Summation Theory in Dirichlet Spaces

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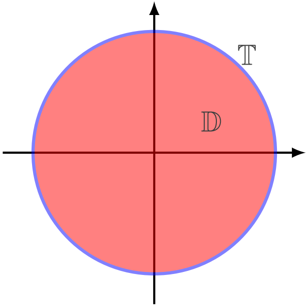
UAM

The Territory



I have lived the greater part of my mathematical life in the unit disk of the complex plane.

D. Sarason

Notations

1. $\mathrm{Hol}(\mathbb{D})$ is the collection of all analytic functions on $\mathbb{D}$.
2. $\mathcal{X} \subset \mathrm{Hol}(\mathbb{D})$ is a Banach space.
3. $\mathcal{P}$ is the set of all analytic polynomials.

Some Banach Function Spaces

1. Hardy Spaces H^p
2. Dirichlet Space $\mathcal{D}$
3. Harmonically Weighted Dirichlet Spaces $\mathcal{D}_\mu$
4. Superharmonically Weighted Dirichlet Spaces $\mathcal{D}_w$
5. Bergman Spaces A^p
6. Model Spaces K_θ
7. de Branges–Rovnyak Spaces $\mathcal{H}(b)$

Approximation Questions:

1. Is the set of polynomials dense in $\mathcal{X}$?
2. Given $f \in \mathcal{X}$, find a sequence of polynomials $(p_n)_{n\geq 1}$ such that $\|p_n - f\|_{\mathcal{X}} \to 0$ as $n \to \infty$.
3. Given $f \in \mathcal{X}$ and $\varepsilon > 0$, find $p \in \mathcal{P}$ such that $\|f - p\|_{\mathcal{X}} < \varepsilon$.

Easy Solution

What is the most natural choice for p_n such that $\|p_n - f\|_{\mathcal{X}} \to 0$?

Taylor Polynomials

Each $f \in \mathcal{X} \subset \mathrm{Hol}(\mathbb{D})$ has the Taylor series representation

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Consider its *Taylor polynomials*

$$S_n f(z) = \sum_{k=0}^{n} a_k z^k, \qquad (n \geq 0).$$

Then we **expect** that

$$\|S_n f - f\|_{\mathcal{X}} \to 0, \qquad (n \to \infty).$$

The Hardy–Hilbert Space H^2

Let

$$H^2 := \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_2 < \infty\},$$

where

$$\|f\|_2 := \left(\sum_{k=0}^{\infty} |a_k|^2\right)^{\frac{1}{2}}.$$

Then, immediately from the definition,

$$\|S_n f - f\|_2 = \left(\sum_{k=n+1}^{\infty} |a_k|^2\right)^{\frac{1}{2}} \to 0.$$

The Dirichlet Space $\mathcal{D}$

Let

$$\mathcal{D} := \{f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}(f) < \infty\},$$

where

$$\mathcal{D}(f) := \left(\sum_{k=0}^{\infty} k|a_k|^2 \right)^{\frac{1}{2}}.$$

The Dirichlet Space $\mathcal{D}$

We define

$$\|f\|_{\mathcal{D}}^2 := \|f\|_2^2 + \mathcal{D}(f) = \sum_{k=0}^{\infty} (k+1)|a_k|^2.$$

Then, again immediately from the definition,

$$\|S_n f - f\|_{\mathcal{D}} = \left(\sum_{k=n+1}^{\infty} (k+1)|a_k|^2 \right)^{\frac{1}{2}} \to 0.$$

The Hardy Space H^p

Let $0 < p < \infty$ and

$$H^p := \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_p < \infty\},$$

where

$$\|f\|_p := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}.$$

Similarly,

$$H^\infty := \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_\infty < \infty\},$$

where

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

The Hardy Space H^p

Then, for $1 < p < \infty$,

$$\|S_n f - f\|_p \to 0, \qquad (n \to \infty).$$

Not an immediate result. It follows from the M. Riesz theorem (1928) on the boundedness of projection

$$P_+ : \begin{array}{ccc} L^p(\mathbb{T}) & \longrightarrow & H^p(\mathbb{T}) \\ \sum_{n=-\infty}^{\infty} a_n z^n & \longmapsto & \sum_{n=0}^{\infty} a_n z^n \end{array}$$

The Remaining Cases

What about H^1 and H^∞?

The Hardy Space H^∞

The polynomials are not dense in H^∞.

Elementary fact: The uniform limit of continuous functions is continuous.

Infinite Blaschke products, or singular inner functions, are in H^∞ but not continuous on $\overline{\mathbb{D}}$. Hence, they cannot be uniformly approximated by polynomials on $\overline{\mathbb{D}}$.

The Disk Algebra $\mathcal{A}(\mathbb{D})$

Let

$$\mathcal{A}(\mathbb{D}) := H^\infty \cap \mathcal{C}(\overline{\mathbb{D}}) = \text{ Closure of polynomials in } H^\infty.$$

Then, by definition, polynomials are dense in $\mathcal{A}(\mathbb{D})$.

For each $f \in \mathcal{A}(\mathbb{D})$, do we have

$$\|S_n f - f\|_\infty \to 0?$$

The First Crack

Not necessarily!

There is an

$$f \in \mathcal{A}(\mathbb{D})$$

such that its Taylor polynomials **do not converge** uniformly to f on $\mathbb{D}$.

Justification

Lebesgue's constants are

$$L_n := \|S_n\|_{\mathcal{A}(\mathbb{D})\to\mathcal{A}(\mathbb{D})} \asymp \log n, \qquad n \geq 1.$$

Thus, in particular,

$$\sup_{n\geq 0} \|S_n\|_{\mathcal{A}(\mathbb{D})\to\mathcal{A}(\mathbb{D})} = \infty.$$

Justification

Hence, by the Banach–Steinhaus Theorem, there is an $f \in \mathcal{A}(\mathbb{D})$ such that

$$\sup_{n \geq 0} \|S_n f\|_{\mathcal{A}(\mathbb{D})} = \infty.$$

In particular,

$$S_n f \not\to f, \qquad \text{in } \mathcal{A}(\mathbb{D}).$$

Historical Note

- Paul du Bois-Reymond (1873) constructed a function whose Fourier series diverges at a point of continuity.
- His construction can be modified to obtain a function f is the disc algebra $\mathcal{A}(\mathbb{D})$ whose Taylor polynomials do not converge uniformly on $\mathbb{D}$, i.e.,

$$\|S_n f - f\|_{H^\infty} \not\to 0.$$

- There is a similar construction to show that

$$\|S_n f - f\|_{H^1} \not\to 0.$$

Divergent Series

How can we transform
a
divergent series
to a
convergent series?

Cesàro Means

The series $\sum_{k=0}^{\infty} a_k$ is C-summable to s if

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k = s.$$

Abel Means

The series $\sum_{k=0}^{\infty} a_k$ is A-summable to s if

$$\lim_{r \to 1^-} \sum_{k=0}^{\infty} r^k a_k = s.$$

Comparison

Summable

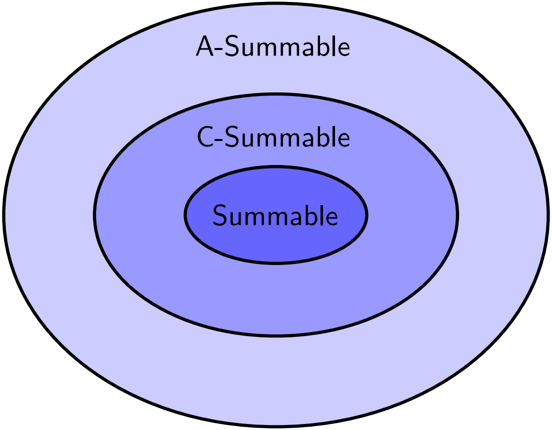
$\Downarrow$

C-summable

$\Downarrow$

A-summable

Summation Methods



More Summation Methods

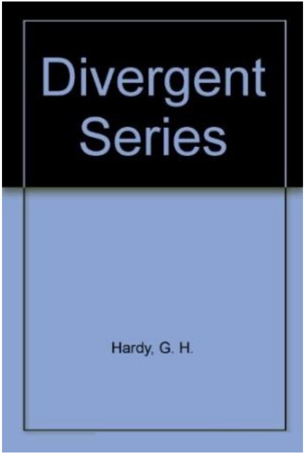
The generalized Cesàro means of order α:

$$\sigma_n^\alpha f(z) = \sum_{k=0}^{n} \frac{\binom{n-k+\alpha}{\alpha}}{\binom{n+\alpha}{\alpha}} a_k z^k,$$

where

$$\binom{n+\alpha}{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \qquad \alpha > -1.$$

Note that $\sigma_n^0 f = S_n f$ and $\sigma_n^1 f = \sigma_n f$.


Divergent Series
Hardy, G. H.

Divergent Series

Summation of Entire Series

Divergent Series

Abel Means

Making $\mathbb{T}$ a Nice Boundary

The Abel means of

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad (f \in \mathrm{Hol}(\mathbb{D})),$$

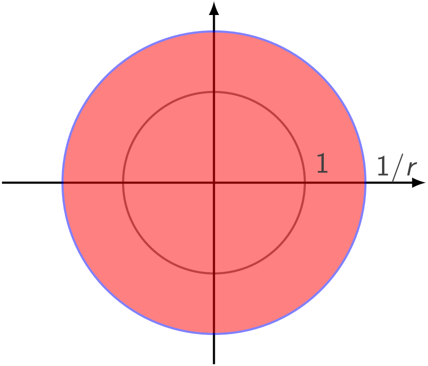
are

$$f_r(z) = \sum_{k=0}^{\infty} r^k \times a_k z^k = f(rz), \qquad (0 < r < 1).$$

Do we have

$$\|f - f_r\|_{\mathcal{X}} \to 0, \qquad (r \to 1^-)?$$

f_r Lives on a Bigger Disc



The main feature of f_r is that it is defined on the disc $|z| < 1/r$ which contains $\overline{\mathbb{D}}$ as a proper subset. In short, f_r is analytic at all points of $\mathbb{T} = \partial\mathbb{D}$.

Good Features

In several function spaces $\mathcal{X} \in \mathrm{Hol}(\mathbb{D})$, we have

i $f_r \in \mathcal{X}$,

ii $\|f - f_r\|_{\mathcal{X}} \to 0$ as $r \to 1$,

for all $f \in \mathcal{X}$.

Good News

Abel summation works for:

i. Hardy Spaces H^p, $0 < p < \infty$.
ii. Disc Algebra $\mathcal{A}(\mathbb{D})$.
iii. Dirichlet Spaces $\mathcal{D}_\mu$.
iv. Bergman Spaces A^p.

Bad News

There are function spaces $\mathcal{X}$, where the dilation technique does not work.

Two essential reasons:

i $\mathcal{X}$ is not star-shaped, i.e., $\exists f \in \mathcal{X}$ but $f_r \notin \mathcal{X}$.

ii $\mathcal{X}$ is star-shaped, i.e.,

$$f \in \mathcal{X} \implies f_r \in \mathcal{X},$$

yet $f_r \not\to f$ in the norm of $\mathcal{X}$.

Bad News

There are function spaces $\mathcal{X}$, where the dilation technique does not work.

Two essential reasons:

i. Model spaces K_θ are not star-shaped.
ii. de Branges-Rovnyak spaces $\mathcal{H}(b)$ are star-shaped, yet it is possible that $f_r \not\to f$ in the norm of $\mathcal{H}(b)$.

Historical Note

- Sarason (1986): Using a dulaity argument, polynomials are dense in $\mathcal{H}(b)$.

- Chevrot–Guillot–Ransford (2010): Dilation fails in $\mathcal{H}(b)$. Construction of b and an $f \in \mathcal{H}(b)$ such that

$$\limsup_{r\to 1} \|f_r\|_{\mathcal{H}(b)} = \infty.$$

Historical Note

- ElFallah–Fricain–Kellay–JM–Ransford (2016): Construction of b and an $f \in \mathcal{H}(b)$ such that

$$\lim_{r \to 1} \|f_r\|_{\mathcal{H}(b)} = \infty.$$

- ElFallah–Fricain–Kellay–JM–Ransford (2016): A semi-constructive solution for polynomial approximation.

Historical Note

- JM–Ransford (2017): Construction of an 'outer' symbol b and an $f \in \mathcal{H}(b)$ such that

$$\lim_{r \to 1} \|f_r\|_{\mathcal{H}(b)} = \infty.$$

- JM–Parisé–Ransford (2021): Even stronger methods like Borel means and logarithmic means fail for $\mathcal{H}(b)$ spaces.

Divergent Series

Cesàro Means

Fejér Polynomials

The Cesàro Means of

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

are (the so-called *Fejér polynomials*)

$$\sigma_n f(z) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k z^k, \qquad (n \geq 1).$$

Do we have

$$\|\sigma_n f - f\|_{\mathcal{X}} \to 0, \qquad (n \to \infty)?$$

Historical Note

- Hardy–Littlewood: In the Hardy space H^1,

$$\|f - \sigma_n f\|_{H^1} = \left\| f - \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k z^k \right\|_{H^1} \to 0.$$

- Hardy–Littlewood: In the disk algebra $\mathcal{A}(\mathbb{D})$,

$$\|f - \sigma_n f\|_{H^\infty} = \left\| f - \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k z^k \right\|_{H^\infty} \to 0.$$

Historical Note - Generalized version

- Hardy–Littlewood: In the Hardy space H^1, for each $\alpha > 0$,

$$\|f - \sigma_n^\alpha f\|_{H^1} = \left\| f - \sum_{k=0}^{n-1} \frac{\binom{n-k+\alpha}{\alpha}}{\binom{n+\alpha}{\alpha}} a_k z^k \right\|_{H^1} \to 0.$$

- Hardy–Littlewood: In the disk algebra $\mathcal{A}(\mathbb{D})$, for each $\alpha > 0$,

$$\|f - \sigma_n^\alpha f\|_{H^\infty} = \left\| f - \sum_{k=0}^{n-1} \frac{\binom{n-k+\alpha}{\alpha}}{\binom{n+\alpha}{\alpha}} a_k z^k \right\|_{H^\infty} \to 0.$$

Historical Note

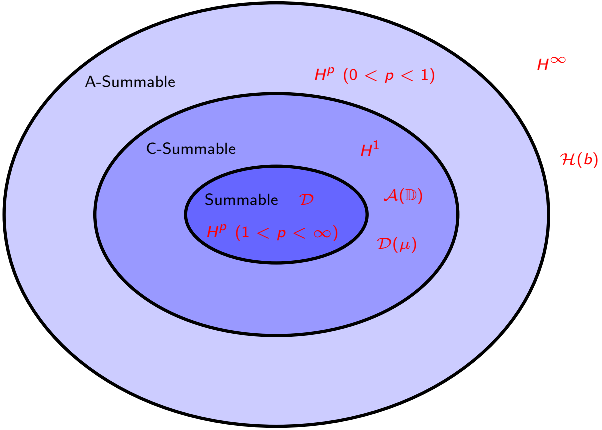
- ElFallah–Fricain–Kellay–JM–Ransford (2016): The Cesàro summation fails in de Branges-Rovnyak spaces $\mathcal{H}(b)$.
- JM–Ransford (2018): The Cesàro summation works in superharmonically weighted Dirichlet spaces $\mathcal{D}_w$.
- JM–Parisé–Ransford (2020): The Cesàro summation or order $> 1/2$ work in superharmonically weighted Dirichlet spaces $\mathcal{D}_w$. Moreover, the order $1/2$ is sharp.

Summability in Function Spaces

AS: $\sum_{k=0}^{\infty} r^k a_k z^k \to f$

CS: $\sum_{k=0}^{n-1} (1 - \frac{k}{n}) a_k z^k \to f$

S: $\sum_{k=0}^{n} a_k z^k \to f$



The Dirichlet Space $\mathcal{D}$

Recall that

$$\mathcal{D} := \{f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}(f) < \infty\},$$

where

$$\|f\|_{\mathcal{D}}^2 := \|f\|_2^2 + \mathcal{D}(f) = \sum_{k=0}^{\infty} (k+1)|a_k|^2.$$

Another Representation for $\mathcal{D}(f)$

We have

$$\mathcal{D}(f) = \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \, dA(z),$$

where $dA(z)$ is the two-dimensional Lebesgue (area) measure.

A Generalization

Let w be a positive superharmonic function on $\mathbb{D}$. We define

$$\mathcal{D}_w(f) = \frac{1}{\pi}\int_{\mathbb{D}} w(z)\,|f'(z)|^2\,dA(z),$$

and

$$\mathcal{D}_w := \{f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}_w(f) < \infty\}.$$

A Generalization

Easy to see that

$$\mathcal{D}_w \subset H^2.$$

We define

$$\|f\|^2_{\mathcal{D}_w} := \|f\|^2_{H^2} + \mathcal{D}_w(f).$$

Then $\mathcal{D}_w$ becomes a reproducing kernel Hilbert Space (RKHS) on the open unit disc $\mathbb{D}$.

Historical Note

- The Dirichlet integral appeared in Dirichlet's method for solving the Laplace equation (the so called Dirichlet principle).
- A. Beurling introduced the classical Dirichlet space in his thesis (1933) and its foundation was laid by him and L. Carleson in subsequent years.
- The harmonically weighted Dirichlet spaces were introduced by S. Richter (1991) in his analysis of shift-invariant subspaces of the classical Dirichlet space (Beurling-type theorem).
- The superharmonic weights were introduced by A. Aleman (1993).

A Special Weight

The weights

$$w(z) = (1 - |z|^2)^\alpha, \qquad (0 \leq \alpha \leq 1).$$

have been extensively studies.

They form a scale linking the classical Dirichlet space $\mathcal{D}$ $(\alpha = 0)$ to the Hardy space H^2 $(\alpha = 1)$.

A Special Weight

The latter is a consequence of the Littlewood–Paley formula:

$$\|f\|_{H^2}^2 = |f(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|}\, dA(z).$$

Note that

$$(1-|z|^2) \asymp \log\frac{1}{|z|} \qquad \text{as } |z|\to 1.$$

A Potential Theory Result

To each positive superharmonic function w corresponds a *unique* positive finite Borel measure μ on $\overline{\mathbb{D}}$ such that

$$w(z) = \int_{\mathbb{D}} \log \left| \frac{1-\bar{\zeta}z}{\zeta - z} \right| \frac{2d\mu(\zeta)}{1-|\zeta|^2} + \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \, d\mu(\zeta)$$

for all $z \in \mathbb{D}$.

$\mathcal{D}_w(f)$

Recall that

$$\mathcal{D}_w(f) = \frac{1}{\pi} \int_{\mathbb{D}} w(z) \, |f'(z)|^2 \, dA(z).$$

$\mathcal{D}_\mu(f)$

Hence,

$$\mathcal{D}_\mu(f) = \int_{\mathbb{D}} \left[\int_{\mathbb{D}} \log\left|\frac{1-\bar{\zeta}z}{\zeta - z}\right| \frac{2d\mu(\zeta)}{1-|\zeta|^2} + \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(\zeta)\right] |f'(z)|^2\, dA(z).$$

Dirac Measures

If $\mu = \delta_\zeta$, then

$$\mathcal{D}_\zeta(f) = \int_{\mathbb{D}} \left[\log \left| \frac{1-\bar{\zeta}z}{\zeta - z} \right| \frac{2}{1-|\zeta|^2} \right] |f'(z)|^2 \, dA(z), \qquad (\zeta \in \mathbb{D}),$$

or

$$\mathcal{D}_\zeta(f) = \int_{\mathbb{D}} \left[\frac{1-|z|^2}{|\zeta - z|^2} \right] |f'(z)|^2 \, dA(z), \qquad (\zeta \in \mathbb{T}).$$

Superposition

By Fubini, we thus have

$$\mathcal{D}_\mu(f) = \int_{\overline{\mathbb{D}}} \mathcal{D}_\zeta(f)\, d\mu(\zeta).$$

J. Douglas Formula (1931)

We also have

$$\mathcal{D}_\zeta(f) = \frac{1}{2\pi}\int_{\mathbb{T}} \left|\frac{f(\lambda)-f(\zeta)}{\lambda-\zeta}\right|^2 |d\lambda|,$$

where

$$f(\zeta) := \lim_{r\to 1^-} f(r\zeta).$$

(J. Douglas and L. Ahlfors are the first Fields Medalists in 1936.)

An Important Application

Define

$$Q_\zeta f(z) := \frac{f(z) - f(\zeta)}{z - \zeta}.$$

Then, by Douglas formula,

$$\|f\|^2_{\mathcal{D}_\zeta} = \|f\|^2_{H^2} + \|Q_\zeta f\|^2_{H^2}.$$

Approximation in Local Dirichlet Spaces
Negative Results

Taylor Polynomials Fail

If

$$f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{D}_\zeta$$

then we **cannot** conclude that

$$\|S_n f - f\|_{\mathcal{D}_\zeta} \to 0.$$

Taylor Polynomials Fail

There is a function

$$f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{D}_1$$

such that

$$\sup_{n\geq 1} \|S_n f\|_{\mathcal{D}_1} = \sup_{n\geq 1} \left\| \sum_{k=0}^{n} a_k z^k \right\|_{\mathcal{D}_1} = \infty.$$

In particular,

$$\|S_n f - f\|_{\mathcal{D}_1} \not\to 0, \qquad (n \to \infty).$$

Justification

Consider

$$h(z) := z^n - z^{n+1}, \qquad (z \in \mathbb{D}).$$

Then $h \in \mathcal{D}_1$ and

$$(S_n h)(z) = z^n, \qquad (z \in \mathbb{D}).$$

Justification

Then $\|h\|^2_{\mathcal{D}_1} = 3$ and $\|S_n h\|^2_{\mathcal{D}_1} = n+1$ and, by Douglas formula,

$$\|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} \geq \left(\frac{n+1}{3}\right)^{\frac{1}{2}}, \qquad (n \geq 0).$$

In particular,

$$\sup_{n \geq 0} \|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} = \infty.$$

Justification

Therefore, by the Banach-Steinhaus theorem, there is an $f \in \mathcal{D}_1$ such that

$$\sup_{n \geq 0} \|S_n f\|_{\mathcal{D}_1} = \infty.$$

In particular, for this specific $f \in \mathcal{D}_1$,

$$\|S_n f - f\|_{\mathcal{D}_1} \not\to 0.$$

Lebesgue-type Constants

As in the classical setting, we define

$$L_n := \|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} = \sup_{f \in \mathcal{D}_1} \frac{\|S_n f\|_{\mathcal{D}_1}}{\|f\|_{\mathcal{D}_1}}.$$

A maximizing function $f \in \mathcal{D}_1$, $f \neq 0$, satisfies

$$L_n = \|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} = \frac{\|S_n f\|_{\mathcal{D}_1}}{\|f\|_{\mathcal{D}_1}}.$$

The Norms

Recall that $\mathcal{D}_1(f)$ is a semi-norm. We need to add an extra term to count the constant term. Here are three popular ways:

$$\begin{aligned}
\|f\|^2_{\mathcal{D}_1} &= |f(0)|^2 + \mathcal{D}_1(f), \\
\|f\|^2_{\mathcal{D}_1} &= |f(1)|^2 + \mathcal{D}_1(f), \\
\|f\|^2_{\mathcal{D}_1} &= \|f\|^2_{H^2} + \mathcal{D}_1(f).
\end{aligned}$$

There are three corresponding Theorems by JM-Withanachchi-Shirazi, 2022.

The Norm

Theorem (MWS 2022)

Assume that $\mathcal{D}_1$ is equipped with the norm

$$\|f\|_{\mathcal{D}_1}^2 = |f(0)|^2 + \mathcal{D}_1(f).$$

Then

$$\|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} = \sqrt{n+1}, \quad n \geq 0.$$

Moreover, the unique maximizing function is

$$f(z) = (n+1)z^n - nz^{n+1}, \quad n \geq 0.$$

The Norm

Theorem (MWS 2022)

Assume that $\mathcal{D}_1$ is equipped with the norm

$$\|f\|_{\mathcal{D}_1}^2 = |f(1)|^2 + \mathcal{D}_1(f).$$

Then

$$\|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} = \sqrt{n+2}, \quad n \geq 0.$$

Moreover, the unique maximizing function is

$$f(z) = (n+2)z^n - (n+1)z^{n+1}, \quad n \geq 0.$$

The Norm

Theorem (MWS 2022)

Let $\rho = \frac{3+\sqrt{5}}{2}$. Assume that $\mathcal{D}_1$ is equipped with the norm

$$\|f\|^2_{\mathcal{D}_1} = \|f\|^2_{H^2} + \mathcal{D}_1(f).$$

Then there are three cases.

- *If $n = 0$,*

$$\|S_0\|_{\mathcal{D}_1 \to \mathcal{D}_1} = 1,$$

with the unique maximizing function $f(z) = 1$.

The Norm

Theorem (Continued)

- *If* $1 \leq n \leq 4$,

$$\|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} = \sqrt{\frac{4(n+1)}{n+3+\rho}},$$

with the unique maximizing function

$$f(z) = \frac{(1-1/\rho)z^{n+1}}{1-z/\rho}.$$

The Norm

Theorem (Continued)

- *If* $n \geq 5$,

$$\|S_n\|_{\mathcal{D}_1 \to \mathcal{D}_1} = \sqrt{\frac{n+1}{\rho}},$$

with the unique maximizing function

$$f(z) = \frac{z^n(1-z)}{1-z/\rho}.$$

Open Question

Recall that $S_n = \sigma_n^0$.

Using the new techniques developed for the proof of the above results, can we evaluate

$$\|\sigma_n^\alpha\|_{\mathcal{D}_1 \to \mathcal{D}_1}?$$

Approximation in Local Dirichlet Spaces
Positive Results

The 'Last' Coefficient

Despite the (possible) unpleasant situation

$$\|S_n f - f\|_{\mathcal{D}_\zeta} \not\to 0,$$

if we properly modify just the last term in the Taylor polynomial $S_n f$, then the new polynomial sequence becomes convergent.

The Modified Taylor Polynomial

Theorem (JM-Ransford, 2018)

Let

$$f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{D}_\zeta.$$

Then there is $a_n' \in \mathbb{C}$ *such that, with*

$$p_n(z) := \sum_{k=0}^{n-1} a_k z^k + a_n' z^n,$$

we have

$$\|p_n - f\|_{\mathcal{D}_\zeta} \to 0.$$

A Convergence Result

For each

$$f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{D}_\zeta$$

the (numerical) series

$$\sum_{k=0}^{\infty} a_k \zeta^k$$

is convergent.

The Modified Taylor Polynomial

Theorem (Explicit Version)

Let

$$f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{D}_\zeta.$$

Put

$$p_n(z) := \sum_{k=0}^{n-1} a_k z^k + \left(\sum_{k=n}^{\infty} a_k \zeta^{k-n} \right) z^n.$$

Then

$$\|p_n - f\|_{\mathcal{D}_\zeta} \to 0.$$

General Case

Is there a constructive method for $\mathcal{D}_\mu$?

Hadamard Product

If

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

and

$$g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

then their **Hadamard product** is

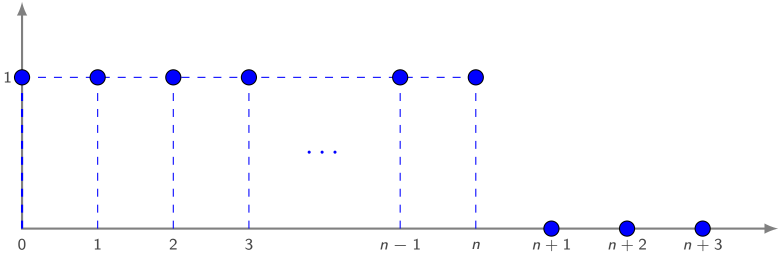
$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

In our applications, $f, g \in \mathrm{Hol}(\mathbb{D})$ and thus $f * g \in \mathrm{Hol}(\mathbb{D})$.

Dirichlet Kernel

The Dirichlet kernel is

$$D_n(z) := \sum_{k=0}^{n} z^k.$$



The Dirichlet Kernel

Dirichlet Kernel

Then

$$(D_n * f)(z) := \sum_{k=0}^{n} a_k z^k.$$

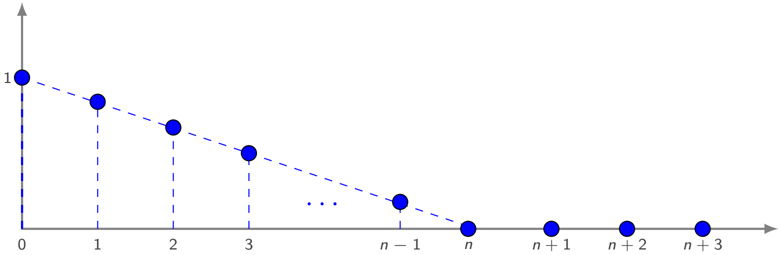
Hence,

$$S_n f = D_n * f.$$

Fejér Kernel

The Fejér kernel is

$$F_n(z) := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) z^k.$$



The Fejér Kernel

Fejér Kernel

Then

$$(F_n * f)(z) := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k z^k.$$

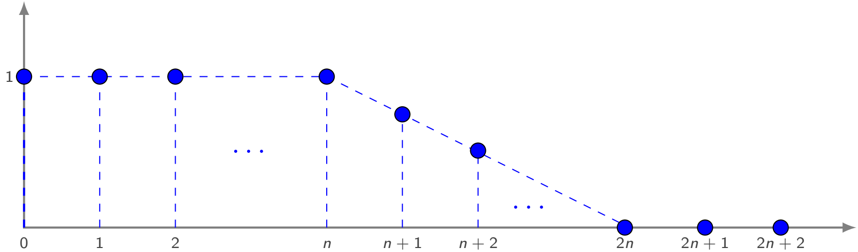
Hence,

$$\sigma_n f = F_n * f.$$

de la Vallée Poussin Kernel

The de la Vallée Poussin kernel is

$$V_n(z) := 2F_{2n}(z) - F_n(z).$$



The de la Vallée Poussin Kernel

The Crucial Estimation

Theorem (JM-Ransford 2018)

Let K be a polynomial of degree d, say

$$K(z) := \sum_{k=0}^{d} c_k z^k.$$

If $f \in \mathcal{D}_\mu$, then $K * f$ is (a polynomial in $\mathcal{D}_\mu$) such that

$$\mathcal{D}_\mu(K * f) \leq \left((d+1) \sum_{k=1}^{d} |c_k - c_{k+1}|^2 \right) \mathcal{D}_\mu(f).$$

Fejér Kernel

Let

$$F_n(z) := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) z^k.$$

Corollary

If $f \in \mathcal{D}_\mu$, then $F_n * f$ is (a polynomial in $\mathcal{D}_\mu$) such that

$$\mathcal{D}_\mu(F_n * f) \le \frac{n}{n+1} \mathcal{D}_\mu(f).$$

Fejér Kernel

Taking

$$f(z) = n - (n+1)z + z^{n+1},$$

we have

$$\mathcal{D}_1(f) = n(n+1) \qquad \text{and} \qquad \mathcal{D}_1(F_n * f) = n^2.$$

Thus the constant $n/n+1$ in the corollary is sharp.

Fejér Kernel

Recall $F_n * f = \sigma_n f$. Since $\mathcal{D}_\mu(F_n * f) \le \mathcal{D}_\mu(f)$, we conclude:

Corollary

Let $f \in \mathcal{D}_\mu$. Then

$$\mathcal{D}_\mu(\sigma_n f - f) \to 0.$$

de la Vallée Poussin Kernel

Let

$$V_n(z) := 2F_{2n}(z) - F_n(z).$$

Corollary

If $f \in \mathcal{D}_\mu$, then $V_n * f$ is (a polynomial in $\mathcal{D}_\mu$) such that

$$\mathcal{D}_\mu(V_n * f) \leq 2\,\mathcal{D}_\mu(f).$$

de la Vallée Poussin Kernel

Taking

$$f(z) = 1 - 2z^n + z^{2n},$$

we have

$$\mathcal{D}_1(f) = 2n \qquad \text{and} \qquad \mathcal{D}_1(V_n * f) = 4n.$$

Thus the constant 2 in the corollary is sharp.

The Estimation Parameter

Let K be a polynomial of degree d, say

$$K(z) := \sum_{k=0}^{d} c_k z^k.$$

In the light of *Estimation Theorem*, we define

$$\delta(K) := \left((d+1)\sum_{k=1}^{d} |c_k - c_{k+1}|^2\right)^{\frac{1}{2}}.$$

Recall - The Crucial Estimation

Theorem

Let K be a polynomial of degree d, say

$$K(z) := \sum_{k=0}^{d} c_k z^k.$$

If $f \in \mathcal{D}_\mu$, then $K * f$ is (a polynomial in $\mathcal{D}_\mu$) such that

$$\mathcal{D}_\mu(K * f) \le \delta^2(K)\, \mathcal{D}_\mu(f).$$

Superposition

Given the polynomials K_n, the idea is to form

$$K(z) := \sum_{n=1}^{\infty} \lambda_n K_n(z)$$

such that K behaves like a kernel.

Superposition

Theorem

Let K_n be a sequence of polynomial kernels, and let $(\lambda_n)_{n\geq 1}$ be any sequence of complex numbers such that

$$\delta(K) := \sum_{k=1}^{\infty} |\lambda_n| \, \delta(K_n) < \infty.$$

Then the (formal) power series

$$K(z) := \sum_{n=1}^{\infty} \lambda_n K_n(z)$$

is well-defined.

Superposition

Theorem (continued)

Moreover, for each $f \in \mathcal{D}_\mu$, *the series*

$$K * f = \sum_{n=1}^{\infty} \lambda_n K_n * f$$

converges in $\mathcal{D}_\mu$ *and*

$$\mathcal{D}_\mu(K * f) \leq \delta^2(K)\, \mathcal{D}_\mu(f).$$

An Application

Take

$$K_n(z) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) z^k.$$

Suppose that c_k, $k \geq 0$, are such that $\lim_{k\to\infty} c_k = 0$ and

$$\delta := \sum_{k=2}^{\infty} \sqrt{k(k-1)}\,|c_{k+1} - 2c_k + c_{k-1}| < \infty.$$

Put

$$\lambda_k = k(c_{k+1} - 2c_k + c_{k-1}), \qquad (k \geq 1).$$

An Application

Then

$$K(z) = \sum_{n=1}^{\infty} \lambda_n K_n(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Moreover,

$$f \in \mathcal{D}_\mu \quad \Longrightarrow \quad K * f \in \mathcal{D}_\mu$$

and

$$\mathcal{D}_\mu(K * f) \leq \delta^2 \mathcal{D}_\mu(f).$$

An Application

Hence, whenever $\delta \leq 1$, we conclude that

$$\mathcal{D}_\mu(K * f - f) \to 0$$

for all $f \in \mathcal{D}_\mu$.

An Application

As a special case, take

$$c_k = r^k, \qquad (k \geq 0),$$

we obtain $\delta^2 = r^2(2-r)$. Thus, if $f \in \mathcal{D}_\mu$, then $f_r \in \mathcal{D}_\mu$ and

$$\mathcal{D}_\mu(f_r) \leq r^2(2-r)\,\mathcal{D}_\mu(f).$$

Historical Note

- Richter–Sundberg (1991), for harmonic weights,

$$\mathcal{D}_\mu(f_r) \le 4\,\mathcal{D}_\mu(f).$$

- Aleman (1993), for superharmonic weights,

$$\mathcal{D}_\mu(f_r) \le \frac{5}{2}\,\mathcal{D}_\mu(f).$$

- Sarason (1997), for harmonic weights,

$$\mathcal{D}_\mu(f_r) \le \frac{2r}{1+r}\mathcal{D}_\mu(f).$$

Historical Note

- ElFallah–Kellay–Klaja–JM–Ransford (2016), for superharmonic weights,

$$\mathcal{D}_\mu(f_r) \leq \frac{2r}{1+r}\mathcal{D}_\mu(f).$$

- JM–Ransford (2018), for superharmonic weights,

$$\mathcal{D}_\mu(f_r) \leq r^2(2-r)\mathcal{D}_\mu(f).$$

Question

Find

$$\phi(r) := \sup_{\mu,f} \frac{\mathcal{D}_\mu(f_r)}{\mathcal{D}_\mu(f)}, \qquad (0 \le r \le 1).$$

By now (2021), we just know that

$$r^2 \le \phi(r) \le r^2(2-r), \qquad (0 \le r \le 1).$$

Recent Progress

The generalized Cesàro means of order α:

$$\sigma_n^\alpha f(z) = \sum_{k=0}^{n} \frac{\binom{n-k+\alpha}{\alpha}}{\binom{n+\alpha}{\alpha}} a_k z^k,$$

where

$$\binom{n+\alpha}{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \qquad \alpha > -1.$$

Note that $\sigma_n^0 f = S_n f$ and $\sigma_n^1 f = \sigma_n f$.

Recent Progress

Theorem (Parisé-JM-Ransford 2020)

If ω is a superharmonic weight on $\mathbb{D}$, if $f \in \mathcal{D}_\omega$ and if $\alpha > \frac{1}{2}$, then

$$\sigma_n^\alpha f \to f$$

in $\mathcal{D}_\omega$. Moreover, there exist ω and an $f \in \mathcal{D}_\omega$ such that

$$\sigma_n^{1/2} f \not\to f.$$

LPAS

Let $\mathcal{X}$ be a Banach space in $\mathrm{Hol}(\mathbb{D})$. A linear polynomial approximation scheme for $\mathcal{X}$ is a sequence of bounded operators

$$T_n : \mathcal{X} \to \mathcal{X}, \qquad (n \geq 1),$$

such that $T_n\mathcal{X} \subset \mathcal{P}$ and

$$\|T_nf - f\|_{\mathcal{X}} \to 0, \qquad (n \to \infty),$$

for all $f \in \mathcal{X}$.

Example

For $\mathcal{X} = H^p$, $1 < p < \infty$, and $\mathcal{X} = \mathcal{D}$,

$$T_n f = S_n f := \sum_{k=0}^{n} a_k z^k, \qquad (n \geq 0),$$

is a linear polynomial approximation scheme.

Example

For $\mathcal{X} = \mathcal{D}_\mu$, $\mathcal{X} = \mathcal{A}$ and $\mathcal{X} = H^1$,

$$T_n f = \sigma_n f := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k z^k, \qquad (n \geq 0),$$

is a linear polynomial approximation scheme.

Question

Which Banach spaces on $\mathbb{D}$ admit a linear polynomial approximation scheme?

AP

A Banach space $\mathcal{X}$ has the *approximation property* (AP) if, given any compact subset $K \in \mathcal{X}$ and $\varepsilon > 0$, there is a finite-rank operator $T : \mathcal{X} \to \mathcal{X}$ such that

$$\|Tx - x\| \leq \varepsilon, \qquad (x \in K).$$

BAP

If in addition, there is a constant M, independent of K and ε, so that $T = T_{K,\varepsilon}$ can be chosen such that

$$\|T_{K,\varepsilon}\| \leq M, \qquad (\forall K, \forall \varepsilon),$$

then we say that $\mathcal{X}$ has the bounded approximation property (BAP).

The Characterization

$$\text{LPAS} \Longleftrightarrow \text{BAP}$$

Proposition

A Banach space $\mathcal{X} \subset Hol(\mathbb{D})$ admits a linear polynomial approximation scheme if and only if
- $\mathcal{X}$ contains a dense subspace of polynomials
- and has the BAP.

Schauder basis

If a Banach space has a Schauder basis, then it has the BAP.

In particular, every separable Hilbert space has the BAP.

Hilbert Space Setting

Corollary

Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$. Then $\mathcal{H}$ admits a linear polynomial approximation scheme if and only if it contains a dense subspace of polynomials.

Open Question

The de Branges–Rovnyak space $\mathcal{H}(b)$, b non-extreme, has a linear polynomial approximation scheme.

Find it, explicitly!

Another Question

Is there a Banach space $\mathcal{X}$ in which polynomials are dense, but it does not admit *any* linear polynomial scheme?

Main Ingredient

There exist separable Banach spaces without BAP (Enflo 1973).

Certain closed subspaces of c_0 and ℓ^p, $p \neq 2$, do not have the BAP.

A Construction

Theorem (JM-Ransford 2019, Bonet 2020)

Let $\mathcal{Y}$ be a separable, infinite-dimensional, complex Banach space, and let $(\alpha_n)_{n\geq 0}$ be a strictly positive sequence such that

$$\lim_{n\to\infty} \alpha_n^{1/n} = 1.$$

Then there is $\mathcal{X} \subset Hol(\mathbb{D})$ such that:

- i $\mathcal{X}$ *is isometrically isomorphic to* $\mathcal{Y}$,
- ii $Hol(\overline{\mathbb{D}}) \subset \mathcal{X}$ *and* $\overline{\mathcal{P}} = \mathcal{X}$,
- iii $\|z^n\|_{\mathcal{X}} = \alpha_n$, *for all* $n \geq 0$.

Strange Phenomenon!

Corollary

There exists a Hilbert holomorphic function space $\mathcal{H}$ *on* $\mathbb{D}$ *such that:*

- i $\mathcal{H}$ *contains the polynomials,*
- ii *the polynomials are dense in* $\mathcal{H}$*,*
- iii *the odd polynomials are not dense in the odd functions in* $\mathcal{H}$*.*

Strange Phenomenon!

Corollary

Despite the fact that polynomials are dense in $\mathcal{H}$, there exists $f \in \mathcal{H}$ lying outside the closed linear span of $S_n f : n \geq 0$. Hence for any sequence of linear maps $T_n : \mathcal{H} \to \mathcal{H}$ of the form

$$T_n f := \sum_{k=0}^{n} \alpha_{nk} S_n f$$

we have

$$T_n f \not\to f.$$

Thank You