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SPACES OF ANALYTIC FUNCTIONS, GEOMETRY OF DOMAINS,
AND SUPERPOSITION OPERATORS

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ABSTRACT. We review some relationships between the growth of univalent functions in classical spaces of analytic functions and the geometry of image domains under such maps and apply this to obtain the characterization of nonlinear superposition operators between function spaces.

CONTENTS

Introduction	2
1. Planar domains	4
1.1. Notation	4
1.2. The Whitney decomposition of a planar domain	4
1.3. A topological theorem	5
2. Classical function spaces and the isoperimetric inequality	7
2.1. A brief review of Hardy spaces	7
2.2. Basic facts on Bergman spaces	8
2.3. The isoperimetric inequality	9
3. Growth of functions in the Dirichlet space	13
3.1. The Dirichlet space	13
3.2. The distribution function	14
3.3. On the theorems of Beurling and Chang-Marshall	14
4. Univalent functions and conformally invariant spaces	19
4.1. Univalent functions	19
4.2. Hyperbolic metric in the disk	19
4.3. The Bloch space	20
4.4. Bloch domains	21
4.5. Analytic Besov spaces	21
4.6. Image domains under univalent Besov functions	22
4.7. Univalent B^p domains	25
5. Superposition operators	28
5.1. Superposition operators	28
5.2. A brief review of entire functions	29
5.3. Superpositions acting between two Bergman spaces	30
5.4. Superpositions from the Bloch space to a Bergman space	32

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5.5. Superpositions between two Besov spaces	35
5.6. Some other superpositions	37
6. Exercises	40
References	42

INTRODUCTION

The purpose of these notes is to show how some geometric features of simply connected planar domains relate to the membership of Riemann maps of the disk onto such domains in certain classical function spaces and classes. In other words, given a conformal map of the unit disk onto a domain, we try to understand its membership in some classical conformally invariant spaces just by “drawing a picture” of the domain.

Next, we study the maximum growth of functions in certain classical spaces of analytic functions. When possible, we show that this maximum growth is achieved by univalent mappings which will allow us to do a “univalent interpolation” of large values when needed. In case of the Dirichlet space, we consider some well-known inequalities of Beurling and Chang-Marshall for their boundary values.

Finally, certain questions of comparative rates of growth, especially the superposition operators between two function spaces, are also considered. Obtaining conclusive results in this area requires not only the basics of entire functions but, above all, a combination of some (or all) of the techniques described above so this seems like a natural topic to end this notes with. The main focus is on the superpositions between pairs of classical spaces of analytic functions such as the Bergman spaces A^p and analytic Besov spaces B^p (including the Dirichlet and Bloch spaces).

This material is based on several of the author’s papers that have appeared over the last 12 years or so, many of them jointly with various coauthors (whose names are mentioned in the references), as well as on the results of several other mathematicians, including some classical theorems. It is my hope that eventually, perhaps within a few years, much of this material (in an expanded form) will find its way to an advanced graduate textbook.

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1. PLANAR DOMAINS

In this section we review the basic concepts and collect the essential facts that will be needed later about domains in the complex plane. We begin by fixing some notation.

1.1. Notation. Throughout the text, \mathbb{C} will denote the complex plane and \mathbb{D} the unit disk: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We will write $\mathcal{H}(\mathbb{D})$ for the algebra of all analytic (holomorphic) functions in \mathbb{D} . The notation $K \Subset \mathbb{D}$ will mean that K is a compact subset of the unit disk.

We will often need estimates on the distance of a point w in a planar domain Ω (other than the plane) to the boundary $\partial\Omega$. Throughout these notes, we will use the following notation:

$$d_\Omega(w) = \text{dist}(w, \partial\Omega) = \min\{|w - z| : z \in \partial\Omega\}.$$

Similarly, for a compact subset Q of Ω (e.g., a closed square), we will also use the notation $d_\Omega(Q)$ for the distance from the set Q to $\partial\Omega$.

We will often need to compare two quantities asymptotically (without being concerned about the exact value of the constants). For two positive functions u and v we will use the notation $u \asymp v$ to denote that $mu \leq v \leq Mu$ for some fixed positive constants m and M . Similarly, the notation $u \lesssim v$ will mean that $u \leq Cv$ for some positive constant C .

1.2. The Whitney decomposition of a planar domain. It is well known that the structure of open sets in \mathbb{R}^n ($n > 1$) and, in particular, in \mathbb{C} (when identified with \mathbb{R}^2), is different from that of open sets on the real line. Namely, every open set in the plane can be represented as a countable union of closed dyadic squares (that is, squares of side length 2^k each, where $k \in \mathbb{Z}$, and with sides parallel to the coordinate axes) with pairwise disjoint interiors.

The Whitney decomposition theorem refines this further by stating that we can simultaneously control the size of these squares and their distances to the boundary, as well as their total number for each possible size. A precise formulation is as follows. The details of proofs can be found in [St70], Chapter VI.

LEMMA 1. *An arbitrary non-empty open set Ω in the plane, other than the plane itself, can be represented as $\Omega = \cup_{n=1}^{\infty} Q_n$, where Q_n are closed dyadic squares with pairwise disjoint interiors, and with the property that*

$$\text{diam } Q_n \leq d_\Omega(Q_n) \leq 4 \text{ diam } Q_n,$$

for all $n \geq 1$.

More can be said about the squares in a Whitney decomposition. We will say that a square Q_n from the decomposition is a *neighbor* of a square Q_m if $Q_m \cap Q_n$ is a line segment (note that the four squares that share

only a vertex with the given square do not qualify as neighbors). Being a neighbor is clearly a symmetric relation. It can also be verified that the sizes of any two neighboring squares are comparable (within a factor of 4). It follows that each square has at most 16 *neighbors*.

1.3. A topological theorem. The purpose of this section is to prove the following result from [DGV10]. Although it is a statement of geometric/topological character, a proof exists using techniques from Analysis.

THEOREM 2. *Let Ω be a planar domain. Then there exists a simply connected domain $\Omega' \subset \Omega$ such that $\Omega \setminus \Omega'$ is a countable union of line segments each of which has finite length.*

The key to our proof will be the Whitney decomposition. Somewhat similar constructions of domains were employed earlier by Peter W. Jones, e.g. in [Jo82] and [Jo95], but in a rather different context of Sobolev spaces. An immediate consequence of the theorem above is the following curious topological/geometric fact.

COROLLARY 3. *A planar domain of finite area contains a simply connected domain of the same area.*

Proof of Theorem 2. Let (Q_n) be a Whitney decomposition of Ω as in Lemma 1. We can partition this collection of squares into countably many *generations* according to the scheme described below.

Choose an arbitrary but fixed square Q^* among the squares in the Whitney decomposition; let us refer to it as the *pivoting square*. For any other square Q from the decomposition, consider a Jordan arc γ in Ω that connects the center of Q^* to the center of Q and does not pass through the vertex of any square in the decomposition. The selection of our squares, together with a simple compactness argument, allows us to assure that there are finitely many squares in our decomposition which intersect γ . It is also clear that for any Q , we can choose a finite sequence of our squares in such a way that the first one is Q^* , the last one is Q and each square is a neighbor of the preceding one. In view of this fact, we can assign to any square Q , the minimum of the cardinal numbers of such sequences. We will refer to such positive integer as the *generation* of Q , understanding that Q^* belongs to the *generation zero*. Thus, *generation 1* will be formed by all the neighbors of this square. Inductively, define the *generation $n + 1$* as the neighbors of the squares of the n -th generation already defined. Using a simple compactness argument and the crucial property that $\text{diam } Q_n \asymp d_\Omega(Q_n)$, it can be shown that no square will be omitted in this process, i.e., every square Q_n in the decomposition belongs to some generation. An inductive argument allows us to deduce that there are only finitely many squares in every generation.

Next, we reorder our sequence of squares (Q_n) , starting with Q^* and continuing with all the squares from the first generation (in any order), then with all the squares of the second generation, and so on. Observe that this new sequence has the property that given a square Q_n , the square Q_{n+1} either belongs to the same generation as Q_n or it belongs to the next one.

Recall that if Q_n and Q_m are neighbors, they share a segment on their boundaries. Let us denote by $C_{n,m}$ a small open subsegment properly included in $Q_n \cap Q_m$, that is, $C_{n,m}$ is a segment without its endpoints, contained in Q_n and Q_m simultaneously. Observe that with this selection, $\text{Int}(Q_n) \cup \text{Int}(Q_m) \cup C_{n,m}$ is a domain.

We will construct the domain Ω' inductively. Let Ω_1 be the interior of Q^* . In order to construct the domain Ω_2 , consider the interior of the square Q_2 . Since Q_2 is a neighbor of Q_1 , we can consider the segment $C_{1,2}$ and define $\Omega_2 = \text{Int}(Q_1) \cup \text{Int}(Q_2) \cup C_{1,2}$.

In this way, our domains Ω_n will consist of the union of the interiors of the squares Q_1, \dots, Q_n and a certain union of some segments $C_{n,m}$ that connect every square Q_k with one of the previous ones which must be its neighbor and must also belong to the preceding generation. That is, if we suppose that Ω_n has been constructed in such a way that it satisfies the properties just mentioned, in order to construct Ω_{n+1} , we proceed as follows. Consider Q_{n+1} and suppose it belongs to the generation N . Then choose one k with $1 \leq k \leq n$ so that Q_k belongs to the generation $N-1$ and Q_k and Q_{n+1} are neighbors. Having made this selection, define $\Omega_{n+1} = \Omega_n \cup \text{Int}(Q_{n+1}) \cup C_{k,n+1}$.

Finally, set $\Omega' = \cup \Omega_n$. Observe that our domain Ω' has the following properties.

- Ω' contains the interiors of all squares Q_n .
- Since segments $C_{n,m}$ have been chosen to be completely contained in the intersection of Q_n and Q_m , every square of the generation N is connected to one square of the previous generation and to at most one of the generation $N+1$. Consequently, Ω_n is simply connected for any n .
- Since the union of an ascending chain of simply connected domains is again simply connected, it follows that Ω' is also simply connected.

It is now clear that $\Omega \setminus \Omega'$ is a subset of the union of the boundaries of all squares Q_n , that is, $\Omega \setminus \Omega'$ is contained in a countable union of segments and the result follows. \square

2. CLASSICAL FUNCTION SPACES AND THE ISOPERIMETRIC INEQUALITY

The main purpose of this section is to prove the classical isoperimetric inequality for Jordan domains with rectifiable boundary by using complex analysis. To be more precise, the proof is based on a quantitative form of the inclusion of a Hardy space into the related Bergman space with the double exponent.

We begin by reviewing the very basics of the classical Hardy and Bergman spaces of analytic functions in the disk. No exhaustive review of these spaces will be given.

2.1. A brief review of Hardy spaces. We briefly review the theory of Hardy spaces and the classical Riesz decomposition. For more details, the reader is advised to consult the classical sources such as [Du70] or [Ko98], for example.

Given a function f in $\mathcal{H}(\mathbb{D})$, consider the standard integral means over the circle of radius r centered at the origin:

$$M_p(r, f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

It is a consequence of the subharmonicity of $|f|^p$ that $M_p(r, f)$ is an increasing function of r . If

$$\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty,$$

we will say that f belongs to the Hardy space H^p and define its Hardy space norm as above. Equipped with this norm, H^p is a Banach space whenever $1 \leq p < \infty$. It is not difficult to check that $H^p \subset H^q$ whenever $p > q$.

It is a well-known fact that whenever f belongs to H^p , the *radial limit* $\tilde{f}(\theta) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$ exists for almost every θ in $[0, 2\pi)$ (in the sense of Lebesgue measure). The H^p -norm can be computed by integrating these boundary values:

$$(1) \quad \|f\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |\tilde{f}(\theta)|^p d\theta \right)^{1/p}.$$

When $p = 2$ and the Taylor series expansion of f in the disk is $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then the norm of f can be computed by the formula

$$(2) \quad \|f\|_{H^2} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.$$

It is standard that a sequence (z_n) in \mathbb{D} is the sequence of zeros of some H^p -function if and only if it satisfies the so-called *Blaschke condition*: $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$. Thus, the zero sets of all the spaces H^p are

the same, independently of p . For any such sequence, the corresponding *Blaschke product* is defined as

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n}z}.$$

We remark that when $z_n = 0$, the term z is to be used instead of the corresponding fraction. It can be proved that this infinite product converges uniformly on every $K \Subset \mathbb{D}$ and the limit function is an H^∞ function whose boundary values have modulus one. It is easily seen that $|B(z)| < 1$ for all z inside \mathbb{D} and $|\tilde{B}(e^{i\theta})| = 1$ for almost all θ in $[0, 2\pi)$. In view of (1), the Blaschke product B that corresponds to the zero set of some function f from H^p is an *isometric zero-divisor*, that is, we have what is known as the *Riesz factorization theorem*:

For every $f \in H^p$, we can write $f = Bg$ where B is the Blaschke product formed by the zeros of f and $g = f/B$ is a zero-free member of H^p such that $\|f/B\|_p = \|f\|_p$.

This result is the key to many proofs in the theory of Hardy spaces. We will see an illustration of it in a proof given below.

2.2. Basic facts on Bergman spaces. Let dA denote Lebesgue area measure in the unit disk \mathbb{D} , normalized so that $A(\mathbb{D}) = 1$: $dA(z) = \pi^{-1}dx dy = \pi^{-1}r dr d\theta$. If $0 < p < \infty$, the *Bergman space* A^p is the set of all analytic functions f in the unit disk \mathbb{D} with finite $L^p(\mathbb{D}, dA)$ norm:

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) = 2 \int_0^1 M_p^p(r, f) r dr < \infty,$$

This readily implies that $H^p \subset A^p$ but more is actually true: $H^p \subset A^{2p}$, as we will see later.

Note that $\|f\|_{A^p}$ is a true norm if and only if $1 \leq p < \infty$ and, in this case, A^p is a Banach space. When $0 < p < 1$, A^p is still a complete space with respect to the translation-invariant metric defined by $d_p(f, g) = \|f - g\|_p^p$ but we shall not be concerned with this case. Also, it is easy to check that $A^p \subset A^q$ whenever $p > q$.

When $p = 2$ and the Taylor series expansion of f in the disk is $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then the A^2 norm of f can be computed by the formula

$$(3) \quad \|f\|_{A^2} = \left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \right)^{1/2}.$$

In what follows, we will essentially need only two basic facts about Bergman space functions. The first one is a standard example of a family of functions in such spaces.

LEMMA 4. *The function f_c given by $f_c(z) = (1 - z)^{-c}$ belongs to A^p if and only if $cp < 2$. Also, $f_c \in H^p$ if and only if $cp < 1$.*

Proof. For A^p , integrate in polar coordinates centered at the point $z = 1$ rather than at the origin. For the Hardy spaces, work with boundary values of f . \square

Another relevant fact is the growth of the functions in Bergman spaces. There is a more precise statement but for our purpose it suffices to know just the correct maximum order of growth.

LEMMA 5. *For every f in A^p and every z in \mathbb{D} , we have*

$$|f(z)| \leq \frac{\|f\|_{A^p}}{(1 - |z|)^{2/p}}.$$

Proof. Follows in a straightforward fashion by applying the sub-mean value property to the subharmonic function $|f|^p$ on a smaller disk of radius $1 - |z|$ centered at z . \square

A detailed account of the theory of Bergman spaces can be found in the texts [HKZ00] and [DS04], for example.

2.3. The isoperimetric inequality. The classical isoperimetric inequality is well known and states that $A(\Omega) \leq (4\pi)^{-1}L(\partial\Omega)^2$, where $A(\Omega)$ is the area of a plane Jordan domain Ω , while $L(\partial\Omega)$ is the length of its boundary $\partial\Omega$ (a simple, closed, and rectifiable curve). Equality holds only when Ω is a disk. This means that of all simple closed curves of given length L , the one that encloses the largest area, namely $L^2/(4\pi)$, is a circle of radius $L/(2\pi)$.

There are many known proofs of this fact. We follow the approach by Carleman which uses complex analysis.

THEOREM 6. *For arbitrary p in $(0, \infty)$, every f in H^p belongs to A^{2p} , and $\|f\|_{A^{2p}} \leq \|f\|_{H^p}$, with equality if and only if f has the form*

$$(4) \quad f(z) = \text{const} \cdot \left(\frac{1}{1 - \lambda z} \right)^{2/p}, \quad |\lambda| < 1.$$

Thus, the injection map $J_p : H^p \rightarrow A^{2p}$, $J_p(f) = f$, has norm one.

Proof. The proof is a typical application of the Riesz factorization technique.

We first consider the case $p = 2$, when we can use the norm formulas in terms of the Taylor coefficients. By grouping the terms in the series multiplication, together with the formulas (3) and (2), and the Cauchy-Schwarz inequality (applied to each term in the appropriate sum), we

get

$$\begin{aligned}
\|f\|_{A^4}^4 &= \|f^2\|_{A^2}^2 = \left\| \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k a_{n-k} \right) z^n \right\|_{A^2}^2 \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \left| \sum_{k=0}^n a_k a_{n-k} \right|^2 = \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_k a_{n-k} \cdot \frac{1}{\sqrt{n+1}} \right|^2 \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^n |a_k|^2 |a_{n-k}|^2 = \left(\sum_{m=0}^{\infty} |a_m|^2 \right) \cdot \left(\sum_{n=0}^{\infty} |a_n|^2 \right) \\
&= \|f\|_{H^2}^4,
\end{aligned}$$

which proves the inequality $\|f\|_{A^4} \leq \|f\|_{H^2}$.

Equality holds in the above chain if and only if for each $n \geq 2$ we have

$$a_k a_{n-k} = \frac{C_n}{\sqrt{n+1}}, \quad k = 0, 1, \dots, n,$$

for some constant C_n that depends only on n . (Note that such equalities for $n = 0$ and $n = 1$ hold trivially.) It is easily observed that if $a_0 = 0$, then from the equalities $a_0 \cdot a_{2n} = a_n^2$, $n = 1, 2, 3, \dots$, we conclude that $a_n = 0$ for all n . Thus $f \equiv 0$, a case of little interest. Therefore we may assume $a_0 \neq 0$. By setting $a_1/a_0 = \lambda$, we derive from the chains of equalities

$$a_0 a_n = a_1 a_{n-1} = \dots$$

the relations

$$a_n = \frac{a_1}{a_0} a_{n-1} = \lambda a_{n-1}$$

for all n , from which we infer $a_n = \lambda^n a_0$. It follows after summation that our extremal function (in the case $p = 2$) is of the form

$$f(z) = \sum_{n=0}^{\infty} a_0 \lambda^n z^n = \frac{a_0}{1 - \lambda z}.$$

We can now prove the statement for arbitrary p , by relying on the Riesz factorization technique. Let f belong to H^p and suppose first that it does not vanish anywhere in \mathbb{D} . Then we can choose an analytic branch of $f^{p/2}$ and apply to it the special case $p = 2$ of the inequality just proved in computing

$$\|f\|_{A^{2p}}^{p/2} = \|f^{p/2}\|_{A^4} \leq \|f^{p/2}\|_{H^2} = \|f\|_{H^p}^{p/2},$$

which shows that $\|f\|_{A^{2p}} \leq \|f\|_{H^p}$. Equality will be possible only for functions of the form

$$f(z) = \left(\frac{a_0}{1 - \lambda z} \right)^{2/p}.$$

Finally, if f has zeros in the disk, then it has a factorization $f = Bg$, where B is the Blaschke product with the same zeros as f and g is zero-free with the same H^p -norm as f . Thus, by the cases considered previously,

$$\frac{\|f\|_{A^{2p}}}{\|f\|_{H^p}} = \frac{\|Bg\|_{A^{2p}}}{\|Bg\|_{H^p}} < \frac{\|g\|_{A^{2p}}}{\|g\|_{H^p}} \leq 1$$

(with strict inequality), and we are done. \square

It should be observed that the inclusion $H^p \subset A^q$ is false whenever $q > 2p$. This can be seen by considering the function $f_\alpha(z) = (1 - z)^{-\alpha}$. It suffices to choose α so that $2/q \leq \alpha < 1/p$ and apply Lemma 4 to see that f_α is in H^p but not in A^q .

The classical isoperimetric inequality now follows easily from our result. The key point is as follows. Let F be a univalent map of \mathbb{D} onto a simply connected domain Ω bounded by a rectifiable Jordan curve. For $0 < r < 1$, let $C_r = \{z : |z| = r\}$. Then

$$L(\partial\Omega) = \lim_{r \rightarrow 1^-} L(F(C_r)) = \lim_{r \rightarrow 1^-} r \int_0^{2\pi} |F'(re^{i\theta})| d\theta = 2\pi \|F'\|_{H^1},$$

by [Du70, Theorem 3.12].

COROLLARY 7. *If a Jordan domain Ω of area $A(\Omega)$ is bounded by a rectifiable curve $\partial\Omega$ of length $L(\partial\Omega)$, then $A(\Omega) \leq (4\pi)^{-1} L(\partial\Omega)^2$. Equality holds if and only if Ω is a disk.*

Proof. Appealing to the Riemann Mapping Theorem, we choose a conformal mapping F of \mathbb{D} onto Ω . Then, as observed above, $L(\partial\Omega) = 2\pi \|F'\|_{H^1}$. Furthermore, $A(\Omega) = \pi \|F'\|_{A_2}^2$ (in view of our normalization of the area measure). By applying the case $p = 1$ of Theorem 6 to $f = F'$, the desired inequality follows.

The discussion of extremal functions in (4) shows that equality is only possible when F' is of the form

$$F'(z) = \frac{C}{(1 - \lambda z)^2},$$

for some constant C . But this means that F is a linear fractional map:

$$F(z) = \frac{D}{1 - \lambda z}.$$

Since such transformations carry disks onto disks or half-planes and all our functions are obviously bounded, it follows that $\Omega = F(\mathbb{D})$ is again a disk. \square

The above approach is essentially due to Carleman [Ca21] although it was only formulated for sufficiently smooth domains in his paper, probably because at that time the theory of Hardy spaces had not yet fully been developed. In full generality, the details were supplied by Mateljević [Mat80]. Unfortunately, his paper remained little known so it was rediscovered later [Vu03].

3. GROWTH OF FUNCTIONS IN THE DIRICHLET SPACE

The main purpose of this section is to review the basic properties of functions in the classical Dirichlet space of the disk. This space being smaller than the Hardy space H^2 and its functions more special in a certain sense, their growth can be controlled in a more precise way. This is made quantitative by theorems of Beurling and Chang-Marshall which will be briefly reviewed. Both will be proved here, except for the most difficult case $\alpha = 1$.

3.1. The Dirichlet space. The Dirichlet space \mathcal{D} is the set of all analytic functions f in \mathbb{D} with the finite Dirichlet integral (*i.e.*, such that $f' \in A^2$), equipped with the norm

$$(5) \quad \|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 dA.$$

The *Dirichlet space* can be proved to be a Hilbert space of analytic functions. The norm of a function f in \mathcal{D} can also be computed as follows:

$$(6) \quad \|f\|_{\mathcal{D}}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2,$$

where f has the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ in \mathbb{D} . The following estimate is well known and useful.

LEMMA 8. *Every function in \mathcal{D} satisfies the following growth estimate:*

$$(7) \quad |f(z) - f(0)| \leq \|f\|_{\mathcal{D}} \sqrt{\log \frac{1}{1 - |z|^2}}.$$

Proof. If the Taylor series of f in \mathbb{D} is $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by applying the Cauchy-Schwarz inequality we get

$$\begin{aligned} |f(z) - f(0)| &= \left| \sum_{n=1}^{\infty} \sqrt{n} a_n \frac{z^n}{\sqrt{n}} \right| \\ &\leq \left(\sum_{n=1}^{\infty} n |a_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{|z|^{2n}}{n} \right)^{1/2} \\ &\leq \|f\|_{\mathcal{D}} \sqrt{\log \frac{1}{1 - |z|^2}}. \end{aligned}$$

□

One of the earliest sources that quotes this fact appears to be [SS62], pp. 218–219.

3.2. The distribution function. It is immediate from (6) and (2) that $\mathcal{D} \subset H^2$, hence every function f in the Dirichlet space has boundary values almost everywhere on the unit circle. Thus, given a function f in \mathcal{D} , it makes sense to consider its *boundary distribution function* $|E_\lambda|$ for $\lambda > 0$, where

$$E_\lambda = \{\theta \in [0, 2\pi] : |f(e^{i\theta})| > \lambda\}$$

and $|E|$ denotes the the normalized arc measure of the set $E \subset \mathbb{T}$.

The following formula is useful and widely used. Applying Fubini's theorem to a function g , increasing on $[0, \infty)$ and absolutely continuous on every closed interval of this semi-axis (as in [Ru87, Theorem 8.16]) or the first page of Chapter VIII of [Ko98]), we get

$$\begin{aligned} \int_0^{2\pi} g(|f(e^{i\theta})|) d\theta - 2\pi g(0) &= \int_0^{2\pi} \left(\int_0^{|f(e^{i\theta})|} g'(\lambda) d\lambda \right) d\theta \\ &= \int_0^{2\pi} \left(\int_0^\infty \chi_{\{\theta: |f(e^{i\theta})| > \lambda\}} g'(\lambda) d\lambda \right) d\theta \\ &= \int_0^\infty \left(\int_0^{2\pi} \chi_{\{\theta: |f(e^{i\theta})| > \lambda\}} d\theta \right) g'(\lambda) d\lambda \\ &= 2\pi \int_0^\infty |E_\lambda| g'(\lambda) d\lambda. \end{aligned}$$

3.3. On the theorems of Beurling and Chang-Marshall. The purpose of this section is to show the power of Green's theorem, which can even be used to deduce some relatively strong statements about analytic functions.

The following deep uniform estimate was proved in Beurling's thesis [Be33].

THEOREM 9. *If $f \in \mathcal{D}$, $\|f\|_{\mathcal{D}} \leq 1$, and $f(0) = 0$, then*

$$(8) \quad |E_\lambda| \leq e^{-\lambda^2+1}.$$

We will omit the full proof of this theorem. However, a weaker version will be deduced at the end of this section.

The general Sobolev imbedding theorem was already known to Hardy and Littlewood in the case of analytic functions and can be stated as follows: if $0 < p < 2$, $f \in \mathcal{H}(\mathbb{D})$, and $f' \in L^p(\mathbb{D}, dA)$, then $f \in L^{\frac{2p}{2-p}}(\mathbb{D}, dA)$.

For the critical exponent $p = 2$, the following is true. If $f \in \mathcal{H}(\mathbb{D})$ and $f' \in L^2(\mathbb{D}, dA)$ (that is, if $f \in \mathcal{D}$), then

$$\int_{\mathbb{D}} e^{|f(z)|^2} dA(z) < \infty.$$

This is usually referred to as an *inequality of Trudinger-Moser type*; see [Mo71] or [Yu61], for example. Although the integral over the unit circle of a function which has boundary values there is bigger than its area integral over the disk, one can actually get a stronger version of the above statement, known as the *Chang-Marshall inequality*:

THEOREM 10.

$$\sup \left\{ \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta : \|f\|_{\mathcal{D}} \leq 1, f(0) = 0 \right\}$$

is finite for all $\alpha \leq 1$.

Its proof in the critical case $\alpha = 1$ was a deep result of Chang and Marshall [CM85] which provided an answer to a question stated on p. 1079 of Moser's influential paper [Mo71]. See also [Mar89] for a simplified proof and [Ch96] for more details and the vast literature on this topic and its relations with geometry.

By the *weak Chang-Marshall inequality* we mean the uniform estimate

$$(9) \quad \sup \left\{ \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta : \|f\|_{\mathcal{D}} \leq 1, f(0) = 0 \right\} < \infty, \quad \alpha < 1.$$

Note that the uniform estimate as in Theorem 10 is false when $\alpha > 1$. However, for every fixed f in \mathcal{D} and for all $\alpha > 0$ we still have

$$(10) \quad \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta < \infty.$$

This can be deduced from the following observation due to Garnett (see p. 1016 of [CM85]): if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, there is obviously a polynomial P and $g \in \mathcal{D}$ such that $f = P + g$, $g(0) = 0$, and $\|\sqrt{3\alpha}g\|_{\mathcal{D}} \leq 1$, whence by (9) we have

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} \leq \int_0^{2\pi} e^{2\alpha(|P|^2+|g|^2)} \frac{d\theta}{2\pi} \leq e^{2\alpha\|P\|_{\infty}^2} \int_0^{2\pi} e^{2\alpha|g(e^{i\theta})|^2} d\theta < \infty,$$

which proves the statement.

It is beyond the scope of these notes to present the proof of the strong Chang-Marshall inequality, *i.e.*, the case $\alpha = 1$. The case $0 < \alpha < 1$ (that is, the weak Chang-Marshall inequality) is certainly easier to prove. However, its proof that one normally encounters in the literature is based on Theorem 9. Together with (8), the formula for the distribution function from Subsection 3.2 applied to $g(\lambda) = e^{\alpha\lambda^2}$ yields

$$(11) \quad \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} = 1 + 2\alpha \int_0^{\infty} \lambda e^{\alpha\lambda^2} |E_{\lambda}| d\lambda < \infty$$

for any $\alpha < 1$. Now (9) follows immediately from Theorem 9.

Instead of this argument which uses Beurling's deep theorem, we give an extremely simple proof as presented in [PV06] which relies only on the growth estimate for Dirichlet functions and Green's formula. (Note that the case $\alpha = 1$ is much harder and requires even more refined arguments than Beurling's theorem.)

THEOREM 11. *For every positive value $\alpha < 1$, we have*

$$\sup \left\{ \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta : \|f\|_{\mathcal{D}} \leq 1, f(0) = 0 \right\} < \infty.$$

Proof. Fix $\alpha < 1$. Let $f \in \mathcal{D}$, $\|f\|_{\mathcal{D}} \leq 1$, and $f(0) = 0$. Consider the function

$$W_f(z) = \exp(\alpha|f(z)|^2) - 1.$$

Its dilatations $W_{f,r}$, defined by $W_{f,r}(z) = W_f(rz)$, vanish at the origin and belong to $\mathcal{C}^\infty(\overline{\mathbb{D}})$, so we may apply the first lemma in Section D.1, Chapter X of [Ko98] to get

$$(12) \quad \int_0^{2\pi} W_f(re^{i\theta}) d\theta = \pi \int_{\mathbb{D}} \log \frac{1}{|z|} \cdot r^2 \cdot (\Delta W_f)(rz) dA(z).$$

A straightforward computation of the Laplacian of W_f yields:

$$\Delta W_f = 4\partial\bar{\partial} \exp(\alpha|f|^2) = 4\alpha|f'|^2(1 + \alpha|f|^2) \exp(\alpha|f|^2).$$

Since by Fatou's lemma we have

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta \leq 2\pi + \liminf_{r \rightarrow 1^-} \int_0^{2\pi} W_f(re^{i\theta}) d\theta,$$

the theorem will follow from (12) if we can show that the integrals over \mathbb{D} of the functions

$$U_{f,r}(z) = \log \frac{1}{|z|} \cdot |f'(rz)|^2 (1 + \alpha|f(rz)|^2) \exp(\alpha|f(rz)|^2)$$

are all finite and bounded by the same constant (independent of r) for each f as specified above. This is actually rather simple.

By (7) and by our assumptions that $f(0) = 0$ and $\|f\|_{\mathcal{D}} \leq 1$, we obtain

$$\begin{aligned} U_{f,r}(z) &\leq \log \frac{1}{|z|} \cdot \frac{1 + \alpha \log \frac{1}{1-r^2|z|^2}}{(1-r^2|z|^2)^\alpha} \cdot |f'(rz)|^2 \\ &\leq \log \frac{1}{|z|} \cdot \frac{1 + \alpha \log \frac{1}{1-|z|}}{(1-|z|)^\alpha} \cdot |f'(rz)|^2. \end{aligned}$$

For R sufficiently close to one, $\log(1/|z|) \asymp 1 - |z|$ whenever $R < |z| < 1$. Since $\alpha < 1$, we get

$$U_{f,r}(z) \leq |f'(rz)|^2 \text{ on some annulus } A_R = \{z : R < |z| < 1\}.$$

It is well known that $M_2^2(r, f') = (2\pi)^{-1} \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta$ is an increasing function of r , hence

$$(13) \quad \int_{A_R} U_{f,r} dA \leq \int_{\mathbb{D}} |f'(rz)|^2 dA(z) = 2 \int_0^1 M_2^2(r\rho, f') \rho d\rho \\ \leq 2 \int_0^1 M_2^2(\rho, f') \rho d\rho = \|f\|_{\mathcal{D}} \leq 1.$$

On the other hand, the area version of the sub-mean value property yields

$$(1 - R)^2 |f'(rz)|^2 \leq (1 - r|z|)^2 |f'(rz)|^2 \leq \int_{D(rz, 1-r|z|)} |f'|^2 dA \leq \|f\|_{\mathcal{D}}^2 \leq 1$$

whenever $|z| \leq R$. Hence

$$(14) \quad U_{f,r}(z) \leq M_R \log(1/|z|) \text{ on the punctured closed disk } \overline{D(0, R)} \setminus \{0\},$$

where M_R is a constant that depends only upon R .

From (13) and (14) we finally obtain

$$\int_{\mathbb{D}} U_{f,r}(z) dA(z) \leq 1 + M_R \int_{\overline{D(0, R)}} \log \frac{1}{|z|} dA$$

for all $r \in (0, 1)$ and all f such that $\|f\|_{\mathcal{D}} \leq 1$ and $f(0) = 0$, which is what was needed. \square

It would require harder work than what was shown above to derive the strong versions of the theorems of Chang-Marshall or Beurling. However, as a corollary of the above proof we obtain a weaker version of Beurling's Theorem 9.

COROLLARY 12. *Let $0 < \alpha < 1$. If $f \in \mathcal{D}$, $\|f\|_{\mathcal{D}} \leq 1$, and $f(0) = 0$, then*

$$|E_\lambda| \leq C_\alpha e^{-\alpha\lambda^2},$$

where the positive constant C_α depends only on α .

Proof. By Theorem 11, we know that $\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta \leq C_\alpha$ for some fixed constant C_α . Hence, keeping in mind that $|E_\lambda|$ is a non-increasing

positive function bounded above by 1, we get

$$\begin{aligned}
C_\alpha &\geq 1 + 2\alpha \int_0^\infty te^{\alpha t^2} |E_t| dt \\
&\geq 1 + 2\alpha \int_0^\lambda te^{\alpha t^2} |E_t| dt \\
&\geq 1 + 2|E_\lambda| \alpha \int_0^\lambda te^{\alpha t^2} dt \\
&\geq 1 + |E_\lambda| (e^{\alpha\lambda^2} - 1) \\
&\geq |E_\lambda| e^{\alpha\lambda^2}
\end{aligned}$$

□

Yet another measure of growth of functions in \mathcal{D} (in terms of the asymptotic behavior of their H^p norms, based also on the Chang-Marshall inequality) can be found in [Vu04R].

4. UNIVALENT FUNCTIONS AND CONFORMALLY INVARIANT SPACES

The Dirichlet space is just one example among the spaces from an increasing scale of conformally invariant spaces B^p in the disk (case $p = 2$). The functions in these spaces allow for a rich interplay with geometric features, in the sense that univalent B^p functions can be described in terms of geometric properties of the image domain in a quantitative way.

4.1. Univalent functions. A function f is said to be *univalent* in \mathbb{D} if $f \in \mathcal{H}(\mathbb{D})$ and is one-to-one. We will denote by \mathcal{U} the set of all such functions. For different topics in the theory of univalent functions the reader may consult the monographs by Duren [Du83] or Pommerenke [Po92], for example.

By the inverse function theorem, every univalent function f is a homeomorphism of the disk onto $f(\mathbb{D})$. Thus, whenever $f \in \mathcal{U}$, the domain $f(\mathbb{D})$ is simply connected. By the Riemann mapping theorem, the converse is also true: for any given simply connected domain Ω (other than the plane itself) there is a function f (called a *Riemann map*) that takes \mathbb{D} onto Ω and the origin to a prescribed point. The Riemann map f has the following basic but important property (Corollary 1.4 of [Po92]) which, recalling the notation from Subsection 1.1, can be written as follows.

LEMMA 13. *If $f \in \mathcal{U}$ then*

$$(15) \quad \frac{1}{4}(1 - |z|^2)|f'(z)| \leq \text{dist}(f(z), \partial\Omega) \leq (1 - |z|^2)|f'(z)|, \quad z \in \mathbb{D}.$$

Hence $(1 - |z|^2)|f'(z)| \asymp d_{f(\mathbb{D})}(f(z))$, for all $z \in \mathbb{D}$.

The proof of this standard result is based on the Schwarz-Pick lemma on the one hand and the Koebe one-quarter theorem on the other hand ([Po92], Section 1.3). This estimate is relevant in geometric function theory.

4.2. Hyperbolic metric in the disk. In what follows we will need a few basic properties of the hyperbolic metric. Recall that the *hyperbolic distance* between two points z and w in the disk is defined as

$$\rho(z, w) = \inf_{\gamma} \int_{\gamma} \frac{|d\zeta|}{1 - |\zeta|^2} = \frac{1}{2} \log \frac{1 + \left| \frac{z-w}{1-\bar{z}w} \right|}{1 - \left| \frac{z-w}{1-\bar{z}w} \right|},$$

where the infimum is taken over all rectifiable curves γ in \mathbb{D} that join z with w . The first formula is the definition and the second can be proved by observing that ρ is invariant under disk automorphism and then, after applying such a map, reducing the general case to special case when $z = 0$ and $w = r \in (0, 1)$ and doing some elementary calculus estimates and integration.

The hyperbolic metric ρ_Ω on an arbitrary simply connected domain Ω (other than the entire plane) is defined via the corresponding pullback to the disk: if f is a Riemann map of \mathbb{D} onto Ω then

$$\rho_\Omega(f(z), f(w)) = \rho(z, w) = \inf_{\Gamma} \int_{f^{-1}(\Gamma)} \frac{|d\zeta|}{1 - |\zeta|^2},$$

where the infimum is taken over all rectifiable curves Γ in Ω from $f(z)$ to $f(w)$. The metric ρ_Ω does not depend on the choice of the Riemann map f . For more details we refer the reader to [Po92, Sections 1.2 and 4.6].

From the definition of hyperbolic metric we notice the following important feature of Riemann maps: if $f(0) = 0$ then

$$(16) \quad \rho_\Omega(0, f(z)) = \rho(0, z) \geq \frac{1}{2} \log \frac{1}{1 - |z|}, \quad z \in \mathbb{D}.$$

Another fundamental property, which is easily deduced from (15), is that (in our earlier notation for the distance to the boundary of the domain)

$$(17) \quad \rho_\Omega(w_1, w_2) \leq \inf_{\Gamma} \int_{\Gamma} \frac{|dw|}{d_\Omega(w)},$$

where the infimum is taken over all rectifiable curves Γ in Ω from w_1 to w_2 .

4.3. The Bloch space. By definition, $f \in \mathcal{B}$ if $f \in \mathcal{H}(\mathbb{D})$ and

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

A modification of the Schwarz-Pick lemma shows that $H^\infty \subset \mathcal{B}$. The inclusion is strict. Actually, there are many univalent functions that belong to $\mathcal{B} \setminus H^\infty$, a simple example being $f(z) = \log \frac{1+z}{1-z}$.

Let $f \in \mathcal{B}$. Integration of the derivative f' from the origin to z along a line segment leads to the following basic growth estimate for arbitrary Bloch function f .

LEMMA 14. *Whenever $f \in \mathcal{B}$ and $z \in \mathbb{D}$, we have*

$$|f(z) - f(0)| \leq \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} \cdot \|f\|_{\mathcal{B}}.$$

In particular, convergence in the Bloch space implies uniform convergence on any $K \Subset \mathbb{D}$.

Note that the function $\log \frac{1+z}{1-z}$ mentioned above achieves maximum possible growth along the entire radius $[0, 1)$. A different and more complicated examples will be exhibited later, in the section on superposition operators from the Bloch into a Bergman space.

One of the basic papers on the subject of Bloch spaces is that of Anderson, Clunie, and Pommerenke [ACP74]; see also Danikas' lecture notes [Da99] or Chapter 5 of Zhu's book [Zh90].

It is clear from the basic estimate that a univalent function f belongs to \mathcal{B} if and only if

$$\sup_{z \in \mathbb{D}} d_{f(\mathbb{D})}(f(z)) < \infty,$$

i.e. if and only if $f(\mathbb{D})$ does not contain arbitrarily large disks.

4.4. Bloch domains. A planar domain Ω is said to be a *Bloch domain* if and only if every $f \in \mathcal{H}(\mathbb{D})$ with the property $f(\mathbb{D}) \subset \Omega$ must belong to \mathcal{B} . (Note that $f(\mathbb{D}) = \Omega$ is not required and no special properties of Ω are being assumed either.) The following result is considered a "folk knowledge".

PROPOSITION 15. *A planar domain is a Bloch domain if and only if it does not contain arbitrarily large disks.*

It should be remarked that similar result exist for other conformally invariant spaces of analytic functions, such as $BMOA$ or Q_p but we will not discuss them here.

4.5. Analytic Besov spaces. It is actually possible to construct a scale of spaces of L^p type that includes both \mathcal{D} and \mathcal{B} . This is done by integrating $(1 - |z|^2)|f'(z)|$ with respect to the hyperbolic area element $dA(z)/(1 - |z|^2)^2$. Thus, when $1 < p < \infty$, a function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the *analytic (diagonal) Besov space* B^p if and only if

$$s_p(f)^p = (p-1) \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty.$$

The above seminorm s_p is invariant under the conformal automorphisms of the disk: $s_p(f \circ \varphi) = s_p(f)$, for every disk automorphism φ . A true norm is usually given by the formula

$$\|f\|_{B^p}^p = |f(0)|^p + s_p(f)^p.$$

It is clear that $B^2 = \mathcal{D}$, the Dirichlet space. An important property of analytic Besov spaces is the following: $B^p \subset B^q$, whenever $1 < p < q \leq \infty$. Also, we may interpret the Bloch space as a limit case: $B^\infty = \mathcal{B}$. There is a way of defining the space B^1 as well but this would require more work and some further considerations, so we will avoid this in the present article.

In view of their conformal invariance, B^p spaces are closely related with hyperbolic metric. Furthermore, they represent the range of the Bergman projection when acting on L^p spaces with respect to the hyperbolic area measure. They also relate naturally with the membership in

Schatten class of Hankel operators, etc. Thus, it is clear that they are very natural and important objects to study.

Note that the space B^p is also invariant under translations $\tau(z) = z + a$ and dilations $d_r(z) = rz$, $0 < r < 1$; that is, if $f \in B^p$ then also $\tau \circ f \in B^p$ and $f \circ d_r \in B^p$.

Besides the invariance properties of B^p spaces, we will only use the following statement in the sequel.

LEMMA 16. *Whenever $f \in B^p$, $1 < p < \infty$, and $z \in \mathbb{D}$, we have*

$$|f(z) - f(0)| \leq C \left(\log \frac{1 + |z|}{1 - |z|} \right)^{1-1/p} \cdot \|f\|_{B^p}.$$

In particular, convergence in B^p norm implies uniform convergence on any $K \Subset \mathbb{D}$.

A statement of this type can be found in Zhu's semi-expository paper [Zh91], for example. Another useful reference on analytic Besov spaces is [Zh90, Chapter 5]. One of the most influential papers on the subject was [AFP85].

4.6. Image domains under univalent Besov functions. Among the simply connected domains, those that are images of the disk under univalent maps in B^p can be characterized in a very convenient way. This was observed and used in [Wa00] and in [BFV01].

PROPOSITION 17. *Let $1 < p < \infty$. A domain Ω has the property that every $f \in \mathcal{U}$ such that $f(\mathbb{D}) = \Omega$ belongs to B^p if and only if Ω is simply connected and $\int_{\Omega} d_{\Omega}^{p-2} dA < \infty$.*

Proof. Since the Jacobian of the change of variable $w = f(z)$ is $|f'(z)|^2$ and $(1 - |z|^2)|f'(z)| \asymp d_{\Omega}(f(z))$ in view of Lemma 13, we readily see that

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) \asymp \int_{\Omega} d_{\Omega}(w)^{p-2} dA(w),$$

which proves the statement. □

This simple but extremely useful result allows us to construct simply connected domains whose Riemann maps lie in the desired B^p space. The following example is from [DGV02].

PROPOSITION 18. *Let (a_n) be infinite sequence of positive numbers. Consider a sequence of open squares Q_n with side lengths a_n respectively and whose one side lies on the positive part of the x -axis so that $\overline{Q_n} \cap \overline{Q_{n+1}}$ is a vertical segment for all n . Define Ω to be the interior of the union $\cup_{n=1}^{\infty} \overline{Q_n}$. Let F be a univalent map of the unit disk onto Ω . Then $F \in B^p$ if and only if $\sum_{n=1}^{\infty} a_n^p < \infty$.*

Thus, there are univalent functions in every B^p and $B^p \neq \cap_{q>p} B^q$.

Proof. We give a proof only when $2 \leq p < \infty$; the case $1 < p < 2$ is handled analogously but with all inequalities reversed. For each point in Q_n we have $d_\Omega(w) \leq a_n/2$, whence

$$\int_{\Omega} d_\Omega(w)^{p-2} dA(w) \lesssim \sum_{n=1}^{\infty} \int_{Q_n} a_n^{p-2} dA(w) \lesssim \sum_{n=1}^{\infty} a_n^p.$$

For the reverse estimate, consider the square Q'_n concentric with Q_n but half the size. Every point w in this square satisfies $d_\Omega(w) \geq a_n/4$; hence

$$\int_{\Omega} d_\Omega(w)^{p-2} dA(w) \gtrsim \sum_{n=1}^{\infty} \int_{Q'_n} a_n^{p-2} dA(w) \gtrsim \sum_{n=1}^{\infty} a_n^p,$$

which proves the desired inequality. \square

Walsh used Proposition 17 to give a couple of interesting examples which were subsequently improved in [DGV02].

Observe first that a univalent map onto a domain of finite area necessarily belongs to the Dirichlet space. But does it have to belong to any smaller B^p space ($p < 2$)? Walsh [Wa00] answered this in the negative by giving the following example.

PROPOSITION 19. *There exists a domain Ω of finite area such that no univalent map $f : \mathbb{D} \rightarrow \Omega$ can belong to $\cup_{p < 2} B^p$.*

Proof. Such a domain Ω can be constructed by making countably many slits in the unit square

$$Q = \{x + iy : 0 < x < 1, 0 < y < 1\}$$

as follows: delete a vertical line segment of height $1/(n+1)$ at the base point $k/2^n$, $k = 1, 3, \dots, 2^n - 1$, for every $n \geq 1$. Then every point $w = x + iy$ in Ω with the imaginary part

$$\frac{1}{n+2} < y < \frac{1}{n+1}, n = 1, 2, \dots$$

also satisfies $d_\Omega(w) \leq 1/2^{n+1}$. Hence, whenever $p < 2$, we have

$$\int_{\Omega} d_\Omega(w)^{p-2} dA(w) \gtrsim \sum_{n=1}^{\infty} \frac{2^{(2-p)n}}{(n+1)(n+2)} = \infty,$$

showing that a univalent map of \mathbb{D} onto Ω cannot belong to A^p , according to Proposition 17. \square

Clearly, the domain from Proposition 19 is far from being a Jordan domain, for many of its boundary points are not simple. The following example was given in [DGV02].

PROPOSITION 20. *There exists a Jordan domain Ω of finite area such that no univalent map $f : \mathbb{D} \rightarrow \Omega$ can belong to $\cup_{p < 2} B^p$.*

Proof. Such a domain can be obtained by gluing to a vertical strip countably many “combs”, each finer and of smaller area than the previous one. To put it in precise terms, let

$$\alpha_n = n^{-2}, n \geq 1, \quad \beta_0 = 0, \beta_n = \sum_{k=1}^n \alpha_k, n \geq 1, \quad \beta = \sum_{n=1}^{\infty} \alpha_n = \frac{\pi^2}{6}.$$

Consider the domains

$$R_n = \{x + iy : 2\beta_{n-1} < y < 2\beta_{n-1} + \alpha_n, 0 \leq x < \alpha_n\}, \quad n \geq 1.$$

$$T_{n,k} = \{x + iy : 2\beta_{n-1} < y < 2\beta_{n-1} + \frac{\alpha_n}{2}, \frac{k}{2^n} \alpha_n \leq x \leq \frac{k}{2^n} \alpha_n + \frac{\alpha_n}{2^{n+1}}\},$$

for $n = 1, 2, 3, \dots$ and $0 \leq k \leq 2^n - 1$. Define

$$S_n = R_n \setminus \left(\bigcup_{k=0}^{2^n-1} T_{n,k} \right)$$

and finally

$$\Omega = \left(\bigcup_{n=1}^{\infty} S_n \right) \cup \{x + iy : -1 < x < 0, 0 < y < 2\beta\}.$$

It is easy to see that Ω is a Jordan domain. By applying a reasoning similar to that of Proposition 19, we arrive at the same conclusion: $\int_{\Omega} d_{\Omega}^{p-2} dA = \infty$ when $p < 2$. We omit the details here. \square

It is clear that a univalent map of the disk onto a simply connected domain whose complement has finite area cannot belong to \mathcal{D} . Can it belong to any larger B^p space ($p > 2$)? Walsh [Wa00] gave the following counterexample.

PROPOSITION 21. *There exists a simply connected domain Ω whose complement has zero area, yet every univalent map of \mathbb{D} onto Ω belongs to $\cap_{p > 2} B^p$.*

His construction consisted in deleting from the plane countably many vertical half-lines, each of them from a point with certain special rational coordinates to infinity. We refer the reader to [Wa00] for the details. The following improvement was found in [DGV02].

PROPOSITION 22. *There exists a slit domain Ω , that is, a simply connected domain whose complement is a single Jordan arc, such that every univalent map of \mathbb{D} onto Ω belongs to $\cap_{p > 2} B^p$.*

Proof. The domain is constructed by deleting from the plane a spiral-like curve that wanders off from the origin to the point at infinity. More precisely, let

$$r_0 = 0, r_n = \sum_{k=1}^n \frac{1}{k}, n \geq 1,$$

and consider the following sequence of Jordan arcs in the form of letter “C”:

$$C_n = \left\{ z : |z| = r_n, |\arg z| > \frac{r_{n+1} - r_n}{2} = \frac{1}{2(n+1)} \right\}$$

and the following sequences of “upper” line segments:

$$U_n = [r_{2n-1}e^{i(r_{2n}-r_{2n-1})/2}, r_{2n}e^{i(r_{2n+1}-r_{2n})/2}]$$

and “lower” segments

$$L_n = [r_{2n}e^{-i(r_{2n+1}-r_{2n})/2}, r_{2n+1}e^{-i(r_{2n+2}-r_{2n+1})/2}].$$

Next, join the upper edges of C_1 and C_2 by U_1 , as well as those of C_3 and C_4 by U_2 , etc. Connect also the lower edges of C_2 and C_3 by L_1 . Continuing this process yields as a result the simple arc

$$\Gamma = (\cup_{n=1}^{\infty} C_n) \cup (\cup_{n=1}^{\infty} U_n) \cup (\cup_{n=1}^{\infty} L_n)$$

that connects the origin with the point at infinity since $r_n \rightarrow \infty$ as $n \rightarrow \infty$. Hence the domain $\Omega = \mathbb{C} \setminus \Gamma$ is simply connected. A procedure similar to the one applied before allows us to estimate the integral $\int_{\Omega} d_{\Omega}^{p-2} dA$ from above by a convergent series whenever $2 < p < \infty$. Again, we refer the reader to [DGV02] for specific details of such estimates. \square

4.7. Univalent B^p domains. One can easily give a natural definition of a B^p domain. By analogy with an unpublished argument due to Hedemalm for Q_p domains, the following was shown in [DGV02].

PROPOSITION 23. *When $1 < p < \infty$, B^p domains do not exist.*

Proof. Suppose there exist such a domain Ω . Then Ω contains some open disk. Since B^p is invariant under translations and dilations, without loss of generality we may assume that $\mathbb{D} \subset \Omega$. Then every analytic function that maps the disk into itself must belong to B^p . In particular, every infinite Blaschke product does. This contradicts a theorem of H.O. Kim [Ki84] which states that the only Blaschke products in B^p are the finite ones. Hence there does not exist a B^p domain. \square

Even though the B^p domains do not exist, there is a reasonable middle ground between this phenomenon and Proposition 17. It is convenient and natural to define the following notion. We will say that Ω is a *univalent B^p domain* if every univalent function f in \mathbb{D} such that $f(\mathbb{D}) \subset \Omega$

must have the property $f \in B^p$. Note that we are not requiring that Ω be simply connected, nor that f be onto (as in the criterion of Walsh).

The following lemma might be interesting in itself.

LEMMA 24. *Whenever $p \geq 2$, every domain Ω contains a simply connected domain Ω' with the property that for any finite $\alpha \geq 0$*

$$\int_{\Omega} d_{\Omega}^{\alpha}(w) dA(w) \asymp \int_{\Omega'} d_{\Omega'}^{\alpha}(w) dA(w),$$

with the constants of comparison depending only on α (but not on the geometry of Ω !).

Proof. Actually, the domain constructed in the proof of Theorem 2 has the desired property. This can be seen as follows. Since Ω and Ω' differ only by a set of Lebesgue measure zero, and $\Omega' \subset \Omega$ we get trivially that

$$\int_{\Omega'} d_{\Omega'}^{\alpha} dA(w) \leq \int_{\Omega} d_{\Omega}^{\alpha}(w) dA(w).$$

For the reverse inequality, we can use the fact that for any square Q ,

$$\int_Q d_Q^{\alpha}(w) dA(w) \geq C_{\alpha}(\text{diam } Q)^{\alpha+2},$$

where the constant C_{α} depends only on α and not on the decomposition and can be computed explicitly.

Recalling the properties of the Whitney decomposition, we get

$$\begin{aligned} \int_{\Omega'} d_{\Omega'}^{\alpha}(w) dA(w) &= \sum_{j=1}^{\infty} \int_{Q_j} d_{\Omega'}^{\alpha}(w) dA(w) \\ &\geq \sum_{j=1}^{\infty} \int_{Q_j} d_{Q_j}^{\alpha}(w) dA(w) \\ &\geq C_{\alpha} \sum_{j=1}^{\infty} (\text{diam } Q_j)^{\alpha+2} \\ &= 2C_{\alpha} \sum_{j=1}^{\infty} \int_{Q_j} (\text{diam } Q_j)^{\alpha} dA(w) \\ &\geq \tilde{C}_{\alpha} \sum_{j=1}^{\infty} \int_{Q_j} d_{\Omega}^{\alpha}(w) dA(w) \\ &= \tilde{C}_{\alpha} \int_{\Omega} d_{\Omega}^{\alpha}(w) dA(w). \end{aligned}$$

This completes the proof. □

Here is one of the principal results of [DGV02]. Note that this statement “interpolates” between the trivial case $p = 2$ (the univalent \mathcal{D} domains clearly being the ones of finite area) and the expected result for the Bloch space: the supremum of the radii of all disks contained in the domain must be finite.

THEOREM 25. *Univalent B^p domains exist if and only if $2 \leq p \leq \infty$. If this is the case, Ω is a univalent B^p domain if and only if $\int_{\Omega} d_{\Omega}^{p-2} dA < \infty$.*

Proof. To show that there are no univalent B^p domains when $1 < p < 2$, suppose that Ω is such a domain. Then Ω contains some disk D_0 . Now, by Walsh’s example (Proposition 19) there is a bounded function $f \in \mathcal{U}$ such that $f \notin B^p$. We can find complex constants α and β so that if $g = \alpha + \beta f$ then $g(\mathbb{D}) \subset D_0 \subset \Omega$. Then $g \in \mathcal{U}$ and $g(\mathbb{D}) \subset \Omega$ but $g \notin B^p$, a contradiction.

Now for the main part: the description of univalent B^p domains when $2 \leq p < \infty$. The proof is similar but easier for the Bloch space (case $p = \infty$) and will be omitted here. We remark that it does not require Whitney squares - one can comfortably work with disks.

The easy implication goes as follows. Let $\int_{\Omega} d_{\Omega}^{p-2} dA < \infty$ and let f be an arbitrary univalent map such that $f(\mathbb{D}) = D \subset \Omega$. It is easy to see that $d_D(w) \leq d_{\Omega}(w)$ for all w in D . Taking into account that $p \geq 2$, we have

$$\int_D d_D^{p-2} dA \leq \int_D d_{\Omega}^{p-2} dA \leq \int_{\Omega} d_{\Omega}^{p-2} dA < \infty,$$

so by Proposition 17 we know that $f \in B^p$. This shows that Ω is a univalent B^p domain.

We finally prove the difficult part. The key point is, of course, Lemma 24. We may reason as follows. Suppose Ω is a univalent B^p domain. We have to show that $\int_{\Omega} d_{\Omega}^{p-2} dA < \infty$. Assume the contrary: $\int_{\Omega} d_{\Omega}^{p-2} dA = \infty$. By Lemma 24, Ω contains a simply connected domain Δ such that we also have $\int_{\Delta} d_{\Delta}^{p-2} dA = \infty$. Then by Proposition 17 there exists a univalent map f of \mathbb{D} onto Δ such that $f \notin B^p$. Since $\Delta \subset \Omega$, this would mean that Ω is not a univalent B^p domain. This contradiction completes the proof. \square

It is clear from the definition that a subdomain of a univalent B^p domain is also a univalent B^p domain. An example of such a domain that is not even finitely connected is obtained by deleting the centers of all squares Q_n in the domain from Proposition 18.

5. SUPERPOSITION OPERATORS

The purpose of this section is, for different given pairs of spaces of analytic functions, to fully describe all superposition operators from one function space into another in terms of the behavior of their symbol (always an entire function). This provides a comparative measure of growth of the functions in two different spaces.

As a rule, the proofs of all results here will be based on elementary properties of entire functions and on two basic elements studied in detail in earlier sections: on growth estimates for functions in different spaces and on geometric construction of domains related to univalent functions in our spaces.

5.1. Superposition operators. We begin by describing the basic problems.

Trivially, if $f \in A^p$ and $n \leq [p/q]$, then $f^n \in A^q$. It follows immediately that $P \circ f \in A^q$, for any polynomial P of degree $\leq [p/q]$. Are such polynomials the only entire functions for which the statement is true? The answer is yes, as found by Cámara and Giménez [CG94] in 1994. It is not surprising that similar phenomena were also observed in Hardy spaces H^p (see [Ca95]).

In general, given two metric function spaces X and Y , where $X, Y \subset \mathcal{H}(\mathbb{D})$, we will say that φ acts by *superposition* from X into Y if $\varphi \circ f \in Y$ whenever $f \in X$. If this is the case, we say that φ defines the *superposition operator*

$$S_\varphi : X \rightarrow Y, \quad S_\varphi(f) = \varphi \circ f.$$

It follows easily that φ must be entire if X contains the linear functions. This is so because the identity function belongs to X in this case, hence $\varphi \in Y$ and, in particular, $\varphi \in \mathcal{H}(\mathbb{D})$. By applying φ to a linear function, it is then easily seen that φ is analytic in any open disk and hence entire. Also (when X and Y are linear spaces), S_φ is a linear operator if and only if $\varphi(z) = cz$, where $c = \text{const}$.

Since S_φ is typically a non-linear operator, we cannot take for granted many of the usual properties the linear operators have, such as closed graph and action \Rightarrow boundedness, boundedness \Leftrightarrow continuity, and so forth. Actually, we should even make sure to formulate the definition of these concepts in this case. Thus, we will say that S_φ is a *bounded operator* if it maps bounded sets into bounded sets (in analogy with the linear case). For example, it is rather easy to see that the function $\varphi(z) = z^n$ and, analogously, any polynomial of degree $n \leq [p/q]$ induces a bounded superposition operator from A^p into A^q (just inspect carefully the obvious estimates used in proving that the operator acts from one space into the other).

QUESTION. Let X and Y be two metric spaces of functions in $\mathcal{H}(\mathbb{D})$ that contain the linear functions.

- (1) For which (entire functions) φ do we have $S_\varphi(X) \subset Y$?
- (2) If $S_\varphi(X) \subset Y$, when is S_φ a bounded operator?

Knowing the answers to the above questions seems like a very natural way of comparing “how much faster” the functions in one space grow than those in the other. Superpositions between various spaces of real functions have been studied extensively. Appel’s and Zabrejko’s monograph [AZ90] reviews the developments up to about 1990 and there have been literally hundreds of papers on the topic published since then. The question on when one function acts between two spaces also comes up often in certain topics of harmonic analysis, functional analysis, or function algebras. However, such a study between classical spaces of analytic functions and in the terms formulated above has begun, surprisingly enough, only quite recently.

In this section we present a small sample of results from [CG94], [BFV01], [AMV04], and [BV08], hoping that they will give the reader the flavor of this recent line of research.

5.2. A brief review of entire functions. As is usual, by an *entire function* we mean one that is analytic in the whole complex plane \mathbb{C} . Two basic types of entire functions will be of interest to us here: the polynomials and the functions of finite positive order.

We recall the *Cauchy estimates*, a standard generalization of Liouville’s theorem, easily deduced from the Cauchy integral formula.

LEMMA 26. Given a non-negative number a and an entire function φ , there exists a positive number R such that

$$|\varphi(w)| \leq M|w|^\alpha, \quad \text{whenever } |w| > R,$$

if and only if φ is a polynomial of degree at most $[\alpha]$, the greatest integer part of α .

Given a non-constant entire function φ , its *order* ρ is defined as

$$(18) \quad \rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r},$$

where $M(r) = \max\{|\varphi(z)| : |z| = r\}$. We will always work with one φ at a time and will therefore suppress any reference to φ in $M(r)$ and ρ in order not to burden the notation. The *type* σ of an entire function φ of order ρ ($0 < \rho < \infty$) is given by

$$(19) \quad \sigma = \limsup_{r \rightarrow \infty} \frac{\log M(r)}{r^\rho}.$$

The possibilities $\sigma = 0$ and $\sigma = \infty$ are not excluded.

The simplest example of a function of integer order ρ and finite positive type σ is $e^{\sigma z^\rho}$. More involved examples (including the case of fractional orders) can be constructed in the form of power series or as integrals of certain complex functions.

Some very basic material on entire functions can be found in Chapter VIII of [Ti88]. The classical monographs [Bo54] and [Le96] contain plenty of information.

5.3. Superpositions acting between two Bergman spaces. The first result we review here concerns the superposition operators between two Bergman spaces. As already mentioned, it was proved in [CG94]. The statement is intuitively quite reasonable to expect but its proof still requires a certain amount of work. The method employed in [CG94] is the most natural one: it consists in choosing the right “test functions” in A^p and making a clever use of the standard Cauchy estimates for entire functions. It should be worth observing that several details in the original proof can be simplified. The result is a somewhat shorter proof we present below that involves choosing only one test function instead of several such functions.

THEOREM 27. *Let $0 < p, q < \infty$ and let φ be entire. Then S_φ maps A^p into A^q if and only if φ is a polynomial whose degree is at most $[p/q]$. If this is the case, then S_φ is actually a bounded operator.*

Proof. We have already commented on the sufficiency of the condition on degree $\leq [p/q]$ at the beginning of the section, so we need only verify the necessity. To this end, suppose φ is not a polynomial of degree $\leq p/q$. We can find $\varepsilon > 0$ such that $[p/q + \varepsilon] < [p/q] + 1$. According to the Cauchy estimates (Lemma 26), we can choose inductively an infinite sequence of points (w_n) in the plane with increasing moduli and the property that

$$(20) \quad |\varphi(w_n)| > n|w_n|^{p/q+\varepsilon}, \text{ for all } n.$$

At least one of the 8 octants

$$\{z : \pi n/4 \leq \arg z < \pi(n+1)/4\}, \quad n = 0, 1, \dots, 7,$$

contains infinitely many points w_n . Without loss of generality, we may assume it is the first octant. The reason for this is the following: φ is a polynomial of degree N if and only if the function given by $\psi(w) = \varphi(\lambda w)$ is also such, whenever $|\lambda| = 1$, so that rotations are allowed.

By the Bolzano-Weierstrass theorem, the sequence $(\arg w_n)$ will have a convergent subsequence. We may choose a further subsequence, denoted again (w_n) , so that the arguments $\arg w_n$ decrease to zero. Again, this is a consequence of the fact that φ is a polynomial of degree N if and only if the function given by $\psi(w) = \overline{\varphi(\overline{w})}$ is also such, so we are allowed to use

reflections across the real axis. Another rotation can be used after this to take the sequence back to the first octant.

Now choose

$$f(z) = \left(\frac{1+z}{1-z} \right)^\alpha, \quad \alpha \left(\frac{p}{q} + \varepsilon \right) = \frac{2}{q}.$$

Since $0 < \alpha < 2/p$, it follows from Lemma 4 that $f \in A^p$.

Let us first consider the case when $p \geq 1$. Then $f \in \mathcal{U}$ and maps the unit disk onto an angle with vertex at the origin and of aperture $< 2\pi$. Note also that it maps any Stolz angle in \mathbb{D} with vertex at $z = 1$ symmetric with respect to the real axis onto a symmetric angle of opening smaller than π centered at the origin. In particular, the pre-image of the first octant is contained in a Stolz angle.

Let z_n be points in the disk such that $f(z_n) = w_n$. The choice of z_n is unique because f is univalent. Moreover, by the mapping properties of f , all z_n will belong to a Stolz angle. Then $1 - |z_n| > c|1 - z_n|$ for some $c \in (0, 1)$. By deleting again finitely many terms, we may also assume that $\operatorname{Re} z_n \geq 0$ for all n (hence $|1 + z_n| \geq 1$). By our choice of ε we have

$$\begin{aligned} \frac{\|\varphi \circ f\|_{A^q}}{(1 - |z_n|)^{2/q}} &\geq |\varphi(f(z_n))| = |\varphi(w_n)| > n|w_n|^{p/q+\varepsilon} \\ &= n \left(\frac{|1+z_n|}{|1-z_n|} \right)^{\alpha(p/q+\varepsilon)} \geq n \left(\frac{c}{1-|z_n|} \right)^{\alpha(p/q+\varepsilon)} \\ &\geq \frac{Mn}{(1-|z_n|)^{2/q}}. \end{aligned}$$

In view of Lemma 5 this contradicts the assumption that $\varphi \circ f \in A^q$.

The case when $0 < p < 1$ is easily taken care of by multiplying both p and q by a sufficiently large positive integer N so as to have $Np \geq 1$ and $Nq \geq 1$, thus reducing the problem to the case already considered. \square

Even when the operator S_φ is not linear, one can understand the concepts of “continuous” and “locally Lipschitz” in the usual terms. The following statement was also proved in [CG94].

THEOREM 28. *When φ acts by superposition from A^p into A^q as in the conditions of Theorem 27, the operator S_φ is continuous and also Lipschitz at every point of A^p .*

We do not give a proof here but instead refer the reader to the original paper [CG94]. Interesting results regarding superpositions from Bergman space into the Nevanlinna area class can also be found there. We also remind the reader that further details regarding Hardy and related spaces can be found in the survey [Ca95].

5.4. Superpositions from the Bloch space to a Bergman space. Here we have a different situation, in the sense that any Bergman space is “much bigger” than the Bloch space; in other words, Bergman functions grow much faster than Bloch functions.

Since Bloch functions grow at most as $\log \frac{1}{1-|z|}$ and A^p functions grow at most as $(1-|z|)^{2/p}$, we may ask whether any function like $\varphi(z) = e^{cz}$, $c \neq 0$, will still have the property

$$f \in \mathcal{B} \Rightarrow \varphi \circ f \in A^p.$$

One immediately notices that, even though $f_c(z) = c \log 1/(1-z)$ is a Bloch function, the function

$$e^{f_c(z)} = \frac{1}{(1-z)^c}$$

will not belong to the Bergman space A^p if we choose $c \geq 2/p$, according to Lemma 4. This should lead to the understanding that entire functions of order one and finite positive type will not serve for mapping \mathcal{B} into A^p by superposition. However, this is precisely where the cut occurs. Before we proceed to our next result, we will need the following lemma.

The auxiliary construction of a conformal map onto a specific Bloch domain with the maximal (logarithmic) growth along a certain broken line displayed below might be of some independent interest. Thus, we state it separately as a lemma. Loosely speaking, such a domain can be imagined as a “highway from the origin to infinity” of width 2δ .

LEMMA 29. *For each positive number δ and for every sequence $\{w_n\}_{n=0}^\infty$ of complex numbers such that $w_0 = 0$, $|w_1| \geq 5\delta$, $\arg w_1 < \pi/2$, $\arg w_n \searrow 0$, and*

$$(21) \quad |w_n| \geq \max \left\{ 3|w_{n-1}|, \sum_{k=1}^{n-1} |w_k - w_{k-1}| \right\} \quad \text{for all } n \geq 2,$$

there exists a domain Ω with the following properties:

- (i) Ω is simply connected;
- (ii) Ω contains the infinite polygonal line $L = \cup_{n=1}^\infty [w_{n-1}, w_n]$, where $[w_{n-1}, w_n]$ denotes the line segment from w_{n-1} to w_n ;
- (iii) any Riemann map f of \mathbb{D} onto Ω belongs to \mathcal{B} ;
- (iv) $\text{dist}(w, \partial\Omega) = \delta$ for each point w on the broken line L .

Proof. It is clear from (21) that $|w_n| \nearrow \infty$ as $n \rightarrow \infty$. We construct the domain Ω as follows. First connect the points w_n by a polygonal line L as indicated in the statement. Let $D(z, \delta) = \{w : |z - w| < \delta\}$ and define

$$\Omega = \bigcup \{D(z, \delta) : z \in L\},$$

i.e. let Ω be a δ -thickening of the polygonal line L . In other words, Ω is the union of simply connected cigar-shaped domains

$$C_n = \bigcup \{D(z, \delta) : z \in [w_{n-1}, w_n]\}.$$

By our choice of w_n , it is easy to check inductively that $|w_n - w_k| \geq 5\delta$ whenever $n > k$. Since our construction implies that

$$C_n \subset \{w : |w_{n-1}| - \delta < |w| < |w_n| + \delta\},$$

we see immediately that

- (a) for all m, n , $C_m \cap C_n \neq \emptyset$ if and only if $|m - n| \leq 1$;
- (b) for all n , $C_n \cap C_{n+1}$ is either $D(w_n, \delta)$ or the interior of the convex hull of $D(w_n, \delta) \cup \{a_n\}$ for some point a_n outside of $\overline{D(w_n, \delta)}$.

Thus, each $\Omega_N = \bigcup_{n=1}^N C_n$ is also simply connected. Since

$$\Omega = \bigcup_{N=1}^{\infty} \Omega_N \quad \text{and} \quad \Omega_N \subset \Omega_{N+1} \quad \text{for all } N,$$

we conclude that Ω is also simply connected (like in [DGV02], Section 4.2, p. 56). By construction, $\text{dist}(w, \partial\Omega) \leq \delta$ for all w in Ω , hence any Riemann map onto Ω will belong to \mathcal{B} . It is also clear that (iv) holds. \square

Both the above lemma and our next result are taken from [AMV04].

THEOREM 30. *Let $0 < p < \infty$ and let φ be entire. Then the following statements are equivalent:*

- (a) $S_\varphi : \mathcal{B} \rightarrow A^p$;
- (b) S_φ maps \mathcal{B} boundedly into A^p ;
- (c) φ either has order less than one, or order one and type zero.

Proof. In order to prove that (c) implies (b), let us suppose that φ is an entire function of order one and type zero. By (18), this means that

$$\limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} = 1$$

and the quantity

$$(22) \quad E(r) = \frac{\log M(r)}{r}$$

tends to zero as $r \rightarrow \infty$.

Let f be an arbitrary function in \mathcal{B} of norm at most K . By Lemma 14, we have

$$(23) \quad |f(z)| \leq \left(\log \frac{1}{1 - |z|} + 1 \right) K.$$

By our assumption on $E(r)$, for some sufficiently large R_0 it follows that

$$(24) \quad E(|w|) < 1/(2pK), \quad \text{whenever } |w| > R_0.$$

Thus, whenever $|f(z)| > R_0$ we get

$$\begin{aligned} |\varphi(f(z))| &\leq M(|f(z)|) \\ &= e^{|f(z)| \cdot E(|f(z)|)} \\ &\leq \exp\left(K \cdot \left(\log \frac{1}{1-|z|} + 1\right) \cdot E(|f(z)|)\right) \\ &\leq \frac{e^{1/(2p)}}{(1-|z|)^{1/(2p)}} \end{aligned}$$

by the definition of $M(r)$ and by (22), (23), and (24) respectively.

If, on the contrary, $|f(z)| \leq R_0$ then $|\varphi(f(z))| \leq M(R_0)$ by the Maximum Modulus Principle. Combining the two possible cases, we obtain

$$\|\varphi \circ f\|_p^p \leq e^{1/2} \int_{\mathbb{D}} \frac{dA(z)}{(1-|z|)^{1/2}} + M(R_0)^p = C,$$

where C depends only on φ , p , and K but not upon f . This shows that S_φ is a bounded operator from \mathcal{B} into A^p .

The reasoning is similar, but simpler, when φ has order $\rho < 1$: use the estimate $M(r) \leq \exp(r^{\rho+\varepsilon})$ for a small enough ε and large enough r .

It is plain that (b) implies (a).

Thus, it is only left to prove that (a) implies (c). This is the crucial part and it suffices to consider only the harder case so assume that $S_\varphi : \mathcal{B} \rightarrow A^p$ but φ has order one and type different from zero, there exists an ε and a sequence $\{r_n\}_{n=1}^\infty$ such that $r_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$(25) \quad \frac{\log M(r_n)}{r_n} \geq \varepsilon > 0, \quad \text{for all } n.$$

In other words, there exists a sequence $\{w_n\}_{n=1}^\infty$ such that $|w_n| = r_n$ and

$$(26) \quad |\varphi(w_n)| = M(r_n) \geq e^{\varepsilon|w_n|}, \quad \text{for all } n.$$

Fix a constant $\delta > 12/(\varepsilon p)$. We can now choose an infinite subsequence, denoted again $\{w_n\}$, so that the sequence $\{\arg w_n\}$ in $[0, 2\pi]$ is convergent and all points w_n lie in an angular sector of opening $\pi/2$. We may further assume that they are all located in the first quadrant and the arguments $\arg w_n$ decrease to 0, by applying symmetries or rotations if necessary. There is no loss of generality in doing this because the entire functions ψ and φ_t defined by $\psi(z) = \overline{\varphi(\bar{z})}$ and $\varphi_t(z) = \varphi(e^{it}z)$ respectively have the same order and type as φ .

Select inductively a further subsequence, labeled again $\{w_n\}$, in such a way that $w_0 = 0$, $|w_1| \geq 5\delta$, and (21) holds. Next, construct a domain Ω with the properties (i)–(iv) indicated in Lemma 29. Let f be a Riemann map of \mathbb{D} onto Ω that fixes the origin.

Now let z_n be the points in \mathbb{D} for which $w_n = f(z_n)$. Since $|w_n| \rightarrow \infty$ as $n \rightarrow \infty$, it follows that $|z_n| \rightarrow 1$. By applying estimate (16) for hyperbolic metric, the triangle inequality, inequality (17) and property (iv) from Lemma 29, as well as the properties (21) of the points w_n respectively, we obtain the following chain of inequalities:

$$\begin{aligned}
\frac{1}{2} \log \frac{1}{1 - |z_n|} &\leq \rho_{\Omega}(0, w_n) \\
&\leq \sum_{k=1}^n \rho_{\Omega}(w_{k-1}, w_k) \\
&\leq \sum_{k=1}^n \int_{[w_{k-1}, w_k]} \frac{|dw|}{d_{\Omega}(w)} \\
&= \sum_{k=1}^n \int_{[w_{k-1}, w_k]} \frac{|dw|}{\delta} \\
&= \frac{1}{\delta} \sum_{k=1}^n |w_k - w_{k-1}| \\
&\leq \frac{3}{\delta} |w_n|.
\end{aligned}$$

This shows that

$$(27) \quad |w_n| \geq \frac{\delta}{6} \log \frac{1}{1 - |z_n|}.$$

It follows from (26) and (27) that

$$(28) \quad |\varphi(w_n)| \geq \exp\left(\frac{\varepsilon\delta}{6} \cdot \log \frac{1}{1 - |z_n|}\right) = \frac{1}{(1 - |z_n|)^{(\varepsilon\delta)/6}}.$$

On the other hand, $f \in \mathcal{B}$, hence by assumption (a): $\varphi \circ f \in A^p$. By Lemma 5, we have

$$(29) \quad |\varphi(w_n)| = |(\varphi \circ f)(z_n)| \leq \frac{\|\varphi \circ f\|_p}{(1 - |z_n|)^{2/p}},$$

for all n . However, (28) and (29) contradict each other since $\varepsilon\delta/6 > 2/p$ by our initial choice of δ . This completes the proof. \square

5.5. Superpositions between two Besov spaces. Our next result is technically more complicated as it requires controlling the derivative of the function in the initial space instead of the function itself. Also, the test function should be chosen as a univalent map in B^p onto a certain domain whose exact shape is known only roughly but not completely. Thus, one needs maps similar to the ones constructed in Subsection 4.6 only slightly more general. The following class of examples was given in [BFV01].

PROPOSITION 31. Let $1 < p < \infty$, (w_n) be a sequence of complex numbers, and let (r_n) and (h_n) be sequences of positive numbers with the following properties:

- (a) $0 \leq \arg w_n < \pi/4$ and $|w_n| \leq |w_{n+1}|/2$, $n \in \mathbb{N}$;
- (b) $r_n < |w_n|/4$ and $|h_n| < \min\{r_n, r_{n+1}\}/3$, $n \in \mathbb{N}$.

Let $D_n = D(w_n, r_n) = \{z : |z - w_n| < r_n\}$ and let R_n be the rectangle whose longer symmetry axis is the segment $[w_n, w_{n+1}]$ and whose shorter side has length $2h_n$. Then the domain $\Omega = \cup_{n=1}^{\infty} (D_n \cup R_n)$ is simply connected and, if f is a Riemann map of \mathbb{D} onto Ω , then $f \in B^p$ if and only if

$$\sum_{n=1}^{\infty} r_n^p + \sum_{n=1}^{\infty} |w_{n+1} - w_n| h_n^{p-1} < \infty.$$

The proof resembles that of Proposition 18.

Armed with this new tool, we are now ready to give a characterization of all entire maps that transform one Besov space into another (or into the Bloch space) via superposition. Intuitively, it is clear that B^p is smaller than B^q when $p < q$, but “not much smaller”. How should this be expressed in terms of the superposition operators acting from one space into another? The answer is fairly simple to state. This result is also from [BFV01].

THEOREM 32. Let $1 < p, q \leq \infty$, where $B^\infty = \mathcal{B}$. Then we have the following conclusions.

- (a) If $p \leq q$, then $S_\varphi : B^p \rightarrow B^q$ if and only if φ is a linear function.
- (b) If $p > q$, then $S_\varphi : B^p \rightarrow B^q$ if and only if φ is a constant function.

Proof. The reasoning we are about to use applies equally to (a) and (b) throughout, except at the end of the proof where we will have to distinguish between the two cases.

We first show that φ must be linear in either case. Assume the contrary: φ' is not identically constant, and let $r_n = 2^{-n}$. In view of Liouville’s theorem, φ' is unbounded, so we can select inductively a sequence (w_n) of complex numbers so that $|w_1| > 2$ and

$$|w_{n+1}| \geq 2|w_n|, \quad |\varphi'(w_n)| \geq r_n^{-2}$$

for all n . As in the proof of Theorem 27, at least one of the eight basic octants contains infinitely many points w_n . By a rotation if necessary, we may therefore assume that $0 \leq \arg w_n < \pi/4$, and so Proposition 31 is applicable. Define

$$h_n = 2^{-n-2} |w_{n+1} - w_n|^{-1/(p-1)}.$$

Let Ω be the domain defined in Proposition 31 using the sequences (w_n) , (r_n) , and (h_n) as data and let $f : \mathbb{D} \rightarrow \Omega$ be a univalent map of \mathbb{D} onto Ω .

By Proposition 31, we know that $F \in B^p$. Let $f(z_n) = w_n$. It is easily seen that $|z_n| \rightarrow 1$ as $n \rightarrow \infty$. Applying Lemma 13, we obtain

$$|\varphi'(w_n)| |f'(z_n)| (1 - |z_n|) \asymp |\varphi'(w_n)| d_\Omega(w_n) \geq C/r_n \rightarrow \infty,$$

which tells us that $\varphi \circ f \notin \mathcal{B}$, a contradiction. This tells us that $S_\varphi(B^p) \subset B^q$ implies that φ is linear, independently of the values of p and q .

(a) Since $B^p \subset B^q$ when $p \leq q$, it is clear that every linear function acts by superposition from B^p into B^q .

(b) We already know that φ has to be linear but it cannot contain the z -term (otherwise it would easily follow that $B^p \subset B^q$, which is not the case. Thus, $\varphi \equiv \text{const}$. \square

The following statement was not recorded in [BFV01] but is rather easy to prove.

PROPOSITION 33. *Let $1 < p, q \leq \infty$. If S_φ acts from B^p into B^q , it is also a bounded operator.*

Proof. In the case (b) of Theorem 32, the range $S_\varphi(B^p)$ is a singleton, hence the superposition operator is trivially bounded.

In the case (a), we have $\varphi(z) = az + b$, where $a, b \in \mathbb{C}$. The key point consists in observing that the injection map from B^p into B^q (allowing the possibility of $B^\infty = \mathcal{B}$) is a bounded linear operator. This is a consequence of the Closed Graph Theorem. Namely, by Lemma 16 the convergence in B^p implies uniform convergence on compact subsets of \mathbb{D} ; this also applies to the Bloch space by Lemma 14. Thus, assuming that $f_n \rightarrow f$ in B^p and $f_n \rightarrow g$ in B^q , $p \leq q$, we deduce that $f_n \rightrightarrows f$, as well as to g , on all $K \Subset \mathbb{D}$, whence $f \equiv g$. This shows that the injection map from B^p into B^q (possibly \mathcal{B}) has closed graph and is therefore a bounded operator: $\|f\|_{B^q} \leq C \|f\|_{B^p}$. Now if $\|f\|_{B^p} \leq M$ then

$$\|S_\varphi(f)\|_{B^q} \leq |a| \|f\|_{B^q} + |b| \leq C M |a| + |b|,$$

showing that S_φ maps bounded sets into bounded sets and is, thus, a bounded operator from B^p into B^q . \square

As was also the case with [DGV02], due to limitations in space we were only able to give here a glimpse of results obtained in [BFV01]. There are further theorems there regarding superpositions between other spaces of Dirichlet type. In certain cases, such results require slightly more involved examples of conformal maps as well as certain inequalities of Trudinger-Moser type due to Chang and Marshall. However, these results will not be presented here.

5.6. Some other superpositions. The three examples given here (Theorem 27, Theorem 32, and Theorem 30) should not mislead the reader to expect that whenever S_φ acts from one space to another it should also be

bounded. Since S_φ is a nonlinear operator if $\varphi(z) \not\equiv cz$, we cannot always expect the continuity and boundedness to be equivalent.

There are actually examples that show that this is clearly not the case. Namely, the above proof was modified in a number of places in [BV08] to prove similar but actually somewhat different results for superpositions operators from the smaller spaces B^p into Bergman spaces. If we denote by $E(t)$ the class of entire functions of order less than a positive number t , or of order t and finite type, the result can be formulated as follows.

THEOREM 34. *Suppose $1 < p < \infty$ and $0 < q < \infty$. Then $S_\varphi(B^p) \subset A^q$ if and only if $\varphi \in E(p/(p-1))$, and $S_\varphi(\mathcal{B}_0) \subset A^q$ if and only if $\varphi \in E(1)$. All superposition operators from B^p or \mathcal{B}_0 to A^q are continuous (as maps between metric spaces).*

Indeed, there exist continuous unbounded superposition operators from B^p to A^q , as can be deduced from the following characterization of boundedness and compactness. Below, we denote by $E_0(t)$ the class of entire functions of order less than t , or of order t and type zero.

THEOREM 35. *Suppose $1 < p < \infty$ and $0 < q < \infty$. Then S_φ is bounded from B^p to A^q if and only if $\varphi \in E_0(p/(p-1))$, and S_φ is bounded from \mathcal{B}_0 to A^q if and only if $\varphi \in E_0(1)$. All such bounded operators are Montel compact (meaning that they map bounded sets into relatively compact sets).*

The proofs are too technical to be presented here as they require certain generalized Trudinger-type inequalities; we refer the reader to the original paper [BV08].

A number of varied results on superpositions between different spaces of Dirichlet type had been obtained earlier in [BFV01]. In some cases the proofs require the Chang-Marshall inequality discussed earlier and the constructions of domains similar to the ones given above. To get a taste of what was proved there, we enunciate the following theorem proved there and prove only one of the two implications. Here D^p will denote the general Dirichlet-type space of analytic functions in \mathbb{D} for which $f' \in A^p$, equipped with the obvious norm, $1 \leq p < \infty$.

THEOREM 36. *If $q < 2$ and φ is entire, then $S_\varphi(\mathcal{D}) \subset D^q$ if and only if $\varphi \in E(2)$.*

Proof. We only prove sufficiency. Let $\varphi \in E(2)$. By the standard fact about entire functions and their derivatives [Ti88, p. 265], we also have $\varphi' \in E(2)$, and so there exists $\beta > 0$ such that $|\varphi'(w)| \leq e^{\beta|w|^2}$ for all sufficiently large $|w|$.

Recall that by Garnett's observation (10) related to the Chang-Marshall theorem, the integral

$$\int_0^{2\pi} \exp(\alpha |f(e^{i\theta})|^2) d\theta$$

is finite for any $\alpha > 0$. Applying first Hölder's inequality and this fact, we obtain

$$\begin{aligned} \int_{\mathbb{D}} |f'|^q |\varphi' \circ f|^q dA &\leq \left(\int_{\mathbb{D}} |f'|^2 dA \right)^{q/2} \left(\int_{\mathbb{D}} |\varphi' \circ f|^{2q/(2-q)} dA \right)^{(2-q)/2} \\ &\leq \|f\|_{\mathcal{D}}^q \left(K + \int_{\mathbb{D}} \exp[2q\beta |f|^2/(2-q)] dA \right)^{(2-q)/2} < \infty, \end{aligned}$$

for all $f \in \mathcal{D}$. Note that above we have also used the following fact:

$$\int_{\mathbb{D}} e^{\beta |f|^2} dA = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \int_{\mathbb{D}} |f|^{2n} dA \leq \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \int_{\mathbb{T}} |f|^{2n} d\theta = \int_{\mathbb{T}} e^{\beta |f|^2} d\theta.$$

The necessity is more involved but the constructions needed are quite similar to the ones given by Proposition 31. \square

The subject of superposition operators has not been exhausted yet and several other papers have appeared in print over the last ten years. We mention, among others, [Xi05], [BV06], and [Xu07]. Nice results on superpositions between Q_p and Hardy spaces have been obtained in [GM10]. Recently, superposition operators between weighted spaces of Hardy type have been studied in [BoV13] and [Ra13]. Relationship of certain superpositions with universal functions, maximal ideals, and composition operators on the disk has been covered in [Mo13].

6. EXERCISES

- (1) Prove the following lemma used implicitly in the proof of Theorem 2: the union of an ascending chain of simply connected domains is again simply connected. That is, if Ω_n are simply connected for all $n \in \mathbb{N}$ and $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \dots$ then $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ is simply connected. (Hint: Use the Cauchy integral formula.)
- (2) In the estimate used in Lemma 24, find a correct value of the constant C_α . (Hint: Consider the square $\frac{1}{2}Q$ of the same center as Q but half the diameter and estimate $d_\Omega(w)$ for the points $w \in \frac{1}{2}Q$.)
- (3) Order the rational numbers in $[0, 1)$ in a sequence $(r_k)_{k=1}^{\infty}$ and consider the sequence of points

$$z_{n,k} = \left(1 - \frac{1}{2^{n+k}}\right) e^{2\pi i r_k}, \quad n, k \in \mathbb{N}.$$

Check that this sequence satisfies the Blaschke condition:

$$\sum_n \sum_k (1 - |z_{n,k}|) < \infty.$$

and accumulates at all points of the unit circle. Why doesn't this contradict the fact that the corresponding Blaschke product must have radial limits of modulus one almost everywhere?

- (4) Prove formula (3), justifying the convergence of the series where necessary.
- (5) Complete the proof of Lemma 4. (Hint: For the A^p lemma, integrate in polar coordinates centered at $z = 1$, so that $z = 1 + re^{i\theta}$. What are the limits of integration? For the H^p lemma, work with the boundary values of f_c and use the fact that $\cos \theta \asymp 1 - \theta^2/2$ for small θ .)
- (6) Supply the details of the proof of Lemma 5.
- (7) Let $f \in H^p$ and let $|E_\lambda|$ be the distribution function of the boundary values of f as defined in Subsection 3.2. Show that

$$|E_\lambda| \leq \frac{\|f\|_{H^p}^p}{\lambda^p}, \quad \lambda > 0.$$

- (8) Give an example of an unbounded univalent function in the Dirichlet space \mathcal{D} . (It is OK to use existence theorems such as the Riemann mapping theorem, for example.)
- (9) Show that $H^\infty \subset \mathcal{B}$. (Hint: Use the Schwarz-Pick lemma for the analytic functions σ from the unit disk into itself: $1 - |\sigma(z)|^2 \leq (1 - |z|^2)|\sigma'(z)|$, for all $z \in \mathbb{D}$.)
- (10) (a) Let φ be an entire function. Show that the $\psi(z) = \overline{\varphi(\bar{z})}$ is also an entire function and has the same order and type as φ .

- (b) Compute the order and type of the entire function $\varphi(z) = e^{\pi z^3}$ and determine the corresponding ψ . Does it differ from φ ?
- (11) Verify that every polynomial P of degree $n \leq [p/q]$ induces a bounded superposition operator from A^p into A^q . In other words, show that for every R with $0 < R < \infty$, there exists a constant $C = C(R)$ such that $\|f\|_{A^p} \leq R$ implies $\|P \circ f\|_{A^q} \leq C$.
- (12) (a) Show that $\mathcal{B} \subset A^q$ for any $q > 0$.
(b) Show that S_φ maps A^p into \mathcal{B} if and only if $\varphi \equiv \text{const.}$ (Hint: Use Theorem 27.)

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