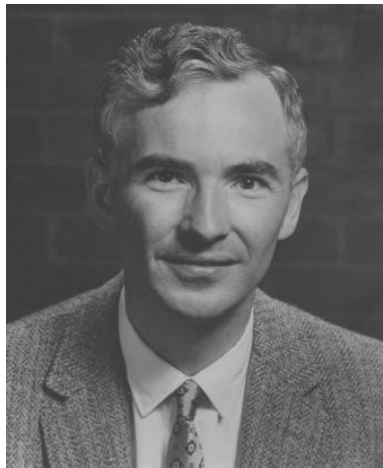# Multiplicative Hankel matrices

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February, 2015

Henry Helson (1927–2010)

Henry Helson is a household name in harmonic analysis through the "Helson–Szegő condition". From Yves Meyer's review of the first edition of Helson's "Harmonic Analysis": "*This book is a piece of art. It can be read like a novel. [...] The reviewer would advise this book to be read as a source of intellectual excitement by any student in mathematics who still does not know whether he is interested in becoming a researcher in the field.*

Helson's work on multiplicative Hankel matrices

In two papers, published in 2006 and (posthumously) in 2010, Henry Helson initiated a study of multiplicative Hankel matrices, which are finite or infinite matrices whose entries $a_{m,n}$ only depend on the product $m \cdot n$.

I will discuss the motivation for these papers, their contents, and subsequent developments.

Hankel matrices

A Hankel matrix is a square matrix (finite or infinite) whose entries $a_{m,n}$ only depend on $m+n$:

$$\begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 \\ \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 \\ \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_7 \\ \rho_4 & \rho_5 & \rho_6 & \rho_7 & \rho_8 \end{pmatrix}.$$

It is constant on skew-diagonals; any (finite or infinite) sequence ρ_n defines a Hankel matrix $(a_{m,n})$ by the recipe $a_{m,n} = \rho_{m+n}$. The most prominent example of an infinite Hankel matrix is Hilbert's matrix:

$$\rho_n = \frac{1}{n+1} \quad \text{i.e.} \quad a_{m,n} = \frac{1}{m+n+1}.$$

Hankel operators on $H^2(\mathbb{T})$

Consider the Hardy space $H^2(\mathbb{T})$, which consists of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|f\|_2^2 := \sum_{n=0}^{\infty} |a_n|^2 = \int_{\mathbb{T}} |f(z)|^2 \frac{|dz|}{2\pi}.$$

We think of $H^2(\mathbb{T})$ as a closed subspace of $L^2(\mathbb{T})$ and let $\overline{P}$ denote orthogonal projection from $L^2(\mathbb{T})$ onto $\overline{H^2}$, the L^2 space of conjugate analytic functions. Every $\psi(z) = \sum_{n=0}^{\infty} \overline{\rho_n} z^n$ in $H^2(\mathbb{T})$ defines a Hankel operator H_ψ on $H^2(\mathbb{T})$ by the rule:

$$H_\psi f = \overline{P}(\overline{\psi} f)$$

which at least makes sense when f is a polynomial. Observe:

- the matrix of H_ψ with respect to the canonical orthogonal basis (z^n) is the Hankel matrix (ρ_{m+n}).

Nehari's theorem (1957)

Nehari's theorem answers the following question:

- For which ψ does H_ψ define a bounded operator from H^2 to $\overline{H^2}$?

To arrive at this result, we start by noting that

$$\|H_\psi f\|_2 = \sup_{g \in H^2(\mathbb{T}), \|g\|_2 = 1} \left| \langle fg, \psi \rangle \right|,$$

hence H_ψ is certainly bounded if ψ defines a bounded linear functional on $H^1(\mathbb{T})$, the closed subspace of analytic functions in $L^1(\mathbb{T})$. An important fact is that every h in $H^1(\mathbb{T})$ can be factored as

$$h = fg, \text{ where } f, g \in H^2(\mathbb{T}) \text{ and } \|f\|_2^2 = \|g\|_2^2 = \|f\|_1.$$

It gives the reverse implication in Nehari's theorem:

Nehari's theorem ctd.

Theorem (Nehari, 1957)

H_ψ is bounded if and only if ψ defines a bounded linear functional on $H^1(\mathbb{T})$. In fact, $\|H_\psi\| = \min\{\|\phi\|_\infty : P\phi = \psi\}$.

Nehari is pre-*BMO*; in modern terms we would say that ψ belongs to *BMOA*. The leading example is the Hilbert matrix which can be dealt with in many different ways, including the following two:

- $\sum_{n=1}^\infty n^{-1} z^n = 2iP(\arg(1-z))$, hence $\|H_\psi\| \le \pi$.
- The same bound is obtained from Hardy's inequality which says that a function $f(z) = \sum_{n=0}^\infty a_n z^n$ in $H^1(\mathbb{T})$ satisfies

$$\sum_{n=0}^\infty \frac{|a_n|}{n+1} \le \pi \|f\|_1.$$

Some additional work shows that the norm of the Hilbert matrix is indeed π.

Nehari's theorem and bounded symbols

Our definition

$$H_\psi f := \overline{P}(\overline{\psi} f)$$

makes sense whenever ψ is an arbitrary function in $L^2(\mathbb{T})$. In fact, any two functions ψ_1 and ψ_2 define the same Hankel operator if $P\psi_1 = P\psi_2$. If ψ is in $H^2(\mathbb{T})$, we refer to it as the analytic symbol of H_ψ. Then any other function ϕ such that $P\phi = \psi$ is also a symbol of H_ψ. Nehari's theorem can now be rephrased as:

Theorem (Alternate version of Nehari)

H_ψ is bounded if and only if it admits a bounded symbol.

Small Hankel operators on $\mathbb{T}^d$

Our definition

$$H_\psi f := \overline{P}(\overline{\psi}f)$$

makes sense if ψ is in $H^2(\mathbb{T}^d)$ (or in $L^2(\mathbb{T}^d)$) and f is an analytic polynomial in d complex variables, where $\overline{P}$ is orthogonal projection from $L^2(\mathbb{T}^d)$ onto $\overline{H^2(\mathbb{T}^d)}$; it gives a so-called small Hankel operator ("small" because $H^2(\mathbb{T}^d)$ and $\overline{H^2(\mathbb{T}^d)}$ together fill up a "small" part of $L^2(\mathbb{T}^d)$).

- Can we extend Nehari's theorem to this setting?

The factorization of h in H^1 into a product of two functions f and g does not hold anymore, but the key observation is that we need less.

The Ferguson–Lacey–Terwilleger theorem

To establish Nehari's theorem for $d > 1$, it would be enough to have $f = \sum_{j=1}^{\infty} g_j h_j$ with

$$\sum_{j=1}^{\infty} \|g_j\|_2 \|h_j\|_2 \leq C \|f\|_1$$

for a constant C independent of f. If this holds, we say that $H^1(\mathbb{T}^d)$ admits weak factorization and set

$$\|f\|_{w,1} := \inf_{\sum_j g_j h_j = f} \sum_{j=1}^{\infty} \|g_j\|_2 \|h_j\|_2.$$

Clearly, $\|f\|_1 \leq \|f\|_{w,1} \leq \|f\|_2$.

Theorem (Ferguson–Lacey (2002), Lacey–Terwilleger (2009))

$H^1(\mathbb{T}^d)$ *admits weak factorization for every finite $d > 1$.*

Interpretation of Nehari's theorem

Nehari's theorem (including its extension by Ferguson–Lacey–Terwilleger for $1 < d < \infty$) yields two equivalent statements:

- H_ψ is a bounded Hankel operator if and only if there exists a function ϕ in $L^\infty(\mathbb{T}^d)$ such that the orthogonal projection of ϕ onto $H^2(\mathbb{T}^d)$ coincides with ψ.
- $H^1(\mathbb{T}^d)$ admits weak factorization.

Helson's project:

- Decide whether these statements remain valid when $d = \infty$.

Hardy spaces on $\mathbb{T}^\infty$ and the Bohr lift

Harald Bohr made the following basic observation:

Let $f(s) = \sum_{n\geq 1}^{N} a_n n^{-s}$ be an ordinary Dirichlet polynomial. We factor each integer n into a product of prime numbers $n = p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ and set $z = (p_1^{-s}, ..., p_d^{-s})$. Then

$$f(s) = \sum_{n=1}^{N} a_n (p_1^{-s})^{\alpha_1} \cdots (p_d^{-s})^{\alpha_d} = \sum_{n=1}^{N} a_n z_1^{\alpha_1} \cdots z_d^{\alpha_d} =: F(z).$$

This so-called Bohr lift is not just formal: it provides an isometric isomorphism between Hardy spaces on $\mathbb{T}^\infty$ and Hardy spaces of Dirichlet series.

H^p spaces of Dirichlet series

We follow Hedenmalm–Lindqvist–Seip (1997) and Bayart (2002), assuming $1 \le p < \infty$.

- On the one hand, we define $\mathscr{H}^p$ as the completion of the set of Dirichlet polynomials $P(s) = \sum_{n=1}^{N} a_n n^{-s}$ with respect to the norm

$$\|P\|_{\mathscr{H}^p} = \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T |P(it)|^p dt \right)^{1/p}.$$

- On the other hand, via the Bohr lift $z_j = p_j^{-s}$, $\mathscr{H}^p$ is isometrically isomorphic to $H^p(\mathbb{T}^\infty)$, the closure in $L^p(\mathbb{T}^\infty)$ of holomorphic polynomials in infinitely many variables (z_j).

Explicitly, in multi-index notation we write $n = (p_j)^{\alpha(n)}$ and obtain

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\alpha(n)},$$

the Bohr lift of $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

A basic fact about $\mathscr{H}^2$

Since for $f(s) = \sum a_n n^{-s}$ and $g(s) = \sum b_n n^{-s}$ in $\mathscr{H}^2$,

$$\langle f, g \rangle_{\mathscr{H}^2} = \sum_{n=1}^{\infty} a_n \overline{b_n},$$

we get that

- The reproducing kernel K_w of $\mathscr{H}^2$ is $K_w(s) = \zeta(s + \overline{w})$, whence

  $$|f(\sigma + it)| \leq (\zeta(2\sigma))^{1/2} \|f\|_{\mathscr{H}^2} \leq ((\sigma - 1/2)^{-1/2} + C)\|f\|_{\mathscr{H}^2}$$

  for every f in $\mathscr{H}^2$ with C an absolute constant.

A lot more is known about the function theory of $\mathscr{H}^2$ (boundary behavior, zeros, interpolating sequences), but for what follows it suffices to know the above fact and consequently that functions in $\mathscr{H}^2$ are analytic in the half-plane $\sigma = \operatorname{Re} s > 1/2$.

Hankel operators and forms on $\mathbb{T}^\infty$

A function $\psi(s) = \sum_{n=1}^\infty \overline{\rho_n} n^{-s}$ in $\mathscr{H}^2$ defines, exactly as before, a small Hankel operator if we set

$$H_\psi F = \overline{P}(\overline{\mathscr{B}\psi} F),$$

where $\overline{P}$ is orthogonal projection onto $\overline{H^2(\mathbb{T}^\infty)}$. It is slightly easier to see that this becomes something concrete and understandable if we consider instead the corresponding Hankel form. If we write $f(s) = \sum_{n=1}^\infty a_n n^{-s}$, $g(s) = \sum_{n=1}^\infty b_n n^{-s}$, then we observe that

$$\langle fg, \overline{\psi} \rangle_{\mathscr{H}^2} = \langle \mathscr{B}f\mathscr{B}g, \overline{\mathscr{B}\psi} \rangle_{H^2(\mathbb{T}^\infty)},$$

which is the corresponding Hankel form acting on $\mathscr{H}^2 \times \mathscr{H}^2$. We will now see how this form can be expressed in terms of its action on the sequences $a := (a_n)$ and $b := (b_n)$.

Multiplicative Hankel forms

If $\psi \in H^2(\mathbb{T})$, then the corresponding Hankel form can be written in discrete form as

$$H_\psi(a,b) = \sum_{j,k=0}^{\infty} a_j b_k \rho_{j+k},$$

where $\psi(z) = \sum_{n=0}^{\infty} \overline{\rho_n} z^n$. For $\psi \in H^2(\mathbb{T}^d)$, we have correspondingly

$$H_\psi(a,b) = \sum_{\alpha,\beta \geq 0} a_\alpha b_\beta \rho_{\alpha+\beta}.$$

Writing again positive integers n in multi-index notation $n = p^\alpha$, we obtain the following infinite dimensional version of a Hankel form:

$$H_\psi(a,b) = \sum_{j,k=1}^{\infty} a_j b_k \rho_{j \cdot k}.$$

Hilbert–Schmidt operators

Operators of Hilbert Schmidt type constitute a distinguished class of compact operators. In general, if T acts on a Hilbert space H and e_n is an orthonormal basis for H, then T is Hilbert–Schmidt if

$$\sum_n \| Te_n \|^2 < \infty.$$

For H_ψ acting on $H^2(\mathbb{T})$, we may choose $e_n = z^n$ to see that H_ψ is Hilbert–Schmidt if and only if

$$\sum_{k=0}^\infty \sum_{n=0}^\infty |\rho_{n+k}|^2 = \sum_{n=0}^\infty |\rho_n|^2 (n+1) < \infty.$$

Equivalently, the condition is that the analytic symbol ψ belong to the so-called Dirichlet space.

Helson's theorem for Hilbert–Schmidt operators

A multiplicative Hankel matrix obtained from the sequence ρ_n will be of Hilbert–Schmidt type if and only if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\rho_{mn}|^2 = \sum_{n=1}^{\infty} |\rho_n|^2 d(n) < \infty,$$

where $d(n)$ is the divisor function. This observation inspired Helson to deduce the following beautiful inequality:

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)} \le \|f\|_{\mathscr{H}^1}^2,$$

valid for functions $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathscr{H}^1$. It led him to:

Theorem (Helson 2006)

Multiplicative Hankel matrices of Hilbert–Schmidt type admit bounded symbols.

A multiplicative version of Nehari's theorem?

In a nice little book on Dirichlet series published in 2005, Helson asked whether for every bounded multiplicative bilinear form there is a bounded function on $\mathbb{T}^\infty$ that defines the form, or, equivalently, whether $H^1(\mathbb{T}^\infty)$ admits weak factorization.

Clearly this fails if we can prove that the best constant C_d in the bound $\|f\|_{w,1} \leq C_d \|f\|_1$ tends to ∞ when $d \to \infty$. Thus we may try to get a lower bound for the constant in the Lacey–Ferguson–Terwilleger theorem.

The Schur test

In his 2010 paper, Helson used the Schur test to estimate the norm of H_ψ. In our case, all we need is a sequence of positive numbers c_j such that

$$\sum_k \rho_{jk} c_k \le B c_j.$$

Then H_ψ will be bounded on $H^2 \times H^2$ with norm $\le B$.

A key point is this: If we view H_ψ as a linear functional acting on H^1 and equip H^1 with the norm $\|\cdot\|_{w,1}$, then this functional will have the same bound B on its norm, and thus we get a lower bound for $\|\psi\|_{w,1}$.

Helson used the following simple and natural form: $\rho_n = 1$ for $n \le N$ and $\rho_n = 0$ otherwise.

Helson's conjecture

By his simple argument involving the Schur test, Helson obtained a large class of Dirichlet polynomials Q for which $\|Q\|_2 \le 2\|Q\|_{w,1}$, namely those with unimodular coefficients. To give a negative answer to the question of whether weak factorization extends to the infinite-dimensional polydisc, which was his main object, he "only" needed a sequence of such polynomials Q_N for which $\|Q_N\|_1 = o(\|Q_N\|_2)$ when $N \to \infty$. He made the following conjecture:

$$\left\| \sum_{n=1}^{N} n^{-s} \right\|_1 = o(\sqrt{N}).$$

Bondarenko and Seip (2014) showed that

$$\left\| \sum_{n=1}^{N} n^{-s} \right\|_1 \ge c(\log N)^{-0.02152}\sqrt{N},$$

and we have come to believe that the conjecture is false.

Solution to Helson's problem

Although the discrete representation of multiplicative Hankel forms suggests a strong link to number theory, we can avoid the difficulties encountered by Helson, while still using the Schur test in the same way. Namely, consider instead the homogeneous polynomial

$$\psi(z) = (z_1 + z_2) \cdot (z_3 + z_4) \cdots (z_{2k-1} + z_{2k}).$$

Theorem (Ortega-Cerdà–Seip (2012))

$\sup_{f \neq 0, f \in H^1(\mathbb{D}^{2k})} \|f\|_{w,1} / \|f\|_1 \geq (\pi^2/8)^{k/2}.$

The proof shows that this particular ψ has optimal weak factorization $\psi \cdot 1$ (it "factors" badly!) and thus $\|\psi\|_{w,1} = \|\psi\|_2$. Hence: Nehari's theorem does not extend to $d = \infty$ or, equivalently, it does not extend to multiplicative Hankel matrices. (See Brevig–Perfekt (2014) for a sharper result.)

A canonical example of a multiplicative Hankel matrix?

This is perhaps all very good, but a very natural request is this:

- Please give us a canonical example of a multiplicative Hankel matrix, to see what this is about and to convince ourselves that this is a subject worth studying!

Responding to this request, we will see number theory returning prominently to our discussion.

Another look at the classical Hilbert matrix

Recall that the prime example of an infinite Hankel matrix:

$$A := \left(\frac{1}{m+n+1} \right)_{m,n \geq 0}.$$

We may view the Hilbert matrix as the matrix of an operator in a different way than we saw above, namely as that of the integral operator

$$\mathbf{H}_a f(z) := \int_0^1 f(t)(1-zt)^{-1} dt \qquad (1)$$

with respect to the standard basis $(z^n)_{n \geq 0}$ for $H^2(\mathbb{T})$. Magnus (1950) used this representation to show that the Hilbert matrix has no eigenvalues and that its continuous spectrum is $[0, \pi]$.

What is the multiplicative Hilbert matrix?

We start from the analogous integral operator:

$$\mathbf{H}f(s) := \int_{1/2}^{+\infty} f(w)(\zeta(w+s)-1)dw$$

acting on Dirichlet series $f(s) = \sum_{n=2}^{\infty} a_n n^{-s}$. We say that f is in $\mathscr{H}_0^2$ if f is in $\mathscr{H}^2$ and $a_1 = 0$. The reproducing kernel K_w of $\mathscr{H}_0^2$ is $K_w(s) = \zeta(s+\overline{w}) - 1$. This implies that

$$\langle \mathbf{H}f, g \rangle_{\mathscr{H}_0^2} = \int_{1/2}^{\infty} f(w)\overline{g(w)}dw. \qquad (2)$$

This makes sense for all f, g in $\mathscr{H}_0^2$ for well-known but nontrivial reasons (involving the notion of Carleson measure). Hence **H** is well defined and bounded on $\mathscr{H}_0^2$. Since $\langle f, \mathbf{H}f \rangle_{\mathscr{H}_0^2} = 0$ if and only if $f \equiv 0$. So (2) also implies that **H** is strictly positive.

The multiplicative Hilbert matrix

Since

$$\int_{1/2}^{\infty} (nm)^{-w} dw = \frac{1}{\sqrt{mn}\log(mn)},$$

the matrix of **H** with respect to the basis $(n^{-s})_{n\geq 2}$ is

$$M := \left(\frac{1}{\sqrt{mn}\log(mn)}\right)_{m,n\geq 2}.$$

We call M the **multiplicative Hilbert matrix**.

Theorem (Brevig, Perfekt, Seip, Siskakis, Vukotić, arXiv 2014)

*The operator **H** is a bounded and strictly positive operator on $\mathscr{H}_0^2$ with $\|\mathbf{H}\| = \pi$. It has no eigenvalues, and the continuous spectrum is $[0,\pi]$.*

The symbol of the multiplicative Hilbert matrix

The multiplicative Hankel matrix has analytic symbol

$$\psi(s) := \sum_{n=2}^{\infty} \frac{n^{-s}}{\sqrt{n}\log n}.$$

We observe that $-\psi$ is, up to a linear term, a primitive of the Riemann zeta function. Returning to Nehari, we ask:

Question
Does the multiplicative Hilbert matrix have a bounded symbol?

Equivalently, we may ask whether we have

$$\left| a_1 + \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{n}\log n} \right| \lesssim \|f\|_{\mathscr{H}^1}$$

when $f(s) = \sum a_n n^{-s}$ is in $\mathscr{H}^1$. If we put absolute values on the terms in this sum, this would—if valid—be the right analogue of Hardy's inequality mentioned earlier in the talk.

Concluding remarks

More generally, we may ask:

- Can we find a way of characterizing bounded multiplicative Hankel matrices (by necessity, something different from Nehari's theorem)?
- Which multiplicative Hankel matrices admit bounded symbols?

Thank you for your attention!