Desigualdades contractivas para espacios de Hilbert de funciones analíticas

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Hardy spaces

As usual, let $\mathbb{D}$ be the open unit disk of $\mathbb{C}$, $\mathbb{T} := \partial\mathbb{D}$ and let $\mathcal{H}(\mathbb{D})$ be the set of all holomorphic functions in $\mathbb{D}$. If $p > 0$ and $f \in \mathcal{H}(\mathbb{D})$, we will write

$$M_p(r, f) := \left(\int_0^{2\pi} |f(re^{it})|^p \, \frac{dt}{2\pi} \right)^{\frac{1}{p}}, \quad 0 < r < 1.$$

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$$M_p(r, f) := \left(\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}, \quad 0 < r < 1.$$

We define the *Hardy space* H^p as

$$H^p := \left\{f \in \mathcal{H}(\mathbb{D}) \ : \ \|f\|_{H^p} := \sup_{0 \leq r < 1} \{M_p(r, f)\} < \infty\right\}.$$

$M_p(r, f)$ as a function of r

In 1915, Hardy proved that $M_p(r, f)$ is an increasing function of r, and then $\|f\|_{H^p} = \lim_{r\to 1^-} M_p(r, f)$.

$M_p(r, f)$ as a function of r

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Theorem (Hardy-Stein identity)

If $f \in \mathcal{H}(\mathbb{D})$ and $p > 0$, then

$$\frac{d}{dr} M_p^p(r, f) = \frac{p^2}{2r} \int_{r\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2}\, dA(z), \quad 0 < r < 1.$$

Radial limits

If $f \in H^p$, then $f(e^{it}) := \lim_{r \to 1^-} f(re^{it})$ exists for almost every t and moreover

$$\|f\|_{H^p} = \left(\int_0^{2\pi} |f(e^{it})|^p \, \frac{dt}{2\pi} \right)^{\frac{1}{p}}.$$

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If $p \geq 1$, $H^p = \left\{ f \in L^p(\mathbb{T}) \ : \ \widehat{f}(k) = 0, \forall k < 0 \right\}$ where

$$\widehat{f}(k) := \int_0^{2\pi} f(e^{it}) e^{-kti} \, \frac{dt}{2\pi}, \quad \forall k \in \mathbb{Z}.$$

Weighted Bergman spaces

Let dA denote the normalized area measure of $\mathbb{D}$. If $p > 0$ and $\alpha > -1$, we say that $f \in \mathcal{H}(\mathbb{D})$ belongs to the *standard weighted Bergman space* A^p_α if

$$\|f\|_{A^p_\alpha} := \left((\alpha+1) \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z) \right)^{\frac{1}{p}} < \infty.$$

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If $f \in H^p$, then

$$\lim_{\alpha \to -1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}.$$

Thus, we set $A^p_{-1} := H^p$.

Point evaluations in A^p_α

If $p > 0$ and $\alpha \geq -1$ the subharmonicity of $|f|^p$ yields that

$$|f(0)| \leq \|f\|_{A^p_\alpha}$$

for all f. If $\zeta \in \mathbb{D}$ and $\varphi_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta} z}$, the application of this estimate to the function $T_\zeta f := (\varphi'_\zeta)^{\frac{\alpha+2}{p}} \cdot (f \circ \varphi_\zeta)$ implies that

$$|f(\zeta)|(1 - |\zeta|^2)^{\frac{\alpha+2}{p}} = |T_\zeta f(0)| \leq \|T_\zeta f\|_{A^p_\alpha} = \|f\|_{A^p_\alpha}.$$

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$$|f(\zeta)|(1 - |\zeta|^2)^{\frac{\alpha+2}{p}} = |T_\zeta f(0)| \leq \|T_\zeta f\|_{A^p_\alpha} = \|f\|_{A^p_\alpha}.$$

Moreover, the equality is possible if and only if f is a constant multiple of

$$k_\zeta(z) = \frac{(1 - |\zeta|^2)^{\frac{\alpha+2}{p}}}{(1 - \bar{\zeta} z)^{\frac{2(\alpha+2)}{p}}}.$$

We will call these functions *(normalized) reproducing kernels* of A^p_α.

Reproducing kernel Hilbert spaces

If $\alpha > -1$, the space A^2_α is a Hilbert space when endowed with the inner product

$$\langle f, g\rangle_{A^2_\alpha} = (\alpha+1)\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha \, dA(z)$$

Reproducing kernel Hilbert spaces

If $\alpha > -1$, the space A^2_α is a Hilbert space when endowed with the inner product

$$\begin{aligned} \langle f, g\rangle_{A^2_\alpha} &= (\alpha+1)\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha \, dA(z) \\ &= \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{c_{\alpha+2}(n)}, \end{aligned}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and

$$c_\beta(n) := \binom{n+\beta-1}{n} = \frac{\Gamma(n+\beta)}{\Gamma(\beta) n!}, \quad \forall n \geq 0.$$

If $\beta > 0$, we define the *weighted Dirichlet space* D_β as the set of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{D_\beta}^2 := \sum_{n=0}^{\infty} |a_n|^2 c_\beta(n) < \infty.$$

Weighted Dirichlet and Besov spaces

If $\beta > 0$, we define the *weighted Dirichlet space* D_β as the set of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{D_\beta}^2 := \sum_{n=0}^{\infty} |a_n|^2 c_\beta(n) < \infty.$$

Remark

The weighted Dirichlet spaces should not be confused with the *weighted Besov spaces*

$$B_\gamma^2 := \left\{ f \in \mathcal{H}(\mathbb{D}) \; : \; f' \in A_\gamma^2 \right\}.$$

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First example

We say that the inclusion between two Banach (or quasi Banach) spaces is *contractive* if the corresponding inclusion operator has norm less than or equal to 1.

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Theorem (Carleman's inequality)

For all $f \in H^1$, *we have that*

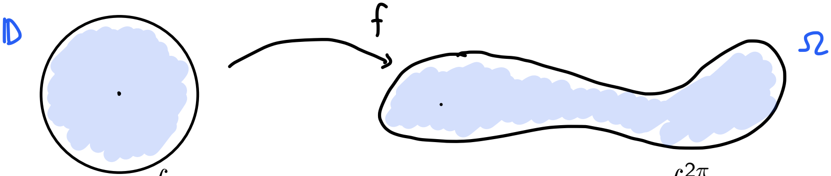
$$\left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \right)^{\frac{1}{2}} = \|f\|_{A^2} \leq \|f\|_{H^1},$$

and equality is possible if and only if f *is a multiple of the reproducing kernel of* H^1.

Isoperimetric inequality

Theorem (Isoperimetric inequality)

If $\Omega \subset \mathbb{C}$ is a simply connected domain bounded by a Jordan curve with finite length, then $\text{Area}(\Omega) \leq \frac{1}{4\pi}\big(\text{Length}(\partial\Omega)\big)^2$.



$$\text{Area}(\Omega) = \pi \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \quad \text{and} \quad \text{Length}(\partial\Omega) = \int_0^{2\pi} |f'(e^{it})| \, dt.$$

The equality is attained if and only if $f'(z) = C(1 - \overline{\zeta} z)^{-2}$, so Ω must be a disk.

Hardy spaces of the polydisk

If $d \geq 1$ and $p > 0$, we define $H^p(\mathbb{T}^d)$ as the closure of polynomials with respect to the norm (or quasi norm when $0 < p < 1$)

$$\|f\|_{L^p(\mathbb{T}^d)} := \left(\int_{(0,2\pi)^d} |f(e^{it_1}, \ldots, e^{it_d})|^p \, \frac{dt_1 \ldots dt_d}{(2\pi)^d} \right)^{\frac{1}{p}} .$$

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If $p \geq 1$, then

$$H^p(\mathbb{T}^d) = \left\{ f \in L^p(\mathbb{T}^d) \, : \, \widehat{f}(k_1, \ldots, k_d) = 0 \text{ if } k_j < 0 \right\}.$$

Hardy spaces of the polydisk

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If $p \geq 1$, then

$$H^p(\mathbb{T}^d) = \left\{ f \in L^p(\mathbb{T}^d) \, : \, \widehat{f}(k_1, \ldots, k_d) = 0 \text{ if } k_j < 0 \right\}.$$

Similarly, we introduce $H^p(\mathbb{T}^\infty)$ as the closure of polynomials in an arbitrary number of variables with respect to the topology of $L^p(\mathbb{T}^\infty)$.

Helson's trick

Let $f(z_1, z_2) = \sum_{n_1, n_2 \geq 0} a_{n_1,n_2} z_1^{n_1} z_2^{n_2} \in H^1(\mathbb{T}^2)$.

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$$\left(\sum_{n_1,n_2\geq 0} \frac{|a_{n_1,n_2}|^2}{(n_1+1)(n_2+1)} \right)^{\frac{1}{2}} = \left(\int_0^{2\pi} \int_0^{2\pi} |T_1 T_2 f(e^{it_1}, e^{it_2})|^2 \frac{dt_1}{2\pi} \frac{dt_2}{2\pi} \right)^{\frac{1}{2}}$$

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$$\left(\sum_{n_1,n_2\geq 0}\frac{|a_{n_1,n_2}|^2}{(n_1+1)(n_2+1)}\right)^{\frac{1}{2}} = \left(\int_0^{2\pi}\int_0^{2\pi}|T_1T_2f(e^{it_1},e^{it_2})|^2\frac{dt_1}{2\pi}\frac{dt_2}{2\pi}\right)^{\frac{1}{2}}$$

$$\leq \left(\int_0^{2\pi}\left(\int_0^{2\pi}|T_2f(e^{it_1},e^{it_2})|\frac{dt_1}{2\pi}\right)^2\frac{dt_2}{2\pi}\right)^{\frac{1}{2}}$$

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$$\begin{aligned}
\left(\sum_{n_1,n_2\geq 0}\frac{|a_{n_1,n_2}|^2}{(n_1+1)(n_2+1)}\right)^{\frac{1}{2}} &= \left(\int_0^{2\pi}\int_0^{2\pi}|T_1T_2f(e^{it_1},e^{it_2})|^2\frac{dt_1}{2\pi}\frac{dt_2}{2\pi}\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{2\pi}\left(\int_0^{2\pi}|T_2f(e^{it_1},e^{it_2})|\frac{dt_1}{2\pi}\right)^2\frac{dt_2}{2\pi}\right)^{\frac{1}{2}} \\
&\leq \int_0^{2\pi}\left(\int_0^{2\pi}|T_2f(e^{it_1},e^{it_2})|^2\frac{dt_2}{2\pi}\right)^{\frac{1}{2}}\frac{dt_1}{2\pi}
\end{aligned}$$

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$$\begin{aligned}
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&\leq \int_0^{2\pi}\left(\int_0^{2\pi}|T_2f(e^{it_1},e^{it_2})|^2\frac{dt_2}{2\pi}\right)^{\frac{1}{2}}\frac{dt_1}{2\pi}\\
&\leq \|f\|_{H^1(\mathbb{T}^2)}.
\end{aligned}$$

Helson's inequality

For any $d \geq 1$, we deduce that

$$\left(\sum_{n_1,\dots,n_d \geq 0} \frac{|a_{n_1,\dots,n_d}|^2}{(n_1+1)\dots(n_d+1)} \right)^{\frac{1}{2}} \leq \|f\|_{H^1(\mathbb{T}^d)}.$$

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Theorem (Helson's inequality)

For every Dirichlet polynomial $f(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ *we have that*

$$\left(\sum_{n=1}^{N} \frac{|a_n|^2}{d_2(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^1},$$

where $d_2(n)$ *denotes the number of divisors of* n*.*

Contractive inequalities between Bergman spaces

If $p \in (0,2]$, it was conjectured (Pavlović (2014); Brevig, Ortega-Cerdà, Seip and Zhao (2017)) that

$$\left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{2/p}(n)} \right)^{\frac{1}{2}} = \|f\|_{A^2_{\frac{2}{p}-2}} \leq \|f\|_{H^p}, \quad \forall f \in H^p.$$

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Theorem (Kulikov)

Let $p > q > 0$ and $\alpha, \beta \geq -1$ such that $\frac{\alpha+2}{p} = \frac{\beta+2}{q}$. Then the inequality

$$\|f\|_{A^q_\beta} \leq \|f\|_{A^p_\alpha}$$

holds for every $f \in A^p_\alpha$. Moreover, the equality is attained if and only if f is a constant multiple of a reproducing kernel.

Extension of Helson's inequality, $0 < p \leq 2$

Theorem (Extended Helson's inequality)

If $0 < p \leq 2$ *and* $f(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ *is a Dirichlet polynomial, we have that*

$$\left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{d_{2/p}(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^p},$$

where

$$d_{2/p}(n) := \prod_{j=1}^{\infty} c_{2/p}(k_j), \quad n = \prod_{j=1}^{\infty} p_j^{k_j} \in \mathbb{N}.$$

Contractive inclusions between Dirichlet and Hardy spaces

Conjecture (Brevig, Ortega-Cerdà, Seip and Zhao)

If $p > 2$, then the inequality
$$\|f\|_{H^p} \leq \|f\|_{D_{p/2}}$$
holds for all $f \in D_{p/2}$.

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Theorem

If $p > 2$, then we have that

$$\|f\|_{H^p} \leq \left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{2/p}(n)} \right)^{\frac{1}{2}} =: \|f\|_{B^2_{2/p}}, \quad \forall f \in B^2_{2/p}.$$

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In particular, the above conjecture is true.

Some relevant remarks

Why $B^2_{2/p}$? An important property used in Kulikov's result is that the operators

$$T_{\gamma,\zeta}f = (\varphi'_\zeta)^\gamma \cdot (f \circ \varphi_\zeta),$$

are isometries both in A^p_α and A^q_β if $\frac{\alpha+2}{p} = \frac{\beta+2}{q} = \gamma$. Aleman and Mas proved that, when $p > 2$, $B^2_{2/p}$ is the unique "reasonable" Hilbert space $X \subset \mathcal{H}(\mathbb{D})$ satisfying that the operators $\{T_{1/p,\zeta}\}_{\zeta\in\mathbb{D}}$ are uniformly bounded.

Some relevant remarks

Why $\|\cdot\|_{B^2_{2/p}}$? This is not the usual norm of $B^2_{2/p}$, but for this choice we have that

$$\|k_\zeta\|_{H^p} = \|k_\zeta\|_{B^2_{2/p}}, \quad \forall \zeta \in \mathbb{D}.$$

In fact, it seems that the reproducing kernels are the unique extremal functions (it is true under some additional conditions).

Proof

Let $n \geq 1$ and $p > 2$. Consider the quantity

$$K_{p,n} := \sup \left\{ \left\| \sum_{j=0}^{n} a_j z^j \right\|_{H^p} : \sum_{j=0}^{n} \frac{|a_j|^2}{c_{2/p}(j)} = 1 \right\} \geq 1.$$

Because of the density of polynomials, it suffices to show that $K_{p,n} = 1$ for all n.

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Because of the density of polynomials, it suffices to show that $K_{p,n} = 1$ for all n. Take $q_n(z) = a_0 + \ldots a_n z^n$ a normalized extremal polynomial (that is, $\|q_n\|_{B^2_{2/p}} = 1$ and $\|q_n\|_{H^p} = K_{p,n}$). We are going to prove that q_n must be constant.

Proof

For $r \in (0,1)$, consider the polynomial $q_r(z) := q_n(rz)$. By extremality of q_n, we have that

$$\|q_r\|_{H^p}^p \leq K_{p,n}^p \|q_r\|_{B^2_{2/p}}^p, \quad \forall r \in (0,1).$$

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$$\|q_r\|_{H^p}^p \leq K_{p,n}^p \|q_r\|_{B^2_{2/p}}^p, \quad \forall r \in (0,1).$$

On the one hand, we have that

$$\|q_r\|_{B^2_{2/p}}^2 = 1 - \sum_{j=1}^{n} (1 - r^{2j}) \frac{|a_j|^2}{c_{2/p}(j)},$$

$$\|q_r\|_{B^2_{2/p}}^p = 1 - \frac{p}{2} \sum_{j=1}^{n} (1 - r^{2j}) \frac{|a_j|^2}{c_{2/p}(j)} + o(1-r),$$

when $r \to 1^-$.

Proof

On the other hand, it is clear that $\|q_r\|_{H^p}^p = M_p^p(r, q_n)$.
Then, we see that

$$M_p^p(r, q_n) - M_p^p(1, q_n) \leq -K_{p,n}^p \frac{p}{2} \sum_{j=1}^{n} (1 - r^{2j}) \frac{|a_j|^2}{c_{2/p}(j)} + o(1 - r),$$

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$$\begin{aligned} M_p^p(r, q_n) - M_p^p(1, q_n) &\leq -K_{p,n}^p \frac{p}{2} \sum_{j=1}^{n} (1 - r^{2j}) \frac{|a_j|^2}{c_{2/p}(j)} + o(1 - r), \\ \frac{M_p^p(1, q_n) - M_p^p(r, q_n)}{1 - r} &\geq K_{p,n}^p \frac{p}{2} \sum_{j=1}^{n} \frac{1 - r^{2j}}{1 - r} \frac{|a_j|^2}{c_{2/p}(j)} + \frac{o(1 - r)}{1 - r}, \end{aligned}$$

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$$\frac{M_p^p(1,q_n) - M_p^p(r,q_n)}{1-r} \geq K_{p,n}^p \frac{p}{2} \sum_{j=1}^{n} \frac{1-r^{2j}}{1-r} \frac{|a_j|^2}{c_{2/p}(j)} + \frac{o(1-r)}{1-r},$$

when $r \to 1^-$. Thus, Hardy–Stein identity yields that

$$\frac{p}{2} \int_{\mathbb{D}} |q_n'|^2 |q_n|^{p-2} \, dA \geq K_{p,n}^p \sum_{j=1}^{n} j \frac{|a_j|^2}{c_{2/p}(j)}.$$

Proof

Assume that q_n is not constant. In other words, there exists an index j_0 in $\{1, \ldots, n\}$ such that $a_{j_0} \neq 0$. Applying Hölder's inequality, we have that

$$K_{p,n}^{p} \sum_{j=1}^{n} j \frac{|a_j|^2}{c_{2/p}(j)} \leq \frac{p}{2} \|q_n'\|^2_{A^{\frac{4p}{p+2}}} \|q_n\|^{p-2}_{A^{2p}}.$$

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$$K_{p,n}^p \sum_{j=1}^n j \frac{|a_j|^2}{c_{2/p}(j)} \leq \frac{p}{2} \|q_n'\|^2_{A^{\frac{4p}{p+2}}} \|q_n\|^{p-2}_{A^{2p}}.$$

Since q_n is not a reproducing kernel, Kulikov's inequality yields that

$$K_{p,n}^p \sum_{j=1}^n j \frac{|a_j|^2}{c_{2/p}(j)} < \frac{p}{2} \|q_n'\|^2_{A^2_{\frac{2}{p}-1}} \|q_n\|^{p-2}_{H^p} = K_{p,n}^{p-2} \sum_{j=1}^n j \frac{|a_j|^2}{c_{2/p}(j)},$$

which is impossible because $K_{p,n} \geq 1$. Thus, we have that $a_j = 0$ for all $1 \leq j \leq n$ and therefore q_n is constant.

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Reverse Helson's inequality, $2 < p$

Corollary

Let $p > 2$ and let $f(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ be a Dirichlet polynomial. Then we have that

$$\|f\|_{\mathcal{H}^p} \leq \left(\sum_{n=1}^{N} \frac{|a_n|^2}{d_{2/p}(n)} \right)^{\frac{1}{2}} .$$

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$$\|f\|_{\mathcal{H}^p} \leq \left(\sum_{n=1}^{N} \frac{|a_n|^2}{d_{2/p}(n)} \right)^{\frac{1}{2}}.$$

If $p = 2k$, the inequality

$$\|f\|_{\mathcal{H}^p} \leq \left(\sum_{n=1}^{N} |a_n|^2 d_{p/2}(n) \right)^{\frac{1}{2}}$$

was already proved by Bondarenko, Brevig, Saksman, Seip and Zhao.

Inequalities for the Riesz projection

The *Riesz projection* is defined as

$$P_+F(e^{it}) := \sum_{k=0}^{\infty} \widehat{F}(k)e^{ikt}, \quad F \in L^1(\mathbb{T}).$$

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- M. Riesz: If $q \in (1, \infty)$, P_+ is bounded from $L^q(\mathbb{T})$ to H^q.
- Hollenbeck and Verbitsky: If $q \in (1, \infty)$, the following sharp inequality holds:

$$\|P_+F\|_{H^q} \leq \csc\left(\frac{\pi}{q}\right) \|F\|_{L^q}, \quad \forall F \in L^q(\mathbb{T}).$$

Contractive inequalities for the Riesz projection

- Marzo and Seip: $P_+ : L^\infty(\mathbb{T}) \to H^4$ is contractive. Interpolation methods yield that P_+ is also contractive from $L^q(\mathbb{T})$ to $H^{\frac{4q}{q+2}}$, $2 \le q \le \infty$.

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Conjecture (Brevig, Ortega-Cerdà, Seip and Zhao)

If $p \in [1, \infty)$ and $p' = \frac{p}{p-1}$, then $P_+ : L^{p'}(\mathbb{T}) \to H^{4/p}$ is contractive.

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Corollary

If $p > 2$, then P_+ is contractive from $L^{p'}(\mathbb{T})$ to $D_{2/p}$.

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Proof.

Take $F \in L^{p'}(\mathbb{T})$ and $g \in B^2_{2/p}$, then

$$\begin{aligned} |\langle P_+F, g\rangle_{H^2}| &= |\langle F, g\rangle_{L^2}| \\ &\leq \|F\|_{L^{p'}} \|g\|_{H^p} \\ &\leq \|F\|_{L^{p'}} \|g\|_{B^2_{2/p}}, \end{aligned}$$

and therefore $\|P_+F\|_{D_{2/p}} \leq \|F\|_{L^{p'}}$. □

Thank you for your attention!