Foundations of Mathematical Analysis, Master Program, UAM

Notes by Dragan Vukotić, 2022-23

Additional topics from measure and integration (version 2.0)

In these brief notes we review some integration theorems formulated in greater generality than in the typical undergraduate courses. They are more general in two ways. On the one hand, we formulate them for arbitrary positive measures instead of just for Lebesgue measure on the real line. On the other hand, we give stronger or more general versions than the ones known from undergraduate courses.

In all the statements that follow, $(X,\mathcal{M},\mu)$ will be a measure space and E a measurable set (that is, $E\in\mathcal{M}$). Convergence almost everywhere (*a.e.*) is always to be understood with respect to the measure μ. It is sometimes convenient to formulate the results in these notes for functions integrable over E, thus generalizing the functions integrable over the entire space X. By a measurable function on $E\in\mathcal{M}$ we will always mean the restriction of a measurable function $f: X\to\mathbb{C}$ (or $f: X\to\mathbb{R}$) to the set E. By an integrable function over E we will mean the restriction of a measurable function $f: X\to\mathbb{C}$ (or $f: X\to\mathbb{R}$) such that the integral $\int_E f\,d\mu=\int_X f\chi_E\,d\mu$ is finite.

On exchanging the limit and Lebesgue integral

We now review several known results and some new ones regarding the exchange of the limit and the integral, analogous to the well-known one for uniform convergence on a closed bounded interval. Thanks to the fact that we are now dealing with the Lebesgue integral rather than with the Riemann integral, we do not need such strong assumption; usually integrability (and something else) is enough instead of continuity.

The chronological order in which the theorems mentioned here were discovered does not necessarily coincide with the order of exposition in books where the theory is presented. Our first two results refer to non-negative functions; the first one is the only one that is an inequality rather than an equality.

Theorem 1. *(Fatou's Lemma, 1906). Let (f_n) be a sequence of non-negative measurable functions on E. Then*

$$\int_E \liminf_{n\to\infty} f_n\,d\mu \le \liminf_{n\to\infty}\int_E f\,d\mu.$$

We omit the proof here. Note that the above inequality can actually be strict, as the following example shows.

Example 1. *Let $E \in \mathscr{M}$ be such that $\mu(E) > 0$ and $\mu(E^c) > 0$, and define $f_n = \chi_E$ for n odd and $f_n = \chi_{E^c} = 1 - \chi_E$ for n even. Then $\int_X f_n\, d\mu = \mu(E)$ for n odd and $\int_X f_n\, d\mu = \mu(E^c)$ for n even, hence*

$$\liminf_{n\to\infty} \int_X f_n\, d\mu = \min\{\mu(E), \mu(E^c\} > 0$$

while $\liminf_{n\to\infty} f_n = 0$ *and, thus,* $\int_E \liminf_{n\to\infty} f_n\, d\mu = 0$.

Theorem 2. *(Monotone Convergence Theorem, Beppo-Levi and Lebesgue). Let (f_n) be a sequence of measurable functions on E such that $0 \le f_1(x) \le f_2(x) \le f_3(x) \le \ldots\infty$ for all $x \in X$ and suppose that also $f_n(x) \to f(x)$ for all $x \in X$. Then $\int_E f_n\, d\mu \to \int_E f\, d\mu$ as $n \to \infty$.*

Observe that we are not assuming integrability of our functions, so technically the limit on the left-hand side (or even all the terms $\int_E f_n\, d\mu$) could be ∞. In this case, the correct interpretation of the result is that then the limit (the term on the right, $\int_E f\, d\mu$) is also ∞.

It is worth noting that there is also a version of Theorem 2 for decreasing functions; the assumptions are $f_1(x) \ge f_2(x) \ge f_3(x) \ge \ldots 0$ for all $x \in X$ and we also have to require the integrability of f over E. Without this additional assumption, the conclusion will not follow. (It is a good exercise to find a counterexample).

Finally, we should stress that instead of assuming convergence at all points it suffices to assume convergence μ-a.e.

The following classical result is known as the *Lebesgue dominated convergence theorem* (typically abbreviated as LDCT).

Theorem 3. *(Lebesgue, 1904). If f and f_n are integrable functions on E, $f_n \to f$ a.e. on E and g is an integrable function on E such that $|f_n(x)| \le g(x)$ for a.e. point $x \in E$, then $\int_E f_n\, d\mu \to \int_E f\, d\mu$ as $n \to \infty$. Actually, a stronger conclusion holds:*

$$\int_E |f_n - f|\, d\mu \to 0, \quad n \to \infty.$$

Observation. The second conclusion is stronger thanks to the basic inequality

$$\left| \int_E f_n\, d\mu - \int_E f\, d\mu \right| \le \int_E |f_n - f|\, d\mu.$$

Exercise 1. *Compute the integral*

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx$$

in more than one way. Justify your answer.

SOLUTION. □ First of all, writing $f_n(x) = \chi_{[0,n]}(x)\left(1+\frac{x}{n}\right)^n e^{-2x}$, note that

$$\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx = \int_0^\infty f_n(x)\,dx\,.$$

Clearly, for each fixed $x \geq 0$, we have $x \in [0,n]$ for all large enough n, say for $n \geq [x]+1$. From here and by elementary Calculus, it follows that $f_n(x) \to e^x e^{-2x} = e^{-x} = f(x)$ for all $x \geq 0$. Since $\int_0^\infty f(x)\,dx = 1$ (this happens because *the value of the Lebesgue integral of a continuous function on* $[0,\infty)$ *coincides with improper Riemann integral*, a good exercise for you to work out), all we have to do is justify the exchange of the limit and the integral to conclude that the limit in question equals one. This can be done in at least two different ways.

(1) One way of doing this is using the LDCT (our Theorem 3). For this, we need an integrable dominant. To find one, it suffices to see that for each $x \geq 0$ we have

$$f_n(x) \leq \left(1+\frac{x}{n}\right)^n e^{-2x} \leq e^{-x}\,.$$

The first inequality is obvious. Thus, we need only prove the second one, which is equivalent to

$$\left(1+\frac{x}{n}\right)^n e^{-x} \leq 1\,, \quad \forall x \geq 0\,.$$

To this end, consider the function $\varphi(x) = \left(1+\frac{x}{n}\right)^n e^{-x}$. Note that

$$\varphi'(x) = \left(1+\frac{x}{n}\right)^{n-1} e^{-x} - \left(1+\frac{x}{n}\right)^n e^{-x} = -\frac{x}{n}\left(1+\frac{x}{n}\right)^{n-1} e^{-x} \leq 0$$

for all $x \geq 0$. This shows that φ is non-increasing, hence $\varphi(x) \leq \varphi(0) = 1$ for all $x \geq 0$, and we are done.

(2) Another solution exists using Theorem 2. To this end, we need to show that $0 \leq f_n(x) \leq f_{n+1}(x)$ for all $x \geq 0$. The first inequality is obvious and it is the second one that requires some work. The statement is trivially true for $x \in (n, n+1]$ since $f_n(x) = 0$ there and for $x > n+1$ since both functions are zero for such values. Thus, we only have to check that the inequality holds in $[0,n]$. In this interval, both characteristic functions are equal to 1 so the inequality $f_n(x) \leq f_{n+1}(x)$ becomes

$$\left(1+\frac{x}{n}\right)^n \leq \left(1+\frac{x}{n+1}\right)^{n+1},$$

which is further equivalent to

$$\psi(x) = (n+1)\log\left(1+\frac{x}{n+1}\right) - n\log\left(1+\frac{x}{n}\right) \geq 0\,.$$

The derivative of this function is

$$\psi'(x) = \frac{n+1}{x+n+1} - \frac{n}{x+n} = \frac{x}{(x+n)(x+n+1)} \geq 0\,.$$

Hence φ is increasing and since $\varphi(0) = 0$, it follows that the function is non-negative in $[0,n]$, which is what we wanted to prove. Now we can apply the Monotone Convergence Theorem to justify the exchange of limit and the integral in another way. ■

The following result is of a different nature and allows us to deduce the convergence of integrals from pointwise convergence and convergence of L^1-norms of the functions involved.

Theorem 4. *(Littlewood, 1925). If f and f_n are integrable functions on E, $f_n \to f$ a.e. on E and*

$$\int_E |f_n|\, d\mu \to \int_E |f|\, d\mu, \quad n \to \infty,$$

then $\int_E f_n\, d\mu \to \int_E f\, d\mu$ as $n \to \infty$. Actually, a stronger conclusion holds:

$$\int_E |f_n - f|\, d\mu \to 0, \quad n \to \infty.$$

Observation. In general (without any hypothesis on convergence a.e.), none of the conditions that appear in the statement:

$$\int_E |f_n|\, d\mu \to \int_E |f|\, d\mu, \quad \int_E f_n\, d\mu \to \int_E f\, d\mu$$

implies the other. This is where the interest in the theorem comes from. It is convenient to look for simple examples to convince yourself that this is so.

Theorem 5. *(Pratt, 1960). If f and f_n, $n \in \mathbb{N}$, are measurable real-valued functions on E and g and g_n, $n \in \mathbb{N}$, are integrable functions on E such that $f_n \to f$ a.e on E, $g_n \to g$ a.e. on E, $|f_n(x)| \le g_n(x)$ a.e. on E and $\int_E g_n\, d\mu \to \int_E g\, d\mu$ as $n \to \infty$, then f es integrable over E and $\int_E f_n\, d\mu \to \int_E f\, d\mu$ as $n \to \infty$. Again, we also have*

$$\int_E |f_n - f|\, d\mu \to 0, \quad n \to \infty.$$

PROOF. Before embarking on the proof, we note that the hypotheses on the comparison between f_n and g_n automatically imply that g_n and g are non-negative *a.e.* They also imply integrability of f_n (formally not assumed in the statement).

We split the proof into three steps.

Step 1. We will first show the weaker statement: $\int_E f_n\, d\mu \to \int_E f\, d\mu$ as $n \to \infty$. This will be helpful later in proving the rest.

To this end, observe that the assumption $|f_n| \le g_n$ (that is, $-g_n \le f_n \le g_n$; note that it is here where we are using explicitly the fact that our functions are real-valued) implies both $g_n - f_n \ge 0$ and $g_n + f_n \ge 0$ almost everywhere. The first inequality, together with Fatou's Lemma and the assumption on convergence: $\int_E g_n\, d\mu \to \int_E g\, d\mu$, $n \to \infty$, allows us to conclude that

$$\int_E (g-f)\, d\mu \le \liminf_{n\to\infty} \int_E (g_n - f_n)\, d\mu = \lim_{n\to\infty} \int_E g_n\, d\mu - \limsup_{n\to\infty} \int_E f_n\, d\mu = \int_E g\, d\mu - \limsup_{n\to\infty} \int_E f_n\, d\mu.$$

Note that we have just used two basic properties known from Calculus:

$$\lim a_n = a \quad \Longrightarrow \quad \liminf_{n\to\infty} (a_n + b_n) = a + \liminf_{n\to\infty} b_n, \quad \liminf_{n\to\infty} (-c_n) = -\limsup_{n\to\infty} c_n.$$

It follows from the inequality obtained that $\limsup_{n\to\infty}\int_E f_n\,d\mu \le \int_E f\,d\mu$.

In a completely analogous fashion, we get

$$\int_E (g+f)\,d\mu \le \liminf_{n\to\infty}\int_E (g_n+f_n)\,d\mu = \lim_{n\to\infty}\int_E g_n\,d\mu + \liminf_{n\to\infty}\int_E f_n\,d\mu = \int_E g\,d\mu + \liminf_{n\to\infty}\int_E f_n\,d\mu.$$

Thus, $\int_E f\,d\mu \le \liminf_{n\to\infty}\int_E f_n\,d\mu$.

It follows from the above inequalities that $\lim_{n\to\infty}\int_E f_n\,d\mu$ exists and is equal to $\int_E f\,d\mu$.

Step 2. Assume that the functions f and f_n, $n\in\mathbb{N}$, satisfy the conditions of the Theorem for some g, g_n, $n\in\mathbb{N}$. It is clear that then $|f|$ and $|f_n|$, $n\in\mathbb{N}$, satisfy the same conditions (with the same g, g_n, $n\in\mathbb{N}$) since $|f_n|\to|f|$ *a.e.*, due to the inequality $\big||f_n|-|f|\big|\le|f_n-f|$. Also, $\big||f_n|\big|=|f_n|\le g_n$. Hence we can apply the conclusions derived in Step 1 to the functions $|f_n|$ instead of f_n and $|f|$ instead of f. This shows that the hypotheses of the Theorem imply that $\int_E|f_n|\,d\mu\to\int_E|f|\,d\mu$ as $n\to\infty$.

Step 3. To complete the proof, suppose that the functions f and f_n, $n\in\mathbb{N}$, satisfy the conditions of the Theorem for certain g and g_n, $n\in\mathbb{N}$. Thanks to Step 2, we know that then $\int_E|f_n|\,d\mu\to\int_E|f|\,d\mu$ as $n\to\infty$.

Now, instead of the functions f_n, f, g_n and g, consider respectively the functions

$$F_n=|f_n-f|,\quad F=0,\quad G_n=|f_n|+|f|,\quad G=2|f|.$$

It turns out that they also satisfy the conditions of the Theorem since $F_n\to F$ and $G_n\to G$ almost everywhere in E, $|F_n|\le G_n$ almost everywhere in E, and

$$\int_E G_n\,d\mu=\int_E(|f_n|+|f|)\,d\mu\to 2\int_E|f|\,d\mu=\int_E G\,d\mu,\quad n\to\infty.$$

By Step 1, it follows that $\int_E F_n\,d\mu\to\int_E F\,d\mu$; in other words, $\int_E|f_n-f|\,d\mu\to 0$, as $n\to\infty$. ■

Observation. First, it is clear that Pratt's theorem is a generalization of LDCT, for it suffices to choose $g_n=g$ for all $n\in\mathbb{N}$ to deduce LDCT from Pratt.

Secondly, Pratt's theorem also implies that of Littlewood. To this end, assume that the functions f and f_n, $n\in\mathbb{N}$, satisfy the conditions of Littlewood's theorem. By choosing $g_n=|f_n|$ and $g=|f|$, se see that the conditions of Pratt's theorem are all fulfilled, hence we deduce that

$$\int_E|f_n-f|\,d\mu\to 0,\quad n\to\infty.$$

Recommended readings for this topic

- J.W. Pratt, On interchanging limits and integrals, *Ann. Math. Statist.* **31** (1960), No. 1, 74–77.
- H.L. Royden: *Real Analysis*, 3rd edition, MacMillan, New York 1988 (Section 4.4).
- W. Rudin: *Real and Complex Analysis*, McGraw-Hill, New York 1987 (3rd edition, Chapter 1).