Sums for Divergent Series:

A Tauberian Adventure

Peter Duren

Sums of Divergent Series

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

Cesàro and Abel Sums

Infinite series: $a_0 + a_1 + a_2 + a_3 + \ldots$

Partial sums: $s_n = a_0 + a_1 + \cdots + a_n$

Cesàro means: $\sigma_n = \dfrac{1}{n+1}\left(s_0 + s_1 + \cdots + s_n\right)$

Abel means: $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

Thm. $s_n \to s \implies \sigma_n \to s.$

Abel's Thm. $s_n \to s \implies f(x) \to s$

as $x \to 1.$

Frobenius' Thm. $\sigma_n \to s \implies f(x) \to s.$

Examples

$$1 - 1 + 1 - 1 + \cdots = \tfrac{1}{2} :$$

$$\sigma_n = \frac{1}{n+1}\left(1 + 0 + 1 + 0 + \cdots + s_n\right) \to \tfrac{1}{2}$$

$$f(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} \to \tfrac{1}{2}$$

$$1 - 2 + 3 - 4 + \cdots = \tfrac{1}{4} :$$

$$1 - 2x + 3x^2 - 4x^3 + \cdots = \frac{1}{(1+x)^2} \to \tfrac{1}{4}$$

But $\displaystyle\liminf_{n\to\infty} \sigma_n = 0$ and $\displaystyle\limsup_{n\to\infty} \sigma_n = \tfrac{1}{2}$.

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots = \infty :$$

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots = \log\frac{1}{1-x} \to \infty$$

Tauberian Theorems

Tauber's Theorem (1897).

$f(x) \to s$ and $na_n \to 0 \implies s_n \to s$.

Hardy's Theorem (1910).

$\sigma_n \to s$ and na_n bounded $\implies s_n \to s$.

Littlewood's Theorem (1911).

$f(x) \to s$ and na_n bounded $\implies s_n \to s$.

Note. A Tauberian theorem also says a series is not summable if its divergence is *too slow*. For instance, L's thm says a divergent series with $a_n = O(1/n)$ can not be Abel summable.

G. H. Hardy

(1877 - 1947)

J.E. LITTLEWOOD

(1885 - 1977)

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DIVERGENT SERIES

BY

G. H. HARDY

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PREFACE

HARDY in his thirties held the view that the late years of a mathematician's life were spent most profitably in writing books; I remember a particular conversation about this, and though we never spoke of the matter again it remained an understanding. The level below his best at which a man is prepared to go on working at full stretch is a matter of temperament; Hardy made his decision, and while of course he continued to publish papers his last years were mostly devoted to books; whatever has been lost, mathematical literature has greatly gained. All his books gave him some degree of pleasure, but this one, his last, was his favourite. When embarking on it he told me that he believed in its value (as he well might), and also that he looked forward to the task with enthusiasm. He had actually given lectures on the subject at intervals ever since his return to Cambridge in 1931, and had at one time or another lectured on everything in the book except Chapter XIII.

The title holds curious echoes of the past, and of Hardy's past. Abel wrote in 1828: 'Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.' In the ensuing period of critical revision they were simply rejected. Then came a time when it was found that something after all could be done about them. This is now a matter of course, but in the early years of the century the subject, while in no way mystical or unrigorous, *was* regarded as sensational, and about the present title, now colourless, there hung an aroma of paradox and audacity.

August 1948

J. E. LITTLEWOOD

Puzzles Column

Elwyn R. Berlekamp and Joe P. Buhler

1. One of the charms of writing this column is corresponding with people who have comments or improvements on the puzzles that we present. Here is a second order version of this: a reader's variation of a puzzle that was itself a reader-proposed variation of an earlier puzzle.

To facilitate comparison (and provide implicit hints) we recall the first two versions before giving the new one. Last year we asked the following:

> 100 ants on a meter stick begin traveling to the right or left at one meter per minute. Colliding ants instantaneously reverse direction; when an ant reaches either end of the meter stick it falls off. What is the longest amount of time one must wait to be sure that no ants remain?

(The origin of this puzzle isn't clear; it can be found on several web sites.)

John Guilford (via Stan Wagon) proposed the following variation, which appeared here last spring:

> 100 ants are placed uniformly randomly on a one-meter stick, and an extra ant, Alice, is placed precisely at the center. Each ant begins moving right or left at 1 meter per minute (the direction being chosen randomly), and instantly reverses directions on a collision as in the earlier problem. But when an ant reaches one of the ends of the stick, it instantly reverses direction. What is the probability that after 1 minute Alice is again exactly at the center of the meter stick?

Finally, Matthew Hubbard writes as follows (we paraphrase):

> I enjoyed working on The Ant Problem in this quarter's Emissary from MSRI. I also solved a variant where Alice and the other ants are on a circle of circumference 1 meter; the question and conditions are otherwise the same. Is the solution to this problem well known? A lot of the thinking is similar to the solution of the ~~~ ~~~oblem, but the answer is slightly different.

2. Solve the following cryptogram:

```
      MAAAMS
   +    SIAM
   + MEETING
   =
         IN
   + ATLANTA
   +      01
   +      06
   +    2005
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"EMISSARY" (MSRI NEWSLETTER) FALL 2004
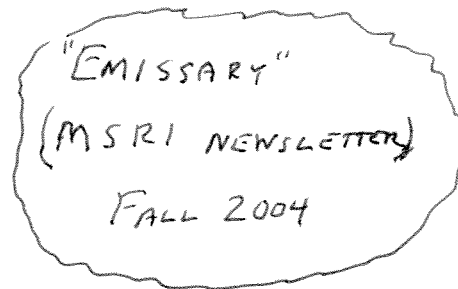
A solution is of course a one-to-one mapping from the letters that occur to decimal digits such that the indicated equation is true. Leading zeroes are not allowed.

3. A set A of positive integers is said to have density d if the fraction of integers in $[1, n]$ that are in A approaches d as n goes to infinity, i.e.,

$$d = \lim_{n \to \infty} A(n)/n,$$

where $A(n)$ is the cardinality of the set of elements in A of size at most n.

Find two sets of density one-half whose intersection does not have a density, i.e., the relevant limit above does not exist.

4. For positive real x less than 1, define

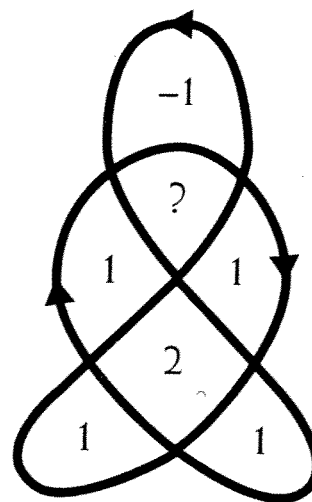$$f(x) = x - x^2 + x^4 - x^8 + x^{16} - \cdots$$

Does $f(x)$ have a limit as x approaches 1 from below? If so, what is the limit?

(We thank Noam Elkies for this question, which he posted to sci.math.research recently.)

Puzzles on Wheels

From October 2004 through February 2005, San Francisco bus riders will enjoy puzzling over a mathematical problem displayed among the ads for health insurance and instant rice. They — and anyone else who wants to participate — have a chance to win $100 by submitting the correct answer online.

This city-wide effort to bring fun and challenging mathematics to citizens of all ages, especially school-age children, is funded as a pilot project by the National Science Foundation. See www.msri.org/pow/ to look at the current problem! (The picture shown here is for the one that will go up in February.)

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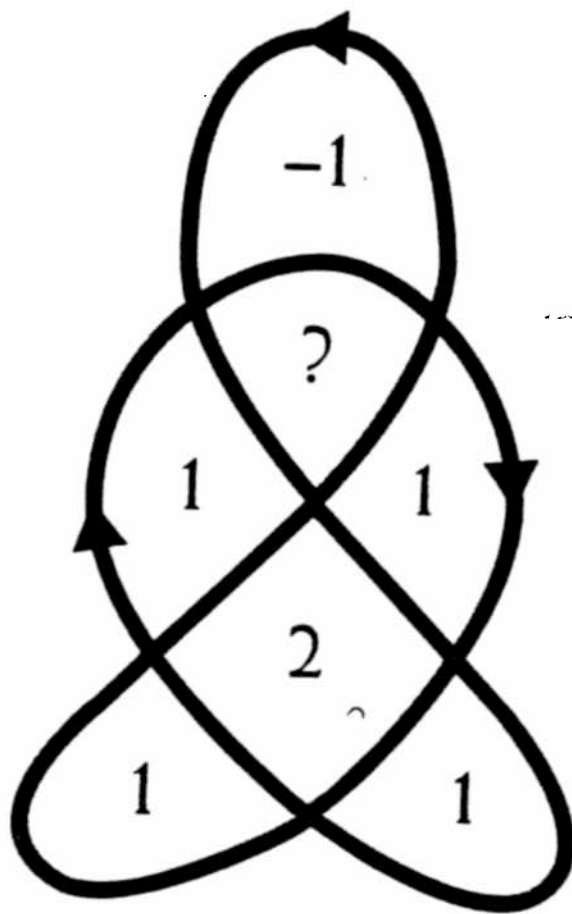
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Preliminary Observations

$$f(x) = x - x^2 + x^4 - x^8 + x^{16} - \dots, \quad 0 < x < 1.$$

Calculation gives

$$\liminf_{n \to \infty} \sigma_n = \frac{1}{3}, \quad \limsup_{n \to \infty} \sigma_n = \frac{2}{3}.$$

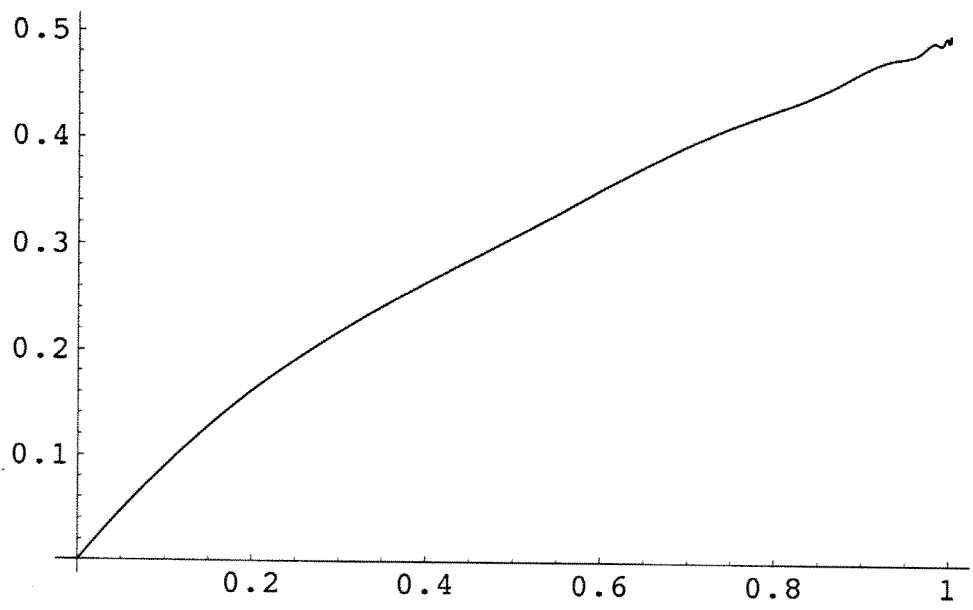$$f(x) = (x - x^2) + (x^4 - x^8) + \dots > 0.$$
$$f(x) = x - (x^2 - x^4) - (x^8 - x^{16}) - \dots < x.$$
$$\therefore \ 0 < f(x) < x < 1, \quad 0 < x < 1.$$

$$f(x) = x - f(x^2), \text{ so IF } f(x) \to s, \text{ then } s = \frac{1}{2}.$$

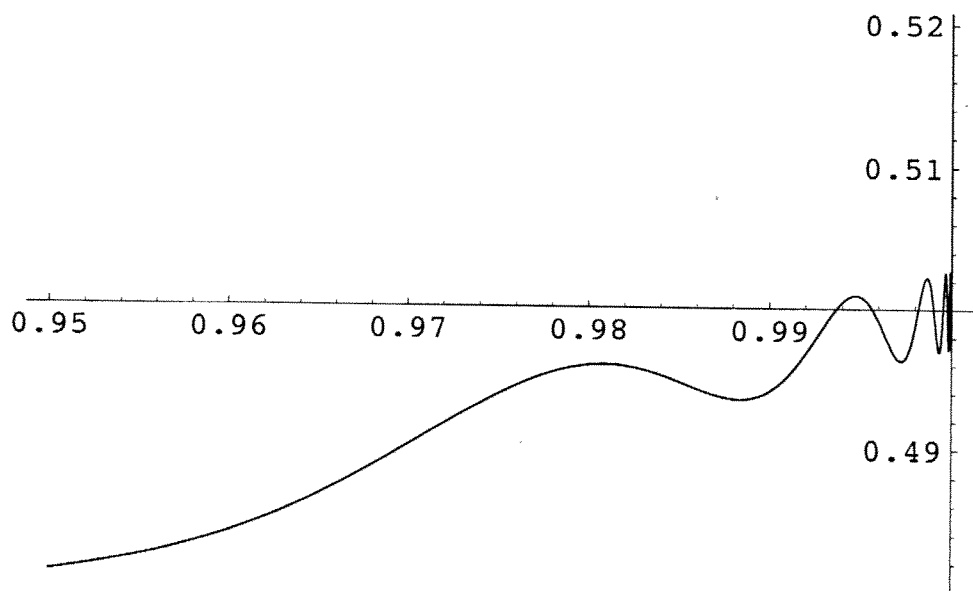$$f(x) = x - x^2 + f(x^4) > f(x^4).$$

Is $f(x)$ increasing in $0 < x < 1$?

The limit does not exist!

Hardy proved in 1907 that as $x \to 1$, the sum

$$f(x) = x - x^2 + x^4 - x^8 + x^{16} - \ldots$$

undergoes tiny but persistent oscillations.

In[1]:= **N[(0.99) ^ (2 ^ (30))]**

Out[1]= $7.6872000637 \times 10^{-4686675}$

In[2]:= **N[(0.999) ^ (2 ^ (30))]**

Out[2]= $3.429389170416 \times 10^{-466554}$

In[3]:= **N[(0.9999) ^ (2 ^ (30))]**

Out[3]= $4.501203223499 \times 10^{-46635}$

In[4]:= **N[2 ^ (30)]**

Out[4]= 1.07374×10^{9}

More Tauberian Theorems

High-Indices Theorem (H & L, 1926).

Let $f(x) = \sum_{k=1}^{\infty} a_{n_k} x^{n_k}$, where $\dfrac{n_{k+1}}{n_k} \geq q > 1$.

Then $f(x) \to s \implies s_n \to s$.

Hardy-Littlewood Tauberian Thm (1914).

$f(x) = \sum_{n=0}^{\infty} a_n x^n \to s$ and $s_n \geq 0 \implies \sigma_n \to s$.

Note that for Hardy's series

$$f(x) = x - x^2 + x^4 - x^8 + x^{16} - \dots ,$$

$0 \leq s_n \leq 1$ and $\{\sigma_n\}$ diverges.

Karamata's Proof

The original proofs of the Tauberian theorems of Littlewood and Hardy-Littlewood were much more difficult than those of Tauber and Hardy. For many years they were considered a real *tour de force.* Then in 1930 a young Serbian mathematician, Jovan Karamata, published a two-page paper outlining clever but completely elementary proofs based only on the Weierstrass approximation theorem. Karamata's approach was to prove the H-L theorem and to deduce L's theorem from it. In 1952, Wielandt found a simple refinement of K's method that avoids the detour through Cesàro summability and proves Littlewood's theorem directly.

Approximation Lemma

Karamata's method uses the Weierstrass approximation theorem only to establish the following lemma.

Lemma. Let $g(x)$ be continuous on the interval $[0, 1]$ except for a jump-discontinuity at a point in $(0, 1)$. Then for each $\varepsilon > 0$ there exist polynomials P and Q such that
$P(x) < g(x) < Q(x)$ for all $x \in [0, 1]$ and
$$\int_0^1 \left[g(x) - P(x) \right] dx < \varepsilon \,,$$
$$\int_0^1 \left[Q(x) - g(x) \right] dx < \varepsilon \,.$$

Hardy-Littlewood Tauberian Theorem.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \to s \ \text{ and } \ s_n \geq 0 \implies \sigma_n \to s \,.$$

Karamata's Proof. By hypothesis,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n \to s \ \text{ as } x \to 1.$$

This implies, more generally, that

$$(1-x) \sum_{n=0}^{\infty} s_n \, x^n \, p(x^n) \to s \int_0^1 p(t) \, dt$$

for every polynomial p. By lemma, it follows that

$$(1-x) \sum_{n=0}^{\infty} s_n \, x^n \, g(x^n) \to s \int_0^1 g(t) \, dt$$

if g is continuous in $[0,1]$ except for a jump.

It is here that the hypothesis $s_n \geq 0$ is needed.

Now choose

$$g(t) = \begin{cases} 0, & 0 \le t < 1/e \\ 1/t, & 1/e \le t \le 1, \end{cases}$$

and note that $\displaystyle\int_0^1 g(t)\, dt = 1$.

Let $x_m = e^{-1/m}$, so that $x_m^n \ge 1/e$ iff $n \le m$. Therefore,

$$\sum_{n=0}^{\infty} s_n x_m^n g(x_m^n) = \sum_{n=0}^{m} s_n = (m+1)\sigma_m.$$

But $\displaystyle (1 - x_m) \sum_{n=0}^{\infty} s_n x_m^n g(x_m^n) \to s$ as $m \to \infty$,

since $x_m \to 1$, so it follows that

$$(m+1)(1 - x_m)\sigma_m \to s \quad \text{as } m \to \infty.$$

Finally, because $(m+1)(1 - x_m) \to 1$,

we conclude that $\sigma_m \to s$ as $m \to \infty$.

Another Proof that $f(x)$ Has No Limit

Recall that
$$f(x) = x - x^2 + x^4 - x^8 + x^{16} - \ldots$$
has the property $\quad f(x) = x - f(x^2)\,,$

which implies that if limit exists, it must be $\dfrac{1}{2}$.

Also implies $\quad f(x^4) < f(x) \quad$ for $\; 0 < x < 1\,.$

In other words,

$$f(x) < f(x^{1/4})) < f(x^{1/16}) < \cdots \text{ for } 0 < x < 1\,,$$

so suffices to show $f(x) > \dfrac{1}{2}$ for some $x \in (0,1)\,.$

Guided by *Mathematica* graph, calculate

$$f(0.995) = 0.50088 \cdots > \frac{1}{2}\,.$$

Hardy (1907) mentions idea without calculation.

Amplitude of Oscillation

J.P. Keating and J.B. Reade (2000) used the Poisson summation formula to analyze Hardy's sum and other alternating gap series.

Poisson Summation Formula.

For $\varphi \in L^1(\mathbb{R})$ continuous, etc.,

$$\sum_{n=-\infty}^{\infty} \varphi(n) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(2\pi n) ,$$

where $\quad \widehat{\varphi}(u) = \int_{-\infty}^{\infty} e^{iut} \varphi(t) \, dt$

is Fourier transform. Apply PSF in form

$$\sum_{n=-\infty}^{\infty} (-1)^n \varphi(n) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}((2n+1)\pi) ;$$

$$\widehat{\varphi}((2n+1)\pi) = \int_{-\infty}^{\infty} e^{i2\pi nt} \left[e^{i\pi t} \varphi(t) \right] dt .$$

Application to Hardy's Sum

To apply the PSF to Hardy's sum, choose
$\varphi(t) = x^{2^{|t|}}$ for fixed $x \in (0, 1)$. Then

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2^n}$$

$$= \frac{1}{2} x + \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n x^{2^{|n|}}$$

$$= \frac{1}{2} x + \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \varphi(n)$$

$$= \frac{1}{2} x + \frac{1}{2} \sum_{n=-\infty}^{\infty} \widehat{\varphi}((2n+1)\pi)$$

$$= \frac{1}{2} x + \mathrm{Re} \left\{ \sum_{n=0}^{\infty} \widehat{\varphi}((2n+1)\pi) \right\},$$

since $\widehat{\varphi}(-u) = \overline{\widehat{\varphi}(u)}$.

Calculation of Fourier Transform $\widehat{\varphi}(u)$

Write $x = e^{-\lambda}$, $0 < \lambda < \infty$. Then

$$\widehat{\varphi}(u) = \int_{-\infty}^{\infty} e^{iut} e^{-\lambda\, 2^{|t|}} \, dt$$

$$= 2\,\mathrm{Re} \left\{ \int_{0}^{\infty} e^{iut} e^{-\lambda\, 2^{t}} \, dt \right\}$$

$$= \frac{2}{\log 2} \,\mathrm{Re} \left\{ \lambda^{-\left(\frac{iu}{\log 2}\right)} \int_{\lambda}^{\infty} s^{\left(\frac{iu}{\log 2}\right)-1} e^{-s} \, ds \right\},$$

where $s = \lambda\, 2^{t}$. As $x \to 1$, or as $\lambda \to 0$, it turns out that

$$\widehat{\varphi}(u) = \frac{2}{\log 2} \,\mathrm{Re} \left\{ \lambda^{-\left(\frac{iu}{\log 2}\right)} \Gamma\left(\frac{iu}{\log 2} \right) \right\} + o(1).$$

By PSF, can now write Hardy's sum $f(x)$ as

$$\frac{1}{2}x + \frac{2}{\log 2}\operatorname{Re}\left\{\sum_{n=0}^{\infty}\lambda^{-\left(\frac{(2n+1)\pi i}{\log 2}\right)}\Gamma\left(\frac{(2n+1)\pi i}{\log 2}\right)\right\}.$$

With $\lambda = 2^{-\mu}$, the sum becomes

$$\sum_{n=0}^{\infty}\Gamma\left(\frac{(2n+1)\pi i}{\log 2}\right)e^{(2n+1)\mu\pi i},$$

which is periodic with period 2 in

$$\mu = -\frac{\log\log(1/x)}{\log 2}.$$

The first term $(n = 0)$ is dominant and contributes an oscillation with amplitude

$$\frac{2}{\log 2}\left|\Gamma\left(\frac{i\pi}{\log 2}\right)\right|.$$

Conclude via relation $|\Gamma(iy)|^2 = \pi/(y\sinh y)$ that $f(x)$ oscillates about $\frac{1}{2}$ with approximate amplitude $0.00275\ldots$ as $x \to 1$. Error is $< 10^{-9}$.

Another Example

The function $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{n^2}$ does converge as $x \to 1$. In fact, it is not hard to see that $\liminf_{n\to\infty} \sigma_n = \limsup_{n\to\infty} \sigma_n = \frac{1}{2}$, so that $\sum a_n$ is Cesàro summable to $\frac{1}{2}$. Hence it is Abel summable and $f(x) \to \frac{1}{2}$ as $x \to 1$. Keating and Reade illustrate use of PSF by showing this directly. Details are much easier. Calculations related to proof of Jacobi's inversion formula for the theta function by Poisson summation.

Jacob Korevaar

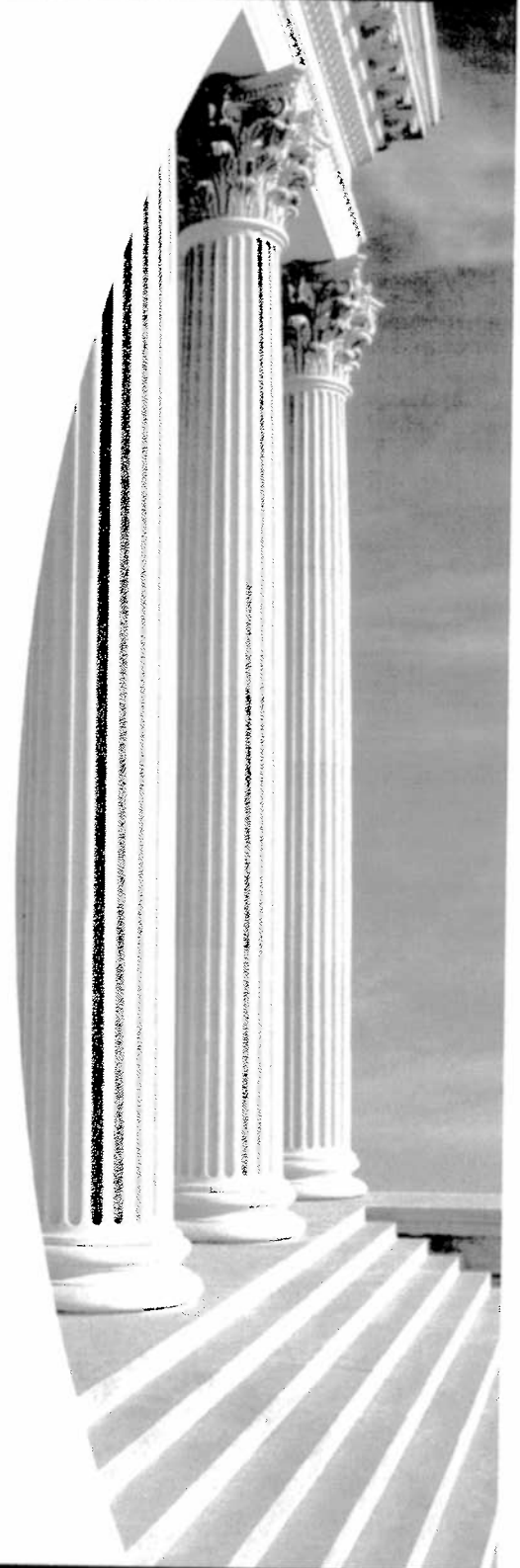# Tauberian Theory

A Century of Developments

Springer

Invitation to Classical Analysis

Peter Duren