case

$$T_t = e^{t\Gamma} = \sum_{n=0}^{\infty} \frac{t^n \Gamma^n}{n!}.$$

This formula, with appropriate interpretation, is valid also for c_0-semigroups. So operator semigroups are, in some sense, the operator analogues of exponential functions. In all cases Γ is a closed operator, i.e. $\{(x, \Gamma(x)) : x \in D(\Gamma)\}$ is a closed subset of $\mathbf{X} \times \mathbf{X}$.

Resolvent and spectrum of Γ:

The resolvent set $\rho(\Gamma)$ consists of all $\lambda \in \mathbb{C}$ such that

$$R(\lambda, \Gamma) = (\lambda I - \Gamma)^{-1} : \mathbf{X} \to \mathbf{X} \text{ is bounded,}$$

and $\sigma(\Gamma) = \mathbb{C} - \rho(\Gamma)$, spectrum of Γ.

In contrast with bounded operators, $\sigma(\Gamma)$ can be unbounded, it can vary in size from empty set to a whole left half-plane.

The *growth bound* of a c_0-semigroup (T_t) is

$$\omega = \lim_{t \to \infty} \frac{\log \|T_t\|}{t},$$

and it satisfies $-\infty \leq \omega < \infty$. For each $a > \omega$ there is $M = M(a) < \infty$ such that

$$\|T_t\| \leq M e^{at},\ t \geq 0.$$

Further if $\mathrm{Re}(\lambda) > \omega$ then $\lambda \in \rho(\Gamma)$ and

$$R(\lambda, \Gamma)(x) = \int_0^\infty e^{-\lambda t} T_t(x)\, dt, \quad x \in \mathbf{X}.$$

The spectral theorem for semigroups says

$$e^{t\sigma(\Gamma)} \subseteq \sigma(T_t) \quad \text{for } t \geq 0.$$

In general the containment is strict. For the point spectrum however there is equality

$$e^{t\sigma_\pi(\Gamma)} = \sigma_\pi(T_t) \setminus \{0\} \quad \text{for } t \geq 0.$$

For $\lambda, \mu \in \rho(\Gamma)$, the resolvent equation

$$R(\lambda, \Gamma) - R(\mu, \Gamma) = (\mu - \lambda) R(\lambda, \Gamma) R(\mu, \Gamma)$$

implies that either $R(\lambda, \Gamma)$ is compact for all $\lambda \in \rho(\Gamma)$ or not compact for any such λ.

2. Semigroups of analytic functions

We will concentrate on $\mathbb{D}$, and denote $\mathcal{H}(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic}\}$.

If $\phi : \mathbb{D} \to \mathbb{D}$ is analytic, we may consider the discrete iterates for $n = 0, 1, 2, ...$,

$$\phi_0(z) \equiv z, \quad \phi_1 = \phi, \quad \phi_n = \phi_{n-1} \circ \phi, ...$$

and wonder if it is possible to embed (ϕ_n) into a continuous parameter family $(\phi_t)_{t \geq 0}$. This is not always possible but if it happens, that family is a semigroup of functions.

Def. A family $\Phi = \{\phi_t : t \geq 0\}$ of analytic self-maps of $\mathbb{D}$ is a *semigroup of functions* if

(i) $\phi_0 \equiv z$, the identity map of $\mathbb{D}$,
(ii) $\phi_s \circ \phi_t = \phi_{s+t}$ for $s, t \geq 0$,
(iii) $\phi_t \to \phi_0$ locally uniformly, as $t \to 0$.

Basic properties of Φ (Berkson-Porta 1978):

- Each ϕ_t is univalent.

- The limit

$$G(z) = \lim_{t \to 0+} \frac{\phi_t(z) - z}{t} = \frac{\partial \phi_t(z)}{\partial t}|_{t=0},$$

exists uniformly on compact subsets of $\mathbb{D}$. It is the *infinitesimal generator* of $\{\phi_t\}$.

- $G(z)$ satisfies

$$G(\phi_t(z)) = \frac{\partial \phi_t(z)}{\partial t} = G(z)\frac{\partial \phi_t(z)}{\partial z},$$

for each $z \in \mathbb{D}$, $t \geq 0$.

- There is a unique $b \in \mathbb{D} \cup \mathbb{T}$ such that
(1) b is a fixed point for each ϕ_t,
(2) $\phi_t(z) \to b$ as $t \to \infty$ for each $z \in \mathbb{D}$, with the exception of the case when (ϕ_t) consists of elliptic automorphisms of $\mathbb{D}$. The unique b is *the Denjoy-Wolff point* of Φ.

• $G(z)$ has unique representation

$$G(z) = (\bar{b}z - 1)(z - b)F(z),$$

with b the DW point of ϕ_t and $F(z)$ analytic on $\mathbb{D}$ with $\text{Re}F(z) \geq 0$.

Constructing semigroups:

Case 1. Let $h : \mathbb{D} \to \mathbb{C}$ a univalent analytic function, $h(0) = 0$, whose range $\Omega = h(\mathbb{D})$ satisfies:
There is a $c \in \mathbb{C}$ with $\text{Re}(c) \geq 0$, such that the spirals $S_w = \{e^{-ct}w : t \geq 0\} \subset \Omega$ for each $w \in \Omega$. Then

$$\phi_t(z) = h^{-1}(e^{-ct}h(z)),\ z \in \mathbb{D}, t \geq 0$$

is a semigroup with DW point $b = 0$. These h are the *spirallike* (*starlike* if c is real) functions and the above geometric condition is equivalent to $\text{Re}\left(\frac{1}{c}\frac{zh'(z)}{h(z)}\right) \geq 0$ for $z \in \mathbb{D}$.

Case 2. Let $h : \mathbb{D} \to \mathbb{C}$ a univalent analytic function, $h(0) = 0$, whose range $\Omega = h(\mathbb{D})$ satisfies:
There is a direction c with $Re(c) \geq 0$ such that the half lines $L_w = \{w + ct : t \geq 0\} \subset \Omega$ for each $w \in \Omega$. Then

$$\phi_t(z) = h^{-1}(h(z) + ct), \quad z \in \mathbb{D}, t \geq 0,$$

is a semigroup with DW point $b = 1$. Such h satisfy $\text{Re}\left(\frac{1}{c}(1-z)^2 h'(z)\right) \geq 0$ for $z \in \mathbb{D}$, and are *close-to-convex*.

It turns out that under a normalization, **every** semigroup falls exactly in one of the preceding two cases depending on the location of its DW point b: Case 1 $\Leftrightarrow b \in \mathbb{D}$, and, Case 2 $\Leftrightarrow b \in \mathbb{T}$.

In conclusion each (ϕ_t) is determined uniquely by any of the following:
1. The generator $G(z) = (\bar{b}z - 1)(z - b)F(z)$,
2. The triple $(b, c, h(z))$

The relation between G and h is

$$G(z) = G'(b)\frac{h(z)}{h'(z)} \quad \text{when } b \in \mathbb{D},$$

and

$$G(z) = \frac{1}{h'(z)} \quad \text{when } b \in \mathbb{T}.$$

The univalent function h is called the *Königs function* or the *associated univalent function* of the semigroup.

3. Semigroups of composition operators

Given a $\Phi = (\phi_t)$ on $\mathbb{D}$, the composition operators

$$T_t(f)(z) = f(\phi_t(z))$$

form a semigroup of linear transformations on $\mathcal{H}(\mathbb{D})$. One can view $(T_t(f))_{t\geq 0}$ as "generalized translates inside $\mathbb{D}$" of $f \in \mathcal{H}(\mathbb{D})$.

Now suppose $\mathbf{X} \subset \mathcal{H}(\mathbb{D})$ is a Banach space of analytic function on $\mathbb{D}$, and (ϕ_t) a semigroup such that the composition operators

$$T_t(f) = f \circ \phi_t : \mathbf{X} \to \mathbf{X}$$

are bounded. Then (T_t) is an operator semigroup on X. Spaces that satisfy the requirements are for example, Hardy spaces H^p, Bergman A^p, the Dirichlet space $\mathcal{D}$, the spaces BMOA and Bloch $\mathcal{B}$, H^∞, and many others.

3. Strong continuity

Since $\phi_t(z) \overset{t\to 0}{\longrightarrow} z$ for each $z \in \mathbb{D}$, it follows that

$$T_t(f)(z) = \lim_{t\to 0} f(\phi_t(z)) = f(z), \quad f \in \mathbf{X}$$

for each $z \in \mathbb{D}$. To show strong continuity of (T_t), one has to upgrade this pointwise convergence to $\lim_{t\to 0} \|f \circ \phi_t - f\|_{\mathbf{X}} = 0$.

A general argument, for spaces $\mathbf{X}$ in which polynomials are dense, is as follows:
By a triangle inequality and the fact that

$$\sup_{t\in(0,\delta)} \|T_t\| < \infty \text{ for some } \delta > 0,$$

which is valid in all known spaces, the question reduces to proving

$$\lim_{t\to 0} \|P \circ \phi_t - P\| = 0,$$

for each polynomial P, and this will follow if we prove it for monomials, i.e. if we show $\lim_{t\to 0} \|\phi_t(z)^k - z^k\| = 0$ for each k. The latter, in classical function spaces, follows from dominated convergence or a similar theorem.

The above argument gives strong continuity of $\{T_t\}$ for every inducing $\{\phi_t\}$ on,
i) the Hardy spaces H^p, $1 \leq p < \infty$,
ii) the Bergman spaces A^p, $1 \leq p < \infty$,
iii) the Dirichlet space $\mathcal{D}$.

Further, assuming $\{T_t\}$ is strongly continuous on $\mathbf{X}$, the infinitesimal generator is

$$\Gamma(f)(z) = \lim_{t \to 0} \frac{f(\phi_t(z)) - f(z)}{t} = \frac{\partial (f \circ \phi_t)(z)}{\partial t}|_{t=0}$$
$$= f'(\phi_t(z)) \frac{\partial \phi_t(z)}{\partial t}|_{t=0} = G(z) f'(z),$$

with domain $D(\Gamma) = \{f \in \mathbf{X} : Gf' \in \mathbf{X}\}$.

Thus the generator is a differential operator

$$\Gamma(f)(z) = G(z) f'(z),$$

and differential operators are usually not bounded on classical spaces. Thus composition semigroups (T_t) are not uniformly continuous unless $G \equiv 0$, i.e. unless $\phi_t(z) = z$ for every t, the trivial semigroup.

On spaces where polynomials are not dense, different things can happen:

• On H^∞, no nontrivial $\{\phi_t\}$ induces a strongly continuous semigroup.

This follows from a more general theorem of H. Lotz (1985), on the geometry of Banach spaces, saying that if $\mathbf{X}$ is a space with certain geometric properties (is a *Grothendieck space* with the *Dunford-Pettis property*) then every c_0-semigroup (T_t) on $\mathbf{X}$ is automatically uniformly continuous and thus it has a bounded infinitesimal generator.

H^∞ is known to be such a space. If $\{T_t\}$ were strongly continuous on H^∞ then it would be uniformly continuous and $\Gamma(f)(z) = G(z)f'(z)$ would be a bounded operator on H^∞, which happens only when (ϕ_t) is trivial.

• The Bloch space $\mathcal{B}$, of analytic functions on $\mathbb{D}$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty,$$

is known to be isomorphic to l^∞, a space which has the same geometric Banach space property as H^∞, and thus so does $\mathcal{B}$. Thus no nontrivial semigroup is strongly continuous on $\mathcal{B}$.

• On BMOA: Recall that BMOA consists of those $f \in H^2$ for which

$$\sup_{a\in\mathbb{D}} \|f(\phi_a(z)) - f(a)\|_{H^2} < \infty, \;\; \phi_a(z) = \frac{a-z}{1-\bar{a}z}$$

with norm

$$\|f\|_* = |f(0)| + \sup_{a\in\mathbb{D}} \|f(\phi_a(z)) - f(a)\|_{H^2}.$$

It is a non separable Banach space, and

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

The subspace VMOA is the closure of the polynomials in BMOA. Using the density of the polynomials, the general argument applies and gives strong continuity of every composition semigroup on VMOA.

The question of strong continuity on BMOA remained open for a long time, and was settled in 2017 by A. Anderson, M. Jovovic, W. Smith. Using interpolating Blaschke products as test functions, they showed that if $\mathbf{X}$ is a space with

$$H^{\infty} \subseteq \mathbf{X} \subseteq \mathcal{B}$$

with both embeddings continuous, then no non-trivial $\{\phi_t\}$ induces a strongly continuous composition semigroup on $\mathbf{X}$.

Similar non strong continuity results hold on other spaces, for example, on Q_p spaces (the Möbius invariant versions of weighted Dirichlet Hilbert spaces), on certain analytic Morey spaces, etc.

It is interesting to note that the form

$$\Gamma(f)(z) = A(z)f'(z), \quad A(z) \text{ analytic on } \mathbb{D},$$

of the generator characterizes composition semigroups. It was shown by W. Arendt, I. Chalendar and independently by E. Gallardo, D. Yakubovich (both 2018) that if $\mathbf{X}$ is a Banach space of analytic functions on $\mathbb{D}$, satisfying some mild conditions, and (T_t) a c_0-operator semigroup on $\mathbf{X}$ with generator of the above form, then (T_t) consists of composition operators.

4. The resolvent operator

Let us restrict to semigroups (ϕ_t) with DW point $b = 0$, and consider the induced operator semigroup on Hardy spaces $\mathbf{X} = H^p$. Finding the resolvent operator

$$R(\lambda, \Gamma) = (\lambda - \Gamma)^{-1}, \ \lambda \in \rho(\Gamma),$$

involves solving for g the differential equation

$$f = \lambda g - Gg'.$$

Choosing conveniently the point $\lambda = 1 \in \rho(\Gamma)$ we find

$$g(z) = R(1, \Gamma)(f)(z) = \frac{1}{h(z)} \int_0^z f(\zeta) h'(\zeta)\, d\zeta,$$

where h is the Königs function of (ϕ_t). Denote this bounded operator by R_h, and let M_z be the usual multiplication by z.

Also consider the auxiliary transformations

$$P_h(f)(z) = \frac{1}{zh(z)} \int_0^z f(\zeta)\zeta h'(\zeta) d\zeta$$

$$Q_h(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{\zeta h'(\zeta)}{h(\zeta)}\, d\zeta.$$

A computation shows that

$$M_z P_h = R_h M_z, \text{ and } Q_h = P_h + Q_h P_h.$$

The first equation says that roughly $R_h \sim P_h$ and the second that $P_h \sim Q_h$. Further since

$$\frac{zh'(z)}{h(z)} = 1 + z \left(\log \frac{h(z)}{z} \right)'$$

we find

$$M_z Q_h = J + L_h M_z$$

where J is the integration operator

$$f \to \int_0^z f(\zeta) d\zeta$$

and

$$L_h(f)(z) = \int_0^z f(\zeta) \left(\log \frac{h(\zeta)}{\zeta} \right)' d\zeta.$$

In particular since R_h P_h Q_h and J are bounded, given a (ϕ_t) with associated Köning function h we have

(i) L_h is bounded on H^p for $1 \leq p < \infty$.

(ii) questions about compactness, or membership of $R(\lambda, \Gamma)$ in operator ideals boil down to the same questions for L_h.

Next, a classical result for univalent functions h on $\mathbb{D}$ with $h(0) = 0$ says

$$\log \frac{h(z)}{z} \in BMOA,$$

and this leads to the question wether the Volterra type operator

$$T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta) \, d\zeta.$$

is bounded on H^p for every $g \in BMOA$.

For $p = 2$, Ch. Pommerenke had shown that

$$T_g : H^2 \to H^2 \text{ is bounded } \Leftrightarrow g \in BMOA,$$

and the same turned out to be true for values of $p \in [1, \infty)$, (A. Aleman, A. S.):
(i) T_g bounded on $H^p \Leftrightarrow g \in BMOA$
(ii) T_g compact on $H^p \Leftrightarrow g \in VMOA$.

As a consequence we have a complete characterization of those inducing semigroups (ϕ_t) with DW point b=0, for which the composition semigroup (T_t) has compact resolvent. With h the Königs function for (ϕ_t), for the induced composition operator semigroup (T_t) on H^p, $1 \leq p < \infty$, the following are equivalent:

(i) $R(\lambda, \Gamma)$ compact.
(ii) $\log \frac{h(z)}{z} \in VMOA$.
(iii) $h \in \cap_{p<\infty} H^p$.

Similar results about T_g and $R(\lambda, \Gamma)$ can be proved on Bergman and other spaces.

6. The Cesaro operator.

A particular example of Volterra type operator is the Cesaro operator

$$C(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. A calculation with power series gives

$$C(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1-\zeta} \, d\zeta,$$

thus $C = \frac{1}{z} T_g$ with $g(z) = \log \frac{1}{1-z}$. Solving the equation

$$g(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1-\zeta} \, d\zeta$$

for f we find

$$f(z) = (1-z)(zg(z))' = -z(1-z)g'(z) - (1-z)g(z)$$

and the differential operator

$$\Delta(g)(z) = -z(1-z)g'(z) - (1-z)g(z)$$

is easily seen to be the generator of the weighted composition semigroup

$$S_t(f)(z) = \frac{\phi_t(z)}{z} f(\phi_t(z))$$

where

$$\phi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}.$$

This (S_t) is strongly continuous of H^p for $p \geq 1$ and some technical calculations give

$$\|S_t\|_{H^p \to H^p} = e^{-t/p}, \text{ for } p \geq 2,$$

and $\sigma(\Delta) = \{z : \text{Re}(z) \leq -\frac{1}{p}\}$. In particular $0 \in \rho(\Delta)$ and

$$C = R(0, \Delta).$$

It follows that $\|C\|_{H^p \to H^p} = p$ for $p \geq 2$. The value of $\|C\|_{H^p \to H^p}$ when $p < 2$ is not known.

7. Maximal subspace of strong continuity

On $BMOA$ a theorem of D. Sarason says:

Theorem. For $f \in BMOA$ the following are equivalent:
(a) $f \in VMOA$.
(b) $\lim_{t\to 0} \|f(e^{it}z) - f(z)\|_* = 0$.
(c) $\lim_{t\to 0} \|f(e^{-t}z) - f(z)\|_* = 0$.

Thus for the rotation group $\phi_t(z) = e^{it}z$ and the dilation semigroup $\phi_t(z) = e^{-t}z$ the corresponding composition semigroup (T_t) has the property:
VMOA is the largest subspace of BMOA on which (T_t) is strongly continuous.

For other semigroups the space of strong continuity can be larger.

Example: If $\phi_t(z) = e^{-t}z + 1 - e^{-t}$ and

$$f(z) = \log \frac{1}{1-z} \in BMOA \setminus VMOA.$$

Then

$$\lim_{t \to 0} \|f \circ \phi_t - f\|_* = \lim_{t \to 0} t = 0,$$

and so $\{\phi_t\}$ induces a strongly continuous $\{T_t\}$ on the space

$$V = \text{span}\{VMOA, f(z)\},$$

strictly larger than VMOA.

In general, given $\{\phi_t\}$, denote by

$$[\phi_t, BMOA].$$

the maximal subspace of $BMOA$, on which $\{T_t\}$ is strongly continuous. Thus we have

$$VMOA \subseteq [\phi_t, BMOA]$$

for every $\{\phi_t\}$, and there are $\{\phi_t\}$ such that

$$VMOA \subsetneqq [\phi_t, BMOA].$$

By Sarason's theorem, for $\phi_t(z) = e^{it}z$ or $\phi_t(z) = e^{-t}z$,

$$[\phi_t, BMOA] = VMOA.$$

It turns out that there are several other semigroups for which the maximal subspace of strong continuity coincides with VMOA

Recall that for a semigroup with DW point 0 the generator has the form $G(z) = -zF(z)$ with Re$F \geq 0$. Assume for simplicity that $F(0) = 1$, then the growth estimate for functions of positive real part $|F(z)| \geq \frac{1-|z|}{1+|z|}$ as $|z| \to 1$, so

$$\frac{1-|z|}{G(z)} = O(1), \quad |z| \to 1.$$

A sufficient condition for $VMOA = [\varphi_t, BMOA]$ was proved by Blasco, Contreras, Diaz-Madrigal, Martinez, S. and it says:

If G is the generator of (φ_t) with DW point 0, and there is $0 < \alpha < 1$ such that

$$\frac{(1-|z|)^\alpha}{G(z)} = O\left(1\right), \quad |z| \to 1,$$

then $VMOA = [\varphi_t, BMOA]$.

Thus in addition to Sarason's cases

$$\varphi_t(z) = e^{it} z, \qquad \varphi_t(z) = e^{-t} z,$$

there are several other semigroups for which $VMOA = [\varphi_t, BMOA]$. For example if $\{\varphi_t\}$ has generator

$$G(z) = -z(1-z)^{\alpha}, \quad 0 < \alpha < 1,$$

then $VMOA = [\varphi_t, BMOA]$.

A partial converse to the above result was also proved: If the DW point of (φ_t) is 0 and $VMOA = [\varphi_t, BMOA]$, then

$$\lim_{|z|\to 1} \frac{1-|z|}{G(z)} = 0.$$

But the question of giving a characterization of those (ϕ_t) for which $[\varphi_t, BMOA] = VMOA$ remained open for some time. It was recently answered (2021) by N. Chalmoukis and V. Daskalogiannis. They proved:

(i) $[\varphi_t, BMOA] = VMOA$ can happen only for elliptic (ϕ_t), and
(ii) for such semigroups the characterization is

$$(\star)\ \lim_{|a|\to 1}\left(\log\frac{e}{1-|a|^2}\right)^2\int_{\mathbb{D}}\frac{1-|\phi_a|^2}{|G(z)|^2}dm(z) = 0$$

where G is the generator of (φ_t).

All the above questions, concerning the maximal subspace of strong continuity also arise for the Bloch space and its subspace $\mathcal{B}_0$, the closure of the polynomials in $\mathcal{B}$. The analogous characterization for the equality $\mathcal{B}_0 = [\varphi_t, \mathcal{B}]$

$$(\star\star) \qquad \lim_{|z|\to 1}\frac{1-|z|^2}{G(z)}\log\frac{1}{1-|z|^2} = 0.$$

was proved by Chalmoukis and Daskalogiannis.

Surprisingly and unexpectedly, the two conditions $(\star)$ and $(\star\star)$ turn out to be equivalent.

8. Weighted composition semigroups

If $w : \mathbb{D} \to \mathbb{C}$ is analytic, the formula

$$S_t(f)(z) = \frac{w(\phi_t(z))}{w(z)} f(\phi_t(z)), \quad f \in \mathbf{X},$$

defines, for suitable w, bounded operators on $\mathbf{X}$, and (S_t) is a semigroup.

For example if the DW point is $b = 0$ then $w(z) = z$ is a good choice and gives

$$S_t(f)(z) = \frac{\phi_t(z)}{z} f(\phi_t(z)).$$

As another example, for $\phi_t(z) = 1 - (1-z)^{e^{-t}}$ use $w(z) = \frac{1}{1-z}$ to obtain

$$S_t(f)(z) = (1-z)^{1-e^{-t}} f(\phi_t(z)),$$

a semigroup of bounded operators on $\mathbf{X} = H^p$.

In a more general setting, given a (ϕ_t), a *cocycle* for (ϕ_t) is a family $(m_t : t \geq 0)$ of analytic functions on $\mathbb{D}$ satisfying

(i) $m_0 \equiv 1$
(ii) $m_{t+s}(z) = m_t(z)m_s(\phi_t(z)) \quad$ for each $z \in \mathbb{D}$ and $t, s \geq 0$.
(ii) The map $t \to m_t(z)$ is continuous for each $z \in \mathbb{D}$.

If (m_t) is a cocycle the formula

$$U_t(f)(z) = m_t(z)f(\phi_t(z)), \quad f \in \mathbf{X}, \tag{1}$$

defines a semigroup (U_t) of operators on $\mathbf{X}$ provided each U_t is bounded.

A large class of cocycles is constructed as follows. Let g be analytic on $\mathbb{D}$ then the functions

$$m_t(z) = \exp\left(\int_0^t g(\phi_s(z))\, ds\right), \quad z \in \mathbb{D},\ t \geq 0, \tag{2}$$

is a cocycle for (ϕ_t).

The question of strong continuity of weighted composition semigroups is much more complex than in the unweighted case. But if it is strongly continuous the infinitesimal generator is given by

$$\Delta(f)(z) = G(z)f'(z) + g(z)f(z),$$

where G is the generator of (ϕ_t).

Appropriate choices of w or more generally of the function g to generate a cocycle, give weighted composition semigroups (or groups) that are related to classical operators.

Thank you for your attention