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Metric concepts. Approximation

In these notes, we often review several basic metric or topological properties which should be well known from earlier courses. However, in view of the many differences between advanced undergraduate courses at different institutions, here we collect in one place some of the most important definitions, examples and facts which will be useful in the rest of the course. This is a rather condensed version of a large number of concepts and facts and only constitutes a very brief review but still includes a number of details, proofs, and examples.

Like in any area of Mathematics, a thorough knowledge of these concepts can only be acquired over a period of time and after reading portions of several texts, starting from simpler texts and ending with advanced ones on Analysis and Topology, in addition to doing appropriate exercises. Should a more detailed study of these topics be needed, some recommendations for reading can be found at the end of these notes, like in other lectures.

Notation. Throughout these lecture notes, $\mathbb{N} = \{1, 2, 3, \dots\}$ will denote the set of positive integers (also called natural numbers), $\mathbb{Z}$ the set of all integers, $\mathbb{Q}$ the set of all rational numbers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{C}$ the set of all complex numbers.

Elementary metric and topological concepts: a review

Metric spaces. This concept was covered in at least some undergraduate courses such as Advanced Calculus, Mathematical Analysis, Real Analysis, or Topology. In what follows, X will always be a non-empty set.

Definition. A *metric* on the set $X \neq \emptyset$ is a function $d : X \times X \to \mathbb{R}$ with the following properties:

- $d(x, y) \geq 0$ for $x, y \in X$;
- $d(x, y) = 0$ if and only if $x = y$;
- $d(y, x) = d(x, y)$, for all $x, y \in X$ (symmetry);
- $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$ (triangle inequality).

The number $d(x, y)$ is called the *distance* between x and y. The ordered pair (X, d) is called a *metric space*.

Of course, it is often possible to define different metrics on a given space X.

Example 1. *Every set X with at least two elements admits the so-called discrete metric defined as $d(x, x) = 0$ and $d(x, y) = 1$, whenever $x \neq y$.*

Example 2. *A metric can be defined in* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ *in many ways. In addition to the well-known Euclidean distance defined by* $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ *it does not take much effort to check that*

$$d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|, \quad d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

are define metrics.

In any subset $Y \subset X$ of a metric space (X, d) the distance between two points can be computed by the formula for their distance in X. Thus, the set Y, together with the restriction of d to $Y \times Y$, is a metric space itself. We refer to Y as a *metric subspace* of X. The metric obtained in this way is called the *metric induced* by d (or *inherited* from (X, d)). For the sake of simplicity, we will write (Y, d) instead of the more formal $(Y, d|_{Y \times Y})$.

Normed spaces. In this section we review briefly a very important kind of metric spaces which deserve a special study. Later on, some lectures will be devoted to the typical techniques in Banach and Hilbert spaces.

In certain vector spaces, in addition to the usual algebraic operations (sums and scalar multiples of vectors that satisfy the standard laws known from Linear Algebra), we can obtain an additional structure by defining a norm.

Definition. Let X be a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *norm* in X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties:

- $\|x\| \geq 0$ for all $x \in X$;
- $\|x\| = 0$ if and only if $x = 0$, the null vector of the space;
- $\|\lambda x\| = |\lambda| \|x\|$, for all $x \in X$ and all $\lambda \in \mathbb{K}$ (homogeneity);
- $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$ (triangle inequality).

The ordered pair (X, d) is called a *normed space.*

Note that a vector space may admit more than one norm. Any norm defined on X induces a natural distance in X, which provides a metric structure (hence automatically a topological structure). This is done simply by defining $d(x, y) = \|x - y\|$. Hence, every normed space is also a metric space. We shall refer to the above metric d as the *natural metric induced by the norm.*

Example 3. *The usual norm on* $\mathbb{R}$ *is the absolute value:* $\|x\| = |x|$, *for all* $x \in \mathbb{R}$. *On the Euclidean space*

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \ldots \times \mathbb{R} = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

we can define a different norm $\|\cdot\|_p$ *for any given number* $p \in [1, \infty)$ *by*

$$\|(x_1, x_2, \ldots, x_n)\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p},$$

generalizing the expressions from Example 2.

Note that, in order to check that $\|\cdot\|_p$ is actually a norm one needs to check the triangle inequality, which is the basic *Minkowski inequality*:

$$\left(\sum_{k=1}^{n}|x_k-y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n}|y_k|^p\right)^{1/p}.$$

Since every normed space has a natural metric, by definition, saying that a sequence of vectors $(x_n)_n$ in X *converges* to an element $x \in X$ in the norm easily translates to $d(x_n, x) = \|x_n - x\| \to 0$ as $n \to \infty$. This amounts to saying that for each $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\|x_n - x\| < \varepsilon$.

In these lectures we will review several familiar normed spaces, in addition to some new ones. We recall one function space that is very important in the questions of uniform approximation.

Example 4. *The space of all real-valued and continuous functions in* $[a, b]$*, where* $-\infty < a < b < \infty$*, will be denoted by* $\mathscr{C}[a, b]$*. A norm on* $\mathscr{C}[a, b]$ *can be defined as* $\|f\| = \max_{a \leq x \leq b}|f(x)|$*. Convergence in this space is equivalent to uniform convergence of* f_n *to* f *in* $[a, b]$*.*

Uniform convergence will be discussed in greater generality and in more detail in these notes.

Open sets. Let (X, d) be a metric space. We review some important types of sets and fix the notation.

Definition. The *open ball* of radius r centered at $x \in X$ is the set $B(x; r) = \{y \in X : d(x, y) < r\}$. The *closed ball* centered at x and of radius r is the set $\overline{B}(x; r) = \{y \in X : d(x, y) \leq r\}$. (The notation $B_r(x)$ and $\overline{B}_r(x)$ is used often as well.)

In the plane (Euclidean space $\mathbb{R}^2$ of dimension two) an open ball is an open disc and the closed ball is a closed disc (a disc with the circle added); in the three-dimensional Euclidean space $\mathbb{R}^3$ with the usual metric it is a ball in the usual sense.

Here is an exercise which you are probably already familiar with: draw the ball $B((0,0); 1)$ in $\mathbb{R}^2$, centered at the origin and of radius one, if d is defined by $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$; do the same for $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Clearly, these sets will be different from the standard Euclidean ball with respect to the metric d_2.

Definition. A set U in a metric spaces is said to be an *open set* if for every $x \in U$ there exists a radius $r = r_x > 0$ such that $B(x; r) \subset U$.

It follows immediately that every open set U is a union of open balls centered at points of U: $U = \bigcup_{x \in U} B(x; r_x)$. Reciprocally, every union of open balls clearly has the property of being an open set: given a point x in the union, it belongs to some ball (not necessarily centered at x and then it is easy to find a smaller ball centered at x and contained in the union. This shows the following:

Proposition 1. *A set in a metric space is open if and only if it is a union of some collection of open balls.*

In particular, every open ball is an open set. Here are some general properties of open sets that are easy to prove from the definition.

Proposition 2. *The open sets in a metric space have the following basic properties.*
(1) $\emptyset$ and X are open sets.
(2) the intersection of any two open sets is again open.
(3) the union of an arbitrary collection (finite, countable or uncountable) of open sets is again an open set.

We recall that any family of subsets of X that has these three properties is called a *topology* and a space equipped with a topology is called a *topological space*. Thus, every metric space is a topological space, when equipped with the topology defined via open balls as above, called the *natural topology* or *topology induced by the metric*.

In an example for $\mathbb{R}^2$ given above, it is not difficult to check that, even though the balls in the metrics d_1 and d_∞ do not coincide, the collections of open sets in these two metrics are the same; that is, a set is open in one of them if and only if it is open in the other. In other words, they induce the same topology in spite of being different metrics. This is so because a ball in one of them contains a ball in the other and is also contained in another ball in that metric. Moreover, the open sets in these two metrics coincide with the open sets in the usual Euclidean metric. We say that the usual topology is *compatible* with any of these metrics.

It is important to keep in mind that there are topological spaces in which no metric can be defined that is compatible with the topology. Such spaces are called *non-metrizable spaces*.

Definition. By an *open neighborhood* of a point $x \in X$ we mean any open set containing x.

In particular, every open ball is an open neighborhood of x.

Interior of a set. We now recall a typical operation that produces an open set starting from an arbitrary set: the operation of taking the interior of a set.

Definition. Let X be a metric space and $A \subset X$. A point x is said to an *interior point* of A if there exists a radius $r > 0$ such that $B(x; r) \subset A$. The set of all interior points of A is called the *interior* of A and is typically denoted by $\text{Int } A$ or by $\mathring{A}$.

Note that the interior of a set can sometimes be empty. The following properties can be inferred from the definition.

Proposition 3. *For every set A in a metric space, the following are true.*
(1) $\text{Int } A$ *is an open set and* $\text{Int } A \subset A$.
(2) Moreover, $\text{Int } A$ *is the largest open set contained in A and, thus, coincides with the union of all open sets contained in A..*
(3) $\text{Int}(\text{Int } A) = \text{Int } A$ *(taking the interior is an idempotent operation).*
(4) A is open if and only $A = \text{Int } A$.

Closed sets. Closure of a set. We recall another important type of sets in metric spaces. The definition below can be used in topological spaces in general.

Definition. A set F in a metric spaces X is said to be a *closed set* if its complementary set $F^c = X \setminus F$ is an open set.

The following statement, dual to Proposition 2, is easy to deduce from the above definition by taking complements and using de Morgan's laws.

Proposition 4. *The closed sets in a metric space have the following basic properties.*
(1) $\emptyset$ and X are closed sets.
(2) the union of any two closed sets is again closed.
(3) the intersection of an arbitrary collection of closed sets is again a closed set.

Here is a basic and very useful property that characterizes closed sets in metric spaces.

Proposition 5. *F is a closed subset of a metric space X if and only if for every sequence (x_n) in F such that $x_n \to x$ for some $x \in X$, we have $\in F$.*

In other words, F is closed if and only if it contains all of its limit points. This is not difficult to prove from the definition. It can be used, for example, to show that every closed ball is actually a closed set.

Definition. The closure of a set A is the smallest closed set that contains A; equivalently, it is the intersection of all closed sets that contain A. Notation: $\overline{A}$ or $\mathrm{Cl}\,A$.
(The above definition is valid both in metric and in topological spaces.) For example, the closure of the open ball $B(x;r)$ in the Euclidean space $\mathbb{R}^n$ coincides with the closed ball $\overline{B}(x;r)$. The following characterization is well known.

Proposition 6. *In a metric space X, the closure $\mathrm{Cl}\,A$ is the set of all $x \in X$ such that $x_n \to x$ for some sequence (x_n) of points in A.*

Note that the sequence mentioned above can be a sequence of distinct points or a stationary sequence, with all points being equal to x (or a mixture of both types).

The following dual statement to Proposition 3 is clear from the definition.

Proposition 7. *For every set A in a metric space, the following are true.*
(1) $\mathrm{Cl}\,A$ is a closed set.
(2) $\mathrm{Cl}\,A \supset A$.
(3) $\mathrm{Cl}\,(\mathrm{Cl}\,A) = \mathrm{Cl}\,A$ (taking the closure is an idempotent operation).
(4) A is closed if and only $A = \mathrm{Cl}\,A$.

Convergence in metric spaces. The convergence of sequences in metric spaces is defined in a way analogous to the one known from Calculus.

Definition. A sequence $(x_n)_{n=1}^{\infty}$ in X *converges* to an element $x \in X$ if

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \in \mathbb{N}, \; n \geq N \Rightarrow d(x_n, x) < \varepsilon \, .$$

The last inequality can be expressed as $x_n \in B(x;\varepsilon)$. We denote the convergence of $(x_n)_n$ to x by writing $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$, as is usual.

In the context of metric spaces, it is easily shown that if a limit of a sequence exists, it must be unique. (To see this, just imitate the familiar proof of calculus and write it in metric terms.)

The above definition can also be formulated in equivalent terms by using open neighborhoods instead of open balls. However, trying to adapt it to the context of topological spaces (using open neighborhoods since no balls are available) causes certain problems, for example, non-uniqueness of limits. Thus, certain additional separation properties are often required from the topology.

The following simple characterization of closed sets in metric spaces is well known and very useful. It is easy to prove using the definition of a closed set.

Proposition 8. *A set F in a metric space X is closed if and only if for every sequence $(x_n)_n$ in F, if $x_n \to x$, $n \to \infty$, then $x \in F$.*

Complete metric spaces. We begin with two definitions. The first one is essentially the same as in the Euclidean space and the second one takes into account that, in a general metric space, there is no "origin" as in $\mathbb{R}^n$.

Definition. A sequence $(x_n)_n$ in a metric space (X, d) is a said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all m, $n \in N$,

$$m > n \geq N \Rightarrow d(x_m, x_n) < \varepsilon \, .$$

Recall that a sequence $(x_n)_n$ in a metric space (X, d) is said to be *bounded* if there exists a point $x \in X$ and a constant $R > 0$ such that $d(x, x_n) \leq R$ for all $n \in \mathbb{N}$; in other words, if $x_n \in \overline{B}(x;R)$ for all $n \in \mathbb{N}$.

It is immediate from the definition that every Cauchy sequence is bounded: given $\varepsilon > 0$, take $n = N$ to see that $d(x_m, x_N) < \varepsilon$ for all $m \geq N$; let $M = \max\{d(x_k, x_N) : 1 \leq k < N\}$. Then for all $m \in \mathbb{N}$ we have $d(x_m, x_N) \leq R = \min\{\varepsilon, M\}$.

Also, it is easy to check that every convergent sequence is a Cauchy sequence but, in general, a Cauchy sequence need not be convergent (it is well known that this happens with various sequences in $\mathbb{Q}$). Those spaces in which this always happens deserve special attention.

Definition. A metric space is said to be *complete* if every Cauchy sequence in it is convergent. A complete normed space is called a *Banach space*.

$\mathbb{R}$ and, more generally, $\mathbb{R}^n$ with the usual metric are standard examples of complete spaces. They are also Banach spaces over the field of scalars $\mathbb{R}$ (that is, they are real Banach spaces). $\mathbb{C}^n$ is a complex Banach space; that is, a Banach space over the scalar field $\mathbb{C}$. $\mathscr{C}[a,b]$ is another Banach space.

Using arguments based on sequences, it is rather easy to prove the following observation (another good review exercise!).

Proposition 9. *Let (X,d) be a complete metric space and $Y \subset X$. Then the space (Y,d) is complete if and only if Y is closed in X.*

The subset $\mathbb{Q}$ of the real line is not a complete metric space, as is well known. However, it can be enlarged to $\mathbb{R}$ to obtain a complete metric space. Actually, every metric space can be completed (in an abstract way, meaning that we may not actually be able to recognize clearly the completion as a specific space). Here is a precise statement.

Theorem 1. *For every metric space (X,d) there exists a complete metric space (Y,D) such that a certain subspace X^* of Y is isometrically isomorphic to X; that is, there is a bijective map $f : X \to X^*$ such that $D(f(x), f(y)) = d(x,y)$ for all $x, y \in X$.*

It is usually said that Y contains an *isometric copy* X^* of X. As a basic example, after completing the metric space $\mathbb{Q}$ with the usual distance, we obtain a space isometrically isomorphic to $\mathbb{R}$.

Continuous functions. The definition of continuity of a function between two metric spaces is quite similar to the one seen earlier for functions between Euclidean spaces.

Definition. A function $f : X \to Y$ between two metric spaces (X,d) and $Y,\rho)$ is said to be *continuous* at the point $a \in X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$, $d(x,y) < \delta$ implies $\rho(f(x), f(a)) < \varepsilon$. We say that f is continuous on X if it is continuous at every point of X.

Here is a simple characterization of continuity in sequential terms. The proof is standard and similar to the one from Calculus.

Proposition 10. *A function $f : X \to Y$ between two metric spaces (X,d) and $Y,\rho)$ is continuous at the point $a \in X$ if and only if for every sequence of points $(x_n)_n$ in X convergent to the point a, the sequence $(f(x_n))_n$ is convergent to $f(a)$.*

Let $f : X \to Y$ and $S \subset Y$. Recall that the *inverse image* or *pre-image* of the set S under f is the set $f^{-1}(S) = \{x \in X : f(x) \in Y\}$. For example, if $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2$, then $f^{-1}([4,9)) = [2,3) \cup (-3,-2]$.

Proposition 11. *A function $f : X \to Y$ between two metric spaces is continuous on X if and only if for every open set V in Y, the inverse image $f^{-1}(V)$ is an open set in X.*

Similarly, $f : X \to Y$ is continuous on X if and only if for every closed set F in Y, the inverse image $f^{-1}(F)$ is an open set in X.

Example 5. *The set $F = \{(x, y) \in \mathbb{R}^2 : x^2 + xy^4 = 1\}$ is closed because the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + xy^4$ is continuous (being a polynomial in x and y), $F = f^{-1}(\{1\})$ and $\{1\}$ is a closed subset of $\mathbb{R}$. Also, the set $U = \{(x, y) \in \mathbb{R}^2 : x^2 + xy^4 > 1\}$ is open since $U = f^{-1}(1,\infty)$ and $(1,\infty)$ is an open subset of $\mathbb{R}$.*

Compact sets. Compact sets can be defined in metric spaces and, more generally, in topological spaces. In the case of metric spaces, the different possible definitions (or descriptions) all coincide.

Definition. A set K in a topological space (and, in particular, in a metric space) X is said to be *compact* (or to have the *Heine-Borel property*) if from every open cover of K we can extract a finite subcover. In other words, if U_α, $\alpha \in I$ are open sets in X such that $K \subset \bigcup_{\alpha \in I} U_\alpha$, then there exist $\alpha_1, \dots, \alpha_n \in I$ such that $K \subset \bigcup_{k=1}^{n} U_{\alpha_k}$.

In a metric space, a set K is said to be *sequentially compact* (or to have the *Bolzano-Weierstrass property*) if every sequence $(x_n)_n$ of points in K has a subsequence $(x_{n_k})_k$ convergent to some point $x \in K$, $k \to \infty$.

A set A (in any topological space, including metric spaces) is said to be *relatively sequentially compact* if its closure $\overline{A}$ is compact.

Example 6. *Every finite set (in any topological space) is compact.*

In $\mathbb{R}$, equipped with the usual metric, the finite closed interval $[0,1]$ is compact. The intervals $(0,1)$ and $[0,1)$ are not compact but are relatively compact since the closure of each of them is $[0,1]$.

Theorem 2. *For a set K in a metric space (X,d), the following statements are equivalent:*

(a) K is sequentially compact;

(b) K is compact in the Heine-Borel sense of open covers.

This results gives us the advantage of using any of the equivalent characterizations of compactness in different proofs, choosing the most convenient one on each occasion.

Proposition 12. *In a metric space, every compact set is closed and bounded.*

In a topological space, defining boundedness in general does not make sense and a compact set (for example, a finite set in certain "pathological topologies") need not be closed.

In Euclidean spaces, the converse to the above property is also true. The reader can easily check that the closure of a bounded set is also bounded (the converse being trivial: if the closure of a set is bounded, so is the set itself).

Theorem 3. *(Heine-Borel). A subset of $\mathbb{R}^n$ (with its usual Euclidean metric) is compact if and only if it is closed and bounded.*

Similarly, a set is relatively compact if and only if it is bounded.

The following property of compact sets is well known and holds in any topological space, with an identical proof.

Lemma 1. *Every closed subset of a compact set in a metric space is also compact.*

PROOF. Let X be the space and F the closed subset. Consider an arbitrary open cover $\{U_\alpha : \alpha \in I\}$ of F. Add to it the complementary set $F^c = X \setminus F$, which is open. This extended cover is now an open cover of X and, thus, has a finite subcover: $U_1, \ldots, U_n$, one of these sets possibly being F^c. Since $F^c \cap F = \emptyset$, we may remove F^c from this cover if necessary; the remaining sets $U_1, \ldots, U_n$ clearly cover F. This shows that every open cover of F has a finite subcover, hence F is compact. ■

The following basic property of compact sets should be familiar from a topology course. It actually holds in any topological space.

Lemma 2. *(Finite intersection property) Let $\mathcal{K} = \{K_\alpha : \alpha \in I\}$ be a collection of compact sets in a metric space X. If $\bigcap_{\alpha \in J} K_\alpha \neq \emptyset$ for every finite subcollection $J \subset I$, then $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$.*

PROOF. Assume the contrary to what is claimed: $\bigcap_{\alpha \in I} K_\alpha = \emptyset$. Then, fixing one set $K_\beta \in \mathcal{K}$, we have

$$K_\beta \subset X = \emptyset^c = \left(\bigcap_{\alpha \in I} K_\alpha \right)^c = \bigcup_{\alpha \in I} K_\alpha^c.$$

Since each K_α is a compact set in the metric space X, it is closed, so each set K_α^c is open. By compactness of K we may extract a finite subcover; that is, there exists a finite subcollection $J \subset I$ such that

$$K_\beta \subset \bigcup_{\alpha \in J} K_\alpha^c = \left(\bigcap_{\alpha \in J} K_\alpha \right)^c.$$

But then $K_\beta \cap \left(\bigcap_{\alpha \in J} K\right) = \emptyset$, meaning that the finite collection $\{K_\alpha : \alpha \in J \cup \{\beta\}\}$ has an empty intersection, contrary to the assumption of the lemma. This contradiction proves that $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$. ■

Note that the property used above: *a compact subset of a metric spaces is closed*, is not valid in general topological spaces. One simple counterexample is given by $X = \{a, b\}$, with $a \neq b$ and the *Sierpiński topology*: $\mathscr{T} = \{\emptyset, \{a\}, X\}$ (after the Polish topologist Wacław Sierpiński, 1882-1969). It is clear that this family is a topology. In this topology, the set $\{a\}$ is compact by being finite. However, it is not closed since its complementary set $\{b\}$ does not belong to $\mathscr{T}$.

Of course, it is very useful to have a necessary and sufficient condition for compactness although this is not easy in an arbitrary space. In other metric spaces, characterizations of compact sets (and also of relatively compact sets) are a bit more involved. In all of them some kind of boundedness appears as a necessary condition. We will review later the Arzelà-Ascoli theorem describing the relatively compact sets in the space $\mathscr{C}[a, b]$. If time permits it, we may also see the Montel theorem describing compactness in the metric space of analytic functions on a planar domain, or the theorem of Kolmogorov characterizing compact sets in the Lebesgue space $L^p(a, b)$.

We now turn to another important property of compact sets. We recall that the product of two metric spaces, (X, d) and (Y, ρ) can be defined as the set $X \times Y$ equipped with the metric

$$D((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2).$$

Alternatively, one can use the maximum instead of the sum, leading to the same open, closed, and compact sets. Products of finitely many spaces are defined by induction. This is all quite similar to defining the metric in $\mathbb{R}^2$ from the metric in $\mathbb{R}$, etc.

Example 7. *If we consider the usual metric $d(x, y) = |x - y|$ in $\mathbb{R}$, the formula*

$$d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

defines a product metric in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Of course, this metric induces the usual topology (the usual open sets) in the plane.

Here is a basic result on compactness of products. Note that the name of the author (after being transliterated from the Russian Cyrillic alphabet) is spelled in many different ways in English, German, or French, a frequent spelling being Tychonoff.

Theorem 4. *(Tikhonov). The Cartesian product of two compact spaces is compact. By induction, the same is true for any finite Cartesian product of compact sets.*

The proof is analogous to that of proving the Bolzano-Weierstrass theorem in $\mathbb{R}^n$. In particular, we have that $[a, b] \times [c, d]$ is a compact set in $\mathbb{R}^2$ and also $[a, b] \times F$ is compact for any finite set $F \subset \mathbb{R}$.

The above important theorem continues to hold for infinite Cartesian products but is rather difficult to prove. To begin with, if we have a product of uncountably many sets, we need to use the Axiom of Choice to justify its existence. Next, one has to define a topology in the Cartesian product of an arbitrary family of topological or metric spaces (the so-called product topology); this can be found in topology text such as the book by Munkres. After taking care of these initial steps, the actual proof is quite non-trivial and is often omitted in a first course in Topology.

Another important feature of compactness is that it is preserved by continuous functions.

Theorem 5. *Let $f : X \to Y$ be a function between two metric spaces. If $K \subset X$ compact and $f \in \mathscr{C}(K)$, then $f(K)$ is compact.*

In particular, when K is compact, every function in $\mathscr{C}(K, \mathbb{R})$ is bounded and attains on K its maximum (and its minimum).

This is a standard result from a Topology or Real Analysis course. Thus, we omit the proof.

Uniform convergence and its applications

Uniform convergence and continuity. We are already familiar with uniform convergence on intervals and, possibly, also on more general sets. Uniform convergence can be generalized in a simple and direct way to the context of functions from an arbitrary set into a metric space as follows.

Definition. Let $E \neq \emptyset$ be a set, let (X, d) be a metric space and $f, f_n : E \to X$, $n \in \mathbb{N}$. We say that the sequence of functions $(f_n)_n$ *converges uniformly* to f on E if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ \forall x \in E,\ d(f_n(x), f(x)) < \varepsilon .$$

To denote uniform convergence, the following notation will be used: $f_n \rightrightarrows_E f$.

The following statement generalizes a known result for real-valued functions.

Theorem 6. *If a sequence of functions $f_n : X \to Y$ between two metric spaces converges to f uniformly on X and all f_n are continuous on X, then f is also continuous on X.*

PROOF. Let d and ρ be the respective metrics on X and Y, all $f_n : X \to Y$ continuous on X, and suppose that $f_n \rightrightarrows_X f$. We will show that f is continuous at every point of X. Let $a \in X$ be arbitrary. Given $\varepsilon > 0$, we need to find a number $\delta > 0$ such that

$$\forall x \in X\ d(x, a) < \delta \Rightarrow \rho(f(x), f(a)) < \varepsilon .$$

Since $f_n \rightrightarrows_X f$, there exists $N \in \mathbb{N}$ such that

$$\forall x \in X,\ \rho(f(x), f_N(x)) < \frac{\varepsilon}{3} .$$

(It suffices to fix just one sufficiently large index N.) In particular, this holds for $x = a$.

Since f_N is continuous on X, it is continuous at the point a in particular, hence

$$\exists \delta > 0\ \forall x \in X\ d(x, a) < \delta \Rightarrow \rho(f_N(x), f_N(a)) < \frac{\varepsilon}{3} .$$

Next, by triangle inequality and the information we have, for every $x \in X$ with $d(x, a) < \delta$ we obtain

$$\rho(f(x), f(a)) \leq \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(a)) + \rho(f_N(a), f(a)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon . \qquad \blacksquare$$

The following important result if well known from undergraduate courses.

Theorem 7. *Let $-\infty < a < b < +\infty$. If a sequence of functions $f_n \in \mathscr{C}[a,b]$ converges to f uniformly on $[a,b]$, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f_n(x)\,dx.$$

PROOF. Taking into account that $f \in \mathscr{C}[a,b]$ by Theorem 6, we know that f is also Riemann-integrable over $[a,b]$. Choosing n large enough so that $|f_n(x) - f(x)| < \varepsilon/(b-a)$ for all $x \in [a,b]$, we have immediately that

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \le \int_a^b |f_n(x) - f_n(x)|\,dx < \varepsilon.$$

for all such n. ■

The supremum norm and uniform convergence. Let $E \neq \emptyset$ be a set; frequently, E will be a subspace of a metric space (X,d) but this is not required in general. We denote by $B(E)$ the collection of all bounded functions from E to $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ (B stands for *bounded*); that is, of all the functions for which there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in E$.

Whenever it is not clear from the context whether the functions are real or complex valued, we shall use the notation $B(E,\mathbb{R})$ or $B(E,\mathbb{C})$ in order to clarify this. The space $B(E)$ can be equipped with the *supremum norm*, given by

$$\|f\|_\infty = \sup\{|f(x)| : x \in E\}.$$

It is easy to interpret the convergence of a sequence of functions in this norm: $f_n \to f$ in $\|\cdot\|_\infty$ means that

$$\|f_n - f\| = \sup_{x\in E} |f_n(x) - f(x)| \to 0, \quad n \to \infty.$$

That is,

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon.$$

This immediately implies that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ \forall x \in E\ |f_n(x) - f(x)| < \varepsilon.$$

The converse is also true; it follows easily by modifying slightly the value of ε. In other words, convergence of f_n to f in the norm of the space means exactly that f_n converges to f uniformly in E.

It is easy to see that the sum of two bounded functions is bounded and a constant multiple of a bounded function is also bounded, hence $B(E)$ is a vector space.

Proposition 13. *$B(E)$ is a complete normed space (a Banach space).*

PROOF. The following is a standard procedure for showing that a space is complete.

Let $(f_n)_n$ be a Cauchy sequence in $B(E)$ and let $\varepsilon > 0$ be given. We can find $N \in \mathbb{N}$ such that, whenever $m > n \ge N$, we have $|f_m(x) - f_n(x)| < \varepsilon$. This means that, for each fixed $x \in E$, the sequence of numbers $(f_n(x))_n$ is a Cauchy sequence (in $\mathbb{R}$ or in $\mathbb{C}$). Since both in $\mathbb{R}$ and $\mathbb{C}$ are complete, any such sequence is convergent. That is, for each $x \in E$ there exists a number (real or complex) $f(x)$ such that $f(x) = \lim_{n\to\infty} f_n(x)$. This defines a real (or complex) function f on E.

Next, f is a bounded function. This can be argued as follows. Since $(f_n)_n$ is a Cauchy sequence in the norm of the space $B(E)$, it is bounded in the norm, hence there exist $M > 0$ such that

$$\|f_n\| = \sup_{x\in E} |f_n(x)| \le M, \quad \forall n \in \mathbb{N}.$$

Thus, taking the limit as $n \to \infty$, we see that $|f(x)| \le M$ for all $x \in E$.

Finally, for arbitrary $n \ge N$, taking the limit in $|f_m(x) - f_n(x)| < \varepsilon$ as $m \to \infty$, we conclude that $|f(x) - f_n(x)| \le \varepsilon$. Since this holds for all $x \in E$, it follows that $\|f - f_n\| \le \varepsilon$. This proves that $f_n \to f$ in the norm of $B(E)$. ■

An important subset of $B(E)$ is $\mathscr{C}(E)$, the space of all bounded and continuous functions on E, equipped with the same norm. (Note that, in general, continuity of a function on a set E does not imply its boundedness on E; however, on certain sets to be reviewed latter this will happen.)

Proposition 14. *$\mathscr{C}(E)$ is a closed subspace of $B(E)$. Hence, it is also a Banach space (in particular, a complete metric space).*

PROOF. If $f_n \in \mathscr{C}(E)$ and $f_n \rightrightarrows_E f$, it follows by Theorem 6 that also $f \in \mathscr{C}(E)$. It is an easy exercise (see Problem Set 1) to check that if f_n are all bounded E and form a convergent sequence (hence, a bounded sequence in the sup norm), the limit function f is also bounded on E. It follows that $f \in \mathscr{C}(E)$. Since $\mathscr{C}(E)$ is a closed subspace of $B(E)$, which is complete in view of Proposition 13, it follows by Proposition 9 that $\mathscr{C}(E)$ is also a complete metric space. ■.

The case of main interest to us will be the case when $E = K$, a compact set in a metric space. In this case, every function in $\mathscr{C}(K)$ is automatically bounded by Theorem 6.

Uniform convergence of function series. The concept of uniform convergence of function sequences is easily extended to functional series, in a natural way.

Definition. Let $E \neq \emptyset$ be a set and $f_n : E \to \mathbb{R}$ (or $f_n : E \to \mathbb{C}$). The series of functions $\sum_{n=1}^{\infty} f_n(x)$ is said to *converge uniformly* on E if the sequence of its partial sums $S_N(x) = \sum_{n=1}^{N} f_n(x)$ converges to some function $S(x)$ uniformly for all $x \in E$. As is usual, $S(x)$ is called the *sum* of the series.

The well-known criterion stated below also works both for real-valued and for complex-valued functions.

Theorem 8. *(Weierstrass' M-test). Suppose that the inequality $|f_n(x)| \le M_n$ holds for all for every $n \in N$ and for all $x \in E$. If $\sum_n M_n$ is a convergent series, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (and absolutely) on E.*

PROOF. For $N \in \mathbb{N}$, let $S_N(x) = \sum_{n=1}^{N} f_n(x)$ be the partial sums of the functional series. We ought to show that this series converges uniformly on E. To this end, by a basic condition for uniform convergence, it suffices to show that S_N is a uniform Cauchy sequence on E.

Since $\sum_n M_n$ is a convergent series, its partial sums form a Cauchy sequence of real (or complex) numbers. Thus, given $\varepsilon > 0$, there exists a positive integer N_0 such that $\sum_{n=N+1}^{M} M_n < \varepsilon$ whenever $M > N \geq N_0$. Then for the same values

$$|S_M(x) - S_N(x)| = \left| \sum_{n=N+1}^{M} f_n(x) \right| \leq \sum_{n=N+1}^{M} |f_n(x)| \leq \sum_{n=N+1}^{M} M_n < \varepsilon,$$

for all $x \in E$.

An inspection of the above argument reveals that actually the series with absolute values: $\sum_{n=1}^{\infty} |f_n(x)|$ converges uniformly on E. ■

The disadvantage of the above criterion is the same as that of the Cauchy criterion: it allows us to establish convergence in many cases but it does not help to identify the sum of the series.

Example 8. *The geometric complex power series $\sum_{n=1}^{\infty} z^n$ converges absolutely and uniformly in every closed disc $\{z : |z| \leq r\}$ in the complex plane with $0 < r < 1$. This is so in view of the Weierstrass test with $M_n = r^n$, since $|z^n| = |z|^n \leq r^n$ and the numerical series $\sum_{n=1}^{\infty} r^n$ converges whenever $0 < r < 1$.*

Dini's test for uniform convergence. After these basic topological preparatory details, we are ready to prove the sufficient condition for uniform convergence due to Dini (Ulisse Dini, Italian mathematician, 1845-1918). Note that it can be applied in one exercise in Problem Set 1. A version of the result for closed bounded intervals on the line may look familiar from an earlier Analysis course. Here we formulate a generalization for metric spaces.

Theorem 9. *(Dini) Let K be a compact metric space and let $f_n \in \mathscr{C}(K)$ be a sequence of real-valued functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in K$, $f \in \mathscr{C}(K)$, and suppose that also either:*

(1) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$ and all $n \in \mathbb{N}$

or

(2) $f_n(x) \leq f_{n+1}(x)$ for all $x \in K$ and all $n \in \mathbb{N}$.

Then $f_n \rightrightarrows_K f$.

PROOF. It suffices to prove the statement only in the first case since in the second case it will follow immediately by considering the functions $-f_n$ instead of f_n.

Next, note that, for any fixed $n \in \mathbb{N}$, condition (1) implies $f_n(x) \geq f_m(x)$ for all $x \in K$ and all $m > n$. Taking the limit as $m \to \infty$ we infer that $f_n(x) \geq f(x)$ for all $x \in K$ and arbitrary $n \in \mathbb{N}$.

Let $g_n = f_n - f$. Then, by our assumptions, $g_n \in \mathscr{C}(K)$, $g_n(x) \geq 0$ for all $x \in K$, $g_n(x) \to 0$ as $n \to \infty$ for all $x \in K$, and $g_n(x) \geq g_{n+1}(x)$ for all $x \in K$. We want to show that $g_n \rightrightarrows_K 0$, which will immediately imply that $f_n \rightrightarrows_K f$.

To this end, let $\varepsilon > 0$ be given and consider the sets

$$K_n = \{x \in K : g_n(x) \geq \varepsilon\} = g_n^{-1}([\varepsilon, +\infty)).$$

Being the inverse image of a closed set under a continuous function, each K_n is closed. Since it is a closed subset of the compact set K, it follows that K_n is actually compact.

Next, in view of $g_n(x) \geq g_{n+1}(x)$ for every $x \in K$, it follows that $K_{n+1} \subset K_n$.

Let $x \in K$ be fixed. Since $g_n(x) \to 0$ as $n \to \infty$, there exists $N_0 \in \mathbb{N}$ such that $g_n(x) < \varepsilon$ for all $n \geq N_0$, hence $x \notin K_n$ for all $n \geq N_0$. In particular, $x \notin \bigcap_{n=1}^{\infty} K_n$. Since this is true for any $x \in K$, it follows that $\bigcap_{n=1}^{\infty} K_n = \emptyset$. By Lemma 2, some finite intersection must satisfy $K_{j_1} \cap K_{j_2} \cap \ldots K_{j_m} = \emptyset$.

However, since the sets K_n form a descending chain: $K_1 \supset K_2 \supset K_3 \supset \ldots$, we see that

$$K_{j_1} \cap K_{j_2} \cap \ldots K_{j_m} = K_{\max\{j_1, j_2 \ldots, j_m\}},$$

hence $K_N = \emptyset$ for some N and, thus, also $K_n = \emptyset$ for all $n \geq N$. This means that for all $x \in K$ and all $n \geq N$ the inequality $0 \leq g_n(x) < \varepsilon$ holds, which proves the statement $g_n \rightrightarrows_K 0$. ■

It should be noted that compactness is essential in the assumptions of Dini's theorem, as the following example shows.

Example 9. *Consider the non-compact subset $I = (0, 1]$ of the real line and let*

$$f_n(x) = \frac{1}{nx+1}, \quad x \in (0, 1], \quad n \in \mathbb{N}.$$

It is clear that, for all $x \in (0, 1]$, $f_n(x) \to 0$ as $n \to \infty$, so all the f_n and the limit function are continuous on I, and $f_n(x) \geq f_{n+1}(x)$ for all $x \in I$ (since $(n+1)x \geq nx$).

However, the convergence of f_n to 0 is not uniform in I since $\frac{1}{n} \in I$ for every $n \in \mathbb{N}$ and

$$\sup_{x \in I} |f_n(x) - 0| = \sup_{x \in I} \frac{1}{nx+1} \geq \frac{1}{n \cdot \frac{1}{n} + 1} = \frac{1}{2} \not\to 0, \quad n \to \infty.$$

A continuous function which is nowhere differentiable: a construction. Later in the course we may see an abstract proof that there exist many functions continuous on, say, an interval of the real line which do not have a derivative at any point. As an application of uniform convergence, we give an explicit construction of such a function due to Weierstrass. First let us recall the following definition.

Definition. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. The function f is said to be a *Lipschitz function* with constant C if

$$|f(s) - f(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R}.$$

The interval I in the definition can be open or closed, finite or infinite; this is not relevant. Clearly, every Lipschitz function is continuous on I. Such functions are named after Rudolph O. Lipschitz (German mathematician, 1832-1903). To mention one important class of examples, by the Lagrange mean value theorem from Calculus:

$$f(s) - f(t) = f'(\xi)(s - t),$$

for some ξ between s and t, any function with a bounded derivative in the interval $[s, t]$ is a Lipschitz function there. (The functions sine and cosine are, thus, Lipschitz functions in all of $\mathbb{R}$ with constant $C = 1$.)

An intuitive explanation of the construction presented below is that the multiplication in the argument, taking $\varphi(4^m x)$ instead of $\varphi(x)$ for larger and larger values of m, creates many oscillations in an interval of arbitrarily small length, which is the reason why the function constructed bellow does not have a derivative anywhere.

Theorem 10. *There exists a real-valued continuous function on the entire real line which is nowhere differentiable.*

Proof. Let $\varphi(x) = |x|$, for $-1 \le x \le 1$. Since $\varphi \in \mathscr{C}[-1,1]$ and $\varphi(-1) = \varphi(1)$, we may extend the function periodically to the entire real axis by the formula $\varphi(x+2) = \varphi(x)$, obtaining a continuous function on $\mathbb{R}$, denoted again by φ for the sake of simplicity. Note that the graph of φ is the union of countably many segments, each one with the slope 1 or -1 (and the slopes alternate after each integer point).

Actually, the function φ has a property that is stronger than continuity. It is a Lipschitz function with constant one:

$$|\varphi(s) - \varphi(t)| \le |s-t|, \quad \forall s, t \in \mathbb{R}.$$

Geometrically, this is clear since the slope of the line passing through any two points $(s, \varphi(s))$ and $(t, \varphi(t))$ of the graph of φ is at least -1 and at most 1, hence

$$-1 \le \frac{\varphi(s) - \varphi(t)}{s-t} \le 1.$$

The Lipschitz condition can also be verified algebraically but the analysis is longer. If both s, $t \in [-1,1]$ then $|\varphi(s) - \varphi(t)| = ||s| - |t|| \le |s-t|$. Otherwise write $s = x + 2m$, $t = y + 2n$, with x, $y \in [-1,1]$ and m, $n \in \mathbb{Z}$ and check that

$$|\varphi(s) - \varphi(t)| = |\varphi(x) - \varphi(y)| \le |x-y| \le |x - y + 2(m-n)| = |s-t|.$$

This requires a little bit more of work, taking into account the values of m and n.

Now define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x).$$

Clearly, $0 \le \varphi(t) \le 1$ for all $t \in \mathbb{R}$, hence the series defining f converges uniformly in $\mathbb{R}$, by Theorem 8 and comparison with the geometric series $\sum_{n=0}^{\infty}(3/4)^n$. Being a composition of continuous functions, $\varphi(4^n x)$ is continuous on $\mathbb{R}$ and the function f, being a uniform limit of functions continuous on $\mathbb{R}$, is also a continuous function on $\mathbb{R}$.

We now ought to show that f does not have a finite derivative at any point. To this end, it will be shown that the differential quotient can be arbitrarily large near any point in $\mathbb{R}$. More specifically, for a fixed $x \in \mathbb{R}$ and a fixed $m \in \mathbb{N}$, let $\delta_m = \pm\frac{1}{2} \cdot 4^{-m}$. The sign $\pm$ can be chosen, and in a unique fashion, so that no integer lies strictly between $4^m x$ and $4^m(x + \delta_m)$. This is possible since

$$|4^m(x+\delta_m) - 4^m x| = 4^m |\delta_m| = \frac{1}{2}.$$

Namely, assuming first that $4^m x \notin \mathbb{Z}$, let k be the closest integer to $4^m x$ so that the distance between these two numbers is $< \frac{1}{2}$. If $k < 4^m x$, choose the plus sign, so that $k < 4^m x < 4^m(x+\delta_m) < k+1$. If, on the contrary, $4^m x < k$, choose the minus sign, so that $k-1 < 4^m(x+\delta_m) < 4^m x < k$. In the case when $4^m x \in \mathbb{Z}$, choose the plus sign.

In this way, both points $(4^m x, \varphi(4^m x))$ and $(4^m(x+\delta_m), \varphi(4^m(x+\delta_m)))$ will belong to the segment of the graph of φ with the same slope (either 1 or -1), which will be fundamental at one point in the further analysis.

For an integer $n \geq 0$, define

$$\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}$$

Let us analyze the values of γ_n depending on the value of n.

Whenever $n > m$, say $n = m+k$, $k \in \mathbb{N}$, we have $4^n \delta_m = \pm\frac{1}{2}4^{n-m} = \pm 2 \cdot 4^{k-1}$, so $4^n \delta_m$ is an even integer. Since the function φ is periodic with period 2, it follows that $\varphi(4^n(x+\delta_m)) = \varphi(4^n x)$ and therefore $\gamma_n = 0$.

When $0 \leq n \leq m$, the Lipschitz property of φ implies

$$|\gamma_n| = \left|\frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}\right| \leq \frac{|4^n(x+\delta_m) - 4^n x|}{|\delta_m|} = 4^n$$

Note that when, in particular, $n = m$, due to our choice of the sign in the definition of the number δ_m (the slope of the graph is the same, 1 or -1, at both points $4^m x$ and $4^m(x+\delta_m)$), we have equality in the above inequality, so that $|\gamma_m| = 4^m$.

Taking this analysis into account, by triangle inequality we obtain

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \gamma_n\right| = \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right| = \left(\frac{3}{4}\right)^m \gamma_m - \sum_{n=0}^{m-1}\left|\left(\frac{3}{4}\right)^n \gamma_n\right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{3^m+1}{2}.$$

It follows that

$$\limsup_{\delta\to 0}\left|\frac{f(x+\delta) - f(x)}{\delta}\right| \geq \limsup_{m\to\infty}\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = +\infty.$$

Hence, $f'(x)$ does not exist. Since x was arbitrary, this happens for all $x \in \mathbb{R}$. ■

We will likely explain later on (after reviewing the Baire category theorem) that such examples are actually not an exception but rather a rule among continuous functions! This is guaranteed by a theorem due to S. Banach, which will give a more precise meaning to the statement just made.

Uniform approximation by polynomials. The Stone-Weierstrass theorem

The great German mathematician Karl T. Weierstrass (1815-1897) is generally considered as the father of rigorous Mathematical Analysis, having created the $\varepsilon - \delta$ language and other formal definitions around 1860. In this chapter we will prove the basic approximation theorem discovered by him as late as 1885. This is one of the basic results in a vast area of Mathematics nowadays known as Approximation Theory. We note in passing that, just a few years later, his student Carl Runge (1856-1927) proved an important theorem on uniform approximation of holomorphic functions by polynomials on compact sets, relevant in Complex Analysis. However, this result will not be discussed here as it belongs to a completely different topic.

A very important generalization of the Weierstrass theorem is due to the US mathematician Marshall H. Stone (1903-1989), who was able to identify a very small number of properties that a collection of continuous functions should posses in order to guarantee its density in the algebra of all continuous functions on a compact set. We first want to become familiar with these properties and then formulate the main theorems.

Statement of Weierstrass' theorem and some consequences. In this section, $[a,b]$ will denote a compact (closed and bounded) interval in $\mathbb{R}$. In other words, it will be assumed that $-\infty < a < b < \infty$. (General compact sets will be reviewed briefly later on.) The space of all real-valued and continuous functions in $[a,b]$ will be denoted by $\mathscr{C}[a,b]$. It is clearly a vector space over the reals $\mathbb{R}$ and, moreover, an algebra (meaning that the product of any two funcions in the space remains in it). Also, a natural norm can be defined on $\mathscr{C}[a,b]$ by setting

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)|.$$

We know from Calculus that the maximum is attained since $|f| \in \mathscr{C}[a,b]$ as well. The convergence in this norm is the well-known uniform convergence; in other words, $\|f_n - f\|_\infty \to 0$ as $n \to \infty$ if and only if $f_n \rightrightarrows_{[a,b]} f$ (The readers are invited to check this simple equivalence.) Formally, this means that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ \forall x \in [a,b]\ |f_n(x) - f(x)| < \varepsilon.$$

(Just like in other similar definitions, the $<$ sign in the last inequality above can be replaced by $\le$ without harm.)

We say that a function f *can be uniformly approximated by polynomials* in $[a,b]$ if there exists a sequence of polynomials $(P_n)_{n=1}^\infty$ such that $P_n \rightrightarrows_{[a,b]} f$. Equivalently, this can be formulated by saying that for each $\varepsilon > 0$ there exists a polynomial P such that $\|P - f\| < \varepsilon$.

Theorem 11. *(Weierstrass). Every real-valued function $f \in \mathscr{C}[a,b]$ can be uniformly approximated in $[a,b]$ by polynomials with real coefficients.*

The proof will be deferred for the time being. We first want to make a few comments about some relevant applications of this fundamental result.

Moments of a function. By definition, the *moments* of a function $f \in \mathscr{C}[a,b]$ are the numbers

$$M_n(f) = \int_a^b x^n f(x)\,dx, \qquad n \in \{0,1,2,\dots\}.$$

They can be defined in a similar fashion for integrable functions (in some sense) on other appropriate sets on the real line.

We encounter them, for example, in Probability Theory: if X is continuous random variable with density function f, we define its *expectation* and *variance*, respectively, as

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx\,, \quad V(X) = E[X^2] - E[X]^2 = \int_{\mathbb{R}} x^2 f(x)\,dx - \left(\int_{\mathbb{R}} x f(x)\,dx\right)^2.$$

That is,

$$E[X] = M_1(f)\,, \quad V(X) = M_2(f) - M_1(f)^2\,.$$

In different areas of Mathematical Analysis and Probability we can find questions called *moment problems*. In their different formulations, such problems consist in finding out how much information we need about the moments of a given function (or a measure) in order to be able to determine completely the function (or measure) in question. The following simple statement was a pioneering result in this direction.

Corollary 1. *(Hausdorff). If $f \in \mathscr{C}[a,b]$ is a real-valued function whose moments are all zero:*

$$\int_a^b x^n f(x)\,dx = 0\,, \quad \forall n \in \{0,1,2,\dots\}\,,$$

then $f \equiv 0$ in $[a,b]$. Thus, if two continuous functions have the same moments, they are identically equal in $[a,b]$.

Proof. Taking finite linear combinations of integrals of above type, we get that $\int_a^b P(x) f(x)\,dx = 0$ for every polynomial P with real coefficients. By Weierstrass' Theorem 11, f is a uniform limit of a certain sequence $(P_n)_n$ of polynomials in $[a,b]$. Since a continuous function is bounded on $[a,b]$, this readily implies that

$$P_n f \rightrightarrows_{[a,b]} f^2\,, \quad n \to \infty\,.$$

(Just estimate the difference $P_n f - f^2$ uniformly in $[a,b]$ to check this fact.) Due to uniform convergence, by a well-known theorem from Calculus we are allowed to exchange the limit and the integral, thus obtaining

$$\int_a^b f(x)^2\,dx = \lim_{n\to\infty} \int_a^b P_n(x) f(x)\,dx = 0\,.$$

Since $f^2 \in \mathscr{C}[a,b]$ and $f(x)^2 \ge 0$ for all $x \in [a,b]$, we conclude that $f(x)^2 \equiv 0$ in $[a,b]$. (If you did not see a formal proof of this fact in your Analysis courses, it may be a good exercise to prove the statement.) Note that this is one place where we use in an essential way the assumptions that f is continuous and only takes real values. It follows immediately that $f(x) \equiv 0$ in $[a,b]$. ■

Separability of $\mathscr{C}[a,b]$. The theorem of Weierstrass has another important consequence. Recall that a metric spaces is called *separable* if it contains a dense and countable set. For example, $\mathbb{R}$ (with its usual distance) is separable since it contains $\mathbb{Q}$ as a dense and countable subset. At this point, in addition to some metric/topological concepts, the reader is also expected to review the basic concepts and results from Set Theory needed here.

Corollary 2.. *The space $\mathscr{C}[a,b]$, equipped with the usual norm, is separable.*

Proof. Following a standard argument, we will show that the polynomials with rational coefficients form a dense countable set in $\mathscr{C}[a,b]$.

First, note that the polynomials with coefficients in $\mathbb{Q}$ form a countable set. This can be shown as follows. First, for any fixed $n \in \{0,1,2,\ldots\}$, the set of all polynomials $Q(x) = b_0 + b_1 x + \ldots + b_n x^n$ of degree n, with b_0, b_1, ..., $b_{n-1} \in \mathbb{Q}$, $b_n \in \mathbb{Q} \setminus \{0\}$, has the same number of elements as $\mathbb{Q}^n \times (\mathbb{Q} \setminus \{0\})$. The latter set is countable because any finite Cartesian product of countable sets is countable. Then, finally, the set of all polynomials with coefficients in $\mathbb{Q}$ is countable by being a countable union of countable sets (by splitting the set of all polynomials into constant polynomials, linear polynomials, quadratic polynomials, and so on).

Next, we show that every polynomial with real coefficients can be uniformly approximated by polynomials with rational coefficients. Let $\varepsilon > 0$ and $P(x) = a_0 + a_1 x + \ldots + a_n x^n$ be an arbitrary polynomial with a_0, a_1,..., $a_n \in \mathbb{R}$. Then we can find a polynomial $Q(x) = b_0 + b_1 x + \ldots + b_n x^n$ with b_0, b_1, ..., $b_n \in \mathbb{Q}$ and such that

$$|a_k - b_k||x|^k < \frac{\varepsilon}{n+1}, \quad \forall x \in [a,b], \quad \forall k \in \{0,1,\ldots,n\},$$

by the density of the rationals in $\mathbb{R}$ and also since $|x|^k$ can be bounded uniformly in $[a,b]$ (for example, by the constant $\max\{|a|^k, |b|\}^k$) for each k. It follows immediately that

$$|P(x) - Q(x)| \le \sum_{k=0}^{n} |a_k - b_k||x|^k < (n+1)\frac{\varepsilon}{n+1} = \varepsilon, \quad \forall x \in [a,b].$$

Finally, it is easy to see that every function $f \in \mathscr{C}[a,b]$ can be uniformly approximated by polynomials with rational coefficients. Given $\varepsilon > 0$, by Theorem 11 we can find a polynomial P with real coefficients with $\|P - f\|_\infty < \varepsilon/2$. Next, we know that we can also find a polynomial Q with rational coefficients such that $\|Q - P\|_\infty < \varepsilon/2$. From the triangle inequality for the norm we obtain

$$\|Q - f\|_\infty \le \|Q - P\|_\infty + \|P - f\|_\infty < \varepsilon,$$

which completes the proof. ■

Lebesgue's proof of Weierstrass' theorem: a sketch. We only outline an idea of a very direct proof due to the famous French mathematician Henry L. Lebesgue (1875-1941), the father of the Lebesgue integral.

We recall a theorem from Calculus due to Cantor: if $-\infty < a < b < \infty$ and $f \in \mathscr{C}[a,b]$ then f is actually uniformly continuous on $[a,b]$. This means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever x, $y \in [a,b]$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. In other words, the value of δ depends only on ε but not on the values of x and y chosen in $[a,b]$. This is clearly a consequence of compactness and can be generalized to other compact metric spaces (see Problem Set # 2).

His proofs begins by observing that any $f \in \mathscr{C}[a,b]$ is uniformly continuous in $[a,b]$ by a theorem of Cantor from Calculus, which allows us to approximate f uniformly in $[a,b]$ by piecewise linear functions. These are functions whose graph is a polygonal path connecting finitely many points on the graph of f. Essentially, all such functions can be decomposed into pieces that look more or less like a letter "V" or an inverted letter "V". The pieces of this kind are obtained by a linear function from the function $u(x) = |x|$. An important point is to note that

$$|x-c| + (x-c) = \begin{cases} 0, & \text{if } x \le c \\ 2(x-c), & \text{if } x \ge c. \end{cases}$$

The problem of approximating f uniformly by polynomials then reduces to approximating uniformly by polynomials certain linear combinations of functions of the above type. The bottom line is that one needs to approximate uniformly by polynomials the the function $u(x) = |x|$ in the interval $[-1,1]$. (Why is the interval irrelevant will be explained in the next section on Landau's proof.)

Thus, the entire proof boils down to finding a sequence of polynomials that converges uniformly to u in $[-1,1]$ and this is done by extending and making more precise a well-known generalized binomial formula from Calculus:

$$|x| = \sqrt{x^2} = (1-(1-x^2))^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-1)^n (1-x^2)^n, \quad -1 \le x \le 1.$$

It can be proved that the partial sums of the above series (obviously, polynomials in x) converge to $|x|$ uniformly in $[-1,1]$. The result from Calculus is usually formulated only for the open interval $(-1,1)$ but holds also on the closed interval and, moreover, it can be proved that the convergence is uniform.

Further details can be consulted in Duren's book mentioned in the references (Sections 6.2 and 3.5; alternatively, to prove uniform convergence, one can use the Monotone Convergence Theorem from the Theory of Measure and Integration, to be reviewed later). The reader is advised to draw pictures and check carefully every detail.

Landau's proof of Weierstrass' theorem. In this section we describe in detail the proof of the theorem of Weierstrass given by Landau (Edmund G. Landau, 1877-1938, a well-known German number theorist and complex analyst). We begin by reviewing various relevant details and simplifications.

Reduction to a special case: the exact interval does not matter. We first observe that the theorem is equivalent to some of its special cases.

First reduction. Proving Theorem 11 for the space $\mathscr{C}[a,b]$ (with arbitrary $-\infty < a < b < \infty$) is equivalent to proving it for $\mathscr{C}[0,1]$; hence, it suffices to consider only the latter case. The reason is that, starting from an arbitrary interval $[a,b]$, we can establish a bijective linear map

$$L : [0,1] \to [a,b], \quad L(t) = a + (b-a)t.$$

Obviously, L is continuous and $Q(t)$ is a polynomial of t with real coefficients if and only if $Q(L(t)) = Q(a+(b-a)t)$ is one, thanks to the Newton binomial formula. Hence, $f \in \mathscr{C}[a,b]$ can be approximated by polynomials uniformly in $[a,b]$ if and only if $f \circ L$ can be uniformly approximated by polynomials in $[0,1]$.

Second reduction. Instead of proving Theorem 11 for all functions in $\mathscr{C}[0,1]$, it suffices to prove it only for those functions in $\mathscr{C}[0,1]$ that satisfy the additional condition $f(0) = f(1) = 0$. Indeed, given $f \in \mathscr{C}[0,1]$, define the function g by

$$g(x) = f(x) - f(0) - x(f(1) - f(0)).$$

Clearly, $g(0) = 0 = g(1)$ and $g \in \mathscr{C}[0,1]$ since $g - f$ es a linear function. Moreover, if a sequence of polynomials (P_n) converges to g, then the sequence of polynomials

$$Q_n(x) = P_n(x) + f(0) + x(f(1) - f(0)),$$

converges uniformly to f in $[0,1]$.

Approximate identity (summability kernel). By the previous discussion, if Weierstrass' theorem is valid for one compact interval, it is also valid for any other such interval (passing from $[0,1]$ to that interval). As we shall see later, it is convenient to work in symmetric intervals $[-a,a]$, where $0 < a < \infty$.

Landau's proof is based only on a few properties that are usually required from what is called an approximate identity or a summability kernel. The argument can be adapted to other proofs in the context of approximation (uniforme, pointwise, or in some mean value). We will likely see examples of this in Fourier Analysis.

Definition. A sequence of functions $Q_n : [-a,a] \to \mathbb{R}$ is called an *approximate identity* (or *summability kernel*) if it satisfies the following conditions:

(1) $Q_n(t) \geq 0$, for all $n \in \mathbb{N}$ and every $t \in [-a,a]$;

(2) $\int_{-a}^{a} Q_n(t)\,dt = 1$, for all $n \in \mathbb{N}$;

(3) $\forall \delta \in (0,a)$, $Q_n \rightrightarrows 0$ in $\{x : \delta \leq |x| \leq a\} = [-a,-\delta] \cup]\delta, a]$.

Sometimes it is convenient to require an additional condition:

(4) Q_n is an even function in $[-a,a]$.

Later on we will explain why is the last condition useful and we will see that there exist different relevant families of functions that have the properties of an approximate identity, among them the Fejér and Poisson kernels.

Proposition 15. *Let $f \in \mathscr{C}[0,a]$, $f(0) = f(a) = 0$, and extend f continuously to the rest of the real line by defining $f(x) = 0$ in $(-\infty,0) \cup (a,\infty)$. Let $(Q_n)_n$ be an approximate identity in $[-a,a]$. If we define the functions P_n according to the formula*

$$P_n(x) = \int_{-a}^{a} f(x+t) Q_n(t)\,dt, \tag{1}$$

then $P_n \rightrightarrows f$ in $[0,a]$.

PROOF. Since $f \in \mathscr{C}[0,a]$, by Cantor's theorem, f is uniformly continuous in $[0,a]$, meaning that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in [0,a]\ \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}.$$

Since $f(x) = 0$ in $(-\infty,0) \cup (a,\infty)$, we can actually conclude that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \mathbb{R}\ \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}.$$

(For example, if $|x-y|<\delta$ and $x\in[0,a]$ but $y>a$, then $|x-a|<\delta$, hence $|f(x)-f(y)|=|f(x)-f(a)|<\frac{\varepsilon}{3}$. The discussion is similar if $|x-y|<\delta$, $x<0$ and $y\in[0,a]$, etc.)

Let $\varepsilon>0$. By the previous discussion, with the value of δ chosen as above (depending on ε), if $t\in(-\delta,\delta)$, $x\in\mathbb{R}$, we obtain that

$$|(x+t)-x|=|t|<\delta \Rightarrow |f(x+t)-f(x)|<\frac{\varepsilon}{3}.$$

For the same value of δ, recalling that $Q_n\geq 0$ by property (1) of approximate identities, for all $x\in[0,a]$ we have

$$\begin{aligned} |P_n(x)-f(x)| &= \left|\int_{-a}^{a} f(x+t)Q_n(t)\,dt-\int_{-a}^{a} f(x)Q_n(t)\,dt\right| \leq \int_{-a}^{a}|f(x+t)-f(x)|Q_n(t)\,dt \\ &= \int_{[-a,-\delta]}+\int_{(-\delta,\delta)}+\int_{[\delta,a]}, \end{aligned}$$

the integrand being repeated in each integral in the last line. By what was said earlier, we estimate the second of the three integrals as follows:

$$\int_{(-\delta,\delta)}|f(x+t)-f(x)|Q_n(t)\,dt\leq\frac{\varepsilon}{3}\int_{(-\delta,\delta)}Q_n(t)\,dt\leq\frac{\varepsilon}{3}\int_{-a}^{a}Q_n(t)\,dt=\frac{\varepsilon}{3},$$

using property (2) of approximate identities.

Let $M=\max\{|f(x)|\,:\,x\in[0,a]\}$; note that it exists by continuity of f and compactness of $[0,a]$). Then

$$|f(x+t)-f(x)|\leq 2M,\quad \text{para todo } x,t\in\mathbb{R}.$$

Property (3) of approximate identities allows us to intechange the limit and the integral to obtain

$$\lim_{n\to\infty}\int_{-a}^{-\delta}Q_n(t)\,dt=0=\lim_{n\to\infty}\int_{\delta}^{a}Q_n(t)\,dt.$$

Thus, given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$ we have

$$\int_{-a}^{-\delta}Q_n(t)\,dt<\frac{\varepsilon}{6M},\quad \int_{\delta}^{a}Q_n(t)\,dt<\frac{\varepsilon}{6M}.$$

Hence,

$$\int_{-a}^{-\delta}|f(x+t)-f(x)|Q_n(t)\,dt<2M\cdot\frac{\varepsilon}{6M}=\frac{\varepsilon}{3},\qquad \int_{\delta}^{a}|f(x+t)-f(x)|Q_n(t)\,dt<\frac{\varepsilon}{3}.$$

Finally, putting together the three estimates with $\frac{\varepsilon}{3}$, we obtain that $|P_n(x)-f(x)|<\varepsilon$, for all $n\geq N$. ■

The above Proposition 15 essentially tells us that convolutions of a function with an approximate identity converge uniformly to the given function. However, the reader familiar with convolutions may recognize that the definition of P_n given above does not exactly coincide with that of a convolution. So why do we talk about convolutions? Here is the reason.

Observation. If we assume the additional condition (4): the function Q_n is even, then P_n is exactly the convolution of f with Q_n (in the sense of the usual definition seen in mathematical texts). This can be seen as follows.

Note that $f(x+t) \neq 0 \Rightarrow 0 \leq x+t \leq a \Leftrightarrow -x \leq t \leq a-x$, meaning that in the definition of P_n the interval of integration gets reduced to $[-x, a-x]$ since the integrand is zero in the rest. After the simple change of variable $x+t=s$, using the fact that $Q_n(s-x) = Q_n(x-s)$, we conclude that

$$P_n(x) = \int_{-x}^{a-x} f(x+t)Q_n(t)\,dt = \int_0^a f(s)Q_n(s-x)\,ds = \int_0^a Q_n(x-s)f(s)\,ds = (Q_n * f)(x),$$

which corresponds exactly to the usual definition of a convolution of two functions.

It is important to stress that later on, when studying some Real Analysis techniques, we will normally have the symmetry of the interval at our disposal (integrating either over $\mathbb{R}$ or $[-\pi,\pi]$, hence we will always obtain results in which $Q_n * f \to f$ in some sense (usually weaker than uniform convergence).

Observation. If each $Q_n(x)$ is a polynomial with real coefficients, then so is $P_n(x)$.

This is also easy to check. If $Q_n(x)$ is a polynomial, then $Q_n(x-s)$ is also such for every value of s. Computing the integrals that define P_n (integrating respect to the variable t), we see that $P_n(x)$ is also a polynomial with real coefficients.

Lemma 3. *(Bernoulli's inequality). If* $0 \leq a \leq 1$, *then* $(1-a)^n \geq 1-na$, *for every positive integer* n.

PROOF. Various simple proofs are possible (see if you can give your own!), one of them by induction on n, for example.

Another typical way of proving the inequality is by using Calculus as follows. Let

$$u(a) = (1-a)^n - (1-na), \quad 0 \leq a \leq 1.$$

It is immediate that

$$u'(a) = n - n(1-a)^{n-1} \geq 0, \forall a \in [0,1],$$

so it follows that the function u is non-decreasing in $[0,1]$. Hence $u(a) \geq u(0) = 0$ for all $a \in [0,1]$. ■

Now we are finally ready to give a proof of the theorem of Weiestrass, using the tools developed above and following Landau's idea.

PROOF. By the two reductions seen earlier, it suffices to consider only the functions $f \in \mathscr{C}[0,1]$ such that $f(0) = f(1) = 0$. Any such function can be extended to a continuous function on all of $\mathbb{R}$ by defining its values on the set $(-\infty,0) \cup (1,\infty)$ to be 0 (drawing a picture will be helpful here). This means that we can apply the case $a=1$ of Proposition 15 already proved. To this end, we need only show the existence of approximate identity consisting of even polynomials. Consider the following even polynomials:

$$Q_n(x) = c_n(1-x^2)^n,$$

where the constants $c_n > 0$ are chosen (after integration) in such a way that $\int_{-1}^{1} Q_n(x)\,dx = 1$. Obviously, we have $Q_n(x) \ge 0$ in $[-1,1]$. In order to see that the sequence $(Q_n)_n$ is an approximate identity, we need only check its uniform convergence to zero on the set $\{x : \delta \le |x| \le 1\}$. Starting from

$$\frac{1}{c_n} = \int_{-1}^{1} (1-x^2)^n\,dx = 2\int_0^1 (1-x^2)^n\,dx \ge 2\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx \ge 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},$$

(using Lemma 3 in the second inequality), it follows that $c_n < \sqrt{n}$. Hence, for $\delta \le |x| \le 1$ we obtain

$$0 \le Q_n(x) \le \sqrt{n}\,(1-x^2)^n \le \sqrt{n}\,(1-\delta^2)^n.$$

The upper bound in the last inequality does not depend on x as long as $\delta \le |x| \le 1$. It is well known from Calculus that, for any fixed $r \in (0,1]$, the sequence $\sqrt{n}r^n$ tends to zero as $n \to \infty$. Taking $r = 1-\delta^2$, we obtain that $Q_n(x) \rightrightarrows 0$ on $\{x : \delta \le |x| \le 1\}$.

Now we can apply Proposition 15 to conclude that f can be approximated uniformly by the functions P_n which, in this case, are polynomials since the functions Q_n are polynomials. ■

Algebras of functions. Our exposition of the theorem of Stone-Weierstrass in these notes follows that of Rudin's basic book (with some additional details explained), which is at a more abstract level than in Duren's basic book but at a less abstract level than in Folland's (more advanced) book.

Definition. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $E \ne \emptyset$ be a set. A set $\mathscr{A}$ of functions $f : E \to \mathbb{K}$ is an *algebra* (more precisely, a $\mathbb{K}$-algebra) if it satisfies the following conditions:

(1) $\mathscr{A}$ is a vector space over the scalar field $\mathbb{K}$ (that is, for all $f, g \in \mathscr{A}$ and $c \in \mathbb{K}$, we have $f+g \in \mathscr{A}$ and $cf \in \mathscr{A}$);

(2) $\mathscr{A}$ is closed under multiplication: if $f, g \in \mathscr{A}$, then $fg \in \mathscr{A}$.

In the case when $\mathbb{K} = \mathbb{R}$, we say that $\mathscr{A}$ is a real algebra. In the case $\mathbb{K} = \mathbb{C}$, we talk about a complex algebra.

Example 10. *Understanding $\mathbb{R}[x]$ as the collection of all polynomials of a real variable $x \in \mathbb{R}$ or $x \in [a,b]$ with real coefficients and $\mathbb{C}[x]$ as polynomials with complex coefficients of a real variable $x \in \mathbb{R}$ or $x \in [a,b]$, where $-\infty < a < b < +\infty$, it is easy to verify directly that*

- *The following sets are real algebras: $\mathbb{R}[x]$, $\mathbb{R}[x]|_{[a,b]}$, as well as $\mathscr{C}(E,\mathbb{R})$ (continuous and real-valued functions on a metric space E).*

- *The sets $\mathbb{C}[x]$, $\mathbb{C}[x]|_{[a,b]}$, as well as $\mathscr{C}(E,\mathbb{C})$ (continuous and complex-valued functions on a metric space) are complex algebras. So is $\mathscr{P}(z,\overline{z})$, the set of all polynomials of z and $\overline{z}$ with complex coefficients:*

$$\sum_{j=0}^{m}\sum_{k=0}^{n} c_{j,k} z^j \overline{z}^k, \quad c_{k,l} \in \mathbb{C}.$$

Observation. If $\mathscr{A}$ is a $\mathbb{K}$-algebra such that $f \in \mathscr{A}$ and the constant function $\mathbf{1} \in \mathscr{A}$ and $P \in \mathbb{K}[x]$ is a polynomial, then the composed function $P \circ f \in \mathscr{A}$. In other words, if $c_0, c_1, \ldots, c_n \in \mathbb{K}$, then

$$c_0 + c_1 f + \ldots c_n f^n \in \mathscr{A}.$$

Definition. Let $\mathscr{A}$ be any set of functions $f : E \to \mathbb{K}$ (not necessarily an algebra), where E is a metric space. The *uniform closure* of $\mathscr{A}$ is the closure of $\mathscr{A}$ in the topology of uniform convergence in E.

We will say that $\mathscr{A}$ is *uniformly closed* if

$$f_n \in \mathscr{A}, \quad f_n \rightrightarrows_E f \Rightarrow f \in \mathscr{A}.$$

We will use the notation $\mathrm{Cl}(\mathscr{A})$ to denote the closure instead of $\overline{\mathscr{A}}$ in order to avoid a confusion with complex conjugation (the relevance of this is to be seen later).

The terminology just introduced here allows us to formulate the Weierstrass in the following language:

$\mathscr{C}([a,b],\mathbb{R})$ *is the uniform closure of the algebra* $\mathbb{R}[x]|_{[a,b]}$.

Definition. Let $\mathscr{A}$ be a family of functions $f : E \to \mathbb{K}$ (not necessarily an algebra), where E is a metric space. We say that $\mathscr{A}$ *separates points* on E if

$$\forall x, y \in E,\ x \neq y \Rightarrow \exists f \in \mathscr{A} \text{ such that } f(x) \neq f(y).$$

We say that $\mathscr{A}$ *vanishes at no point* of E if

$$\forall x \in E, \exists f \in \mathscr{A} \text{ s.t. } f(x) \neq 0.$$

Example 11. *Here are some relevant examples.*

- $\mathscr{A} = \mathbb{R}[x]$ *and* $\mathscr{A} = \mathbb{R}[x]|_{[a,b]}$ *are algebras that separate points (on* $\mathbb{R}$ *and on* $[a,b]$*, respectively) since the polynomial* $P(x) = x$ *belongs to both of them. They are algebras that vanish at no point since they both contain non-zero constant functions.*
- *The set of even polynomials, obviously, does not separate points on* $\mathbb{R}$.
- $\mathscr{C}[a,b]$ *is a uniformly closed algebra (since the uniform limit of continuous functions in* $[a,b]$ *is another continuous function) that separates points on* $[a,b]$ *and vanishes at no point of* $[a,b]$.

The Stone-Weierstrass theorem. We are now ready to formulate the theorem of Stone-Weierstrass in the real case. The result was proved by M.H. Stone in 1937.

Theorem 12. *(Stone-Weierstrass, real case). Let* K *be a compact metric space and* $\mathscr{A} \subset \mathscr{C}(K,\mathbb{R})$ *a real algebra that separates points on* K *and vanishes at no point of* K*. Then the uniform closure of* $\mathscr{A}$ *is* $\mathrm{Cl}(\mathscr{A}) = \mathscr{C}(K,\mathbb{R})$.

The theorem of Weierstrass follows as a corollary of Theorem 12 since the polynomials with real coefficients (restricted to the interval $[a,b]$) belong to $\mathscr{C}[a,b]$ and form a real algebra that separates points on $[a,b]$ and vanishes at no point of the interval.

Lemma 4. *Let* $0 < M < \infty$. *Then there exists a sequence of polynomials* $(P_n)_n$ *with real coefficients and* $P_n(0) = 0$ *for all* $n \in \mathbb{N}$ *and such that* $P_n(x) \rightrightarrows |x|$ *in* $[-M, M]$ *as* $n \to \infty$.

PROOF. Thanks to Weierstrass' theorem, we know that there exist polynomials R_n with real coefficients and such that $R_n(x) \rightrightarrows |x|$ in $[-M, M]$ as $n \to \infty$. (This fact can also be proved directly, without resorting to the theorem of Weierstrass, although that requires some work; see Problem Set 1.) Note that this implies that $R_n(0) \to 0$. Now it suffices to consider the polynomials P_n defined by

$$P_n(x) = R_n(x) - R_n(0)\,.$$

Obviously, they converge to $|x|$ uniformly in $[-M, M]$ and take on value 0 at the origin. ■

Lemma 5. *If* $\mathscr{A}$ *is a (real or complex) algebra of bounded functions on a set* $E \neq \emptyset$, *then its uniform closure* $\mathrm{Cl}(\mathscr{A})$ *is a (real or complex) algebra which is uniformly closed.*

PROOF. Being a closure of a set in the supremum norm, we know that $\mathrm{Cl}(\mathscr{A})$ is uniformly closed. Thus, it is only left to check that it is an algebra.

Let f, $g \in \mathrm{Cl}(\mathscr{A})$ and $c \in \mathbb{K}$ (the scalar field, $\mathbb{R}$ or $\mathbb{C}$). Then, by our assumptions, there exist f_n, $g_n \in \mathscr{A}$ such that

$$f_n \rightrightarrows_E f\,, \quad g_n \rightrightarrows_E g\,, \quad n \to \infty\,.$$

Then one checks trivially that

$$f_n + g_n \rightrightarrows_E f + g\,, \quad c f_n \rightrightarrows_E c f\,, \quad n \to \infty\,.$$

Hence, $f + g$, $cf \in \mathrm{Cl}(\mathscr{A})$.

To check that $fg \in \mathrm{Cl}(\mathscr{A})$ requieres a little more care. In the first place, it is an easy exercise to see that g, being a uniform limit on E of bounded functions g_n, is also bounded on E:

$$\exists N > 0 \ \forall x \in E \ |g(x)| \leq N\,.$$

(Hint: remember that every convergent sequence is a Cauchy sequence and every such sequence is bounded in the norm - see notes on the review of metric properties.)

Using this fact, we now observe that convergence in the supremum norm on E implies uniform boundedness of the sequence $(f_n)_n$ in E:

$$\exists M > 0 \ \forall n \in \mathbb{N} \ \forall x \in E \ |f_n(x)| \leq M\,.$$

It follows that

$$|f_n g_n - fg| = |f_n(g_n - g) + g(f_n - f)| \leq |f_n||g_n - g| + |g||f_n - f| \leq M|g_n - g| + N|f_n - f| \rightrightarrows_E 0$$

due to our hypotheses. Hence, $f_n g_n \rightrightarrows_E fg$, and we conclude that $fg \in \mathrm{Cl}(\mathscr{A})$. ■

The following interpolation lemma will be useful too.

Lemma 6. *If* $\mathscr{A}$ *is an algebra (over the scalar field* $\mathbb{K}$, *real o complex) of functions defined on a set* $E \neq \emptyset$ *such that* $\mathscr{A}$ *separates points of* E *and vanishes at point of* E, *then*

$$\forall x, y \in E\,, \ x \neq y \Rightarrow \ \forall \alpha, \beta \in \mathbb{K}\,, \ \exists f \in \mathscr{A} \text{ s.t. } f(x) = \alpha\,, \ f(y) = \beta\,.$$

PROOF. By our assumptions, there exist $g,\ h,\ k \in \mathscr{A}$ such that

$$g(x) \neq g(y), \quad h(x) \neq 0, \quad k(y) \neq 0.$$

Let the functions u and v be defined on E in the following way:

$$u = gk - g(x)k, \quad v = gh - g(y)h.$$

Since $\mathscr{A}$ is an algebra, it is clear that $u,\ v \in \mathscr{A}$. Moreover,

$$u(x) = 0 = v(y), \quad u(y) \neq 0, \quad v(x) \neq 0.$$

Hence, it makes sense to define the function

$$f = \frac{\alpha v}{v(x)} + \frac{\beta u}{u(y)}.$$

It is now clear that $f \in \mathscr{A}$, $f(x) = \alpha$, and $f(y) = \beta$. ■

With the three lemmas at hand, we can proceed to prove Theorem 12 of Stone-Weierstrass.

PROOF. The proof consists of four key steps.

Step (A). We will first check that if $f \in \mathrm{Cl}(\mathscr{A})$, then also $|f| \in \mathrm{Cl}(\mathscr{A})$.

Let $M = \sup_{x\in K} |f(x)| = \max_{x\in K} |f(x)|$ (since K is compact and f is continuous in K) and $\varepsilon > 0$. Thanks to Lema 4, we can find a polynomial $Q(y) = \sum_{k=1}^{n} c_k y^k$, with $c_1,\ c_2,\ \dots,\ c_n \in \mathbb{R}$ and such that

$$\left| \sum_{k=1}^{n} c_k y^k - |y| \right| < \varepsilon, \quad \forall y \in [-M, M].$$

By Lema 5, $\mathrm{Cl}(\mathscr{A})$ is a real algebra and, by our assumptions, $f \in \mathrm{Cl}(\mathscr{A})$, hence $Q \circ f = \sum_{k=1}^{n} c_k f^k \in \mathrm{Cl}(\mathscr{A})$, by a modification of an earlier Observation. (Note that the absence of the constant term of Q is fundamental since we do not have any information on whether constant functions belong to $\mathscr{A}$ or its closure.)

Keeping in mind that, by our choice of M, for all $x \in K$ we have $f(x) \in [-M, M]$, it follows that

$$\big| Q(f(x)) - |f(x)| \big| = \left| \sum_{k=1}^{n} c_k f(x)^k - |f(x)| \right| < \varepsilon, \quad \forall y \in [-M, M].$$

Taking the supremum over $x \in E$, we obtain that $\|Q \circ f - |f|\|_\infty \leq \varepsilon$. Since this can be done for arbitrary ε, it follows that $|f|$ can be uniformly approximated by functions in $\mathrm{Cl}(\mathscr{A})$ and, thus,

$$|f| \in \mathrm{Cl}(\mathrm{Cl}(\mathscr{A})) = \mathrm{Cl}(\mathscr{A}).$$

Step (B). We will now show that if $f_1,\ f_2,\ \dots,\ f_n \in \mathrm{Cl}(\mathscr{A})$, then

$$\min\{f_1, f_2, \dots, f_n\}, \ \max\{f_1, f_2, \dots, f_n\} \in \mathrm{Cl}(\mathscr{A}).$$

Since $\max\{f_1, f_2, \dots, f_n, f_{n+1}\} = \max\{\max\{f_1, f_2, \dots, f_n\}, f_{n+1}\}$ (with an analogous formula for the minimum), if we can prove the statement for two functions, the general case will follow by induction. The case of two functions is easy since

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}.$$

If f, $g \in \mathrm{Cl}(\mathscr{A})$, Lema 5 tells us that $f+g$, $f-g \in \mathrm{Cl}(\mathscr{A})$, hence by Step (A) it follows that $|f-g| \in \mathrm{Cl}(\mathscr{A})$. Therefore, $\max\{f,g\} \in \mathrm{Cl}(\mathscr{A})$.

The reasoning is completely analogous for the minimum by virtue of the formula

$$\min\{f,g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

Step (C). If $f \in \mathscr{C}(K,\mathbb{R})$, $x \in K$ and $\varepsilon > 0$, then there exists $g_x \in \mathrm{Cl}(\mathscr{A})$ s.t. $g_x(x) = f(x)$ and $\forall t \in K$, $g_x(t) > f(t) - \varepsilon$.

To check this, we can argue as follows. By Lema 5, since $\mathscr{A}$ is a real algebra, $\mathrm{Cl}(\mathscr{A})$ is also such. Knowing that $\mathscr{A}$ separates points on K and $\mathscr{A} \subset \mathrm{Cl}(\mathscr{A})$, we know that $\mathrm{Cl}(\mathscr{A})$ also separates points on K. Since $\mathscr{A}$ vanishes at no point of K, for the same reason $\mathrm{Cl}(\mathscr{A})$ also vanishes at no point of K.

We have, thus, verified that the algebra $\mathrm{Cl}(\mathscr{A})$ fulfills all conditions of Lema 6. Hence, for $y \in K$ arbitrary, choosing $\alpha = f(x)$, $\beta = f(y)$, we obtain

$$\exists h_y \in \mathrm{Cl}(\mathscr{A}) \quad \text{s.t.} \quad h_y(x) = f(x)\,,\ h_y(y) = f(y)\,.$$

Since $h_y \in \mathrm{Cl}(\mathscr{A}) \subset \mathscr{C}(K,\mathbb{R})$, due to continuity of h_y and of f at the point y and also to the fact that $h_y(y) = f(y)$, we can find an open subset U_y of K such that $y \in U_y$ and

$$\forall t \in U_y \qquad -\frac{\varepsilon}{2} < h_y(t) - h_y(y) = h_y(t) - f(y) < \frac{\varepsilon}{2}, \quad -\frac{\varepsilon}{2} < f(y) - f(t) < \frac{\varepsilon}{2}.$$

In particular, for every $t \in U_y$ we have $h_y(t) > f(t) - \varepsilon$.

This can be done for all $y \in K$. Obviously, $K = \cup_{y\in K}\{y\} \subset \cup_{y\in K} U_y$ and, since K is compact, there exist points y_1, y_2, ..., $y_n \in K$ such that $K \subset \cup_{k=1}^{n} U_{y_k}$.

By Step (B), we know that $g_x = \max\{h_{y_1}, h_{y_2}, \ldots, h_{y_n}\} \in \mathrm{Cl}(\mathscr{A})$. Obviously,

$$g_x(x) = \max_{1\le k\le n} h_{y_k}(x) = \max_{1\le k\le n} f(x) = f(x)\,.$$

We also know that

$$g_x(t) \ge h_{y_k}(t) > f(t) - \varepsilon\,, \quad \forall k \in \{1,2\ldots,n\}\,,\ \forall t \in U_{y_k}\,.$$

Since the sets U_{y_k} cover K, we have that $\forall t \in K$, $g_x(t) > f(t) - \varepsilon$.

Step (D). $f \in \mathscr{C}(K,\mathbb{R})$, $\varepsilon > 0 \Rightarrow \exists h \in \mathrm{Cl}(\mathscr{A})$ s.t. $|h(x) - f(x)| < \varepsilon$, $\forall x \in K$.

If we prove this, it will follow that $f \in \mathrm{Cl}(\mathrm{Cl}(\mathscr{A})) = \mathrm{Cl}(\mathscr{A})$ and the proof will be complete.

Given $x \in K$, consider the functions g_x constructed in the previous step. Since f, $g_x \in \mathscr{C}(K,\mathbb{R})$ and $g_x(x) = f(x)$, it follows (like before) that for each $x \in K$ there exists an open set V_x such that $x \in V_x$ and

$$\forall t \in V_x\,, \quad g_x(t) < f(t) + \varepsilon\,.$$

Since $K \subset \cup_{x\in K} V_x$ and K is compact, there exist $x_1, \ldots x_m$ s.t. $K \subset \cup_{k=1}^{m} V_{x_k}$.

Let $h = \min\{g_{x_1}, g_{x_2}, \ldots, g_{x_m}\}$. By Step (B), we know that $h \in \mathrm{Cl}(\mathscr{A})$.

Since

$$\forall t \in K\,,\ \forall k \in \{1,2\ldots,m\}\,, \quad f(t) - \varepsilon < g_{x_k}(t)\,,$$

it follows that $\forall t \in K$, $f(t) - \varepsilon < h(t)$. Hence

$$\forall t \in K\,, \quad -\varepsilon < h(t) - f(t) < \varepsilon\,,$$

and we are done. ■

Example 12. *Consider* $\mathscr{A} = \{f \in \mathscr{C}[a,b] : f(b) = 0\}$*. This is a uniformly closed subalgebra of* $\mathscr{C}[a,b]$ *that separates points (since it contains the linear function* $f(x) = x - b$*). Nonetheless, all the functions in* $\mathscr{A}$ *vanish at the point* b*, so one condition in the theorem is not satisfied. We can actually see that* $\mathscr{A}$ *is not dense in the space since it is closed and does not coincide with the entire space.*

It turns out that the statement of the Stone-Weierstrass theorem as given by Theorem 12 is false for complex algebras. Thus, we should add at least another assumption to those considered in the real case in order to obtain a correct result. This can actually be done easily. The following example shows the path we should follow.

Example 13. *Let our compact set* K *be the unit circle:* $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{it} : t \in [0, 2\pi]\}$ *(in other areas of Mathematics often denoted by* $\mathbb{S}^1$*) and consider the following collection of functions*

$$\mathscr{A} = \{f(e^{it}) = \sum_{n=0}^{N} c_n e^{int} : t \in [0, 2\pi], c_n \in \mathbb{C}\}.$$

It is easy to see that $\mathscr{A}$ *is a complex algebra that separates points on* $\mathbb{T}$ *since it contains the identity function (restricted to* $\mathbb{T}$*):* $f(e^{it}) = e^{it}$*. Also, it vanishes at no point of* $\mathbb{T}$ *since it contains the constant functions.*

However, there exist continuous functions on $\mathbb{T}$ *that do not belong to the uniform closure of* $\mathscr{A}$*. In order to check this, the following observation is crucial: every function* $g \in \mathscr{A}$ *satisfies the identity*

$$\int_0^{2\pi} g(e^{it}) e^{it} dt = 0.$$

Then, if $f \in \mathrm{Cl}\mathscr{A}$*, let* $f_n \in \mathscr{A}$ *be such that* $f_n \rightrightarrows_{\mathbb{T}} f$*. Thanks to uniform convergence, by Theorem 7 we can exchange the limit and the integral so as to obtain that*

$$\int_0^{2\pi} f(e^{it}) e^{it} dt = \lim_{n\to\infty} \int_0^{2\pi} f_n(e^{it}) e^{it} dt = 0.$$

It is now immediate that the function $f(e^{it}) = e^{-it}$ *does not have this property since the integral above yields the value* 2π*. This means that this function, although continuous on* $\mathbb{T}$*, cannot belong to* $\mathrm{Cl}(\mathscr{A})$*.*

What was the problem with the algebra $\mathscr{A}$ from the above example? One easily observes that $f(e^{it}) = e^{-it}$ is the complex conjugate of the identity function on $\mathbb{T}$, which does belong to $\mathscr{A}$. This inspires the following approach.

For a complex function $f = u + iv$, where u and v take on real values, denote by $\overline{f}$ its complex conjugate function $\overline{f} = u - iv$.

Definition. Let $\mathscr{A}$ an algebra of functions $f : E \to \mathbb{C}$, where E is a metric space. The algebra $\mathscr{A}$ is said to be *self-adjoint* (or *closed under conjugation*) if for every function $f \in \mathscr{A}$, the function $\overline{f} \in \mathscr{A}$.

Example 14. *The complex algebras* $\mathscr{C}[x]$, $\mathscr{C}[x]|_{[a,b]}$ *and* $\mathscr{P}(z, \overline{z})$ *considered earlier are self-adjoint.*

Now we can formulate correctly the complex version of Theorem 12, adding only one hypothesis.

Theorem 13. *(Stone-Weierstrass, complex case). Let K a compact metric space and let $\mathscr{A} \subset \mathscr{C}(K,\mathbb{R})$ a self-adjoint algebra that separates points on K and vanishes at no point of K. Then* $\mathrm{Cl}(\mathscr{A}) = \mathscr{C}(K,\mathbb{C})$.

PROOF. Let $\mathscr{A}_{\mathbb{R}} = \{f \in \mathscr{A} : f : K \to \mathbb{R}\}$. In the first place, it is important to note that $\mathscr{A}_{\mathbb{R}} \neq \emptyset$. In fact, if $f \in \mathscr{A}$, with $f = u + iv$, u, $v : K \to \mathbb{R}$, then by our assumptions, $\overline{f} = u - iv \in \mathscr{A}$. Hence the functions

$$\mathrm{Re}\, f = \frac{f + \overline{f}}{2}, \qquad \mathrm{Im}\, f = \frac{f - \overline{f}}{2i}$$

since both are in $\mathscr{A}$ and take on only real values.

It is also easy to see that $\mathscr{A}_{\mathbb{R}}$ is a real algebra. For example, if f, $g \in \mathscr{A}_{\mathbb{R}}$, then f, $g \in \mathscr{A}$, hence $fg \in \mathscr{A}$; on the other hand, $fg : K \to \mathbb{R}$ and therefore $fg \in \mathscr{A}_{\mathbb{R}}$. (The remaining conditions are easily checked in the same way.)

$\mathscr{A}_{\mathbb{R}}$ separates points on K: if x, $y \in K$ and $x \neq y$, then according to Lema 5 (valid also for complex algebras) there exists an interpolating function $f \in \mathscr{A}$ such that $f(x) = 1$, $f(y) = 0$. If we write $f = u + iv$, where u, v are real-valued functions, comparing the real and imaginary parts it follows that $u(x) = 1$, $u(y) = 0$. Obviously, $u \in \mathscr{A}_{\mathbb{R}}$.

$\mathscr{A}_{\mathbb{R}}$ vanishes at no point of K: by assumption, if $x \in K$, we can find a function $g \in \mathscr{A}$ with $g(x) \neq 0$. Then there exists a number $\lambda \in \mathbb{C}$ such that $\lambda g(x) > 0$ (it suffices to choose the argument of λ to be the opposite of the argument of $g(x)$). Now consider the function $f = \lambda g$ and let $f = u + iv$. Then $u \in \mathscr{A}_{\mathbb{R}}$ and, recalling that $f(x) > 0$, it follows that $u(x) > 0$.

Finally, by Theorem 12, $\mathscr{C}(K,\mathbb{R}) = \mathrm{Cl}(\mathscr{A}_{\mathbb{R}}) \subset \mathrm{Cl}(\mathscr{A})$. If $f \in \mathscr{C}(K,\mathbb{C})$, writing again $f = u + iv$, it follows that u, $v \in \mathscr{C}(K,\mathbb{R}) \subset \mathrm{Cl}(\mathscr{A})$. Since $\mathrm{Cl}(\mathscr{A})$ is a complex algebra (by Lema 5), it follows that $f = u + iv \in \mathrm{Cl}(\mathscr{A})$. ■

Corollary 3. *Every function in the complex space $\mathscr{C}([a,b],\mathbb{C})$, equipped with the usual norm, can be uniformly approximated by polynomials with complex coefficients.*

PROOF. The polynomials with complex coefficients (considered as functions of the real variable $x \in [a,b]$) obviously form a complex algebra of continuous functions that separates points on $[a,b]$ and vanishes at no point of the interval. Moreover, this algebra is self-adjoint since if $p(x) = a_n x^n + \dots a_1 x + a_0$, with $a_k \in \mathbb{C}$, then

$$\overline{p(x)} = \overline{a_n} x^n + \dots \overline{a_1} x + \overline{a_0}$$

since the values of $x \in [a,b]$ are real. Hence, we may apply Theorem 13 directly. ■

After reviewing the proofs of Theorem 12 and Theorem 13, we observe very few features of metric spaces have been used in them. The main property used was the Heine-Borel one (covering). It actually turns out that it is not difficult to extend the result to the case of compact Hausdorff topological spaces. Such spaces share many properties with compact metric space. Recall that a topological space X is said to be a *Hausdorff space* if, any two distinct points in X have disjoint open neighborhoods. (Of course, in a metric space, two sufficiently small open balls will suffice.)

There also exist more general and more precise versions in the sense that it is not even necessary to assume that the algebra vanishes at no point, obtaining a dichotomy: if all the functions vanish at a given point, then the closure is the set of functions in the space that vanish at that point. Folland's book mentioned in the Bibliography is recommended for such generalizations.

The Weierstrass theorem for trigonometric polynomials. Again, let

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{it} : t \in [0, 2\pi]\}$$

be the unit circle (in Topology and Geometry usually denoted by $\mathbb{S}^1$). The espacio $\mathscr{C}(\mathbb{T})$ of continuous functions on $\mathbb{T}$ can be identified with the space of periodic continuous functions

$$\tilde{C}[0, 2\pi] = \{f \in \mathscr{C}[0, 2\pi] : f(0) = f(2\pi)\}.$$

(Periodic in the sense that they can be extended continuously to the entire real line $\mathbb{R}$ to be periodic with period 2π.) The canonical form of establishing a bijection between the two spaces is the following. Denote by E the complex exponential-trigonometric function $E(t) = e^{it}$. It is easy to see that every $f \in \tilde{C}[0, 2\pi]$ can be written as $f(t) = g(E(t))$, where $g \in \mathscr{C}(\mathbb{T})$ and is unique. Conversely, for every $g \in \mathscr{C}(\mathbb{T})$, the function f defined in this way is unique and $f \in \tilde{C}[0, 2\pi]$.

Typical examples of function in $\mathscr{C}(\mathbb{T})$ are the *complex trigonometric polynomial.* These are functions of the form

$$P(z) = \sum_{k=-N}^{N} c_k z^k, \qquad c_k \in \mathbb{C}\ z \in \mathbb{T},$$

which can also be written as

$$P(e^{it}) = \sum_{k=-N}^{N} c_k e^{ikt}, \qquad c_k \in \mathbb{C},\ t \in [0, 2\pi].$$

Note that this collection includes the *real trigonometric polynomials*, which are functions of the form

$$Q(t) = \sum_{k=0}^{N} a_k \cos kt + \sum_{k=1}^{N} b_k \sin kt, \qquad a_k, b_k \in \mathbb{R},\ t \in [0, 2\pi].$$

The explanation for this is very simple. Starting from the known formulas

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \qquad \sin t = \frac{e^{it} - e^{-it}}{2i} = i\frac{e^{-it} - e^{it}}{2},$$

we see that

$$\begin{aligned} Q(t) &= \sum_{k=0}^{N} a_k \cos kt + \sum_{k=1}^{N} b_k \sin kt = \sum_{k=0}^{N} a_k \frac{e^{ikt} + e^{-ikt}}{2} + i \sum_{k=1}^{N} b_k \frac{e^{-ikt} - e^{ikt}}{2} \\ &= \sum_{k=0}^{N} \frac{a_k - ib_k}{2} e^{ikt} + \sum_{k=1}^{N} \frac{a_k + ib_k}{2} e^{-ikt} = \sum_{k=0}^{N} \frac{a_k - ib_k}{2} e^{ikt} + \sum_{k=-N}^{-1} \frac{a_{-k} + ib_{-k}}{2} e^{ikt} = \sum_{k=-N}^{N} c_k e^{ikt} \end{aligned}$$

for the obvious choice of the numbers $c_k \in \mathbb{C}$.

Here is the analogue of Weierstrass' theorem for trigonometric polynomials.

Theorem 14. *(Weierstrass-Fejér). Every function $f \in \mathscr{C}(\mathbb{T})$ with real values can be approximated uniformly on $\mathbb{T}$ by real trigonometric polynomials.*

Analogously, every function $f \in \mathscr{C}(\mathbb{T})$ with complex values can be approximated uniformly on $\mathbb{T}$ by complex trigonometric polynomials.

PROOF. We prove only the complex case, the real one being an easy (and completely analogous) exercise.

Let $\mathscr{A}$ be the colección of all complex trigonometric polynomials. Obviously, $\mathscr{A}$ is an algebra of continuous functions on $\mathbb{T}$ and is self-adjoint. Moreover, $\mathscr{A}$ separates points on $\mathbb{T}$ since it contains the identity function on $\mathbb{T}$ and vanishes at no point of $\mathbb{T}$ since it contains all constant functions. The conclusion follows by Theorem 13. ■

Characterizations of relatively compact sets in some metric spaces

We expand upon our earlier notes by proving some general results regarding compactness such as the result (stated without proof in the review notes) relating total boundedness and compactness. Next, we characterize the relatively compact sets in some specific metric spaces. More specifically, we prove a general version of the theorem of Arzelà-Ascoli for $\mathscr{C}(K)$ (where K is a compact metric space rather than just the interval $[a,b]$ in a typical undergraduate course) and the theorem of Frechét for the sequence spaces ℓ^p, $1 \le p < \infty$.

ℓ^p spaces. We introduce an important family of normed (hence, also metric) spaces, beginning with an important special case.

Definition. The space ℓ^2 is defined as the set of all sequences $x = (x_1, x_2, \dots)$ for which $\sum_{n=1}^{\infty} |x_n|^2 < \infty$.

Many texts often (in a rather informal way) refer to such sequences as *square-summable sequences*. However, one should be aware that, as always, $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ implies that the series $\sum_{n=1}^{\infty} x_n^2$ is convergent (its sum could be a real or complex number) but not the other way around. For example, if $x_n = \frac{i^n}{\sqrt{n}}$ then $\sum_{n=1}^{\infty} x_n^2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is a convergent alternating series by Leibniz' criterion, while $\sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ is the well-known harmonic series which is divergent.

Proposition 16. *The expression $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$ defines an inner product in ℓ^2. This inner product, in turn, induces the norm defined by $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$.*

PROOF. We omit the proof since it can be found in most of the recommended texts. Moreover, in some of the earlier courses we have already seen at least the finite versions (in $\mathbb{R}^n$ and $\mathbb{C}^n$) of the following inequality, called the *Cauchy-Schwarz inequality*:

$$|\langle x, y\rangle| = \left|\sum_{n=1}^{\infty} x_n \overline{y_n}\right| \le \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{1/2} = \|x\|_2 \|y\|_2$$

Note that this is somewhat stronger than just the inequality $|\langle x, y\rangle| \le \|x\|_2\|y\|_2$ (which is the usual statement of the Cauchy-Schwarz inequality for inner product spaces). The reason for this is that in $\mathbb{R}^n$ and $\mathbb{C}^n$ we have more structure, hence there is another term in between these two quantities. ■

Generalizing the definition of square-summable sequences ℓ^2, we can consider p-summable sequences for any finite positive p.

Definition. Let $0 < p < \infty$. The space ℓ^p is defined as the set of all infinite sequences $x = (x_1, x_2, \ldots)$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < +\infty.$$

The space ℓ^∞ is defined as the set of all bounded infinite sequences.

As an application of the usual finite *Minkowski inequality*:

$$\left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p},$$

valid for $1 \le p < \infty$ (but not for $0 < p < 1$), by letting $n \to \infty$ we obtain the infinite version of the same inequality, which yields the following statement.

Proposition 17. *The expression* $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$ *defines a norm on* ℓ^p*, whenever* $1 \le p < \infty$.

The natural norm on ℓ^∞ is defined by $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

Proposition 18. *If* $0 < p < q \le \infty$ *then* $\ell^p \subset \ell^q$.

PROOF. Let $x \in \ell^p$ be arbitrary. Since the series $\sum_{n=1}^{\infty} |x_n|^p$ converges, it follows that $\lim_{n\to\infty} |x_n|^p = 0$, hence $\lim_{n\to\infty} |x_n| = 0$. In particular, there exists $N \in \mathbb{N}$ such that $|x_n| \le 1$ for all $n \ge N$. But then $|x_n|^q \le |x_n|^p$ since $p < q$ so by Comparison Test we conclude that the series $\sum_{n=1}^{\infty} |x_n|^q$ converges.

Note that the above argument also shows that x is a bounded sequence, so in particular $x \in \ell^\infty$. ■

Example 15. *The sequence* $x = (\frac{1}{n})_{n=1}^{\infty}$ *belongs to* ℓ^p *if only if* $p > 1$*. This is so because of the well-known fact from Calculus that* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges whenever* $p > 1$ *and diverges for* $p \le 1$ *(to see this, use for example the integral test for positive series).*

In view of Proposition 18, we have $\ell^1 \subset \bigcap_{p>1} \ell^p$*. Note that this example also shows that* $\ell^1 \ne \bigcap_{p>1} \ell^p$.

The following property justifies the notation ℓ^∞.

Exercise 1. *For each fixed sequence* $x = (x_1, x_2, \ldots)$, *show that* $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

Finally, the following important result will be proved later in much greater generality.

Theorem 15. ℓ^p *is a complete metric space for all* $p \in (0, \infty]$. *In particular, it is a Banach space when* $1 \le p \le \infty$ *and a Hilbert space when* $p = 2$.

Total boundedness and relative compactness. In this section we establish a new characterization of compactness and relative compactness.

Diameter and bounded sets. The concept of a bounded set, well known from Euclidean spaces, is easy to extend to the setting of normed spaces and even metric spaces.

Definition. Let (X, d) be a metric space. A set $A \subset X$ is said to be *bounded* if there exists a point $a \in X$ such that $\sup_{x\in X} d(a, x) < +\infty$.

Of course, this is the same as saying that there exist a constant $M > 0$ and a point $a \in X$ such that $d(x, a) \le M$ for all $x \in X$.

Definition. The *diameter* of the set A in a metric space (X, d) is $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$.

Note that the value $+\infty$ is allowed.

Example 16. *(a) In every metric space, any finite set has finite diameter. (If it consists of n points, there are only* $\binom{n}{2}$ *different values of the distances between its points* $x_1, \ldots, x_n$ *and a finite maximum among such distances is attained).*

(b) If $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, *then* $\operatorname{diam} A = 1$ *and this supremum is not attained.*

(c) The x*-axis in the Euclidean plane with the usual metric (and any line in general) has infinite diameter.*

Proposition 19. *Every compact set has a finite diameter. Moreover, the infimum in the above definition is attained.*

PROOF. We leave this is an exercise (see sample questions for the Midterm Exam). Hint: first prove and then use the continuity of $d(x, y)$ as a function of two variables. ■

Proposition 20. *(a) Let* (X, d) *be a metric space. A set* $A \subset X$ *is bounded if and only if* $\operatorname{diam}(A) < +\infty$.

(b) If X *is a normed space and* $A \subset X$, *then* A *is bounded if and only if* $\sup_{x\in A} \|x\| < +\infty$.

PROOF. (a) This is a simple exercise. See Problem Set 2.

(b) This is also easy since the induced metric in X is given by $d(x,y) = \|x-y\|$. If $\sup_{x\in A}\|x\| < +\infty$, this means that for the point $a = 0$ we have $\sup_{x\in A} d(x,a) < +\infty$. Conversely, if there is a point $a \in X$ such that $R = \sup_{x\in A}\|x-a\| < +\infty$, then for every $x \in A$ we have $\|x\| \le \|x-a\| + \|a\| \le R + \|a\|$, a fixed positive number. ■

Totally bounded sets. We now discuss a concept that is fundamental in describing compact sets.

Definition. Let $\varepsilon > 0$, (X,d) a metric space and $A \subset X$. A collection M_ε of points is said to be an ε-*net* for A if $A \subset \bigcup_{x\in M_\varepsilon} B(x;\varepsilon)$.

A subset of X is said to be *totally bounded* if for every $\varepsilon > 0$ it has a finite ε-net.

Note that in the definition we did not specify whether we require $M_\varepsilon \subset A$ or simply $M_\varepsilon \subset X$. It turns out that this is completely irrelevant for the definition of a totally bounded set. In other words, the following two ways of defining a totally bounded set are equivalent:

$$(1)\quad \forall\varepsilon > 0\ \exists x_1,\dots,x_n \in X \text{ s.t.} \qquad A \subset \bigcup_{k=1}^{n} B(x_k;\varepsilon),$$

$$(2)\quad \forall\varepsilon > 0\ \exists x_1,\dots,x_n \in A \text{ s.t.} \qquad A \subset \bigcup_{k=1}^{n} B(x_k;\varepsilon).$$

(Clearly,) (2) $\Longrightarrow$ (1). We leave this as an exercise for Problem Set 2.

We review some properties related to total boundedness.

Proposition 21. *Every totally bounded set is bounded.*

PROOF. Given $\varepsilon > 0$, let $M_\varepsilon = \{x_k : 1 \le k \le n\}$ be a finite ε-net for A. Then $\operatorname{diam} M_\varepsilon < +\infty$. Let x, $y \in A$ be arbitrary. Since $A \subset \bigcup_{k=1}^{n} B(x_k;\varepsilon)$, there exist integers l, $m \in \{1,\dots,n\}$ such that $x \in B(x_l;\varepsilon)$ and $y \in B(x_m;\varepsilon)$. Then, by the triangle inequality,

$$d(x,y) \le d(x,x_l) + d(x_l,x_m) + d(x_m,y) < \operatorname{diam} M_\varepsilon + 2\varepsilon.$$

This proves that $\operatorname{diam} A \le \operatorname{diam} M_\varepsilon + 2\varepsilon < +\infty$. ■

Proposition 22. *In the Euclidean space $\mathbb{R}^n$, boundedness and total boundedness are equivalent.*

PROOF. We need only show that boundedness implies total boundedness, in view of Proposition 21. Let A be a bounded set in $\mathbb{R}^n$. Then there exists $R > 0$ such that for each $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and each $k \in \{1,\dots,n\}$ we have $|x_k| \le R$. In other words, $A \subset [-R,R]^n = Q$, an n-dimensional cube. Let $\varepsilon > 0$. By dividing Q into finitely many smaller cubes (of equal and sufficiently small side lengths each), the vertices of these cubes will form a finite ε-net for A. (The reader is invited to check that a quantity smaller than $\varepsilon/\sqrt{n}$ is a "sufficiently small size".) ■

Here is an example of a totally bounded set in a non-Euclidean space.

Example 17. *Consider the set* $A = \{x = (x_n)_{n=1}^{\infty}, : \forall n \in \mathbb{N}, |x_n| \leq \frac{1}{2^n}\}$. *(Check by Comparison test that any member of A is an* ℓ^2 *sequence.) Then A is totally bounded in* ℓ^2. *This can be seen as follows. Given* $\varepsilon > 0$, *pick* $N \in \mathbb{N}$ *such that* $\frac{1}{2^N} < \frac{\varepsilon}{2}$. *Given* $x = (x_1, x_2, \ldots) \in A$, *consider* $y = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$. *Then*

$$\|x - y\|_2 = \left(\sum_{n=N+1}^{\infty} |x_n|^2\right)^{1/2} \leq \left(\sum_{n=N+1}^{\infty} \frac{1}{4^n}\right)^{1/2} = \frac{1}{2^n}\left(\sum_{k=1}^{\infty} \frac{1}{4^k}\right)^{1/2} < \frac{1}{2^N} < \frac{\varepsilon}{2}.$$

The set B of all such y is an $\varepsilon/2$*-net for A. Since we can identify B with a subset of* $\mathbb{R}^N$ *which is bounded, B is totally bounded, hence there exists a finite* $\varepsilon/2$*-net for B in* $\mathbb{R}^N$. *This same set is clearly a finite* ε*-net for A.*

The sequences $e_1 = (1,0,0,0,\ldots)$, $e_2 = (0,1,0,0,\ldots)$, $e_3 = (0,0,1,0,\ldots)$, etc. are important elements of ℓ^2 since they form an orthonormal basis; this will be explained later when we discuss Hilbert space techniques. The following example involving these elements shows that the converse to Proposition 21 is false.

Example 18. *The set* $\{e_n : n \in \mathbb{N}\}$ *is bounded in* ℓ^2 *since* $\|e_n\|_2 = 1$ *for each* $n \in \mathbb{N}$. *However, this set is not totally bounded: for every two distinct elements* e_m *and* e_n *we have* $\|e_m - e_n\|_2 = \sqrt{2}$, *which shows that for* $0 < \varepsilon < \sqrt{2}/2$, *no ball of radius* ε *can contain more than one element* e_n. *Hence no* ε*-net for such values of* ε *can possibly be finite (since it cannot even cover the set* $\{e_n : n \in \mathbb{N}\}$*).*

There is an alternative formulation of total boundedness. Namely, instead of considering covers by open balls $B(x;\varepsilon)$ it is often convenient to use instead arbitrary sets of diameter ε (or at most ε). For a cover by sets of diameter ε, the term ε-*cover* is also used often.

Proposition 23. *A set is totally bounded if and only if for every* $\varepsilon > 0$ *it admits a finite cover by sets of diameter at most* ε.

PROOF. On the one hand, if a set is totally bounded, for any $\varepsilon > 0$ it admits a finite cover by open balls of radius $\varepsilon/2$ each. It is readily seen that each such ball is a set of diameter at most ε: if $x, y \in B(c;r)$ then $d(x,y) \leq d(x,c) + d(c,y) < \varepsilon$, hence $\operatorname{diam} B(c;r) = \sup\{d(x,y) : x,y \in B(c;r)\} \leq \varepsilon$.

On the other hand, suppose that for every $\varepsilon > 0$ a set A has a finite cover by sets of diameter at most ε. Consider an arbitrary but fixed $\varepsilon_0 > 0$. Then A can be covered by sets of diameter at most $\frac{\varepsilon_0}{2}$. Since any such set, say V, is contained in a an open ball of radius ε_0 (fix any point $x \in V$ and observe that for all $y \in V$ we have $d(x,y) \leq \frac{\varepsilon_0}{2} < \varepsilon_0$, hence $V \subset B(x;\varepsilon_0)$). Thus, A admits a finite cover by open balls of radius ε_0. Since this is true for arbitrary $\varepsilon_0 > 0$, we conclude that A is totally bounded. ■

The reader is invited to check that we can also use covers by sets of diameter strictly smaller than ε instead of ε-covers (for arbitrary $\varepsilon > 0$). Such alternative definition of total boundedness is also equivalent to the standard one. In summary, in the proofs we do not have to worry about whether we use the radius of a ball or a diameter of a set, or whether we work with the $<$ or $\leq$ sign. Thus, from now on we will change from one formulation of total boundedness to another rather freely.

Exercise 2. *In every metric space, we have* $\operatorname{diam} B(x; \frac{\varepsilon}{2}) = \varepsilon$. *True or false? Explain your answer by giving a very short proof or a simple counterexample.*

Total boundedness and compactness. The following result explains the relevance of total boundedness by establishing a direct relationship between this concept and relative compactness. In what follows, we assume the equivalence between compactness (expressed in terms of open covers) and sequential compactness in a metric space, which should be known from some earlier course.

Theorem 16. *Let* (X, d) *be a complete metric space. A set* $F \subset X$ *is relatively compact if and only if it is totally bounded.*

PROOF. ($\Longrightarrow$): One typical way of proving this is by showing that if a set is not totally bounded, then it contains a sequence which has no Cauchy subsequence. We leave this as an exercise (see Problem Set 2).

($\Longleftarrow$): Let $F \subset X$ be totally bounded. We will show that every sequence $(x_n)_n$ in F has a subsequence convergent to some element of X; this will imply relative compactness of F.

Let $(x_n)_n$ be an infinite sequence in F. For every $n \in \mathbb{N}$ we can cover F by finitely many sets of diameter at most $1/n$. Thus, in particular, $F \subset \bigcup_{j=1}^{m_1} V_j^{(1)}$, with $\operatorname{diam} V_j^{(1)} \leq 1$. At least one of the sets $V_j^{(1)}$ contains an infinite subsequence of $(x_n)_n$, say $(x_n^{(1)})_n$. Also, $F \subset \bigcup_{j=1}^{m_2} V_j^{(2)}$ with $\operatorname{diam} V_j^{(1)} \leq \frac{1}{2}$, hence at least one of the sets $V_j^{(2)}$ contains an infinite subsequence of $(x_n^{(1)})_n$, say $(x_n^{(2)})_n$. Next, $F \subset \bigcup_{j=1}^{m_3} V_j^{(3)}$ where $\operatorname{diam} V_j^{(3)} \leq \frac{1}{3}$, and therefore at least one of these sets contains an infinite subsequence of $(x_n^{(2)})_n$, say $(x_n^{(3)})_n$. Continuing in this fashion, we obtain a countable number of sequences:

$$\begin{array}{lll} x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, & \ldots & \text{with } d(x_m^{(1)}, x_n^{(1)}) < 1 \\ x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, & \ldots & \text{with } d(x_m^{(2)}, x_n^{(2)}) < \dfrac{1}{2} \\ x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, & \ldots & \text{with } d(x_m^{(3)}, x_n^{(3)}) < \dfrac{1}{3} \\ \ldots\ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Here each one of the subsequences is a subsequence of the one preceding it. Thanks to this, we know that in the list of elements of the sequence $(x_n)_n$, the term $x_1^{(1)}$ comes before the term $x_2^{(2)}$, which in turn comes before $x_3^{(3)}$, and so on. Thus, the *diagonal sequence* $(x_n^{(n)})_n$ is a subsequence of $(x_n)_n$. Moreover, its elements satisfy the inequalities $d(x_m^{(m)}, x_n^{(n)}) < \frac{1}{n}$ whenever $m > n$, since each $x_m^{(m)}$ is one of the members of the sequence $x_1^{(n)}$, $x_2^{(n)}$,... Hence, the diagonal sequence $(x_n^{(n)})_n$ is a Cauchy sequence in X. But X is complete by assumption, hence $(x_n^{(n)})_n$ is convergent (to some element of X). We have, thus, been able to extract a convergent sequence of our initial sequence $(x_n)_n$. This shows that F is relatively compact. ■

Corollary 4. *A metric space X is compact if and only if X is complete and totally bounded.*

PROOF. ($\Longrightarrow$): Every compact metric space is complete (Exercise, Problem Set 2). Total boundedness follows from Theorem 16.

($\Longleftarrow$): For the entire space X compactness coincides with relative compactness, which is implied by total boundedness, so the statement again follows from Theorem 16. ■

Example 19. *We know from Example 18 that the set $E = \{e_n : n \in \mathbb{N}\}$ in ℓ^2 is bounded but not totally bounded. We can now argue this in a slightly different way: whenever $m \neq n$, we know that $\|e_m - e_n\|_1 = \sqrt{2}$, hence E does not contain a Cauchy sequence. Therefore it cannot contain a convergent sequence and so it cannot be compact. In view of Theorem 16, it cannot be totally bounded.*

Characterizations of relatively compact sets in some well-known spaces. We are now ready for the main goal of these notes: to characterize the relatively compact subsets of certain known metric spaces (actually, Banach spaces). We prove two such theorems now and may mention some further ones as the course advances.

If one compares the typical proofs of results that characterize relative compactness of certain sets in different metric spaces that can be found in the literature, certain similarities can be observed between them, in the sense that they all apply similar ideas. The following basic property is the essence of most such proofs of total boundedness and was formulated as a general phenomenon in an expository article published in 2010 by the Norwegian mathematicians Hanche-Olsen and Holden.

Lemma 7. *Let (X,d) be a metric space. If for every $\varepsilon > 0$ there exists $\delta > 0$ and there exist another metric space (Y,ρ) and a mapping $\Phi : X \to Y$ such that:*

(1) $\Phi(X)$ is totally bounded in Y;

(2) for every $x, y \in X$, if $\rho(\Phi(x),\Phi(y)) < \delta$ then $d(x,y) < \varepsilon$,

then X is totally bounded.

PROOF. Since $\Phi(X)$ is totally bounded by assumption (1), for the value $\delta = \delta(\varepsilon) > 0$ chosen as in the statement, there exists a finite δ-cover of $\Phi(X)$, say $V_1,\dots,V_n$.

By assumption (2), we see that $\Phi^{(-1)}(V_1)$, ..., $\Phi^{(-1)}(V_1)$ is an ε-cover of X. This is easily checked as follows. If $x \in X$ then $\Phi(x) \in \Phi(X)$, hence for at least some $k \in \{1,2,\dots,n\}$ we have $\Phi(x) \in V_k$, so $x \in \Phi^{(-1)}(V_k)$. This shows that $X \subset \bigcup_{k=1}^{n} \Phi^{(-1)}(V_k)$.

Also, if x, $y \in \Phi^{(-1)}(V_k)$ for some $k \in \{1,2,\dots,n\}$, then $\Phi(x)$, $\Phi(y) \in V_k$, so that $\rho(\Phi(x),\Phi(y)) < \delta$. By (2), it follows that $d(x,y) < \varepsilon$. This shows that also $\operatorname{diam}\Phi^{(-1)}(V_k) < \varepsilon$. This shows that X is totally bounded. ■

We shall see in two proofs that follow how Lemma 7 is applied in a typical way (always in one of the two implications that have to proved when characterizing compactness in different spaces).

Arzelà-Ascoli theorem.The reader is probably familiar with the statement of this classical result for $\mathscr{C}[a,b]$ from an earlier course. It is useful to know that proving the generalization for continuous functions on general compact sets in metric spaces does not require much more work. Let us first recall an important definition.

Definition. Let (X,d) be a metric space and $A \subset X$. A family of functions $\mathscr{F} \subset \mathscr{C}(A)$ is said to be *equicontinuous* if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in \mathscr{F}\ \forall x, y \in A \quad d(x,y) < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon .$$

Example 20. *Let (X,d) be a metric space. Consider any family $\mathscr{F}$ of functions $f : X \to \mathbb{R}$ (or $\mathbb{C}$) that satisfy the Lipschitz condition with the same constant: $|f(x) - f(y)| \leq Md(x,y)$. Clearly, $\mathscr{F}$ is equicontinuous; to see this, simply choose $\delta = \varepsilon/M$.*

In particular, if $I \subset \mathbb{R}$ is an interval and $\mathscr{F}$ is a family of functions $f : I \to \mathbb{R}$ with a uniformly bounded derivative, say, $|f'(t)| \leq M$ for all $t \in \mathbb{R}$, then by Lagranges's mean value theorem known from Calculus: $f(x) - f(y) = f'(\xi)(x-y)$ for some ξ between x and y, we see that $\mathscr{F}$ is an equicontinuous family.

Theorem 17. *(Arzelà-Ascoli) Let K be a compact metric space and $\mathscr{F} \subset \mathscr{C}(K)$. Then $\mathscr{F}$ is relatively compact if and only if it is equicontinuous and uniformly bounded.*

PROOF. ($\Longrightarrow$): Suppose $\mathscr{F}$ is relatively compact; then it is totally bounded by Theorem 16. By Proposition 21 it follows that $\mathscr{F}$ is bounded in the norm of $\mathscr{C}(K)$, meaning that it is a uniformly bounded family.

Thus, it is only left to show that it is equicontinuous. To this end, let $\varepsilon > 0$ and let $f_1, f_2, \ldots, f_n$ be an $\varepsilon/3$-net for $\mathscr{F}$. For an arbitrary function $f \in \mathscr{F}$, there exists at least one $k \in \{1,2,\ldots,n\}$ such that $\|f - f_k\| = \max_{x\in K} |f(x) - f_k(x)| < \varepsilon/3$. Hence, for any $x, y \in K$, we obtain

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < \frac{2\varepsilon}{3} + |f_k(x) - f_k(y)|.$$

Since K is compact and all $f_k \in \mathscr{C}(K)$, by generalized Cantor's theorem each f_k is uniformly continuous on K. Thus, there exists $\delta > 0$ such that

$$x, y \in K \text{ and } d(x,y) < \delta \Rightarrow |f_k(x) - f_k(y)| < \frac{\varepsilon}{3}, \quad \text{for } k \in \{1,2,\ldots,n\}.$$

(To obtain such a δ, first choose δ_1 corresponding to f_1, then δ_2 corresponding to f_2, ..., δ_n corresponding to f_n, and finally take $\delta = \min\{\delta_1,\ldots,\delta_2\}$.) Finally, we see that for such $\delta > 0$ and for arbitrary $x, y \in K$ with $d(x,y) < \delta$ we have $|f(x) - f(y)| < \varepsilon$. This shows that $\mathscr{F}$ is an equicontinuous family.

($\Longleftarrow$): Assume that $\mathscr{F}$ is a uniformly bounded and equicontinuous family in $\mathscr{C}(K)$. We will now show that $\mathscr{F}$ is totally bounded by using Lemma 7.

Let $\varepsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that for all $x, y \in K$ and all $f \in \mathscr{F}$, $d(x, y) < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{3}$.

Since $K \subset \bigcup_{x \in K} V_x$, with all V_x open and $\operatorname{diam} V_x < \delta$ (for example, taking $V_x = B(x, \frac{\delta}{2})$), by compactness of K we know that there exist $V_1, \dots, V_n$ such that $K \subset \cup_{k=1}^n V_k$, where each V_k is a neighborhood of some x_k and $\operatorname{diam} V_k < \delta$. If $x \in V_k$ then $d(x, x_k) < \delta$, hence by the previous paragraph $|f(x) - f(x_k)| < \frac{\varepsilon}{3}$.

Now define $\Phi : \mathscr{F} \to \mathbb{R}^n$ by $\Phi(f) = (f(x_1), \dots, f(x_n))$, where $\mathbb{R}^n$ is equipped with the $\|\cdot\|$ norm. Since $\mathscr{F}$ is a uniformly bounded family, it follows that $\Phi(\mathscr{F})$ is a bounded subset of $\mathbb{R}^n$ (note that each vector in it is bounded by the same constant in every coordinate), hence it is totally bounded by Proposition 22. Furthermore, if $f, g \in \mathscr{F}$ and

$$\|\Phi(f) - \Phi(g)\|_\infty = \sup_{1 \le k \le n} |f(x_k) - g(x_k)| < \frac{\varepsilon}{3},$$

for any $x \in K$ we can find V_k such that $x \in V_k$, so by $d(x, x_k) < \delta$ we get

$$|f(x) - g(x)| \le |f(x) - f(x_k)| + |f(x_k) - g(x_k)| + |g(x_k) - g(x)| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Hence the norm in $\mathscr{C}(K)$ of $f - g$ satisfies $\|f - g\| \le \varepsilon$. We have, thus, proved that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) = \varepsilon/3$ and there exists a metric space (Y, ρ), namely $(\mathbb{R}^n, \|\cdot\|)$ such that $\Phi(\mathscr{F})$ is totally bounded and for all $f, g \in \mathscr{F}$ we have $\|\Phi(f) - \Phi(g)\|_\infty < \delta$ implies $\|f - g\| < \varepsilon$. We can now apply Lemma 7 to conclude that $\mathscr{F}$ is totally bounded, hence relatively compact.

We have given a proof only for real-valued functions but it can easily be modified for complex-valued functions, simply by replacing $\mathbb{R}^n$ by $\mathbb{C}^n$. ■

Frechét's theorem. We will now discuss the result, proved by Maurice Frechét around 1908 in the special case $p = 2$, which describes the relatively compact subsets of the sequence space ℓ^p when $1 \le p < \infty$. Note that the statement would not even make sense in the case $p = \infty$ since there is no series involved.

Definition. Let $F \subset \ell^p$, $1 \le p < \infty$. We say that the series $\sum_{n=1}^\infty |x_n|^p$ *converges uniformly* in F (or uniformly in the variable $x = (x_1, x_2, \dots) \in F$) if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall x \in F \;\; \|x - (x_1, \dots, x_N, 0, 0, \dots)\|_p = \left(\sum_{n > N} |x_n|^p \right)^{1/p} < \varepsilon.$$

Theorem 18. *Let $1 \le p < \infty$ and $F \subset \ell^p$. Then F is relatively compact if and only if it satisfies the following two conditions:*

(1) F is uniformly bounded in each coordinate;

and

(2) the series $\sum_{n=1}^\infty |x_n|^p$ converges uniformly in $x = (x_1, x_2, \dots) \in F$.

PROOF. ($\Longleftarrow$): Suppose that F satisfies conditions (1) and (2). We will show that it is totally bounded, hence relatively compact.

Let $\varepsilon > 0$. Choose N as in (2) corresponding to the value $\varepsilon/3$ instead of ε and then define the mapping $\Phi : F \to \mathbb{R}^N$ by $\Phi(x) = (x_1, \dots, x_N)$. Condition (1) then implies that $\Phi(F)$ is bounded, hence totally bounded, by Proposition 22.

Now, if $x, y \in F$ and

$$\|\Phi(x) - \Phi(y)\|_p = \left(\sum_{n=1}^{N} |x_n - y_n|^p\right)^{1/p} < \frac{\varepsilon}{3} = \delta\,,$$

then by Minkowski's inequality

$$\|x - y\|_p = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p} \le \left(\sum_{n=1}^{N} |x_n - y_n|^p\right)^{1/p} + \left(\sum_{n=N+1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=N+1}^{\infty} |y_n|^p\right)^{1/p} < 3 \cdot \frac{\varepsilon}{3} = \varepsilon\,,$$

By Lemma 7 with $\delta = \varepsilon/3$, F is totally bounded.

($\Longrightarrow$): We leave this as an exercise. See Problem Set 2. ■

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