Aplicaciones armónicas y la derivada Schwarziana

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The *Schwarzian derivative* is defined by

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

for any holomorphic f that is locally univalent ($f' \neq 0$).

Simplest diff. operator invariant under Möbius transformations

$$T(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0,$$

i.e., satisfies $S(T \circ f) = Sf$.

Chain rule: $S(g \circ f) = Sg \circ f\,(f')^2 + Sf$

$Sf \equiv 0$ *iff* $f = T$, a Möbius transformation

Let $\mathbb{D}$ be the unit disk.

Nehari (1949): If $f : \mathbb{D} \to \mathbb{C}$ satisfies

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

then f is univalent. Hille (1949): The constant 2 is sharp.

Ahlfors-Weill (1962): If $f : \mathbb{D} \to \mathbb{C}$ satisfies

$$|Sf(z)| \leq \frac{2t}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

for some $t < 1$ then f admits a $\frac{1+t}{1-t}$-qc extension to $\overline{\mathbb{C}}$.

K-quasiconformal (qc): $\dfrac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \leq K, \qquad K \geq 1$

On a hyperbolic domain in $D \subset \overline{\mathbb{C}}$ (*i.e.*, has at least three boundary points) the *hyperbolic metric* is defined by

$$\lambda_D\big(\pi(z)\big)|\pi'(z)||dz| = \lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}, \qquad z \in \mathbb{D},$$

where $\pi : \mathbb{D} \to D$ is a universal covering map.

For $f : D \to \mathbb{C}$ the *Schwarzian norm* is given by

$$\|Sf\|_D = \sup_{z \in D} \frac{|Sf(z)|}{\lambda_D(z)^2}.$$

There exists (f_n) in $D = \mathbb{D} \backslash (-1, 0]$ with $\|Sf_n\|_D \to 0$, yet f_n is not univalent.

A *quasidisk* is the image of $\mathbb{D}$ under a qc self-map of $\overline{\mathbb{C}}$. It has neither inner nor outer cusps.

A *quasicircle* is the boundary of quasidisk.

Ahlfors(1963): Let D be a quasidisk. Then there exists a constant $c = c(D) > 0$ such that if $f : D \to \mathbb{C}$ has $\|Sf\|_D \le c$ then f is univalent and admits a qc extension to $\overline{\mathbb{C}}$.

Gehring(1977): This holds for no other simply connected domain.

Osgood(1980)-Beardon,Gehring(1980): For a finitely connected domain D such a theorem holds *iff* every component of ∂D is either a point or a quasicircle.

Overview of results.

	univalence	qc extension
$\mathbb{D}$	Nehari (1949)	Ahlfors-Weill (1962)
simply conn.	Ahlfors (1963)	Ahlfors (1963)
finitely conn.	Osgood (1980)	Osgood (1980)
uniform	Martio-Sarvas (1979)	Astala-Heinonen (1988)

Let f be a *harmonic mapping* (*i.e.* $\Delta f = 0$) on some domain D.

Lewy (1936): f is locally univalent *iff* its Jacobian $J_f = |f_z|^2 - |f_{\bar{z}}|^2$ does not vanish.

Its *dilatation* $\omega = \overline{f_{\bar{z}}}/f_z$ satisfies $|\omega| < 1$ *iff* f is sense-preserving.

For f loc. univalent this is equivalent to $J_f = |f_z|^2(1 - |\omega|^2) > 0$.

Hernández-Martín (2015); definition of the *Schwarzian derivative*:

$$S_f = (P_f)_z - \frac{1}{2}(P_f)^2,$$

where $P_f = (\log J_f)_z$ is the *pre-Schwarzian derivative*.

Overview in the harmonic setting.

	univalence	qc extension
$\mathbb{D}$	Hernández-Martín (2015)	H-M (2015)
simply conn.	E.1 (2021)	E.1 (2021)
finitely conn.	E.1 (2021)	E.2 (2021)
uniform	E.2 (2021)	?

[E.1] I. Efraimidis, Criteria for univalence and quasiconformal extension for harmonic mappings on planar domains, *Ann. Fenn. Math.* **46** (2021), no. 2, 1123-1134.

[E.2] I. Efraimidis, Quasiconformal extension for harmonic mappings on finitely connected domains, *C. R. Math. Acad. Sci. Paris*, **359** (2021), no. 7, 905-909.

Theorem (E.1)

Let D be a quasidisk. Then there exists a constant $c = c(D) > 0$, such that if f is harmonic in D with $\|S_f\|_D \leq c$ then f is univalent in D and admits a homeomorphic extension to $\overline{\mathbb{C}}$.

Proof.

Let $f = h + \overline{g}$. If $\|S_f\|_D$ is sufficiently small then $\|Sh\|_D$ is small enough, so that h is univalent in D by Ahlfors' theorem.

Use the affine invariance $S_{f+a\bar{f}} \equiv S_f$ to get that

$$h_a = h + ag$$

is univalent in D for every $a \in \mathbb{D}$.

Letting $|a| \to 1^-$ we get that h_a is univalent for every $a \in \mathbb{T}$ by Hurwitz' theorem.

We show that $f = h + \overline{g}$ is injective in D by contradiction:

Let $z_1, z_2 \in D$ be distinct, and such that $f(z_1) = f(z_2)$, hence

$$h(z_1) - h(z_2) = \overline{g(z_2)} - \overline{g(z_1)}.$$

Setting $\theta = \arg\big(h(z_1) - h(z_2)\big)$ we see that

$$\mathbb{R} \ni e^{-i\theta}\big(h(z_1) - h(z_2)\big) = e^{-i\theta}\big(\overline{g(z_2)} - \overline{g(z_1)}\big) = e^{i\theta}\big(g(z_2) - g(z_1)\big),$$

from which we get that $h + e^{2i\theta} g$ is not injective, a contradiction.

For the homeomorphic extension of f to $\overline{\mathbb{C}}$ we use Ahlfors' theorem to get that $h_a = h + ag$ has a qc extension $\widetilde{h_a}$ for every $a \in \overline{\mathbb{D}}$.

We set $F = H + \overline{G}$, where

$$H = \widetilde{h_0} \qquad \text{and} \qquad G = \widetilde{h_1} - \widetilde{h_0},$$

and argue again by contradiction. We omit the details.

Theorem (E.1)

Let D be a quasidisk and let $d \in [0,1)$. Then there exists a constant $c = c(D,d) > 0$, such that if f is harmonic in D with $\|S_f\|_D \leq c$ and $\sup_{z\in D}|\omega(z)| \leq d$ then f admits a quasiconformal extension to $\overline{\mathbb{C}}$.

Proof.

Consider the dilation D_r (*i.e.*, the image of $|z| < r$, for $r < 1$, under a Riemman mapping of D) and use quasiconformal reflections to obtain a K-quasiconformal extension of $f\big|_{D_r}$ to $\overline{\mathbb{C}}$.

Prove that K is independent of r by showing that the cross-ratio of points on $f(\partial D_r)$ is bounded by a uniform constant, independent of r.

Get the desired extension as the limit for $r \to 1$.

Overview

	univalence	qc extension
$\mathbb{D}$	Hernández-Martín (2015)	H-M (2015)
simply conn.	E.1 (2021)	E.1 (2021)
finitely conn.	E.1 (2021)	E.2 (2021)
uniform	E.2 (2021)	?

Theorem (E.2)

Let D be a finitely connected domain whose boundary components are either points or quasicircles and let also $d \in [0,1)$. Then there exists a constant $c = c(D,d) > 0$ such that if f is harmonic in D with $\|S_f\|_D \leq c$ and with dilatation ω satisfying $|\omega(z)| \leq d$ for all $z \in D$ then f admits a quasiconformal extension to $\overline{\mathbb{C}}$.

Proof.

Isolated boundary points are removable for qc mappings.

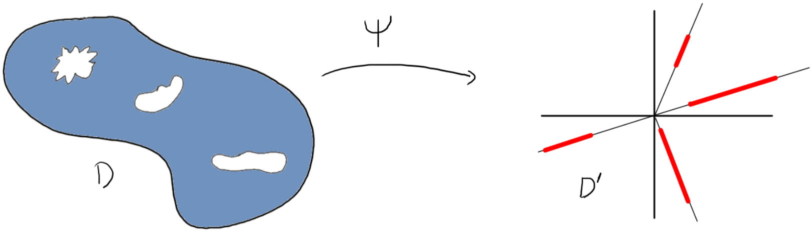
Assume that ∂D consists of n non-degenerate quasicircles C_j, $1 \leq j \leq n$.

Springer (1964): Let D and Ω be two n-tuply connected domains whose boundary curves are quasicircles. Then every qc mapping of D onto Ω can be extended to a qc mapping of the whole plane.

We will show that $\partial f(D)$ consists of quacicircles.

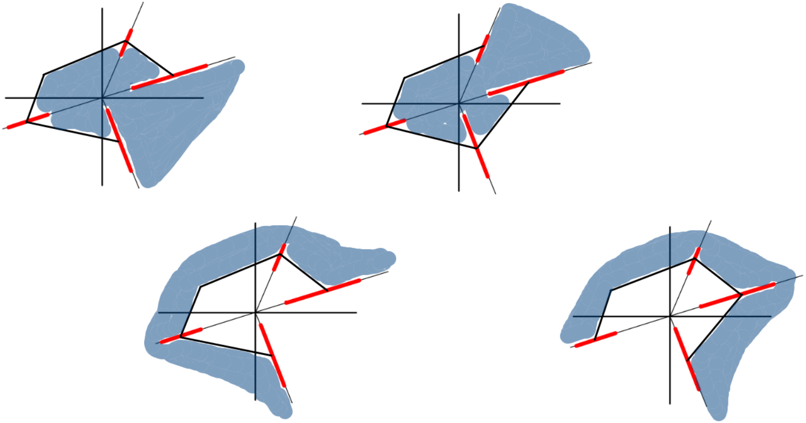
A collection $\mathfrak{D}$ of domains $\Delta \subset D$ is called a *qc decomposition* of D if each Δ is a quasidisk and any two points $z_1, z_2 \in D$ lie in the closure of some $\Delta \in \mathfrak{D}$.

Osgood (1980): If D is a finitely connected domain and each component of ∂D is either a point or a quasicircle then D is qc decomposable. The collection $\mathfrak{D}$ can be taken to be finite.



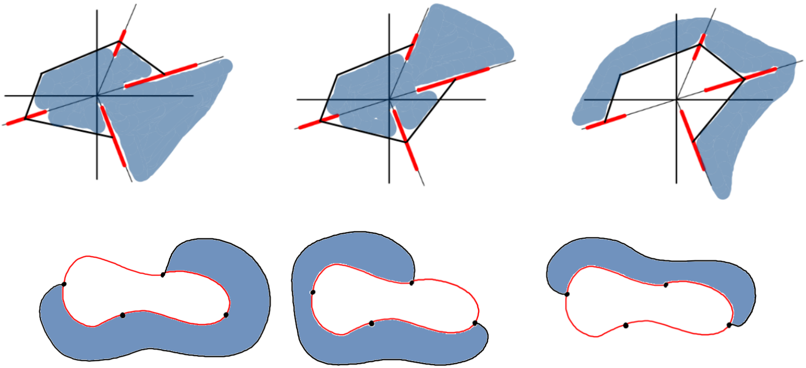
Let Ψ be a conformal mapping of D onto $D' = \overline{\mathbb{C}} \setminus \cup_{k=1}^{n} C'_j$, where C'_j are rectilinear slits lying on rays emanating from the origin.

We join (in order) the midpoints of the slits by a polygonal arc and consider the following domains.



Their pre-images under Ψ constitute a qc decomposition of D.

We wish to show that $f(C_j)$ is a Jordan curve. Consider the following three domains in D' and their pre-images in D, all of whose boundaries intersect C_j.



The restriction of f on each of these admits a qc extension to $\overline{\mathbb{C}}$.

The Jordan curve $f(C_j)$ is actually a quasicircle since each of its points belongs to a quasi-arc.

Overview

	univalence	qc extension
$\mathbb{D}$	Hernández-Martín (2015)	H-M (2015)
simply conn.	E.1 (2021)	E.1 (2021)
finitely conn.	E.1 (2021)	E.2 (2021)
uniform	E.2 (2021)	?

Problem

Let D be a uniform domain. Let also $d \in [0,1)$.

Does there exist a constant $c > 0$ such that if f is harmonic in D with $\|S_f\|_D \leq c$ and $\sup_{z\in D} |\omega(z)| \leq d$ then f admits a qc extension to $\overline{\mathbb{C}}$?

Yes, for d sufficiently small!

Ahlfors-Weill (1962): If $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and satisfies $\|Sf\| \leq 2t$, for $t < 1$, then

$$F(z) = \begin{cases} f(z), & \text{if} \quad |z| \leq 1, \\ E_f(1/\bar{z}), & \text{if} \quad |z| > 1, \end{cases}$$

where

$$E_f(\zeta) = f(\zeta) + \frac{(1-|\zeta|^2)f'(\zeta)}{\overline{\zeta} - \frac{1}{2}(1-|\zeta|^2)Pf(\zeta)}, \qquad \zeta \in \mathbb{D},$$

is a $\frac{1+t}{1-t}$-qc extension of f to $\overline{\mathbb{C}}$. Here $Pf = \frac{f''}{f'}$.

The Beltrami coefficient $\mu_F = F_{\bar{z}}/F_z$ of the extension is

$$\mu_F(z) = -\frac{1}{2}\left(\frac{\zeta}{\bar{\zeta}}\right)^2 (1-|\zeta|^2)^2 Sf(\zeta), \qquad |z|>1,\ \zeta = 1/\bar{z}.$$

Hence, $|\mu_F| \leq t < 1$. Moreover, $F_z \neq 0$.

Therefore, the Jacobian of F satisfies

$$J_F = |F_z|^2 - |F_{\bar{z}}|^2 \geq |F_z|^2(1-t^2) > 0$$

and so F is locally homeomorhic in $|z|>1$.

Lemma

A locally homeomorhic mapping $F : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is a homeomorphism.

A geometric interpretation: Setting $\Omega = f(\mathbb{D})$, we have that

$$E_f(\zeta) = w + \frac{1}{\partial_w \log \lambda_\Omega(w)}, \qquad w = f(\zeta),\ \zeta \in \mathbb{D}.$$

Since $\partial_z u = \frac{1}{2}\overline{\nabla u}$, the complex numbers

$$\big(\partial_w \log \lambda_\Omega(w)\big)^{-1} \qquad \text{and} \qquad \nabla \log \lambda_\Omega(w)$$

have the same argument, so that the extension moves away from $f(\zeta)$ in the direction of maximal growth of λ_Ω.

Minda (1997): The circle of curvature of each hyperbolic geodesic through $f(\zeta)$ at the point $f(\zeta)$ passes through $E_f(\zeta)$.

Best Möbius approximation: If f is analytic at a point ζ then there exists a unique Möbius transformation $M = M(f, \zeta)$ that agrees with f at ζ up to second order, *i.e.*, it satisfies

$$M(\zeta) = f(\zeta), \qquad M'(\zeta) = f'(\zeta), \qquad M''(\zeta) = f''(\zeta).$$

Actually,

$$M(f, \zeta)(z) = f(\zeta) + \frac{(z-\zeta)f'(\zeta)}{1 - \frac{1}{2}(z-\zeta)Pf(\zeta)}.$$

The Ahlfors-Weill extension can be obtained by setting

$$E_f(\zeta) = M(f, 1/\bar{z})(z), \qquad \zeta = 1/\bar{z},\ \ |z| > 1.$$

Harmonic Möbius transformations: $M = T + \alpha\overline{T}$, where $\alpha \in \mathbb{D}$ and T is a (holomorphic) Möbius transformation.

If $f = h + \overline{g}$ is harmonic at a point ζ then the harmonic Möbius transformation that best approximates f at ζ is the unique mapping $M = M(f, \zeta)$ that satisfies

$$M(\zeta) = f(\zeta), \quad M_z(\zeta) = f_z(\zeta), \quad M_{zz}(\zeta) = f_{zz}(\zeta), \quad M_{\overline{z}}(\zeta) = f_{\overline{z}}(\zeta).$$

We have that

$$M(f,\zeta)(z) = f(\zeta) + \frac{(z-\zeta)h'(\zeta)}{1 - \frac{1}{2}(z-\zeta)Ph(\zeta)} + \overline{\omega(\zeta)}\,\overline{\left(\frac{(z-\zeta)h'(\zeta)}{1 - \frac{1}{2}(z-\zeta)Ph(\zeta)}\right)}.$$

An extension of f to $\overline{\mathbb{C}}$ is obtained by setting

$$E_f(\zeta) = M(f, 1/\overline{z})(z), \qquad \zeta = 1/\overline{z},\ |z| > 1.$$

Theorem (I.E., R. Hernández, M.J. Martín)

Let $d \in [0,1)$ *and* $f = h + \overline{g}$ *be a harmonic mapping in* $\mathbb{D}$ *whose dilatation* ω *satisfies* $|\omega(z)| \leq d$ *for all* $z \in \mathbb{D}$*. Let also* τ *be either*

(A) *the pre-Schwarzian* Ph *or*

(B) *the pre-Schwarzian* P_f*.*

Then, for either case (A) *or* (B)*, there exists a constant* $c = c(d) > 0$ *such that if* $\|S_f\| \leq c$ *then the mapping*

$$F(z) = \begin{cases} f(z), & \text{if} \quad |z| \leq 1, \\ E_f(1/\bar{z}), & \text{if} \quad |z| > 1, \end{cases}$$

where

$$E_f(\zeta) = f(\zeta) + \Phi(\zeta) + \overline{\omega(\zeta)\Phi(\zeta)}, \qquad \zeta \in \mathbb{D},$$

and

$$\Phi(\zeta) = \frac{(1-|\zeta|^2)h'(\zeta)}{\overline{\zeta} - \frac{1}{2}(1-|\zeta|^2)\tau(\zeta)}, \qquad \zeta \in \mathbb{D},$$

is a quasiconformal extension of f *to* $\overline{\mathbb{C}}$*.*

Proof.

The extension F_r of the dilation $f_r(z) = f(rz)$, for $r < 1$, is continuous in $\overline{\mathbb{C}}$ with respect to the spherical metric.

It holds that $|\mu_F| \leq d + \varepsilon < 1$, where $\varepsilon = \varepsilon(c) > 0$.

Apply the Lemma.

Let $r \to 1$ to obtain the extension F.

Define the harmonic inner radius of a domain D as

$$\sigma_H(D) = \sup\{c \geq 0 : f \text{ harmonic}, \|S_f\|_D \leq c \Rightarrow \text{univalent}\}$$

Problem

We know that $\sigma_H(\mathbb{D}) \in (0, \frac{3}{2}]$. Compute or, at least, give a positive lower bound for $\sigma_H(\mathbb{D})$.

¡Gracias!

[E.1] I. Efraimidis, Criteria for univalence and quasiconformal extension for harmonic mappings on planar domains, *Ann. Fenn. Math.* **46** (2021), no. 2, 1123-1134.

[E.2] I. Efraimidis, Quasiconformal extension for harmonic mappings on finitely connected domains, *C. R. Math. Acad. Sci. Paris*, **359** (2021), no. 7, 905-909.

[EHM] I. Efraimidis, R. Hernández, M.J. Martín, Ahlfors-Weill extensions for harmonic mappings, preprint, arXiv:2105.07492.