Deddens Algebras and Compact Composition Operators

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(1990s) $T \in \mathcal{D}_A$ if there exists $M = M(T) > 0$ such that

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- Infinite multiplicity [Petrovic, Sievewright]

$\mathbb{D}$ – the unit disk,
Hardy space $H^2 = H^2(\mathbb{D})$,
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the composition operator C_φ: $C_\varphi f = f \circ \varphi$, for $f \in H^2$.

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WLOG: $\varphi(0) = 0$.

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Recall: $T \in \mathcal{D}_A$ if $\exists M$ such that

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In particular, the unilateral shift belongs to $\mathcal{D}_{C_\varphi}$.

If $\mathcal{M}$ is invariant for $\mathcal{D}_{C_\varphi}$ then $\mathcal{M} = \theta H^2$ for some inner function (Beurling). Let $\theta = BS$, with B a Blaschke product and S a singular inner function.

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Fact: $VM_{g'}$ is a bounded operator on H^2 if $g \in \mathrm{BMOA}$ (Pommerenke).

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Let $\theta = BS$*. If* $\alpha \neq 0$ *is a zero of* B*, then the subspace* $\mathcal{M} = \theta H^2$ *cannot be invariant for* $\mathcal{D}_{C_\varphi}$*.*

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Proof:

If $\mathcal{M}$ is invariant for $\mathcal{D}_{C_\varphi}$, and $n \geq 0$, then $\theta(z)z^n \in \mathcal{M}$, so

$$\int_0^z \theta(w)w^n\,dw \in \mathcal{M} \quad \Rightarrow \quad \int_0^\alpha \theta(w)w^n\,dw = 0.$$

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Change of variables $w = s\alpha$:

$$\int_0^1 \theta(s\alpha)\,s^n\,ds = 0, \text{ for } n = 0, 1, 2, \ldots.$$

Contradiction!

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Let $\theta(z) = z^n S(z)$*. If* S *is not a constant then the subspace* $\mathcal{M} = \theta H^2$ *cannot be invariant for* $\mathcal{D}_{C_\varphi}$*.*

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Proof:

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t) \right\},$$

μ a finite measure on $[0, 2\pi]$, $\mu \perp m$ (Lebesgue measure).
Fact: $\exists E \subset \mathbb{T}$, $\mu(E) = 0$, $\forall \zeta \in \mathbb{T} \setminus E$, $\lim_{r\uparrow 1} S(r\zeta) = 0$.
Therefore, if $f \in \mathcal{M}$, and I_f is its inner factor, then $\lim_{r\uparrow 1} I_f(r\zeta) = 0$.

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If $m \in \mathbb{N}$, $g_m(z) = z^{n+m} S(z) \in \mathcal{M}$, so $F_m = V g_m \in \mathcal{M}$. Therefore,

$$\lim_{r\uparrow 1} I_{F_m}(r\zeta) = 0, \quad \text{for all } \zeta \in \mathbb{T} \setminus E, \text{ and all } m \in \mathbb{N}.$$

Proof: (cont'd)

Fact: the outer factor of F_m is given by

$$O_{F_m}(r\zeta) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+r\zeta}{e^{it}-r\zeta}\log|F_m(e^{it})|\,dt\right\}.$$

If $\zeta = e^{it_0}$ and P is the Poisson kernel, then

$$|O_{F_m}(r\zeta)| = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} P_r(t-t_0)\log|F_m(e^{it})|\,dt\right\}.$$

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$F_m = V(z^{n+m}S) \Rightarrow \|F_m\|_\infty \leq 1 \Rightarrow \log|F_m(e^{it})| \leq 0$, for a.e. $t \in [0, 2\pi]$.

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Now, $\lim_{r\uparrow 1} I_{F_m}(r\zeta) = 0$ implies

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Substitution: $w = s\zeta$,

$$F_m(r\zeta) = \int_0^{r\zeta} w^{m+n} S(w)\, dw$$
$$= \int_0^{r} s^{m+n} S(s\zeta)\zeta^{m+n+1}\, ds$$
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Let $r \uparrow 1$. Obtain that $S(s\zeta) \equiv 0$, for all $\zeta \in \mathbb{T} \setminus E$ and all $s \in (0, 1)$. It follows that $E = \mathbb{T}$. Recall that $\mu(E) = 0$, so $\mu(\mathbb{T}) = 0$, so μ is the zero measure, which means that S must be constant.

Conclusion: if $\mathcal{M}$ is a n.i.s for $\mathcal{D}_{C_\varphi}$, then $\mathcal{M} = z^n H^2$ for some $n \in \mathbb{N}$.

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Answer:

$\mathcal{D}_{C_\varphi}$ is weakly dense in (LT), the algebra of all lower triangular matrices.

Future research:

- Generalize to the case $\varphi'(0) = 0$
- Replace H^2 by another space of analytic functions (Bergman, Dirichlet, ...)
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