Universidad Autónoma de Madrid UAM

Master in Mathematics and Applications

Master Thesis 2015/16

Weak Solutions of the Incompressible Euler Equations

Francisco Mengual Bretón

Supervised by Daniel Faraco Hurtado

and Ángel Castro Martínez

ABSTRACT

...

Index

1 INTRODUCTION TO FLUID MECHANICS

1.1 The Navier-Stokes equations

Given a fluid element $\alpha \in \mathbb{R}^d$ we consider the map

$$\begin{aligned} X(\alpha,\cdot) : \mathbb{R}^+ &\to \mathbb{R}^d \\ t &\mapsto X(\alpha,\cdot) \end{aligned}$$

which is the **particle trajectory** of α. If v is the velocity vector field of the fluid, the particle trajectory is the solution of the ODE

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) = v(X(\alpha,t),t), & (\alpha,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ X(\alpha,0) = \alpha, & \alpha \in \mathbb{R}^d. \end{cases}$$

Lemma 1.1.

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(\nabla_\alpha X)(\alpha,t) = \mathrm{div}_x v(X(\alpha,t),t) \det(\nabla_\alpha X)(\alpha,t).$$

Proof. Call $J = \det(\nabla_\alpha X)$. On the one hand

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \det(\partial_{\alpha_j} X_i) = \sum_{i,j=1}^{d} a_{ij} \frac{\mathrm{d}}{\mathrm{d}t} \partial_{\alpha_j} X_i$$

for some functions $a_{ij}((X^k)_{k\neq i})$ on $\mathbb{R}^d \times \mathbb{R}^+$. On the other hand

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{d} (-1)^{i+j} M_{ij} \partial_{\alpha_j} X_i \\ &= \sum_{j=1}^{d} (-1)^{i+j} M_{ij} \frac{\mathrm{d}}{\mathrm{d}t} \partial_{\alpha_j} X_i + \sum_{j=1}^{d} (-1)^{i+j} \partial_{\alpha_j} X_i \frac{\mathrm{d}M_{ij}}{\mathrm{d}t} \end{aligned}$$

where $M_{ij}((X^k)_{k\neq i})$ is the minor of $\nabla_\alpha X$. Therefore, it must be $a_{ij} = (-1)^{i+j} M_{ij}$. Recall the determinant formula by minors

$$\sum_{j=1}^{d} (-1)^{i+j} M_{ij} \partial_{\alpha_j} X_k = \delta_{ik} J.$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t} \partial_{\alpha_j} X_i(\alpha,t) = \partial_{\alpha_j} v_i(X(\alpha,t),t) = \sum_{k=1}^{d} \partial_{x_k} v_i(X(\alpha,t),t) \partial_{\alpha_j} X_k(\alpha,t).$$

Finally,

$$\begin{aligned}
\frac{\mathrm{d}J}{\mathrm{d}t}(\alpha,t) &= \sum_{i,j=1}^{d}(-1)^{i+j}M_{ij}(\alpha,t)\frac{\mathrm{d}}{\mathrm{d}t}\partial_{\alpha_j}X_i(\alpha,t)\\
&= \sum_{i,j,k=1}^{d}(-1)^{i+j}M_{ij}(\alpha,t)\partial_{x_k}v_i(X(\alpha,t),t)\partial_{\alpha_j}X_k(\alpha,t)\\
&= \sum_{i,k=1}^{d}\partial_{x_k}v_i(X(\alpha,t),t)\delta_{ik}J(\alpha,t)\\
&= \mathrm{div}_x v(X(\alpha,t),t)J(\alpha,t).
\end{aligned}$$

□

Proposition 1.2 (The transport formula). *Let Ω a bounded open subset of $\mathbb{R}^d$ with smooth boundary. Then, for every $f\in C^1(\Omega\times[0,\infty))$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X(\Omega,t)} f(x,t)\,\mathrm{d}x = \int_{X(\Omega,t)}\Big[\partial_t f + \mathrm{div}_x(fv)\Big](x,t)\,\mathrm{d}x.$$

Proof. By making the change of variables $x = X(\alpha,t)$ we obtain

$$\int_{X(\Omega,t)} f(x,t)\,\mathrm{d}x = \int_{\Omega} f(X(\alpha,t),t)J(\alpha,t)\,\mathrm{d}\alpha.$$

Hence, since $f\in C^1(\Omega\times[0,\infty))$ (integrabilidad) and $X\in C^2(\Omega\times[0,\infty))$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{X(\Omega,t)} f(x,t)\,\mathrm{d}x &= \int_{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}\Big(f(X(\alpha,t),t)J(\alpha,t)\Big)\,\mathrm{d}\alpha\\
&= \int_{\Omega}\Big[\Big(\nabla_x f\cdot\frac{\mathrm{d}X}{\mathrm{d}t}+\partial_t f\Big)J + f\frac{\mathrm{d}J}{\mathrm{d}t}\Big]\,\mathrm{d}\alpha\\
&= \int_{\Omega}\Big[\nabla_x f\cdot v + \partial_t f + f\,\mathrm{div}_x v\Big]J\,\mathrm{d}\alpha\\
&= \int_{X(\Omega,t)}\Big[\partial_t f + \mathrm{div}_x(fv)\Big](x,t)\,\mathrm{d}x.
\end{aligned}$$

□

1.1.1 Conservation of mass

The law of conservation of mass states that for any system closed to all transfers of mass and energy, the mass of the system must remain constant over time, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X(\Omega,t)}\rho(x,t)\,\mathrm{d}x = 0$$

for every Borel subset Ω of $\mathbb{R}^d$, where $\rho : \mathbb{R}^d \to \mathbb{R}$ measures the infinitesimal density of mass of the fluid. By transport formula 1.2, it must be

$$\int_{X(\Omega,t)} \Big[\partial_t\rho + \mathrm{div}_x(\rho v)\Big](x,t)\,\mathrm{d}x = 0$$

for every Borel subset Ω of $\mathbb{R}^d$. The continuity implies that this holds if and only if

$$\partial_t\rho + \mathrm{div}_x(\rho v) = 0$$

in $\mathbb{R}^d \times \mathbb{R}^+$.

1.1.2 Conservation of momentum

The law of conservation of momentum states that for any system closed to all transfers of mass and energy, the momentum of the system must remain constant over time. This is developed from Newton's second law which says that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X(\Omega,t)} \rho v(x,t)\,\mathrm{d}x = \int_{X(\Omega,t)} \rho g(x,t)\,\mathrm{d}x + \int_{\partial X(\Omega,t)} f(n,\sigma,t)\,\mathrm{d}\sigma$$

for every subdomain Ω of $\mathbb{R}^d$ with smooth boundary, where g and f are the vector field of the infinitesimal volume and surface forces acting on the flow. The surface force is expressed as

$$f(n,x,t) = \tau(x,t)n$$

where τ is a matrix field called the **Cauchy stress tensor**.

On the one hand, Gauss divergence theorem implies

$$\int_{\partial X(\Omega,t)} \tau_i(\sigma,t)\cdot n\,\mathrm{d}\sigma = \int_{X(\Omega,t)} \mathrm{div}_x\tau_i(x,t)\,\mathrm{d}x.$$

On the other hand, transport formula 1.2 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X(\Omega,t)} \rho v_i(x,t)\,\mathrm{d}x = \int_{X(\Omega,t)} \Big[\partial_t(\rho v_i) + \mathrm{div}_x(\rho v_i v)\Big](x,t)\,\mathrm{d}x.$$

Therefore, conservation of momentum means

$$\int_{X(\Omega,t)} \Big[\partial_t(\rho v_i) + \mathrm{div}_x(\rho v_i v)\Big](x,t)\,\mathrm{d}x = \int_{X(\Omega,t)} \Big[\rho g_i + \mathrm{div}_x\tau_i\Big](x,t)\,\mathrm{d}x$$

for every subdomain Ω of $\mathbb{R}^d$ with smooth boundary. As in conservation of momentum, this is equivalent to

$$\partial_t(\rho v_i) + \mathrm{div}_x(\rho v_i v) = \rho g_i + \mathrm{div}_x\tau_i$$

in $\mathbb{R}^d \times \mathbb{R}^+$. Applying conservation of mass to the left hand side we obtain

$$\partial_t(\rho v_i) + \mathrm{div}_x(\rho v_i v) = \rho\partial_t v_i + v_i\Big(\partial_t\rho + \mathrm{div}_x(\rho v)\Big) + \rho v\cdot\nabla_x v_i = \rho \mathrm{D}_t v_i$$

Therefore, conservation of momentum is equivalent to

$$\rho \mathrm{D}_t v = \rho g + \mathrm{div}_x \tau$$

in $\mathbb{R}^d \times \mathbb{R}^+$. An useful notation is derived from the observation

$$\rho \mathrm{D}_t v_i = \partial_t(\rho v_i) + \mathrm{div}_x(\rho(v \otimes v)_i),$$

that is, we can write conservation of momentum as

$$\partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) = \rho g + \mathrm{div}_x \tau$$

in $\mathbb{R}^d \times \mathbb{R}^+$.

Constitutive equations for Newtonian fluids

Assuming that there are not body-forces couples proportional to the mass of the fluid element (such as those exerted by an electric field on polarized fluid molecules) it can be shown that the Cauchy stress tensor τ is symmetric, $\tau_{ij} = \tau_{ji}$ (see [KCD]).

In a fluid at rest there are only normal components of stress on a surface, independently of the orientation of the surface, that is, the stress is **isotropic**. The only (up to a constant) isotropic $(0, 2)$-tensor is δ_{ij}. Hence, τ must be

$$\tau_{ij} = -p\delta_{ij}$$

where p is called the **pressure** of the fluid.

A moving fluid develops additional stress components, σ, because of **viscosity**. More precisely, this is a resistance, due to internal molecular forces, of the flow to be deformed. A simple extension of the last case is

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}.$$

Since this force appears due to movement, σ depends on the quantities $\partial^\alpha v$, α every multiindex. By simplicity, we make the assumption that we can skip the dependence on derivatives of second order and beyond. Invariancy under galilean transformations implies that σ cannot depend explicitly of v, so it must depend on ∇v (see [KCD]). We can express ∇v as direct sum of symmetric and antisymmetric parts $\nabla v = S + R$ where

$$S = \frac{1}{2}(\nabla v + \nabla v^\dagger) \quad \text{and} \quad R = \frac{1}{2}(\nabla v - \nabla v^\dagger)$$

are the **strain rate tensor** and the **rotation tensor** respectively. These correspond to infinitesimal deformation and rotation in the flow. By definition, stresses only develop in fluid elements that change shape. Therefore, only the symmetric part S should be considered in the fluid constitutive equation. The most general linear relation between σ and S is

$$\sigma_{ij} = k_{ijmn} S_{mn}.$$

Isotropy forces that tensor K be isotropic, hence (see [KCD])

$$k_{ijmn} = \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm} + \lambda\delta_{ij}\delta_{mn}$$

where λ, μ, γ depend on the thermodynamic local state. Since τ is symmetric, $\gamma = \mu$. Therefore,

$$\begin{aligned}\sigma_{ij} &= [\mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \lambda\delta_{ij}\delta_{mn}]S_{mn} \\ &= \mu(S_{ij} + S_{ji}) + \lambda\delta_{ij}S_{mm} \\ &= 2\mu S_{ij} + \lambda\delta_{ij}\mathrm{div}_x v.\end{aligned}$$

In this way

$$\tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda\delta_{ij}\mathrm{div}_x v.$$

Taking the trace of the above relation we obtain

$$p = \bar{p} + \beta\mathrm{div}_x v$$

where

$$\bar{p} = -\frac{1}{d}\mathrm{Tr}(\tau) \quad \text{and} \quad \beta = \frac{2}{d}\mu + \lambda$$

are known as the **mean pressure** and the **coefficient of bulk viscosity** respectively of the flow. Finally

$$\tau_{ij} = -p\delta_{ij} + 2\mu\Big(S_{ij} - \frac{1}{d}\delta_{ij}\mathrm{div}_x v\Big) + \beta\delta_{ij}\mathrm{div}_x v.$$

Flows satisfying the above relation are called **Newtonian fluids**. Examples of such fluids are air, water, oil, gasoline, etc.

Returning to conservation of momentum equation, observe that

$$\mathrm{div}_x(\tau_i) = \sum_{j=1}^{d} \partial_{x_j}\tau_{ij} = -\partial_{x_i}p + \sum_{j=1}^{d} \partial_{x_j}\Big(\mu(\partial_{x_j}v_i + \partial_{x_i}v_j) + \Big(\beta - \frac{2}{d}\mu\Big)\delta_{ij}\mathrm{div}_x v\Big).$$

Since μ, β depend of the temperature, if the variation of temperature is neglected, we obtain

$$\mathrm{div}_x(\tau_i) = -\partial_{x_i}p + \mu\Delta_x v_i + \Big(\beta + \frac{d-2}{d}\mu\Big)\partial_{x_i}(\mathrm{div}_x v_i).$$

Stokes assumption is that β is neglected, which is reasonable in many situations. In this way, **Navier-Stokes conservation of momentum equations** are

$$\rho\mathrm{D}_t v = -\nabla_x p + \mu\Delta_x v + \frac{d-2}{d}\mu\nabla_x(\mathrm{div}_x v) + \rho g$$

in vector form.

Euler considered the case of inviscid fluids, that is, $\mu = 0$. In this way, **Euler conservation of momentum equations** are

$$\rho \mathrm{D}_t v = -\nabla_x p + \rho g$$

in vector form.

1.1.3 Homogeneous flows

Definition 1.3. A flow X is said to be **incompressible** if for all subdomains the flow is volume preserving, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(X(\Omega, t)) = 0$$

for every Borel subset Ω of $\mathbb{R}^d$.

Proposition 1.4. *For smooth flows these conditions are equivalents:*

i) X *is incompressible.*

ii) $\det(\nabla_\alpha X) = 1$ *in* $\mathbb{R}^d \times \mathbb{R}^+$.

iii) $\mathrm{div}_x v = 0$ *in* $\mathbb{R}^d \times \mathbb{R}^+$.

Proof. By transport formula 1.2

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{X(\Omega,t)} \mathrm{d}x = \int_\Omega \frac{\mathrm{d}J}{\mathrm{d}t}(\alpha, t)\,\mathrm{d}\alpha = \int_{X(\Omega,t)} \mathrm{div}_x v(x,t)\,\mathrm{d}x$$

for every Borel subset Ω of $\mathbb{R}^d$. On the one hand, continuity implies that this holds if and only if

$$\frac{\mathrm{d}J}{\mathrm{d}t} = 0$$

in $\mathbb{R}^d \times \mathbb{R}^+$, that is, J is constant. Indeed, since

$$\mathcal{L}(X(\Omega, t)) = \int_\Omega J(\alpha, t)\,\mathrm{d}\alpha = J\mathcal{L}(\Omega),$$

this is constant is 1. On the other hand, continuity implies that this holds if and only if

$$\mathrm{div}_x v = 0$$

in $\mathbb{R}^d \times \mathbb{R}^+$. □

Definition 1.5. A flow X is said to be **homogeneous** if it has constant density (we may assume that $\rho \equiv 1$). In such flows, conservation of mass and incompressibility is equivalent by observing

$$\partial_t \rho + \mathrm{div}_x(\rho v) = \rho \mathrm{div}_x v$$

in $\mathbb{R}^d \times \mathbb{R}^+$.

Homogeneous Navier-Stokes Equations are

$$\begin{cases} \mathrm{D}_t v = -\nabla_x p + \mu \Delta_x v + g & \mathbb{R}^d \times \mathbb{R}^+, \\ \mathrm{div}_x v = 0 & \mathbb{R}^d \times \mathbb{R}^+. \end{cases}$$

In the present work, we assume that there are not volume forces acting on the fluid, that is, $g = 0$. In particular, our purpose is to study the solutions of the homogeneous Euler equations without volume forces, which are often called the **Incompressible Euler Equations** (IEE)

$$\begin{cases} \partial_t v + \mathrm{div}_x(v \otimes v) + \nabla_x p = 0 & \mathbb{R}^d \times \mathbb{R}^+, \\ \mathrm{div}_x v = 0 & \mathbb{R}^d \times \mathbb{R}^+. \end{cases}$$

1.2 Conserved quantities

2 WEAK SOLUTIONS

In this section we fix an open subset $\mathscr{D}$ of $\mathbb{R}^d$, and a final time $0 < T \leq \infty$ (with out lost of generality, if $T = \infty$, $[0, T]$ will denote $\mathbb{R}^+$).

2.1 Weak solutions of the incompressible Navier-Stokes equations

...[Motivación para rebajar la regularidad de las soluciones buscadas]

2.1.1 The space of time dependent divergence-free vector fields

Definition 2.1. A vector field $v \in L^2(\mathscr{D}, \mathbb{R}^d)$ is said to be **divergence-free** if $\mathrm{div} v = 0$ in the sense of distributions, that is,

$$\int_{\mathscr{D}} \nabla\phi \cdot v \, \mathrm{d}x = 0 \quad \text{for all } \phi \in C_c^\infty(\mathscr{D}).$$

We will denote the space of such divergence-free vector fields by $\mathcal{H}(\mathscr{D})$, and we will use the notation $\mathcal{H}_w(\mathscr{D})$ to specify that $\mathcal{H}(\mathscr{D})$ is endowed with the weak topology of $L^2(\mathscr{D}, \mathbb{R}^d)$.

The next proposition is immediate by definition.

Proposition 2.2. *The space $\mathcal{H}_w(\mathscr{D})$ is a closed linear subspace of $L^2_w(\mathscr{D}, \mathbb{R}^d)$.*

(... explicar mejor)We saw that classical solutions of the IEE conserves kinetic energy. However, ... The space that we are talking about is time dependent divergence-free velocity fields with bounded kinetic energy, that is, $L^\infty([0, T], H(\mathscr{D}))$.

Proposition 2.3. *Let $v \in L^\infty([0, T]; L^2(\mathbb{R}^d; \mathbb{R}^d))$, $u \in L^1_{loc}([0, T] \times \mathbb{R}^d; \mathbb{R}^{d\times d}))$ and $q \in L^1_{loc}([0, T] \times \mathbb{R}^d)$ be a distributional solution of*

$$\partial_t v + \mathrm{div} u + \nabla q = 0.$$

Then, we can redefine v in a set of times of measure zero such that

$$v \in C_b([0, T]; L^2_w(\mathbb{R}^d; \mathbb{R}^d))$$

Proof. Take a finite time $0 < s \leq T$. Consider a countable set $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_c(\mathbb{R}^d; \mathbb{R}^d)$ dense in $L^2(\mathbb{R}^d; \mathbb{R}^d)$. Denote

$$\Psi_k(t) = \langle \varphi_k, v(t) \rangle = \int_{\mathbb{R}^d} \varphi_k(x) \cdot v(x, t) \, \mathrm{d}x, \quad t \in [0, T], \, k \in \mathbb{N},$$

which are in $L^1([0, s])$ by Hölder inequality. By hypothesis, for every $\phi \in C_c^\infty([0, s])$,

$$\begin{aligned}
\langle \phi, \partial_t \Psi_k \rangle_{[0,T]} &= -\langle \partial_t \phi, \Psi_k \rangle_{[0,T]} = -\langle \partial_t \phi \varphi_k, v \rangle_{\mathbb{R}^d \times [0,T]} \\
&= \langle \phi \varphi_k, \partial_t v \rangle_{\mathbb{R}^d \times [0,T]} = -\langle \phi \varphi_k, \mathrm{div} u + \nabla q \rangle_{\mathbb{R}^d \times [0,T]} \\
&= \langle \phi, \langle \nabla \varphi_k, u(t) \rangle_{\mathbb{R}^d} + \langle \mathrm{div} \varphi_k, q(t) \rangle_{\mathbb{R}^d} \rangle_{[0,T]}.
\end{aligned}$$

Hence

$$\partial_t \Psi_k = \langle \nabla \varphi_k, u(t) \rangle_{\mathbb{R}^d} + \langle \mathrm{div}\varphi_k, q(t) \rangle_{\mathbb{R}^d}$$

in the sense of distributions, so $\partial_t \Psi_k \in L^1([0,s])$ and $\Psi_k \in W^{1,1}([0,s])$. By Morrey's inequality we known that we can redefine Ψ_k in a set of times of measure zero $\tau_k \subset [0,s]$ such that $\Psi_k \in C([0,s])$. Hence, $\tau = \cup_k \tau_k$ is a negligible Borel subset of $[0,s]$, and

$$\Psi_k(t) = \langle \varphi_k, v(t) \rangle = \int_{\mathbb{R}^d} \varphi_k(x) \cdot v(x,t) \, \mathrm{d}x, \quad t \in [0,T] \setminus \tau, \; k \in \mathbb{N}.$$

If T is finite we are done. If $T = \infty$, taking a sequence $s_N \uparrow \infty$ we obtain $\Psi_k \in C([0,T])$ and the above equality still being truth *a.e.* $[0,T]$. Notice that

$$|\Psi_k(t)| \leq \|v\|_{L^\infty_t L^2_x} \|\varphi_k\|_{L^2_x} \quad a.e.\, t \in [0,T],$$

but continuity implies that it must be on $[0,T]$. For every $t \in [0,T]$ denote

$$\Lambda_t(\varphi_k) = \Psi_k(t), \quad k \in \mathbb{N}.$$

Density and the above estimate allows us to extend it naturally to all $L^2(\mathbb{R}^d; \mathbb{R}^d)$. For every $\varphi \in L^2(\mathbb{R}^d; \mathbb{R}^d)$, taking $\varphi_{k_j} \to \varphi$ in $L^2(\mathbb{R}^d; \mathbb{R}^d)$, the quantity

$$\Lambda_t(\varphi) = \lim_{j \to \infty} \Lambda_t(\varphi_{k_j})$$

is well defined (it is a Cauchy sequence in $\mathbb{R}$) and it does not depend on the election of the subsequence. Moreover, the operator Λ_t acting on $L^2(\mathbb{R}^d; \mathbb{R}^d)$ is linear and bounded with $\|\Lambda_t\| = \|v\|_{L^\infty_t L^2_x}$. Hence, by theorem [Riesz], there exists a unique $\bar{v}(\cdot, t) \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$\Lambda_t(\varphi) = \langle \varphi, \bar{v}(t) \rangle = \int_{\mathbb{R}^d} \varphi(x) \cdot \bar{v}(x,t) \, \mathrm{d}x.$$

Moreover,

$$\|\bar{v}(t)\|_{L^2} = \|\Lambda_t\| \leq \|v\|_{L^\infty_t L^2_x}, \quad t \in [0,T],$$

and uniqueness implies that

$$\bar{v}(t) = v(t) \quad a.e.\, t \in [0,T].$$

Finally we check that $\bar{v} \in C([0,T], L_w(\mathbb{R}^d; \mathbb{R}^d))$. Fix $t_0 \in [0,T]$. Let $\varphi \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ and take $\varphi_{k_j} \to \varphi$ in $L^2(\mathbb{R}^d; \mathbb{R}^d)$. Then

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \varphi(x) \cdot \bar{v}(x,t) \, \mathrm{d}x - \int_{\mathbb{R}^d} \varphi(x) \cdot \bar{v}(x,t_0) \, \mathrm{d}x \right| = |\Lambda_t(\varphi) - \Lambda_{t_0}(\varphi)| \\ &\quad \leq |\Lambda_t(\varphi) - \Lambda_t(\varphi_{k_j})| + |\Lambda_t(\varphi_{k_j}) - \Lambda_{t_0}(\varphi_{k_j})| + |\Lambda_{t_0}(\varphi_{k_j}) - \Lambda_{t_0}(\varphi)| \\ &\quad \leq 2\|v\|_{L^\infty_t L^2_x} \|\varphi_{k_j} - \varphi\|_{L^2} + |\Psi_{k_j}(t) - \Psi_{k_j}(t_0)| \end{aligned}$$

and making $t \to t_0$ and after $j \to \infty$ we obtain the result. □

2.1.2 Weak conservation of momentum

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{D}} \varphi\cdot v\,\mathrm{d}x &= \int_{\mathscr{D}} \mathrm{D}_t(\varphi\cdot v)\,\mathrm{d}x = \int_{\mathscr{D}}\Big(\mathrm{D}_t\varphi\cdot v+\varphi\cdot\mathrm{D}_t v\Big)\,\mathrm{d}x\\
&= \int_{\mathscr{D}}\Big(\mathrm{D}_t\varphi\cdot v+\varphi\cdot(-\nabla p+\mu\Delta v)\Big)\,\mathrm{d}x\\
&= \int_{\mathscr{D}}\Big(\mathrm{D}_t\varphi\cdot v+p\,\mathrm{div}\varphi-\mu\nabla\varphi:\nabla v\Big)\,\mathrm{d}x\\
&= \int_{\mathscr{D}}\Big(\Big(\mathrm{D}_t\varphi+\mu\Delta\varphi\Big)\cdot v+p\,\mathrm{div}\varphi\Big)\,\mathrm{d}x\\
&= \int_{\mathscr{D}}\Big((\partial_t\varphi+\mu\Delta\varphi)\cdot v+\nabla\varphi:v\otimes v+p\,\mathrm{div}\varphi\Big)\,\mathrm{d}x.
\end{aligned}
$$

Integrating on $[0,T]$

$$
-\int_{\mathscr{D}}\varphi(x,0)\cdot v_0(x)\,\mathrm{d}x=\int_0^T\int_{\mathscr{D}}\Big((\partial_t\varphi+\mu\Delta\varphi)\cdot v+\nabla\varphi:v\otimes v+p\,\mathrm{div}\varphi\Big)\,\mathrm{d}x.
$$

2.2 Subsolution criterion and non-uniqueness

2.2.1 Nash-Kuiper method

2.2.2 Tartar framework

Given $\mathscr{B}$ a bounded open domain of $\mathbb{R}^m$, $A_1,\dots,A_m$ constant matrices of $\mathbb{R}^{m\times d}$, and K a compact subset of $\mathbb{R}^m$, we consider the general problem consisted on find functions $z:\mathscr{B}\to\mathbb{R}^d$ satisfying

$$
\begin{cases}
\sum_{i=1}^m A_i\partial_i z=0 & \text{in the sense of distributions,}\\
\quad z(y)\in K & a.e.\,y\in\mathscr{B}.
\end{cases}
\tag{2.1}
$$

Assumptions.

Λ) <u>The wave cone.</u> There exists a closed cone $\Lambda\subset\mathbb{R}^d$ and a constant $C>0$ satisfying: for every $\overline{z}\in\Lambda$ there exists a sequence $(z^k)_{k\in\mathbb{N}}\subset C_c^\infty(B,\mathbb{R}^d)$ satisfying

- $\sum_{i=1}^m A_i\partial_i z^k=0$ in B,

- $\mathrm{dist}(z^k(y),[-\overline{z},\overline{z}])\to 0$ uniformly in $y\in B$,

- $z^k\rightharpoonup 0$ (weakly) in $L^2_w(B,\mathbb{R}^n)$,

- $\int_B |z^k|^2 dy > C|\overline{z}|^2$

The more general candidate for the wave cone is

$$\Lambda = \left\{ z \in \mathbb{R}^d \,:\, \exists \xi \in \mathbb{S}^{m-1} \ni \left(\sum_{i=1}^{m} \xi_i A_i \right) z = 0 \right\}.$$

K^Λ) The Λ-convex hull. There exixts a bounded open set $K^\Lambda \subset \mathbb{R}^d$ which doesn's intersect K satisfying: for every $\alpha > 0$ there exists $\beta(\alpha) > 0$ such that, for every $z \in K^\Lambda$ with $\operatorname{dist}(z, K) \geq \alpha > 0$ there exists $\overline{z} \in \Lambda \cap \mathbb{S}^{d-1}$ such that

$$z + (-\beta, \beta)\overline{z} = (z - \beta\overline{z}, z + \beta\overline{z}) \subset K^\Lambda.$$

Lemma 2.4. *The Λ-convex hull K^Λ is contained in the usual convex hull K^{co}.*

$\mathcal{S}$) The space of subsolutions. There exists a nonempty bounded subset $\mathcal{S}$ of $L^2(\mathscr{B})$ consisting of perturbable functions, that is, any $z \in \mathcal{S}$ is continuous with

$$z(y) \in K^\Lambda \quad \text{for all } y \in \mathscr{B},$$

and moreover, for any $z \in \mathcal{S}$ and $w \in C_c(\mathscr{B})$ such that $z+w$ satisfies (2.1) and $(z+w)(y) \in K^\Lambda$ for all $y \in \mathscr{B}$, then $z + w \in \mathcal{S}$.

We consider also the closure of $\mathcal{S}$ in the topology of $L^2_w(\mathscr{B})$, and we denote it by $\overline{\mathcal{S}}$, which is a complete metric space.

Lemma 2.5. *There exists a continuous function $\Phi : \overline{K^\Lambda} \to [0, \infty)$ with*

$$\Phi^{-1}(0) \subset K,$$

such that, for every $z \in \mathcal{S}$ there exixts a sequence $(z^k)_{k\in\mathbb{N}} \subset \mathcal{S}$ tending weakly to z in $L^2_w(\mathscr{B})$ satisfying

$$\int_{\mathscr{B}} |z - z^k|^2(y)\,\mathrm{d}y \geq \int_{\mathscr{B}} \Phi(z(y))\,\mathrm{d}y.$$

Theorem 2.6. *Under the assumptions $(\Lambda, K^\Lambda, \mathcal{S})$, the set*

$$\{z \in \overline{\mathcal{S}} \,:\, z(y) \in K \ a.e.\, y \in \mathscr{B}\}$$

is residual in $\overline{\mathcal{S}}$.

Proof. Let us consider the functional

$$\begin{array}{rcl} \mathcal{J} : \overline{\mathcal{S}} & \to & [0, \infty) \\ z & \mapsto & \displaystyle\int_{\mathscr{B}} \Phi(z(y))\,\mathrm{d}y. \end{array}$$

... □

Comparison with IEE

[we want $\forall t$ instead of $a.e.\,t$, this leads to consider the functional with a supreme in the time variable; we want to consider possible not bounded domains, this leads to an exhausting argument; as the compact set depends on y, we have to be careful with the perturbation to be sure that the perturbed subsolution belongs to $\mathcal{S}$; ...]

$$\begin{aligned} \mathcal{J}_{\mathscr{B}} : \mathcal{S}_q(\mathscr{D}_T, v_0, e) &\to [0,\infty) \\ v &\mapsto \sup_{t\in I} \int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x. \end{aligned}$$

Ir comparando Annals y ARMA a lo largo de este capítulo.

2.2.3 Subsolutions of the incompressible Euler equations

Definition 2.7 (Subsolution)**.** We call a **subsolution** to the IEE a triple (v,u,q), where $v \in C_b([0,T],\mathcal{H}_w(\mathscr{D}))$, $u \in L^\infty([0,T],L^1(\mathscr{D},S_0^d))$, and q a distribution such that

$$\partial_t v + \mathrm{div} u + \nabla q = 0$$

in the sense of distributions.

Proposition 2.8. *Given (v,p) a weak solution to the IEE on $\mathscr{D}_T$, then $(v, v\bigcirc v, p+\frac{1}{d}|v|^2)$ is a subsolution to the IEE on $\mathscr{D}_T$. Reciprocally, if (v,u,q) is a subsolution to the IEE on $\mathscr{D}_T$ such that $u = v\bigcirc v$, then $(v, q-\frac{1}{d}|v|^2)$ is a weak solution of the IEE on $\mathscr{D}_T$.*

Proof. It follows immediately from the following equality in the sense of distributions

$$\partial_t v + \mathrm{div}(v\otimes v) + \nabla p = \partial_t v + \mathrm{div}\Big(v\otimes v - \frac{1}{d}|v|^2 I\Big) + \nabla\Big(p + \frac{1}{d}|v|^2\Big).$$

□

The set of Euler states of speed r is

$$K_r = \{(v, v\bigcirc v) \,:\, v \in r\mathbb{S}^{d-1}\}.$$

These spaces are, by the above proposition, the compact sets were $u(y)$ must be for each $v(y)$ to be a weak solution instead of only a subsolution.

In order to measure the relaxation, we introduce the **generalised energy density**

$$\begin{aligned} \rho : \mathbb{R}^d \times S_0^d &\to \mathbb{R} \\ (v,u) &\mapsto \frac{d}{2}\lambda_{\max}(v\otimes v - u) \end{aligned}$$

where $\lambda_{\max}$ denotes the largest eigenvalue on symmetric matrices. The below lemma justifies why it is a good measurer of the relaxation of subsolutions.

Lemma 2.9 (Properties of ρ)**.** *For all $(v,u) \in \mathbb{R}^d \times S_0^d$*

(a) ρ is convex.

(b) $\frac{1}{2}|v|^2 \leq \rho(v,u)$ with equality if and only if $u = v \bigcirc v$.

(c) $|u|_\infty \leq 2\frac{d-1}{d}\rho(v,u)$.

(d) $\rho(v,u) \leq \frac{d}{2}(|v|^2 + |u|)$.

(e) The $\frac{r^2}{2}$-sublevel set of ρ is the is the convex hull of K_r

$$K_r^{co} = \left\{(v,u) \in \mathbb{R}^d \times S_0^d \,:\, \rho(v,u) \leq \frac{r^2}{2}\right\} = \rho^{-1}\left(\left[0, \frac{r^2}{2}\right]\right)$$

Proof. (a) Note that

$$\begin{aligned}\rho(v,u) &= \frac{d}{2}\max_{\xi\in\mathbb{S}^{d-1}}\langle\xi,(v\otimes v-u)\xi\rangle = \frac{d}{2}\max_{\xi\in\mathbb{S}^{d-1}}\langle\xi,\langle\xi,v\rangle v-u\xi\rangle \\ &= \frac{d}{2}\max_{\xi\in\mathbb{S}^{d-1}}(\langle\xi,v\rangle^2-\langle\xi,u\xi\rangle),\end{aligned}$$

Since $\langle\cdot,v\rangle^2$ is convex and $\langle\cdot,u\cdot\rangle$ is linear, it follows that ρ is convex.

(b) As $v \otimes v = v \bigcirc v + \frac{1}{d}|v|^2 I$,

$$\begin{aligned}\rho(v,u) &= \frac{d}{2}\max_{\xi\in\mathbb{S}^{d-1}}\langle\xi,(v\bigcirc v+\frac{1}{d}|v|^2 I-u)\xi\rangle = \frac{d}{2}\max_{\xi\in\mathbb{S}^{d-1}}\langle\xi,(v\bigcirc v-u)\xi\rangle+\frac{1}{2}|v|^2 \\ &= \frac{d}{2}\lambda_{\max}(v\bigcirc v-u)+\frac{1}{2}|v|^2.\end{aligned}$$

Since $v\bigcirc v-u$ is traceless, $\lambda_{\max}(v\bigcirc v-u)\geq 0$ with equality if and only if $u = v\bigcirc v$.

(c) By *(a)* and *(b)*

$$\rho(v,u) \geq \rho(0,u) = \frac{d}{2}\lambda_{\max}(-u) = -\frac{d}{2}\lambda_{\min}(u).$$

Since u is traceless, $\lambda_{\min}(u) \leq 0$, so

$$\rho(v,u) \geq \frac{d}{2}|\lambda_{\min}(u)|.$$

Recall that the spectral radius of u is $\max\{\lambda_{max}(u), |\lambda_{min}(u)|\}$. If $\lambda_1 \leq \ldots \leq \lambda_d$ are the eigenvalues of u, since u is traceless,

$$\lambda_{\max}(u) = -\sum_{i=1}^{d-1}\lambda_i = \sum_{i=1}^{d-1}(-\lambda_i) \leq \sum_{i=1}^{d-1}(-\lambda_{\min}(u)) = (d-1)|\lambda_{min}(u)|.$$

Therefore (...),

$$|u|_\infty \leq (d-1)|\lambda_{min}(u)| \leq 2\frac{d-1}{d}\rho(v,u).$$

(d) Applying that $\lambda_{\max}$ is dominated by the Frobenius norm (recall that the Frobenius norm is the usual Euclidean norm in $\mathbb{R}^{d\times d}$) we obtain

$$\rho(v,u) = \frac{d}{2}\lambda_{\max}(v\otimes v - u) \leq \frac{d}{2}\|v\otimes v - u\|_F \leq \frac{d}{2}(\|v\otimes v\|_F + \|u\|_F) = \frac{d}{2}(|v|^2 + |u|)$$

when we have used that

$$\|v\otimes v\|_F = \left(\sum_{i=1}^{d}\sum_{j=1}^{d}|v_i v_j|^2\right)^{\frac{1}{2}} = |v|^2.$$

(e) Denote

$$S_r = \rho^{-1}\Big(\Big[0, \frac{r^2}{2}\Big]\Big)$$

which is convex by *(a)*. Since $\rho(v,u) = \frac{r^2}{2}$ for all $(v,u) \in K_r$, it is immediate that $K_r^{\mathrm{co}} \subset S_r$. ...

□

Definition 2.10 (Space of Subsolutions)**.** Let $v_0 \in \mathcal{H}(\mathscr{D})$ an initial divergence-free velocitiy field and

$$e \in C(\mathscr{D}\times(0,T)) \cap C_b([0,T], L^1(\mathscr{D})),$$

an energy density candidate. We consider the space consisted of smooth velocity fields

$$v \in C_b([0,T], \mathcal{H}_w(\mathscr{D}))$$

satisfying:

(i) v attains the initial condition

$$v(0) = v_0 \quad \text{in } \mathcal{H}(\mathscr{D}).$$

(ii) For a fixed (independent of v) smooth scalar function $q : \mathscr{D}_T \to \mathbb{R}$, there exists a smooth matrix field

$$u \in L^\infty([0,T], L^1(\mathscr{D}, S_0^d))$$

such that (v,u,q) is a subsolution of the IEE and

$$\rho(v(x,t), u(x,t)) < e(x,t) \quad \text{for all } (x,t) \in \mathscr{D}\times(0,T).$$

This is the space of smooth strict subsolutions of the IEE on $\mathscr{D}_T$ with corrected pressure q, initial velicity field v_0 and energy density bounded by e. We denote it by

$$\mathcal{S}_q(\mathscr{D}_T, v_0, e).$$

We consider also the closure of $\mathcal{S}_q(\mathscr{D}_T, v_0, e)$ in the topology of $C_b([0,T], \mathcal{H}^2_w(\mathscr{D}))$, and we denote it by

$$\overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e).$$

The original space considered in [Ref] was

$$\mathcal{S}(\mathscr{D}_T, v_0, e) = \bigcup_q \mathcal{S}_q(\mathscr{D}_T, v_0, e),$$

which only requires the existence of some pressure q for every v, but not the same for all. In [ref] .. ... modified the argument to obtain weak solutions at constant pressure, which we will follow in this work.

Corollary 2.11 (Boundness of Subsolutions). *If $v \in \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ and u is the associated smooth matrix field, then*

$$\frac{1}{2}|v|^2 < e, \quad |u|_\infty < 2\frac{d-1}{d}e, \quad \text{in } \mathscr{D} \times (0,T).$$

Proposition 2.12. *$\overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ is a complete metric subspace of $C_b([0,T], \mathcal{H}_w(\mathscr{D}))$. Also, each $v \in \overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ attains the initial condition*

$$v(\cdot, 0) = v_0 \quad \text{in } H(\mathscr{D}).$$

Proof. As $e \in C([0,T], L^1(\mathscr{D}))$, there exists $R > 0$ such that

$$R = 2 \max_{t \in [0,T]} \int_{\mathscr{D}} e(x,t)dx.$$

Since norms are convex, by the... it holds that

$$\int_{\mathscr{D}} |v|^2(x,t)dx \leq 2 \int_{\mathscr{D}} e(x,t)dx \leq R \quad \text{for all } v \in \overline{\mathcal{S}}(\mathscr{D}_T, v_0, q, e),$$

i.e., $\overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ is in B, the L^2-closed ball of radius R of $L^2(\mathscr{D}, \mathbb{R}^d)$. By the sequential Banach-Alaoglu theorem we know that B is a metrizable subspace of $L^2_w(\mathscr{D}, \mathbb{R}^d)$. If d_B is such a metric, then (B, d_B) is a complete compact metric subspace of $L^2_w(\mathscr{D})$. This induces naturally a metric d on $C([0,T], B)$ via

$$d(v,w) = \max_{t \in [0,T]} d_B(v(\cdot,t), w(\cdot,t)), \quad v, w \in C([0,T], B).$$

Also, $C([0,T], B)$ inherits the completeness of B. The topology induced by d on $C([0,T], B)$ is equivalent to its topology inherited as a subspace of $C([0,T], L^2_w(\mathscr{D}, \mathbb{R}^d))$. As $\mathcal{S}_q(\mathscr{D}_T, v_0, e) \subset C([0,T], B)$, and this is closed, $\overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ is a complete metric subspace of $C([0,T], H_w(\mathscr{D}))$. $\square$

Theorem 2.13 (Subsolution criterion). *If $\mathcal{S}(\mathscr{D}_T, v_0, e)$ is non-empty, then there exists infinitely many weak solutions (v, p) of the IEE on $\mathscr{D}_T$ such that*

- *$v \in C_b([0,T], \mathcal{H}_w(\mathscr{D}))$ and $p = q - \frac{1}{d}|v|^2$.*
- *$v(0) = v_0 \quad$ in $\mathcal{H}(\mathscr{D})$.*
- *$\frac{1}{2}|v|^2(x,t) = e(x,t) \quad$ a.e. $x \in \mathscr{D} \quad \forall t \in (0,T)$.*

2.2.4 Measuring the relaxation

Comparar con Nash-Kuiper. La sucesión que se consigue no garantiza la convergencia fuerte. Necesidad de recurrir a Baire (u otros).

Let $\mathscr{B} = \Omega \times I$ a bounded subset of $\mathscr{D}_T$, where Ω is an open bounded subset of $\mathscr{D}$ and $I = [s, t]$ with $0 < s < t < T$. We associate to $\mathscr{B}$ the functional

$$\begin{aligned} \mathcal{J}_{\mathscr{B}} : \overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e) &\to \mathbb{R} \\ v &\mapsto \sup_{t \in I} \int_\Omega \left[e - \frac{1}{2}|v|^2 \right](x, t)\,\mathrm{d}x. \end{aligned}$$

This is well-defined because $\mathcal{S}_q(\mathscr{D}_T, v_0, e) \subset L^\infty([0, T], H(\mathscr{D}))$ and $e \in C([0, T], L^1(\mathscr{D}))$,

Proposition 2.14 (Properties of $\mathcal{J}_{\mathscr{B}}$)**.** *The functional $\mathcal{J}_{\mathscr{B}}$ is:*

(a) Upper-semicontinuous.

(b) Bounded from below by zero, that is,

$$\mathcal{J}_{\mathscr{B}} : \mathcal{S}_q(\mathscr{D}_T, v_0, e) \to [0, \infty).$$

Moreover, if $v \in \overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ satisfies $\mathcal{J}_{\mathscr{B}}(v) = 0$, then

$$\frac{1}{2}|v|^2(x, t) = e(x, t) \quad a.e.\, x \in \Omega \quad \forall t \in I.$$

(c) 1-Baire.

Proof. (a) We prove it by contradiction. Suppose that there exists a sequence $(v^k)_{k \in \mathbb{N}}$ and a function v in $\overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ such that $v^k \overset{d}{\to} v$ but

$$\mathcal{J}_{\mathscr{B}}(v) < \limsup_{k \to \infty} \mathcal{J}_{\mathscr{B}}(v^k),$$

that is,

$$\sup_{t \in I} \int_\Omega \left[e - \frac{1}{2}|v|^2 \right](x, t)\,\mathrm{d}x < \limsup_{k \to \infty} \sup_{t \in I} \int_\Omega \left[e - \frac{1}{2}|v^k|^2 \right](x, t)\,\mathrm{d}x. \tag{2.2}$$

On the other hand, for every $k \in \mathbb{N}$ there exists $t_k \in I$ such that

$$\sup_{t \in I} \int_\Omega \left[e - \frac{1}{2}|v^k|^2 \right](x, t)\,\mathrm{d}x < \int_\Omega \left[e - \frac{1}{2}|v^k|^2 \right](x, t_k)\,\mathrm{d}x + 2^{-k}. \tag{2.3}$$

Taking the lim sup on (2.3) and using (2.2) we obtain

$$\sup_{t \in I} \int_\Omega \left[e - \frac{1}{2}|v|^2 \right](x, t)\,\mathrm{d}x < \limsup_{k \to \infty} \int_\Omega \left[e - \frac{1}{2}|v^k|^2 \right](x, t_k)\,\mathrm{d}x. \tag{2.4}$$

Since I is compact, we may assume (taking a subsequence if necessary) that $t_k \to t_0$ for some $t_0 \in I$. Now, note that

$$\begin{aligned} d_B(v^k(\cdot,t_k), v(\cdot,t_0)) &\leq d_B(v^k(\cdot,t_k), v(\cdot,t_k)) + d_B(v(\cdot,t_k), v(\cdot,t_0)) \\ &\leq d(v^k, v) + d_B(v(\cdot,t_k), v(\cdot,t_0)). \end{aligned}$$

Since $v^k \xrightarrow{d} v$ and $v \in C([0,T], B)$, we conclude

$$v^k(\cdot,t_k) \to v(\cdot,t_0) \quad \text{in } H(\mathscr{D}).$$

Therefore, since L^2-norm is wslsc and $e \in C([0,T], L^1(\mathscr{D}))$,

$$\limsup_{k\to\infty} \int_\Omega \left[e - \frac{1}{2}|v^k|^2\right](x,t_k)\,\mathrm{d}x \leq \int_\Omega \left[e - \frac{1}{2}|v|^2\right](x,t_0)\,\mathrm{d}x,$$

which contradicts (2.4).

(b) Now, if $v \in \mathcal{S}_q(\mathscr{D}_T, v_0, e)$, by corollary 2.11

$$\int_\Omega \left[e - \frac{1}{2}|v|^2\right](x,t)\,\mathrm{d}x \geq 0 \quad \text{for all } t \in I, \tag{2.5}$$

therefore

$$\mathcal{J}_{\mathscr{B}}(v) = \sup_{t\in I} \int_\Omega \left[e - \frac{1}{2}|v|^2\right](x,t)\,\mathrm{d}x \geq 0.$$

Finally, for every $v \in \overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$, taking a sequence $(v^k)_{k\in\mathbb{N}} \subset \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ tending to v, applying the upper-semicontinuity of the functional we conclude

$$\mathcal{J}_{\mathscr{B}}(v) \geq \limsup_{k\to\infty} \mathcal{J}_{\mathscr{B}}(v^k) \geq 0.$$

Assume now that we have $v \in \overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ such that $\mathcal{J}_{\mathscr{B}}(v) = 0$. Let $(v^k)_{k\in\mathbb{N}} \subset \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ a sequence tending to v, and u^k the associated smooth matrix field of each v^k. Since

$$|u^k|_\infty < 2\frac{d-1}{d}\|e\|_{C(\mathscr{B})} \quad \text{in } \mathscr{B},$$

(u^k) is a bounded sequence in $L^\infty(\mathscr{B}, S_0^d)$. Therefore, we may assume (taking a subsequence if necessary) that it converges (weakly) to a function u in $L_w^\infty(\mathscr{B}, S_0^d)$. Now, since L^2-norm is wslsc and $e \in C([0,T], L^1(\mathscr{D}))$, the hypothesis and (2.5) imply

$$0 = \mathcal{J}_{\mathscr{B}}(v) \geq \int_\Omega \left[e - \frac{1}{2}|v|^2\right](x,t)\,\mathrm{d}x \geq \limsup_{k\to\infty} \int_\Omega \left[e - \frac{1}{2}|v^k|^2\right](x,t)\,\mathrm{d}x \geq 0 \quad \text{for all } t \in I,$$

that is,

$$\int_\Omega \left[e - \frac{1}{2}|v|^2\right](x,t)\,\mathrm{d}x = 0 \quad \text{for all } t \in I. \tag{2.6}$$

But now, since ρ is convex, lemma 2.9 and Mazur theorem imply

$$\frac{1}{2}|v|^2(x,t) \leq \rho(v(x,t),u(x,t)) \leq \liminf_{k\to\infty} \rho(v^k(x,t),u^k(x,t)) \leq e(x,t) \quad \text{a.e.}\, x \in \Omega \quad \forall t \in I.$$

This means that (2.6) is

$$\left\| \left[e - \frac{1}{2}|v|^2 \right](\cdot,t) \right\|_{L^1(\Omega)} = 0 \quad \forall t \in I,$$

which is precisely statement.

(c) ... □

Proposition 2.15 (The perturbation property)**.** *For all $\alpha > 0$ there exists $\beta(\alpha,\mathscr{B}) > 0$ such that, whenever $v \in \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ satisfies*

$$\mathcal{J}_{\mathscr{B}}(v) > \alpha,$$

there exists a sequence $(v^k)_{k\in\mathbb{N}} \subset \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ such that $v^k \xrightarrow{d} v$ and

$$\mathcal{J}_{\mathscr{B}}(v) \geq \limsup_{k\to\infty} \mathcal{J}_{\mathscr{B}}(v^k) + \beta.$$

We postpone the proof of this property to the section...

Now, we fix an exhausting sequence $(\mathscr{B}_n)_{n\in\mathbb{N}}$ of $\mathscr{D} \times (0,T)$, where $\mathscr{B}_n = \Omega_n \times I_n$. This means that $\mathscr{B}_n \subset \mathscr{B}_{n+1}$ for all $n \in \mathbb{N}$ and

$$\bigcup_{n\in\mathbb{N}} \mathscr{B}_n = \mathscr{D} \times (0,T).$$

Corollary 2.16. *Given $v \in \overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$, if*

$$\mathcal{J}_{\mathscr{B}_n}(v) = 0 \quad \textit{for all } n \in \mathbb{N},$$

then (v,p) is a weak solution of the IEE on $\mathscr{D}_T$ with initial data $v(\cdot,0) = v_0$, pressure $p = q - \frac{1}{d}|v|^2$ and energy $\frac{1}{2}|v|^2 = e$ a.e. on $\mathscr{D}_T$.

Proof. Given $v \in \overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$, assume that

$$v \in \bigcap_{n\in\mathbb{N}} \mathcal{J}_{\mathscr{B}_n}^{-1}(0).$$

By ... we know that

$$\frac{1}{2}|v|^2(x,t) = e(x,t) \quad \text{a.e.}\, x \in \Omega_n \quad \forall t \in I_n.$$

Since this is true for all $n \in \mathbb{N}$ and $(\mathscr{B}_n)_{n\in\mathbb{N}}$ is an exhausting sequence of $\mathscr{D}_T$,

$$\frac{1}{2}|v|^2(x,t) = e(x,t) \quad \text{a.e.}\, x \in \mathscr{D} \quad \forall t \in (0,T).$$

... □

2.2.5 Proof of the subsolution criterion

Proof. As $\mathcal{J}_{\mathscr{B}_n}$ is 1-Baire, its points of continuity form a residual set of $\overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$. Now we claim that, if v is a point of continuity of $\mathcal{J}_{\mathscr{B}_n}$, then $\mathcal{J}_{\mathscr{B}_n}(v) = 0$. We prove it by contradiction. Suppose that $\mathcal{J}_{\mathscr{B}_n}(v) > \alpha$ for some $\alpha > 0$. Take a sequence $(v^k)_{k\in\mathbb{N}} \subset \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ such that $v^k \xrightarrow{d} v$. As v is a point of continuity,

$$\mathcal{J}_{\mathscr{B}_n}(v^k) \to \mathcal{J}_{\mathscr{B}_n}(v),$$

so we may assume that (starting the sequence sufficiently late)

$$\mathcal{J}_{\mathscr{B}_n}(v^k) > \alpha, \quad \text{for all } k \in \mathbb{N}.$$

The perturbation property gives, for each $k \in \mathbb{N}$, a sequence $(v^{k,j})_{j\in\mathbb{N}} \subset \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ such that $v^{k,j} \xrightarrow{d} v^k$ and

$$\mathcal{J}_{\mathscr{B}_n}(v^k) \geq \limsup_{j\to\infty} \mathcal{J}_{\mathscr{B}_n}(v^{k,j}) + \beta.$$

By a diagonal sequence argument, we can construct a sequence $(v^{k,j(k)})_{k\in\mathbb{N}} \subset \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ such that $v^{k,j(k)} \xrightarrow{d} v$ and

$$\mathcal{J}_{\mathscr{B}_n}(v) \geq \limsup_{k\to\infty} \mathcal{J}_{\mathscr{B}_n}(v^{k,j(k)}) + \beta,$$

but this contradicts the continuity of $\mathcal{J}_{\mathscr{B}_n}$ at v because

$$\exists \lim_{k\to\infty} \mathcal{J}_{\mathscr{B}_n}(v^{k,j(k)}) = \mathcal{J}_{\mathscr{B}_n}(v).$$

In summary, the set

$$\bigcap_{n\in\mathbb{N}} \mathcal{J}_{\mathscr{B}_n}^{-1}(0)$$

is residual in $\overline{\mathcal{S}_q}(\mathscr{D}_T, v_0, e)$ because is a intersection of residual sets in a complete metric space, and, by prop..., we know that all the functions of this set are weak solutions of the IEE. $\square$

2.2.6 Convex integration proves the perturbation property

(...)

<u>Localized waves</u>

Explicar porque hay que meter ondas localizadas. Comparar con Nash-Kuiper. Necesidad de introducir el potencial.

Lemma 2.17 (Maximizing the perturbation)**.** *Let $r > 0$ and $z = (v, u) \in \mathrm{Int}K_r^{\mathrm{co}}$. Then there exists a vector $\overline{z} = (\overline{v}, \overline{u}) \in \mathbb{R}^d \times S_0^d$ and a sufficiently small radius $\epsilon > 0$ such that the line segment*

$$\sigma = [-\overline{z}, \overline{z}]$$

satisfies

$$|\overline{v}| \geq \frac{C}{r}(r^2 - |v|^2) \quad \text{and} \quad z + \overline{B(\sigma, \epsilon)} \subset \mathrm{Int}K(r)^{\mathrm{co}},$$

where $C = \frac{1}{4(D_* - 1)}$ and $B(\sigma, \epsilon) = \{w \in \mathbb{R}^d \times S_0^d : \mathrm{dist}(w, \sigma) < \epsilon\}$. Furthermore,

$$\frac{r^2}{2} - \rho(z + w) > \frac{1}{4}\Big(\frac{r^2}{2} - \rho(z)\Big) \quad \text{for all } w \in \overline{B(\sigma, \epsilon)}.$$

Proof. Let $z = (v, u) \in \mathrm{Int}K_r^{\mathrm{co}}$. We know, by Caratheodory's convex hull theorem [ref], that z lies in the interior of a simplex spanned by D_* elements of $K(r)$, and, we may assume (perturbing slightly such vertices inside K_r, which is possible since z belongs to the interior zone) that $v_i \neq \pm v_j$ whenever $i \neq j$ [aquí se podría explicar más esta parte]. That is, there exists $z_i = (v_i, v_i \bigcirc v_i)$ with $v_i \in r\mathbb{S}^{d-1}$ for $i = 1, \ldots, D_*$ such that $v_i \neq \pm v_j$ whenever $i \neq j$ and

$$z = \sum_{i=1}^{D_*} \lambda_i z_i$$

for some $\lambda_i \in [0, 1)$ for $i = 1, \ldots, D*$ with $\sum \lambda_i = 1$. Assume (relabeling if necessary) that $\lambda_1 = \max \lambda_i$. Notice that, for each $j > 1$, the points

$$z \pm \lambda_j(z_j - z_1) = (\lambda_1 \mp \lambda_j)z_1 + (1 \pm 1)\lambda_j z_j + \sum_{\substack{i=2 \\ i \neq j}}^{D_*} \lambda_i z_i$$

belongs to K_r^{co} because they are convex combination of elements of K_r. Now, since

$$z - z_1 = \sum_{i=2}^{D_*} \lambda_i(z_i - z_1),$$

then

$$|v - v_1| \leq (D_* - 1) \max_{i=2}^{D_*} \lambda_i |v_i - v_1|.$$

Fix the index $j > 1$ for which the above maximum is attained. Finally, we consider the vector

$$\overline{z} = (\overline{v}, \overline{u}) = \frac{\lambda_j}{2}(z_j - z_1) = \frac{\lambda_j}{2}(v_j - v_1, v_j \ominus v_1),$$

and the line segment

$$\sigma = [-\overline{z}, \overline{z}]$$

which is an admissible segment because $v_1 \neq \pm v_j$. Furthermore, let $\overline{w} = \lambda \frac{\lambda_j}{2}(z_j - z_1) \in \sigma$ ($\lambda \in [-1,1]$). Recall that $z + \lambda\lambda_j(z_j - z_1) \in K_r^{\text{co}}$ because we have seen that the extremal points ($\lambda = \pm 1$) belongs to $K(r)^{\text{co}}$. Applying the convexity of ρ we obtain

$$\begin{aligned}
\rho(z+\overline{w}) &= \rho\Big(\frac{1}{2}z + \frac{1}{2}(z + \lambda\lambda_j(z_j - z_1))\Big) \\
&\leq \frac{1}{2}\rho(z) + \frac{1}{2}\rho(z + \lambda\lambda_j(z_j - z_1)) \\
&\leq \frac{1}{2}\rho(z) + \frac{r^2}{4} \\
&= \frac{1}{2}\rho(z) + \frac{r^2}{2} - \frac{r^2}{4}
\end{aligned}$$

hence

$$\frac{r^2}{2} - \rho(z+\overline{w}) \geq \frac{1}{2}\Big(\frac{r^2}{2} - \rho(z)\Big).$$

Now consider the set

$$\mathcal{C} = \Big\{\widetilde{z} \in \mathbb{R}^d \times S_0^d \,:\, \frac{r^2}{2} - \rho(\widetilde{z}) > \frac{1}{4}\Big(\frac{r^2}{2} - \rho(z)\Big)\Big\}$$

which is open and satisfies $z + \sigma \subset \mathcal{C} \subset \operatorname{Int} K_r^{\text{co}}$. Finally, we can choose a sufficiently small radius $\epsilon > 0$ such that $z + \overline{B(\sigma,\epsilon)} \subset \mathcal{C}$.

On the other hand, since $|v| < |v_1| = r$,

$$\begin{aligned}
|\overline{v}| &= \frac{\lambda_j}{2}|v_j - v_1| \geq \frac{1}{2(D_* - 1)}|v - v_1| \\
&\geq \frac{1}{2(D_* - 1)}(r - |v|) \geq \frac{1}{2(D_* - 1)}\frac{r + |v|}{2r}(r - |v|) \\
&= \frac{1}{4(D_* - 1)r}(r^2 - |v|^2) = \frac{C_d}{r}(r^2 - |v|^2).
\end{aligned}$$

□

Explicar el potencial de Annals.
Define the spaces of matrices

$$\begin{aligned}
\mathfrak{S}^{d+1} &= \{U \in S^{d+1} \,:\, U_{(d+1),(d+1)} = 0\} \\
\mathfrak{S}_0^{d+1} &= \{U \in S_0^{d+1} \,:\, U_{(d+1),(d+1)} = 0\}
\end{aligned}$$

Consider the following map

$$\begin{aligned}
U : C^\infty(\mathscr{D} \times (0,T), \mathbb{R}^d \times S_0^d \times \mathbb{R}) &\to C^\infty(\mathscr{D} \times (0,T), \mathfrak{S}^{d+1}) \\
(v,u,q) &\mapsto \begin{pmatrix} u + qI_d & v \\ v & 0 \end{pmatrix}
\end{aligned}$$

Denoting $y = (x, t)$, by definition it is clear that

$$\mathrm{div}_y U = 0 \quad \Leftrightarrow \quad \begin{cases} \partial_t v + \mathrm{div}_x u + \nabla q = 0 \\ \mathrm{div}_x v = 0 \end{cases}$$

Proposition 2.18. *Let $a, b \in \mathbb{R}^d$ such that $|a| = |b| = r$ with $a \neq \pm b$. Then, there exists a matrix-valued, constant coefficient, homogeneous linear differential operator of order three*

$$\mathcal{L} : C^\infty(\mathbb{R}^{d+1}) \to C^\infty(\mathbb{R}^{d+1}, \mathfrak{S}_0^{d+1})$$

and a space-time vector $\eta \in \mathbb{R}^{d+1}$ not parallel to e_{d+1} with the following properties:

- *For all $\phi \in C^\infty(\mathbb{R}^{d+1})$, $U = \mathcal{L}(\phi)$ satisfies*

$$\mathrm{div}_y U = 0, \quad \mathrm{supp}\, U \subset \mathrm{supp}\, \phi.$$

- *If $\phi(y) = \psi(\eta \cdot y)$ for some $\psi \in C^\infty(\mathbb{R})$, then*

$$\mathcal{L}(\phi)(y) = (U_a - U_b)\psi'''(\eta \cdot y), \quad y \in \mathbb{R}^{d+1}.$$

Proof. The construction of this potential comes from the election of a suitable matrix-valued, homogeneous polynomial

$$\begin{array}{rcl} P : \mathbb{R}^{d+1} & \to & \mathbb{R}^{(d+1)\times(d+1)} \\ y & \mapsto & (P_{ij}(y))_{i,j=1}^{d+1} \end{array}$$

taking $\mathcal{L} = P(\partial)$. Since

$$(P(\partial)\phi)^\dagger = P^\dagger(\partial)\phi, \quad (P(\partial)\phi)_{(d+1),(d+1)} = P_{(d+1),(d+1)}(\partial)\phi, \quad \mathrm{Tr}(P(\partial)\phi) = (\mathrm{Tr}P)(\partial)\phi,$$

for all $\phi \in C^\infty(\mathbb{R}^{d+1})$, we must set

$$P^\dagger = P, \quad P_{(d+1),(d+1)} = 0, \quad \mathrm{Tr}P = 0,$$

that is, $P : \mathbb{R}^{d+1} \to \mathfrak{S}_0^{d+1}$. On the other hand, as we want

$$0 = \mathrm{div}_y(P(\partial)\phi)_i = \left(\sum_{j=1}^{d+1} P_{ij}(\partial)\partial_j\right)\phi = (P(\partial)\partial)_i\phi$$

we must set

$$P(y)y = 0 \quad \forall y \in \mathbb{R}^{d+1}.$$

(Obviously, as a linear differential operator, the support of U is contained in the support of ϕ). Consider the matrices

$$R = \hat{a} \ominus \hat{b} = \left(\begin{array}{c|c} a \ominus b & 0 \\ \hline 0 & 0 \end{array}\right), \quad Q_y = y \ominus e_{d+1} = \left(\begin{array}{c|c} 0 & \check{y} \\ \hline -\check{y} & 0 \end{array}\right), \quad y \in \mathbb{R}^{d+1}.$$

We claim that, for all $y \in \mathbb{R}^{d+1}$,

- $\langle Ry, y\rangle = 0, \quad \langle Q_y y, y\rangle = 0$
- $\langle Ry, Q_y y\rangle = 0$

Both follows from the next observation, if $M \in \mathbb{R}^{n\times n}$ is an antisymmetric matrix with null diagonal

$$\langle M\xi, \xi\rangle = \sum_{i<j} m_{ij}\xi_i\xi_j + \sum_{i>j}(-m_{ji})\xi_i\xi_j = 0, \quad \xi \in \mathbb{R}^n.$$

For the second one observe that

$$Ry = ((a \ominus b)\check{y}, 0), \quad Q_y y = (y_{d+1}\check{y}, -|\check{y}|^2),$$

hence

$$\langle Ry, Q_y y\rangle = y_{d+1}\langle (a\ominus b)\check{y}, \check{y}\rangle = 0.$$

Let

$$P(y) = \frac{1}{2}(Ry \otimes Q_y y + Q_y y \otimes Ry), \quad y \in \mathbb{R}^{d+1}.$$

Let us see that this satisfies the desired properties. By definition, it is an homogeneous polynomial of degree three, and P is symmetric. Moreover, for all $y \in \mathbb{R}^{d+1}$,

$$P(y)_{(d+1),(d+1)} = (Ry)_{d+1}(Q_y y)_{d+1} = 0,$$

$$\mathrm{Tr}(P(y)) = \sum_{i=1}^{d+1}(Ry)_i(Q_y y)_i = \langle Ry, Q_y y\rangle = 0$$

and

$$P(y)y = \frac{1}{2}(Ry \otimes Q_y y + Q_y y \otimes Ry)y = \frac{1}{2}(Ry\langle Q_y y, y\rangle + Q_y y\langle Ry, y\rangle) = 0.$$

Now observe that, if $\eta \in \mathbb{R}^{d+1}$ and $\phi(y) = \psi(y\cdot\eta)$ for $\psi \in C^\infty(\mathbb{R}^{d+1})$, then

$$(P(\partial)(\phi))_{ij} = P_{ij}(\partial)(\phi) = P_{ij}(\eta)\psi''',$$

that is

$$\mathcal{L}(\phi) = P(\eta)\psi'''.$$

Therefore, we must find $\eta \in \mathbb{R}^{d+1}$ such that

$$P(\eta) = U_a - U_b = \left(\begin{array}{c|c} a\ominus b & a-b \\ \hline a-b & 0 \end{array}\right)$$

On the one hand, we want

$$(a-b)_i = P(\eta)_{i,d+1} = \frac{1}{2}(R\eta)_i(Q_\eta\eta)_{d+1} = -\frac{1}{2}|\check{\eta}|^2((a\ominus b)\check{\eta})_i.$$

Observe that

$$\begin{aligned}(a \ominus b)(a+b) &= (a \otimes b)(a+b) - (b \otimes a)(a+b) \\ &= a\langle a, b\rangle + a|b|^2 - b|a|^2 - b\langle a, b\rangle \\ &= (a-b)(r^2 + \langle a, b\rangle) \\ &= \frac{1}{2}(a-b)|a+b|^2.\end{aligned}$$

Hence, taking $\check{\eta} = C(a+b) \neq 0$ $(a \neq -b)$,

$$-\frac{1}{2}|\check{\eta}|^2 (a \ominus b)\check{\eta} = -\frac{C^3}{4}|a+b|^4(a-b),$$

therefore, the constant must be

$$C = -\left(\frac{2}{|a+b|^2}\right)^{\frac{2}{3}} = -\frac{1}{(r^2 + \langle a, b\rangle)^{\frac{2}{3}}}.$$

Call $\lambda = \frac{1}{C}\eta_{d+1}$. Then,

$$\begin{aligned}(a \ominus b)_{ij} &= P(\eta)_{ij} \\ &= \frac{1}{2}((R\eta)_i(Q_\eta \eta)_j + (Q_\eta \eta)_i (R\eta)_j) \\ &= \frac{1}{2}(((a \ominus b)\check{\eta})_i(\eta_{d+1}\eta_j) + (\eta_{d+1}\eta_i)((a \ominus b)\check{\eta})_j) \\ &= \frac{\lambda}{2}C^3(((a \ominus b)(a+b))_i(a+b)_j + (a+b)_i((a \ominus b)(a+b))_j) \\ &= \frac{\lambda}{4}C^3|a+b|^2((a-b)_i(a+b)_j + (a+b)_i(a-b)_j) \\ &= \frac{\lambda}{4}C^3|a+b|^2 2(a_i a_j - b_i b_j) \\ &= \frac{\lambda}{2}C^3|a+b|^2(a \ominus b)_{ij},\end{aligned}$$

therefore, must be

$$\lambda = \frac{2}{C^3|a+b|^2} = -\frac{1}{2}|a+b|^2 = -(r^2 + \langle a, b\rangle).$$

Finally, the direction is

$$\eta = -\frac{1}{(r^2 + \langle a, b\rangle)^{\frac{2}{3}}}(a+b, -(r^2 + \langle a, b\rangle)).$$

□

Lemma 2.19. *If $\eta \in \mathbb{R}^{d+1}$ is a direction not parallel to e_{d+1}, then*

$$\lim_{k\to\infty}\int_{\mathcal{B}} \sin^2(k\eta\cdot(x,t))\,\mathrm{d}x = \frac{1}{2}|\mathcal{B}|$$

uniformly in $t \in \mathbb{R}$, for every bounded open set $\mathcal{B} \subset \mathbb{R}^{d+1}$.

Proof. Notice that

$$\begin{aligned}
\sin^2(k\eta\cdot(x,t)) &= \Big(\sin(k\check{\eta}\cdot x)\cos(k\eta_{d+1}t)+\cos(k\check{\eta}\cdot x)\sin(k\eta_{d+1}t)\Big)^2\\
&= \sin^2(k\check{\eta}\cdot x)\cos^2(k\eta_{d+1}t)+\cos^2(k\check{\eta}\cdot x)\sin^2(k\eta_{d+1}t)\\
&\quad + 2\sin(k\check{\eta}\cdot x)\cos(k\check{\eta}\cdot x)\sin(k\eta_{d+1}t)\cos(k\eta_{d+1}t)\\
&= \sin^2(k\check{\eta}\cdot x)+\cos(2\check{\eta}\cdot x)\sin^2(k\eta_{d+1}t)+\frac{1}{2}\sin(2k\check{\eta}\cdot x)\sin(2k\eta_{d+1}t)
\end{aligned}$$

Since $\check{\eta}\neq 0$ and $\mathcal{B}$ is an open bounded set, the first term satisfies

$$\lim_{k\to\infty}\int_{\mathcal{B}} \sin^2(k\check{\eta}\cdot x)\,\mathrm{d}x = \frac{1}{2}|\mathcal{B}|.$$

The other terms

$$\left|\int_{\mathcal{B}}\cos(2\check{\eta}\cdot x)\sin^2(k\eta_{d+1}t)\,\mathrm{d}x\right| \leq \left|\int_{\mathcal{B}}\cos(2\check{\eta}\cdot x)\,\mathrm{d}x\right| \underset{k\to\infty}{\longrightarrow} 0$$

and

$$\left|\int_{\mathcal{B}}\sin(2k\check{\eta}\cdot x)\sin(2k\eta_{d+1}t)\,\mathrm{d}x\right| \leq \left|\int_{\mathcal{B}}\sin(2k\check{\eta}\cdot x)\cdot x)\,\mathrm{d}x\right| \underset{k\to\infty}{\longrightarrow} 0$$

uniformly in $t\in\mathbb{R}$. (referencia) □

The brick grid

For $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$ and $s\in(0,1]$ denote

$$\mathcal{Q}_n(k,s) = k + s\Big[-\frac{1}{2},\frac{1}{2}\Big)^n$$

by the n-dimensional cube of center k and side s. For $i=(i_1,\dots,i_d)\in\mathbb{Z}^d$ we consider the reduced x-cubes $\varepsilon\mathcal{Q}_d(i,s)=\mathcal{Q}_d(\varepsilon i, s\varepsilon)$. Note that $\{\varepsilon\mathcal{Q}(i,1) : i\in\mathbb{Z}^d\}$ form a partition of $\mathbb{R}^d$. Now, for $(i,j)\in\mathbb{Z}^d\times\mathbb{Z}$ denote

$$\mathcal{C}(i,j,s) = \begin{cases} \mathcal{Q}_{d+1}((i,j+\frac{1}{2}),s), & \text{if } |i| \text{ is even},\\ \mathcal{Q}_{d+1}((i,j),s), & \text{if } |i| \text{ is odd},\end{cases}$$

by the shifted t-cylinders. We denote $y_{i,j}$ by the center of each cylinder $\mathcal{C}(i,j,s)$. We consider the reduced t-cylinders $\varepsilon\mathcal{C}(i,j,s)=\mathcal{C}(\varepsilon i,\varepsilon j,s\varepsilon)$ with center $\varepsilon y_{i,j}$. Note that

$\{\varepsilon\mathcal{C}(i,j,1) \,:\, (i,j) \in \mathbb{Z}^d \times \mathbb{Z}\}$ form a partition of $\mathbb{R}^d_x \times \mathbb{R}_t$. Now, let $\varphi^\varepsilon : \mathbb{R}^d_x \times \mathbb{R}_t \to [0,1]$ a smooth function with supp$\varphi \subset \varepsilon\mathcal{Q}_{d+1}(0,1)$ and $\varphi^\varepsilon \equiv 1$ on $\varepsilon\mathcal{Q}_{d+1}(0,\frac{3}{4})$. Denote $\varphi^\varepsilon_{i,j}$ by the translated function φ^ε to $\varepsilon\mathcal{C}(i,j,1)$

$$\Psi^\varepsilon = \sum_{(i,j)\in\mathbb{Z}^d\times\mathbb{Z}} \varphi^\varepsilon_{i,j}.$$

Denote

$$\Omega^\gamma_\varepsilon(s) = \bigcup_{\substack{|i|\in 2\mathbb{Z}+\gamma \\ \varepsilon\mathcal{Q}_d(i,1)\subset\Omega}} \varepsilon\mathcal{Q}_d(i,s).$$

Lemma 2.20. *Given a function $f : \mathscr{B} \to \mathbb{R}$, define the simple function*

$$\boxdot_\varepsilon f = \sum_{(i,j)\in\mathbb{Z}^d\times\mathbb{Z}} f(\varepsilon y_{i,j})\mathbb{1}_{\varepsilon\mathcal{C}(i,j,1)}$$

which is a discretization of f in the middle points of each cylinder $\varepsilon\mathcal{C}(i,j,1)$. Then, if f is uniformly continuous on $\mathscr{B}$,

$$\int_{\Omega^\gamma_\varepsilon(s)} \boxdot_\varepsilon f(x,t)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \frac{s^d}{2}\int_\Omega f(x,t)\,\mathrm{d}x$$

for $\gamma = 1,2$ uniformly in $t \in I$.

Proof. Fix $t \in I$. Observe that, as $\boxdot_\varepsilon f$ is a simple function,

$$\int_{\Omega^\gamma_\varepsilon(s)} \boxdot_\varepsilon f(x,t)\,\mathrm{d}x = s^d \int_{\Omega^\gamma_\varepsilon(1)} \boxdot_\varepsilon f(x,t)\,\mathrm{d}x \quad \text{for } l = 1,2.$$

...

$$\int_{\Omega^1_\varepsilon(1)\cup\Omega^2_\varepsilon(1)} \boxdot_\varepsilon f(x,t)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega f(x,t)\,\mathrm{d}x.$$

... $\square$

From now on, we will denote simply $\Omega^\varepsilon_\varepsilon \equiv \Omega^\gamma_\varepsilon(\frac{3}{4})$. Now, for every fixed time t, observe that the set

$$\{x \in \Omega \,:\, \Psi^\varepsilon(x,t) = 1\}$$

contains at least one of the sets $\Omega^\gamma_\varepsilon$. Moreover, if

$$\tau^1_\varepsilon = \bigcup_{j\in\mathbb{Z}} \varepsilon\mathcal{Q}_1(j+\tfrac{1}{2},\tfrac{1}{2}), \quad \tau^2_\varepsilon = \bigcup_{j\in\mathbb{Z}} \varepsilon\mathcal{Q}_1(j,\tfrac{1}{2}),$$

then $\tau^1_\varepsilon \dot\cup \tau^2_\varepsilon = \mathbb{R}$ and

$$\Psi^\varepsilon(x,t) \equiv 1 \quad \text{in } \Omega^\gamma_\varepsilon \times \tau^\gamma_\varepsilon.$$

The perturbation

Fix $\alpha > 0$. Let $v \in \mathcal{S}_q(\mathscr{D}_T, v_0, e)$ such that $\mathcal{J}_{\mathscr{B}}(v) > \alpha$. Since v and e are uniformly continuous on $\mathscr{B}$ by lemma 2.20

$$\int_{\Omega_\varepsilon^\gamma} \boxdot_\varepsilon \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2}\Big(\frac{3}{4}\Big)^d \int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x$$

for $\gamma = 1, 2$ uniformly in $t \in I$. Therefore, taking $\delta = \frac{\alpha}{8}(\frac{3}{4})^d$, there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon \le \varepsilon_0$,

$$\int_{\Omega_\varepsilon^\gamma} \boxdot_\varepsilon \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x \ge \frac{1}{2}\Big(\frac{3}{4}\Big)^d \int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x - \delta \ge \frac{\alpha}{8}\Big(\frac{3}{4}\Big)^d,$$

for any $t \in I$ for which

$$\int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x \ge \frac{\alpha}{2}.$$

Consider now the associated smooth matrix field $u : \mathscr{D}_T \to S_0^d$. Recall that

$$\rho(z(y)) < e(y) \quad \text{for all } y \in \mathscr{D} \times (0,T).$$

In particular, for some $\delta > 0$,

$$e(y) - \rho(z(y) + w) > \frac{1}{4}(e(y) - \rho(z(y))) \ge \delta \quad \text{for all } y \in \overline{\mathscr{B}},\ w \in \overline{B(\sigma(y), \epsilon(y))}.$$

Recall now that, since ρ is convex on $\mathbb{R}^D$, it is locally Lipschitz. Hence, since the image of z on $\overline{\mathscr{B}}$ with the possible perturbation are bounded, for some $M > 0$

$$|\rho(z(y) + w) - \rho(z(\widetilde{y}) + w)| \le M|z(y) - z(\widetilde{y})|$$

for all $y, \widetilde{y} \in \overline{\mathscr{B}}$ and $w \in \overline{B(\sigma(y), \epsilon(y))}$. On the other hand, since z and e are uniformly continuous on $\overline{\mathscr{B}}$, there exists $\varepsilon_1 > 0$ such that

$$|z(y) - z(\widetilde{y})| \le \frac{\delta}{4M}, \quad |e(y) - e(\widetilde{y})| \le \frac{\delta}{4}$$

for all $y \in \mathcal{Q}(y, \varepsilon)$ and $\widetilde{y} \in \overline{\mathscr{B}}$ and every $0 < \varepsilon \le \varepsilon_1$. Finally

$$\begin{aligned} e(\widetilde{y}) - \rho(z(\widetilde{y}) + w) &= e(\widetilde{y}) - e(y) + \rho(z(y) + w) - \rho(z(\widetilde{y}) + w) + e(y) - \rho(z(y) + w) \\ &\ge -\frac{\delta}{4} - \frac{\delta}{4} + \delta = \frac{\delta}{2} > 0 \end{aligned}$$

for all $y \in \mathcal{Q}(y, \varepsilon)$, $\widetilde{y} \in \overline{\mathscr{B}}$ and $w \in \overline{B(\sigma(y), \epsilon(y))}$ and every $0 < \varepsilon \le \varepsilon_1$. From now on we fix $0 < \varepsilon \le \min\{\varepsilon_0, \varepsilon_1\}$ and we miss the index ε to simplify the notation.

For each $(i, j) \in \mathbb{Z}^d \times \mathbb{Z}$ such that $\varepsilon\mathcal{C}(i, j, 1) \subset \mathscr{B}$, denote

$$z_{i,j} = (v(\varepsilon y_{i,j}), u(\varepsilon y_{i,j})),$$

which is the value taken by (v, u) in the middle point of each cylinder. By definition of the space of subsolutions, each $z_{i,j}$ belongs to $\mathrm{Int}K(r_{i,j})^{\mathrm{co}}$ for $r_{i,j} = \sqrt{2e(\varepsilon y_{i,j})}$. Therefore, by lemma 2.17, there exists $\overline{z_{i,j}} = (\overline{v_{i,j}}, \overline{u_{i,j}}) \in \mathbb{R}^d \times S_0^d$ and a radius $\epsilon > 0$ such that the segment

$$\sigma_{i,j} = [-\overline{z_{i,j}}, \overline{z_{i,j}}]$$

satisfies

$$|\overline{v_{i,j}}| \geq \frac{C_d}{\sqrt{2e(\varepsilon y_{i,j})}}[2e - |v^\varepsilon_{i,j}|^2](\varepsilon y_{i,j}) \geq C_{d,e,\mathscr{B}} \boxdot \left[e - \frac{1}{2}|v|^2\right](\varepsilon y_{i,j}),$$

where $C_{d,e,\mathscr{B}} = \sqrt{\frac{2}{\|e\|_{C(\mathscr{B})}}}C_d$, and, as we have observed before,

$$\rho(z(y) + w) < e(y)$$

for all $y \in \varepsilon\mathcal{C}(i, j, 1)$, $w \in \overline{B(\sigma_{i,j}, \epsilon_{i,j})}$.

Let us consider the operator $\mathcal{L}_{i,j}$ and the direction $\eta_{i,j} \in \mathbb{R}^{d+1}$ (not parallel to e_{d+1}) associated to the segment $\sigma_{i,j}$. Define the perturbation in each cylinder for some frequency $k \in \mathbb{N}$ via

$$z^k_{i,j} = (v^k_{i,j}, u^k_{i,j}) = \mathcal{L}_{i,j}\Big(\varphi_{i,j}\frac{1}{k^3}\cos(k\eta_{i,j} \cdot (x, t))\Big), \quad (x, t) \in \mathscr{D}_T,$$

which is supported in the cylinder $\varepsilon\mathcal{C}(i, j, 1)$. Recall that $(v_{i,j}, u_{i,j}, 0)$ is a subsolution. Moreover, we can choose the frequency $k \geq k_0$ for some $k_0 \in \mathbb{N}$ sufficiently large to guarantees that

$$z^k_{i,j}(x, t) \in B(\sigma_{i,j}, \epsilon),$$

for all $(x, t) \in \mathscr{D}_T$, because, as

$$\mathcal{L}_{i,j}\Big(\frac{1}{k^3}\cos(k\eta_{i,j} \cdot (x, t))\Big) = \overline{z^\varepsilon_{i,j}}\sin(k\eta_{i,j} \cdot (x, t)), \quad (x, t) \in \mathscr{D}_T,$$

then

$$\begin{aligned}
\mathrm{dist}(z^k_{i,j}(x, t), \sigma_{i,j}) &\leq |z^k_{i,j}(x, t) - \varphi_{i,j}\overline{z_{i,j}}\sin(k\eta_{i,j} \cdot (x, t))| \\
&\leq \|z^k_{i,j} - \varphi_{i,j}\overline{z_{i,j}}\sin(k\eta_{i,j}\cdot)\|_{C(\varepsilon\mathcal{C}(i,j,1))} \\
&= \Big\|\mathcal{L}_{i,j}\Big(\varphi_{i,j}\frac{1}{k^3}\cos(k\eta_{i,j}\cdot)\Big) - \varphi_{i,j}\mathcal{L}_{i,j}\Big(\frac{1}{k^3}\cos(k\eta_{i,j}\cdot)\Big)\Big\|_{C(\varepsilon\mathcal{C}(i,j,1))} \\
&\leq C(\mathcal{L}_{i,j}, \varphi^\varepsilon_{i,j})\frac{1}{k} < \epsilon,
\end{aligned}$$

for all $(x, t) \in \mathscr{D}_T$. Define the total perturbation as the (finite) sum of all these localized waves

$$\widetilde{z}^k = (\widetilde{v}^k, \widetilde{u}^k) = \sum_{\varepsilon\mathcal{C}(i,j,1)\subset\mathscr{B}} (v_{i,j}, u_{i,j}),$$

and the perturbed function

$$z^k = (v^k, u^k) = (v, u) + (\widetilde{v}^k, \widetilde{u}^k).$$

Therefore, for each $k \geq k_0$, $v^k \in \mathcal{S}_q(\mathscr{D}_T, v_0, e)$.

Proposition 2.21. $\widetilde{v}^k \xrightarrow{d} 0$.

The property

Since $\Psi \equiv 1$ in $\Omega^\gamma \times \tau^\gamma$,

$$|\widetilde{v}^k(x,t)|^2 = |\overline{v_{i,j}}|^2 \sin^2(k\eta_{i,j} \cdot (x,t)), \quad (x,t) \in \varepsilon\mathcal{C}(i,j,\tfrac{3}{4}),$$

for all cylinders. Using lemma ... ($\eta_{i,j} \nparallel e_{d+1}$)

$$\lim_{k\to\infty} \int_{\varepsilon\mathcal{Q}_d(i,\frac{3}{4})} |\widetilde{v}^k(x,t)|^2 \,\mathrm{d}x = \frac{1}{2}|\varepsilon\mathcal{Q}_d(i,\tfrac{3}{4})||\overline{v_{i,j}}|^2 = \frac{1}{2}\int_{\varepsilon\mathcal{Q}_d(i,\frac{3}{4})} |\overline{v_{i,j}}|^2 \,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
\lim_{k\to\infty} \int_{\Omega^\gamma} |\widetilde{v}^k(x,t)|^2 \,\mathrm{d}x &= \lim_{k\to\infty} \sum_{\substack{|i|\in 2\mathbb{Z}+\gamma \\ \varepsilon\mathcal{Q}_d(i,1)\subset\Omega}} \int_{\varepsilon\mathcal{Q}_d(i,\frac{3}{4})} |\widetilde{v}^k(x,t)|^2 \,\mathrm{d}x \\
&= \frac{1}{2} \sum_{\substack{|i|\in 2\mathbb{Z}+\gamma \\ \varepsilon\mathcal{Q}_d(i,1)\subset\Omega}} \int_{\varepsilon\mathcal{Q}_d(i,\frac{3}{4})} |\overline{v_{i,j}}|^2 \,\mathrm{d}x \\
&\geq \frac{1}{2} C_{d,e,\mathscr{B}}^2 \sum_{\substack{|i|\in 2\mathbb{Z}+\gamma \\ \varepsilon\mathcal{Q}_d(i,1)\subset\Omega}} \int_{\varepsilon\mathcal{Q}_d(i,\frac{3}{4})} \boxdot\Big[e - \frac{1}{2}|v|^2\Big]^2(x,t) \,\mathrm{d}x \\
&= \frac{1}{2} C_{d,e,\mathscr{B}}^2 \int_{\Omega^\gamma} \boxdot\Big[e - \frac{1}{2}|v|^2\Big]^2(x,t) \,\mathrm{d}x \\
&\geq \frac{1}{2|\Omega|} C_{d,e,\mathscr{B}}^2 \bigg(\int_{\Omega^\gamma} \boxdot\Big[e - \frac{1}{2}|v|\Big](x,t) \,\mathrm{d}x \bigg)^2
\end{aligned}$$

uniformly in $t \in \tau^\gamma$. In general,

$$\begin{aligned}
\liminf_{k\to\infty} \int_{\Omega} |\widetilde{v}^k(x,t)|^2 \,\mathrm{d}x &\geq \liminf_{k\to\infty} \min_{\gamma=\{0,1\}} \int_{\Omega^\gamma} |\widetilde{v}^k(x,t)|^2 \,\mathrm{d}x \\
&\geq \frac{1}{2|\Omega|} C_{d,e,\mathscr{B}}^2 \min_{\gamma=\{0,1\}} \bigg(\int_{\Omega^\gamma} \boxdot\Big[e - \frac{1}{2}|v|\Big](x,t) \,\mathrm{d}x \bigg)^2
\end{aligned}$$

uniformly in $t \in \tau^1 \cup \tau^2 = I$.

Writting $v^k = v + \widetilde{v}^k$

$$\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t)\,\mathrm{d}x = \int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\widetilde{v}^k|^2(x,t)\,\mathrm{d}x - \int_\Omega (\widetilde{v}^k \cdot v)(x,t)\,\mathrm{d}x$$

for each $t \in I$. ...

$$\begin{aligned}
\limsup_{k\to\infty} &\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t)\,\mathrm{d}x \\
&\leq \int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x - \frac{1}{2}\liminf_{k\to\infty}\int_\Omega |\widetilde{v}^k|^2(x,t)\,\mathrm{d}x - \liminf_{k\to\infty}\int_\Omega (\widetilde{v}^k \cdot v)(x,t)\,\mathrm{d}x \\
&\leq \int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x - \frac{1}{4|\Omega|}C^2_{d,e,\mathscr{B}} \min_{\gamma\in\{0,1\}} \left(\int_{\Omega^\gamma} \boxdot\Big[e - \frac{1}{2}|v|\Big](x,t)\,\mathrm{d}x\right)^2
\end{aligned}$$

uniformly in $t \in I$. Now, given $t \in I$, if

$$\int_\Omega \Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x \leq \frac{\alpha}{2},$$

then, since $\mathcal{J}_{\mathscr{B}}(v) > \alpha$,

$$\limsup_{k\to\infty}\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t)\,\mathrm{d}x \leq \frac{\alpha}{2} \leq \mathcal{J}_{\mathscr{B}}(v) - \frac{\alpha}{2}.$$

Otherwise, by ...,

$$\int_{\Omega^\gamma} \boxdot\Big[e - \frac{1}{2}|v|^2\Big](x,t)\,\mathrm{d}x \geq \frac{\alpha}{8}\Big(\frac{3}{4}\Big)^d,$$

hence

$$\limsup_{k\to\infty}\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t)\,\mathrm{d}x \leq \mathcal{J}_{\mathscr{B}}(v) - \frac{3^{2d}}{2^{4d+8}|\Omega|}C^2_{d,e,\mathscr{B}}\alpha^2.$$

In general,

$$\limsup_{k\to\infty}\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t)\,\mathrm{d}x \leq \mathcal{J}_{\mathscr{B}}(v) - \beta$$

uniformly in $t \in I$ where

$$\beta(\mathcal{J}_{\mathscr{B}}, \alpha) = \min\Big\{\frac{\alpha}{2}, D_{d,e,\mathscr{B}}\alpha^2\Big\}$$

and

$$D_{d,e,\mathscr{B}} = \frac{3^{2d}}{2^{4d+11}(D_* - 1)^2|\Omega|\|e\|_{C(\mathscr{B})}}.$$

The uniformity brings the conclusion

$$\limsup_{k\to\infty}\mathcal{J}_{\mathscr{B}}(v^k) = \limsup_{k\to\infty}\sup_{t\in I}\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t)\,\mathrm{d}x \leq \mathcal{J}_{\mathscr{B}}(v) - \beta.$$

Otherwise, if $(t_k)_{k\in\mathbb{N}} \in I$ is the sequence for which

$$\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t_k)\,\mathrm{d}x = \sup_{t\in I}\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t)\,\mathrm{d}x,$$

there would be $\epsilon > 0$ and a subsequence $(t_{k_j})_{j\in\mathbb{N}}$ for which

$$\int_\Omega \Big[e - \frac{1}{2}|v^k|^2\Big](x,t_{k_j})\,\mathrm{d}x > \mathcal{J}_{\mathscr{B}}(v) - \beta + \epsilon \quad \text{for all } j \in \mathbb{N},$$

contradicting the uniformity.

2.3 Constructions

2.3.1 Global existence and non-uniqueness on $\mathbb{T}^d$

In this section we consider the d-dimensional torus $\mathbb{T}^d$ as the domain $\mathscr{D}$, equipped with its group structure. At the end we will see that the solution of the (divergence-free) half heat equation provides a subsolution. For that reason, before constructing such subsolution we will recall the basic properties of the fractional heat equation on the torus. For our purpose, we focus on the divergence-free case. [ref]

Consider, for $s \geq 0$, the fractional Sobolev space on the torus

$$H^s(\mathbb{T}^d) = \Big\{ f \in L^2(\mathbb{R}^d) \,:\, \sum_{k\in\mathbb{Z}^d} |k|^{2s}|\hat{v}(k)|^2 < \infty \Big\}.$$

Since

$$\int_{\mathbb{T}^d} \nabla\phi\cdot v\,\mathrm{d}x = (2\pi)^d \sum_{k\in\mathbb{Z}^d} \hat{\phi}(k)k\cdot\hat{v}(k), \quad \phi\in C_c^\infty(\mathbb{T}^d), v\in L^2(\mathbb{T}^d),$$

the divergence-free fractional Sobolev space on the torus is

$$\mathcal{H}^s(\mathbb{T}^d) = \Big\{ v \in H^s(\mathbb{T}^d,\mathbb{R}^d) \,:\, k\cdot\hat{v}(k) = 0 \quad \forall\, k\in\mathbb{Z}^d \Big\}.$$

We also denote

$$H^\infty(\mathbb{T}^d) = \bigcap_{s\geq 0} H^s(\mathbb{T}^d), \quad \mathcal{H}^\infty(\mathbb{T}^d) = \bigcap_{s\geq 0} \mathcal{H}^s(\mathbb{T}^d).$$

For $0 \leq \alpha \leq 1$, the fractional Laplacian is the linear operator defined by

$$\begin{aligned} (-\Delta)^\alpha : H^{2\alpha}(\mathbb{T}^d) &\to L^2(\mathbb{T}^d) \\ f &\mapsto \Big(x \mapsto \sum_{k\in\mathbb{Z}^d} |k|^{2\alpha}\hat{f}(k)e^{ik\cdot x} \Big) \end{aligned}$$

This is an interpolation between the identity map on $L^2(\mathbb{T}^d)$ ($\alpha = 0$) and the usual Laplacian on $H^2(\mathbb{T}^d)$ ($\alpha = 1$). From now on, we focus on the case $\frac{1}{2} \leq \alpha \leq 1$.

Given $v_0 \in \mathcal{H}^d(\mathbb{T}^d)$, the (divergence-free) fractional heat equation (FHE) is

$$\begin{cases} \partial_t v + (-\Delta)^\alpha v = 0, & \mathbb{T}^d \times \mathbb{R}^+ \\ \operatorname{div} v = 0, & \mathbb{T}^d \times \mathbb{R}^+ \\ v(0) = v_0, & \mathbb{T}^d \end{cases}$$

which expression in the Fourier side is

$$\begin{cases} \partial_t \hat{v} + |k|^{2\alpha} \hat{v} = 0, & \mathbb{Z}^d \times \mathbb{R}^+ \\ k \cdot \hat{v} = 0, & \mathbb{Z}^d \times \mathbb{R}^+ \\ \hat{v}(0) = \hat{v}_0, & \mathbb{Z}^d \end{cases}$$

The solution of the above system is

$$\hat{v}(k,t) = \hat{v}_0(k) e^{-|k|^{2\alpha} t}, \quad (k,t) \in \mathbb{Z}^d \times \mathbb{R}^+,$$

hence, we can represent the solution of the FHE as its Fourier series

$$v(x,t) = \sum_{k \in \mathbb{Z}^d} \hat{v}_0(k) e^{-|k|^{2\alpha} t} e^{ik\cdot x}, \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}^+ \tag{2.7}$$

or by the convolution

$$v(x,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} v_0(y) \left(\sum_{k \in \mathbb{Z}^d} e^{-|k|^{2\alpha} t} e^{ik\cdot (x-y)} \right) \mathrm{d}y = (v_0 * K_{\alpha,t})(x), \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}^+ \tag{2.8}$$

with the fractional heat kernel

$$K_{\alpha,t}(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} e^{-|k|^{2\alpha} t} e^{ik\cdot x}, \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}^+. \tag{2.9}$$

The above change of the order of integration is justified by Fubini's theorem since the properties of the heat kernel that we will see now. Notice that, for all $s \geq 0$,

$$\sum_{k \in \mathbb{Z}^d} |k|^{2s} \left| e^{-|k|^{2\alpha} t} \right|^2 \leq \sum_{k \in \mathbb{Z}^d} |k|^{2s} (e^{-t})^{2|k|} < \infty$$

and

$$\sum_{k \in \mathbb{Z}^d} |k|^{2s} \left| \hat{v}_0(k) e^{-|k|^{2\alpha} t} \right|^2 \leq \left\| |k|^s (e^{-t})^{|k|} \right\|^2_{\ell^\infty(\mathbb{Z}^d)} \|v_0\|_2^2 < \infty,$$

therefore

$$K_{\alpha,t}, v_0 * K_{\alpha,t} \in \mathcal{H}^\infty(\mathbb{T}^d).$$

For example, for $d = 1$ and $\alpha = \frac{1}{2}$, this is the well known Poisson Kernel

$$K_{\frac{1}{2},t}(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-|k|t} e^{ikx} = \frac{1}{2\pi} \frac{1 - e^{-2t}}{1 - 2e^{-t}\cos(x) + e^{-2t}} = \frac{1}{2\pi} \frac{\sinh(t)}{\cosh(t) - \cos(x)}.$$

The fractional heat semigroup is

$$\begin{array}{rcl} S_{\alpha,t} : \mathcal{H}(\mathbb{T}^d) & \to & \mathcal{H}^\infty(\mathbb{T}^d) \subset \mathcal{D}((-\Delta)^\alpha) \\ v_0 & \mapsto & v_0 * K_{\alpha,t} \end{array}$$

The previous estimates justifies that we can derive under the integral sign. Furthermore, all the derivatives exists and are continuous, that is,

$$S_{\alpha,t}(v_0) \in C^\infty(\mathbb{T}^d \times (0,\infty)).$$

In particular,

$$\exists \frac{\mathrm{d}}{\mathrm{d}t} S_{\alpha,t}(v_0) = v_0 * (\partial_t K_{\alpha,t}) = v_0 * (-(-\Delta)^\alpha K_{\alpha,t}) = -(-\Delta)^\alpha S_{\alpha,t}(v_0).$$

From (2.8) it is clear that it is a contractive semigroup on $\mathcal{H}(\mathbb{T}^d)$

$$\|S_{\alpha,t}(v_0)\|_2 \leq \|v_0\|_2$$

and contractive of exponential type on $\widetilde{\mathcal{H}}(\mathbb{T}^d) = \{v \in \mathcal{H}(\mathbb{T}^d) \,:\, \hat{v}(0) = 0\}$

$$\|S_{\alpha,t}(v_0)\|_2 \leq e^{-t}\|v_0\|_2.$$

For $t_0, t \in \mathbb{R}^+$, applying the dominated convergence we observe

$$\|S_{\alpha,t}(v_0) - S_{\alpha,t_0}(v_0)\|_2^2 = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d} |\hat{v}_0(k)|^2 \left| e^{-|k|^{2\alpha}t} - e^{-|k|^{2\alpha}t_0} \right|^2 \xrightarrow[t\to t_0]{} 0,$$

that is,

$$S_{\alpha,t}(v_0) \in C(\mathbb{R}^+, \mathcal{H}(\mathbb{T}^d)).$$

The idea to relate it with the incompressible Euler equations is to rewrite the divergence as a fractional Laplacian. For that reason we must introduce the fractional gradient

$$\begin{array}{rcl} \nabla^\alpha : H^s(\mathbb{T}^d) & \to & H^{s-(2\alpha-1)}(\mathbb{T}^d, \mathbb{R}^d) \\ f & \mapsto & \left(x \mapsto i \displaystyle\sum_{k\in\mathbb{Z}^d} \frac{k}{|k|^2} |k|^{2\alpha} \hat{f}(k) e^{ik\cdot x} \right) \end{array}$$

for $s \geq 2\alpha - 1$, which generalizes the usual gradient ($\alpha = 1$). Let $v \in H^\infty(\mathbb{T}^d, \mathbb{R}^d)$. On the one hand,

$$\mathrm{div}(\nabla_j^\alpha v) = -\sum_{n=1}^{d} \sum_{k\in\mathbb{Z}^d} \frac{k_j k_n}{|k|^2} |k|^{2\alpha} \hat{v}_n(k) e^{ik\cdot x} = \nabla_j^\alpha(\mathrm{div} v). \tag{2.10}$$

On the other hand,

$$\mathrm{div}(\nabla^\alpha v_j) = -\sum_{k\in\mathbb{Z}^d} \frac{k\cdot k}{|k|^2} |k|^{2\alpha} \hat{v}_n(k) e^{ik\cdot x} = -(-\Delta)^\alpha v_j. \tag{2.11}$$

Although we could play with this relation for arbitrary α, since we will need continuity at $t=0$, we must take $\alpha = \frac{1}{2}$. This special case is the well known Riesz transform $\mathcal{R} = \nabla^{\frac{1}{2}}$, that is,

$$\begin{aligned} \mathcal{R} : H^s(\mathbb{T}^d) &\to H^s(\mathbb{T}^d, \mathbb{R}^d) \\ f &\mapsto \left(x \mapsto i \sum_{k\in\mathbb{Z}^d} \frac{k}{|k|} \hat{f}(k) e^{ik\cdot x} \right) \end{aligned}$$

which satisfies

$$\|\mathcal{R}v_j\|_2 \leq \|v_j\|_2 \quad \text{and} \quad \|\mathcal{R}_j v\|_2 \leq \|v\|_2 \tag{2.12}$$

Theorem 2.22. *For every $v_0 \in \mathcal{H}(\mathbb{T}^d)$ there exists infinitely many weak solutions (v,p) of the IEE on $\mathbb{T}^d \times \mathbb{R}^+$ such that*

- $v \in C_b(\mathbb{R}^+, \mathcal{H}_w(\mathbb{T}^d))$ *and* $p = -\frac{1}{d}|v|^2$.
- $v(\cdot, 0) = v_0$ *in* $\mathcal{H}(\mathbb{T}^d)$.

If in addition $v_0 \in \widetilde{\mathcal{H}}(\mathbb{T}^d)$, then $v \in C_0(\mathbb{R}^+, \mathcal{H}_w(\mathbb{T}^d))$.

Proof. In view of the Subsolution crierion 2.13 it is enough to find suitable subsolution (v,u,q) and energy density e. Let start taking

$$v(t) = S_{\frac{1}{2},t}(v_0), \quad t \geq 0.$$

This is a good candidate for the velocity field because it is smooth,

$$v \in C_b(\mathbb{R}^+, \mathcal{H}(\mathbb{T}^d))$$

and attains the initial condition $v(0) = v_0$. Applying the properties of the Riesz transform (half gradient (2.10) (2.11)) we observe

$$0 = \partial_t v_j + (-\Delta)^{\frac{1}{2}} v_j = \partial_t v_j + \operatorname{div}(-\mathcal{R}v_j - \mathcal{R}_j v) + \nabla 0$$

as we were looking for. Therefore, we can try with $q \equiv$ and $u_j = -\mathcal{R}v_j - \mathcal{R}_j v$ which are smooth. Moreover, u is symmetric (by definition) and traceless

$$\operatorname{Tr} u = -2 \sum_{j=1}^{d} \mathcal{R}_j v_j = -2i \sum_{k\in\mathbb{Z}^d} \frac{k}{|k|} \cdot \hat{v}(k,t) e^{ik\cdot x} = 0.$$

On the other hand, applying (2.12) for $t, t_0 \in \mathbb{R}^+$ we deduce

$$\begin{aligned} \|u_j(t) - u_j(t_0)\|_2 &\leq \|\mathcal{R}v_j(t) - \mathcal{R}v_j(t_0)\|_2 + \|\mathcal{R}_j v(t) - \mathcal{R}_j v(t_0)\|_2 \\ &\leq \|v_j(t) - v_j(t_0)\|_2 + \|v(t) - v(t_0)\|_2 \underset{t\to t_0}{\longrightarrow} 0, \end{aligned}$$

that is,

$$u \in C_b(\mathbb{R}^+, L^2(\mathbb{T}^d, S_0^d)) \subset C_b(\mathbb{R}^+, L^1(\mathbb{T}^d, S_0^d)).$$

Finally, consider

$$e(x,t) = \frac{d}{2}(|v|^2 + |u|)(x,t) + \min\{t, \frac{1}{t}\}, \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}^+.$$

This satisfies

$$e \in C(\mathbb{T}^d \times (0,\infty)) \cap C_b(\mathbb{R}^+, L^1(\mathbb{T}^d))$$

and (recall proposition 2.9)

$$\rho(v(x,t), u(x,t)) < e(x,t) \quad \text{for all } (x,t) \in \mathbb{T}^d \times (0,\infty).$$

Finally, we have just seen that $\mathcal{S}_0(v_0, e)$ is non-empty, and this concludes the proof.

If in addition $v_0 \in \widetilde{\mathcal{H}}(\mathbb{T}^d)$, since the fractional heat kernel is contractive of exponential type, it follows that

$$e \in C_0(\mathbb{R}^+, L^1(\mathbb{T}^d)).$$

$\square$

3 YOUNG MEASURES AND ADMISSIBLE SOLUTIONS

3.1 Parametrized Measures

Definition 3.1. Given X and Z two Radon spaces and μ a nonnegative Radon measure on X, a **parametrized measure** from (X, μ) to Z is a map

$$\begin{aligned} \nu : (X, \mu) &\to \mathcal{M}(Z) \\ x &\mapsto \nu_x \end{aligned}$$

which is **weakly* μ-measurable**, that is, for every bounded Borel function f on $X \times Z$, the map

$$x \in (X, \mu) \mapsto \int_Z f(x, z)\, \mathrm{d}\nu_x(z)$$

is measurable. We will denote

$$L^\infty_{w^*}(X, \mu; \mathcal{M}(Z))$$

for the space consisting of parametrized measures which are μ-uniformly bounded in $\mathcal{M}(Z)$.

Fix $\mathscr{U}$ a nonempty Borel subset of $\mathbb{R}^n$. In our case it will be $\mathscr{U} = \mathbb{R}^d \times [0, T]$ and $n = d + 1$.

3.1.1 Young Measures

Definition 3.2. Given X and Z two Radon spaces and $\mu \in \mathcal{R}^+(X)$, a **Young measure** (YM) from (X, μ) to Z is a parametrized measure which range lies in $\mathrm{Prob}(Z)$. The space of such Young measures is denoted by

$$\mathbb{Y}(X, \mu; Z) = L^\infty_{w^*}(X, \mu; \mathrm{Prob}(Z)).$$

Proposition 3.3 (Disintegration theorem)**.** *Let $\mathfrak{X}$ and X two Radon spaces. Given $\mu \in \mathcal{M}^+(\mathfrak{X})$ and $\pi : \mathfrak{X} \to X$ a Borel map, consider the pushforward measure $\pi_*\mu \in \mathcal{M}^+(X)$. Then, there exists a $[\pi_*\mu]$ a.e. uniquely determined YM*

$$\mu \in \mathbb{Y}(X, \pi_*\mu; \mathfrak{X})$$

such that μ_x is concentrated on the fiber $\pi^{-1}(x)$ for $[\pi_*\mu]$ a.e. $x \in X$ and

$$\int_{\mathfrak{X}} f(\mathrm{x})\, \mathrm{d}\mu(\mathrm{x}) = \int_X \left(\int_{\pi^{-1}(x)} f(\mathrm{x})\, \mathrm{d}\mu_x(\mathrm{x}) \right) \mathrm{d}\pi_*\mu(x)$$

fore every $f \in L^1(\mathfrak{X}, \mu)$. That is,

$$\mu = \mu_x \otimes \pi_*\mu \quad \text{in } \mathcal{M}(\mathfrak{X}).$$

(demo Ambrosio...)

Corollary 3.4. *If X and Y are two Radon spaces and $\mu \in \mathcal{M}^+(X \times Y)$, denote $\mu_X = \pi_* \mu \in \mathcal{M}^+(X)$ where $\pi : X \times Y \to X$ is the canonical projection. Then there exists a $[\mu_X]$ a.e. uniquely determined YM*

$$\mu \in \mathbb{Y}(X, \mu_X; X \times Y)$$

such that μ_x is concentrated on the fiber $\pi^{-1}(x) = \{x\} \times Y$ for $[\mu_X]$ a.e. $x \in X$ and

$$\int_{X \times Y} f(x,y) d\mu(x,y) = \int_X \left(\int_Y f(x,y) d\mu_x(y) \right) \mathrm{d}\mu_X(x)$$

fore every $f \in L^1(X \times Y, \mu)$. That is,

$$\mu = \mu_x \otimes \mu_X \quad \textit{in } \mathcal{M}(X \times Y).$$

Theorem 3.5 (Fundamental Theorem for YM)**.** *Let $(w^k)_{k \in \mathbb{N}}$ a bounded sequence in $L^\infty(\mathscr{U}, \mathbb{R}^d)$. Then, for a subsequence (not relabeled) there exists a YM*

$$\nu \in \mathbb{Y}(\mathscr{U}; \mathbb{R}^d)$$

such that, for every $f \in C_b(\mathscr{U} \times \mathbb{R}^d)$,

$$f(y, w^k(y)) \overset{*}{\rightharpoonup} \langle \nu_y, f(y, \cdot) \rangle$$

(weak) in $L^\infty_{loc}(\mathscr{U})$.*
In particular, if $\mathscr{U}$ is bounded, the weak convergence is in $L^\infty(\mathscr{U})$.*

Proof. Let Ω a bounded Borel subset of $\mathscr{U}$. Take

$$R = \sup_{k \in \mathbb{N}} \|w^k\|_{L^\infty(\Omega), \mathbb{R}^d}$$

and denote $B = \overline{\mathbb{B}}_d(0, R)$. Notice that the operator

$$\begin{array}{rcl} \Lambda_k : C_0(\Omega \times \mathbb{R}^d) & \to & \mathbb{R} \\ h & \mapsto & \displaystyle\int_\Omega h(y, w^k(y)) \, \mathrm{d}y \end{array}$$

is linear, positive and bounded

$$|\Lambda_k(h)| \le \int_\Omega |h(y, w^k(y))| \, \mathrm{d}y \le |\Omega| \|h\|_{L^\infty(\Omega \times \mathbb{R}^d)},$$

hence, by theorem [Riesz] we know that there exists a unique positive Radon measure $\mu^k \in \mathcal{M}^+(\Omega \times \mathbb{R}^d)$ such that

$$\int_\Omega h(y, w^k(y)) \, \mathrm{d}y = \Lambda_k(h) = \langle \mu^k, h \rangle = \int_{\Omega \times \mathbb{R}^d} h(\xi) \, \mathrm{d}\mu^k(\xi)$$

for every $h \in C_0(\Omega \times \mathbb{R}^d)$. Moreover, such measure is concentrated on $\Omega \times B$ and they are uniformly bounded, $\|\mu^k\| \leq |\Omega|$. Therefore, by theorem [Banach-Alaoglu], for a subsequence (not relabeled) there exists $\mu \in \mathcal{M}^+(\Omega \times B)$ such that $\mu^k \overset{*}{\rightharpoonup} \mu$ (weak*) in $\mathcal{M}(\Omega \times \mathbb{R}^d)$, that is,

$$\langle \mu^k, h\rangle \to \langle \mu, h\rangle, \quad h \in C_0(\Omega \times B).$$

By corollary 3.4

$$\langle \mu, h\rangle = \int_{\Omega\times B} h(y,w)\,\mathrm{d}\mu(y,w) = \int_\Omega \left(\int_B h(y,w)\,\mathrm{d}\mu_y(w)\right)\mathrm{d}\mu_\Omega(y)$$

for every $h \in C_0(\Omega \times B)$. In particular, for every $f \in C_b(\mathscr{U} \times \mathbb{R}^d)$ and $\phi \in C_0(\Omega)$, taking $h = f\phi \in C_0(\Omega \times B)$, we have seen that

$$\int_\Omega f(y, w^k(y))\phi(y)\,\mathrm{d}y \to \int_\Omega \langle \mu_y, f(y,\cdot)\rangle \phi(y)\,\mathrm{d}\mu_\Omega(y),$$

that is,

$$f(y, w^k(y))\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle \mu_y, f(y,\cdot)\rangle\,\mathrm{d}\mu_\Omega(y) \tag{3.1}$$

(weak*) in $\mathcal{M}(\Omega)$. For $f(y,w) \equiv 1$, we deduce that

$$\mathrm{d}y = \mu_\Omega$$

in $\mathcal{M}(\Omega)$. Finally, for every $\phi \in L^1(\Omega)$, taking $\phi_n \to \phi$ in $L^1(\Omega)$

$$\begin{aligned}
&\left|\int_\Omega f(y,w^k(y))\phi(y)\,\mathrm{d}y - \int_\Omega \langle \nu_y, f(y,\cdot)\rangle \phi(y)\,\mathrm{d}y\right| \\
&\quad \leq \left|\int_\Omega f(y,w^k(y))(\phi(y)-\phi_n(y))\,\mathrm{d}y\right| + \left|\int_\Omega \langle \nu_y, f(y,\cdot)\rangle(\phi_n(y)-\phi(y))\,\mathrm{d}y\right| \\
&\qquad + \left|\int_\Omega f(y,w^k(y))\phi_n(y)\,\mathrm{d}y - \int_\Omega \langle \nu_y, f(y,\cdot)\rangle \phi_n(y)\,\mathrm{d}y\right| \\
&\quad \leq 2\|f\|_{L^\infty(\Omega\times B)}\|\phi_n - \phi\|_{L^1(\Omega)} + \|f(y,w^k(y))\,\mathrm{d}y - \langle \nu_y, f(y,\cdot)\rangle\,\mathrm{d}y\|_{\mathcal{M}(\Omega)} \sup_{n\in\mathbb{N}} \|\phi_n\|_{L^1(\Omega)} \\
&\quad \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Therefore,

$$f(y, w^k(y)) \overset{*}{\rightharpoonup} \langle \nu_y, f(y,\cdot)\rangle$$

(weak*) in $L^\infty(\Omega)$.

If $\mathscr{U}$ is bounded the proof is done. If $\mathscr{U}$ is unbounded, take

$$\Omega_N = \mathbb{B}_n(0,N) \cap \mathscr{U}.$$

First, for Ω_1 we obtain a YM $\mu_1 \in \mathbb{Y}(\Omega_1;\mathbb{R}^d)$ satisfying (3.1). After that, we can apply the result on Ω_2 for the subsequence of $(w^k)_{k\in\mathbb{N}}$ generated in the previous step obtaining a YM $\mu_2 \in \mathbb{Y}(\Omega_2; \mathbb{R}^d)$ satisfying (3.1) which agrees with μ_1 (*a.e.*) on Ω_1 by construction. Iterating, we obtain a subsequence of $(w^k)_{k\in\mathbb{N}}$ which generates a YM

$$\nu \in \mathbb{Y}(\mathscr{U}; \mathbb{R}^d).$$

Hence, for every $f \in C_b(\mathscr{U} \times \mathbb{R}^d)$,

$$f(y, w^k(y)) \overset{*}{\rightharpoonup} \langle \nu_y, f(y, \cdot)\rangle$$

(weak*) in $L^\infty_{loc}(\mathscr{U})$. □

3.1.2 Generalized Young Measures

Let V be a d-dimensional real vector space[1] equipped with a continuous functional

$$[\cdot] : V \to \mathbb{R}^+$$

which is positive definite

$$[z] = 0 \quad \Leftrightarrow \quad z = 0$$

and positive homogeneous

$$[\alpha z] = \alpha[z], \quad \alpha \in \mathbb{R}^+,\, z \in V.$$

For $p \in \mathbb{R}^+$ define the p-dilatation

$$\begin{array}{rcl} \mathfrak{d}_p : V & \to & V \\ w & \mapsto & w[w]^{p-1} \end{array}$$

($\mathfrak{d}_p(0) = 0$). For every $p, q \in \mathbb{R}^+$, if $w \neq 0$

$$\mathfrak{d}_p\mathfrak{d}_q(w) = w[w]^{q-1}[w[w]^{q-1}]^{p-1} = w[w]^{pq-1} = \mathfrak{d}_{pq}(w)$$

and $\mathfrak{d}_p\mathfrak{d}_q(0) = 0 = \mathfrak{d}_{pq}(0)$. Hence, for every $p \in \mathbb{R}^+$,

$$\mathfrak{d}_{\frac{1}{p}} = \mathfrak{d}_p^{-1},$$

so $\mathfrak{d}_p$ is a bijection. For $p \geq 1$ it is clear that $\mathfrak{d}_p$ is continuous. For $\mathfrak{d}_{\frac{1}{p}}$ the only doubt is at $w = 0$, but observe that

$$[w[w]^{\frac{1}{p}-1}] = [w]^{\frac{1}{p}} \to 0$$

when $w \to 0$, so it must be $w[w]^{\frac{1}{p}-1} \to 0$ when $w \to 0$. Looked at another way, in a compact neighborhood of 0 in V the map $\mathfrak{d}_p$ is a continuous bijection from a compact set to a Hausdorff set, hence it is also an homeomorphism. Finally,

$$\mathfrak{d} : (\mathbb{R}^+, \cdot) \to (\mathrm{Hom}(V), \circ)$$

[1] The results still being truth if we take, instead of V, a closed cone C inside V. Anyway, we only need the vector case.

is a group homomorphism.

A well know fact of these functionals is that they are equivalent to the Euclidean norm. Since $[\cdot]$ is positive definite, $[\hat{z}] > 0$ for every $\hat{z} \in \mathbb{S}^{d-1}$. Since $[\cdot]$ is continuous and $\mathbb{S}^{d-1}$ is compact, by theorem [Weierstrass], there exists $c, C > 0$ such that

$$c \leq [\hat{z}] \leq C \quad \forall \hat{z} \in \mathbb{S}^{d-1}.$$

Now, for every $z \in V$ non null we can take $\hat{z} = \frac{z}{|z|} \in \mathbb{S}^{d-1}$. Finally, positive homogeneity implies that $[\hat{z}] = \frac{[z]}{|z|}$, hence

$$c|z| \leq [z] \leq C|z| \quad \forall z \in V.$$

This guarantees that we can compare $[w^k]$ with $|w^k|$ to obtain the usual L^p norm.

Furthermore, every $[\cdot]$ is characterized by its restriction to $\mathbb{S}^{d-1}$, that is, there is a bijective correspondence between positive definite and homogeneous map from V to $\mathbb{R}^+$ with strictly positive functions on $C(\mathbb{S}^{d-1})$ via

$$[z] = |z|[z/|z|], \quad z \in V$$

($[0] = 0$ by continuity). Denote

$$\mathbb{B}_V = \{z \in V \,:\, [z] < 1\} \quad \text{and} \quad \mathbb{S}_V = \{z \in V \,:\, [z] = 1\}$$

for the unit "ball" and "sphere"of $(V, [\cdot])$. Notice that both are bounded. Moreover, $\mathbb{B}_V$ is open in V and $\mathbb{S}_V$ and $\overline{\mathbb{B}}_V$ are both compact in V.

In general $[\cdot]$ will be the Euclidean norm, so we will write simply V instead of $(V, |\cdot|)$. This is the situation of the original theorem of Alibert and Bouchitté in [AB]. On the other hand, for measure-valued subsolutions in section... we will need to consider general $[\cdot]$ which were not included explicitly in [AB]. However, as we shall see, it is a simple generalization because we have not had to modified the proof but realise that it still being truth for these more general $[\cdot]$.

Definition 3.6. A triple $(\nu, \lambda, \nu^\infty)$ is called a **Generalized Young Measure** (GYM) from $\mathscr{U}$ to V if

$$\nu \in \mathbb{Y}(\mathscr{U}, \mathcal{L}; V), \quad \lambda \in \mathcal{R}^+(\mathscr{U}), \quad \nu^\infty \in \mathbb{Y}(\mathscr{U}, \lambda; \mathbb{S}_V).$$

The space of such GYM is denoted by

$$\mathbb{GY}(\mathscr{U}; V).$$

Consider the homeomorphism

$$\begin{aligned} \mathfrak{D}_p : V &\to \mathbb{B}_V \\ w &\mapsto \frac{w[w]^{p-1}}{1 + [w]^p} \end{aligned}$$

which inverse is

$$\begin{array}{rcl}\mathfrak{D}_p^{-1} : \mathbb{B}_V & \to & V \\ z & \mapsto & \dfrac{z[z]^{\frac{1}{p}-1}}{(1-[z])^{\frac{1}{p}}}\end{array}$$

($\mathfrak{D}_p(0) = 0$). More clearly, these are obtained from

$$\mathfrak{D}_p = \mathfrak{D}_1 \circ \mathfrak{d}_p \quad \text{and} \quad \mathfrak{D}_p^{-1} = \mathfrak{d}_{\frac{1}{p}} \circ \mathfrak{D}_1^{-1}$$

by observing

$$\mathfrak{D}_1^{-1}\mathfrak{D}_1(w) = \frac{\frac{w}{1+[w]}}{1-\left[\frac{w}{1+[w]}\right]} = w.$$

Notice that, if $z = \mathfrak{D}_p(w)$, then

$$1-[z] = 1 - \left[\frac{w[w]^{p-1}}{1+[w]^p}\right] = \frac{1}{1+[w]^p}. \tag{3.2}$$

Definition 3.7. We define $\mathcal{F}_p(\mathscr{U}, V)$ as the class of continuous functions f on $\mathscr{U} \times V$ such that the mapping

$$\begin{array}{rcl}T_p f : \mathscr{U} \times \mathbb{B}_V & \to & \mathbb{R} \\ (x,z) & \mapsto & (1-[z])f(x, \mathfrak{D}_p^{-1}(z))\end{array}$$

can be extended into a bounded continuous function on $\mathscr{U} \times \overline{\mathbb{B}}_V$. That is, a function f belongs to $\mathcal{F}_p(\mathscr{U}, V)$ if $f \in C(\mathscr{U} \times V)$ and

$$\begin{array}{rcl}T_p f : \mathscr{U} \times \overline{\mathbb{B}}_V & \to & \mathbb{R} \\ (x,z) & \mapsto & \begin{cases} (1-[z])f(x, \mathfrak{D}_p^{-1}(z)) & \text{if } z \in \mathbb{B}_V, \\ f^\infty(x,z) & \text{if } z \in \mathbb{S}_V. \end{cases}\end{array}$$

is a well defined bounded continuous function, where f^∞ is the p-**recession function**

$$f^\infty(x,z) = \lim_{z' \to z} (1-[z'])f(x, \mathfrak{D}_p^{-1}(z')), \quad (x,z) \in \mathscr{U} \times \mathbb{S}_V.$$

Proposition 3.8 (Properties of $\mathcal{F}_p(\mathscr{U}, V)$)**.**

a) The p-recession function of $f \in \mathcal{F}_p(\mathscr{U}, V)$ can be calculated by

$$f^\infty(x,z) = \lim_{\substack{z' \to z \\ s \to \infty}} \frac{f(x, sz')}{s^p}, \quad (x,z) \in \mathscr{U} \times \mathbb{S}_V.$$

b) The map T_p is a linear bijection from $\mathcal{F}_p(\mathscr{U}, V)$ to $C_b(\mathscr{U} \times \overline{\mathbb{B}}_V)$ with inverse

$$\begin{aligned} T_p^{-1} : C_b(\mathscr{U} \times \overline{\mathbb{B}}_V) &\to \mathcal{F}_p(\mathscr{U}, V) \\ g &\mapsto \Big((x,w) \mapsto (1 + [w]^p) g(x, \mathfrak{D}_p(w)) \Big). \end{aligned}$$

Furthermore, the vector space $\mathcal{F}_p(\mathscr{U}, V)$ with the norm

$$\|f\|_{\mathcal{F}_p(\mathscr{U},V)} = \sup_{(x,w) \in \mathscr{U} \times V} \frac{|f(x,w)|}{1 + [w]^p}$$

is a Banach space isomorphic to $C_b(\mathscr{U} \times \overline{\mathbb{B}}_d)$ via T_p.

Proof. a) It is clear by (3.2) setting

$$w' = \mathfrak{D}_p(z') = sz' \quad \text{where} \quad s = \frac{[z']^{\frac{1}{p}-1}}{(1 - [z'])^{\frac{1}{p}}}.$$

b) The map

$$\begin{aligned} T_p : \mathcal{F}_p(\mathscr{U}, V) &\to C_b(\mathscr{U} \times \overline{\mathbb{B}}_V) \\ f &\mapsto T_p f \end{aligned}$$

is a linear map with inverse T_p^{-1} by (3.2). Furthermore, by definition is an isometry. □

Theorem 3.9 (Fundamental Theorem for GYM)**.** *Assume that $\mathscr{U}$ is locally compact. Let $(w^k)_{k \in \mathbb{N}}$ a bounded sequence in $L^p(\mathscr{U}; V)$. Then, for a subsequence (not relabeled) there exists a GYM*

$$(\nu, \lambda, \nu^\infty) \in \mathbb{GY}(\mathscr{U}; V)$$

such that, for every $f \in \mathcal{F}_p(\mathscr{U}, V)$,

$$f(y, w^k(y))\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle \nu_y, f(y, \cdot) \rangle\,\mathrm{d}y + \langle \nu_y^\infty, f^\infty(y, \cdot) \rangle\,\mathrm{d}\lambda(y)$$

(weak) in $\mathcal{R}(\mathscr{U})$. Moreover,*

$$\int_{\mathscr{U}} \langle \nu_y, [\cdot]^p \rangle\,\mathrm{d}y + \lambda(\mathscr{U}) \le \sup_{k \in \mathbb{N}} \int_{\mathscr{U}} [w^k(y)]^p\,\mathrm{d}y.$$

Observe that $\lambda \in \mathcal{M}^+(\mathscr{U})$.
In particular, if $\mathscr{U}$ is bounded, the weak convergence is in $\mathcal{M}(\mathscr{U})$.*

Proof. Let Ω a bounded Borel subset of $\mathscr{U}$. Notice that the operator

$$\begin{aligned} \Lambda_k : C_0(\Omega \times \overline{\mathbb{B}}_V) &\to \mathbb{R} \\ h &\mapsto \int_\Omega T_p^{-1} h(y, w^k(y))\,\mathrm{d}x = \int_\Omega (1 + [w^k(x)]^p) h(y, \mathfrak{D}_p(w^k(y)))\,\mathrm{d}y \end{aligned}$$

is linear, positive and bounded

$$|\Lambda_k(h)| \leq \Big(\mathcal{L}(\Omega) + \|w^k\|_{L^p(\Omega;\mathbb{R}^d)}\Big)\|h\|_{L^\infty(\Omega\times\overline{\mathbb{B}}_V)},$$

hence, by theorem [Riesz] we know that there exists a unique positive Radon measure $\mu^k \in \mathcal{M}^+(\Omega \times \overline{\mathbb{B}}_V)$ such that

$$\int_\Omega T_p^{-1}h(y, w^k(y))\,\mathrm{d}y = \Lambda_k(h) = \langle \mu^k, h\rangle = \int_{\Omega\times\overline{\mathbb{B}}_V} h(y,z)\,\mathrm{d}\mu^k(y,z)$$

for every $h \in C_0(\Omega \times \overline{\mathbb{B}}_V)$. Moreover, such measures are uniformly bounded

$$\|\mu^k\| = \|\Lambda_k\| \leq \mathcal{L}(\Omega) + \sup_{k\in\mathbb{N}} \|w^k\|_{L^p(\Omega;\mathbb{R}^d)},$$

hence, by theorem [BA], for a subsequence (not relabeled) there exists $\mu \in \mathcal{M}^+(\Omega \times \overline{\mathbb{B}}_V)$ such that $\mu^k \overset{*}{\rightharpoonup} \mu$ (weak*) in $\mathcal{M}(\Omega \times \overline{\mathbb{B}}_V)$, that is,

$$\langle \mu^k, h\rangle \to \langle \mu, h\rangle, \quad h \in C_0(\Omega \times \overline{\mathbb{B}}_V).$$

By disintegration theorem 3.4

$$\langle \mu, h\rangle = \int_{\Omega\times\overline{\mathbb{B}}_V} h(y,z)\,\mathrm{d}\mu(y,z) = \int_\Omega \left(\int_{\overline{\mathbb{B}}_V} h(y,z)\,\mathrm{d}\mu_y(z)\right)\mathrm{d}\mu_\Omega(y)$$

for every $h \in C_0(\Omega \times \overline{\mathbb{B}}_V)$. In particular, for every $f \in \mathcal{F}_p(\mathscr{U}, V)$ and $\phi \in C_0(\Omega)$, taking $\varphi = T_p(f\phi) = (T_pf)\phi \in C_0(\Omega \times \overline{\mathbb{B}}_V)$, we have seen that

$$\int_\Omega f(y, w^k(y))\phi(y)\,\mathrm{d}y \to \int_\Omega \langle \mu_y, T_pf(y,\cdot)\rangle\phi(y)\,\mathrm{d}\mu_\Omega(y),$$

that is,

$$f(y, w^k(y))\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle \mu_y, T_pf(y,\cdot)\rangle\,\mathrm{d}\mu_\Omega(y)$$

(weak*) in $\mathcal{M}(\Omega)$. For $f(x,w) = 1 + [w]^p$, since $T_pf(x,z) \equiv 1$, we deduce that

$$(1 + [w^k]^p)\,\mathrm{d}y \overset{*}{\rightharpoonup} \mathrm{d}\mu_\Omega$$

(weak*) in $\mathcal{M}(\Omega)$. For $f(x,w) \equiv 1$, since $T_pf(x,z) = 1 - [z]$, we deduce that

$$\mathrm{d}y = \langle \mu_y, 1 - [z]\rangle\,\mathrm{d}\mu_\Omega(y)$$

in $\mathcal{M}(\Omega)$. Let

$$\mathrm{d}\mu_\Omega = p(y)\,\mathrm{d}y + \mathrm{d}\mu_\Omega^s \tag{3.3}$$

the Lebesgue-Radon-Nikodým decomposition of μ_Ω with respect to $\mathrm{d}y$, where p is the Lebesgue-Radon-Nikodým derivative $\frac{\mathrm{d}\mu_\Omega}{\mathrm{d}y}$ and $\mathrm{d}\mu_\Omega^s \perp \mathrm{d}y$. Observe that

$$(1 - p(y)\langle \mu_y, 1 - [z]\rangle)\,\mathrm{d}y = \langle \mu_y, 1 - [z]\rangle\,\mathrm{d}\mu_\Omega^s \tag{3.4}$$

in $\mathcal{M}(\Omega)$. Therefore, it must be

$$p(y)\langle \mu_y, 1-[z]\rangle = 1 \quad (\mathrm{d}y)\, a.e.\, y \in \Omega \tag{3.5}$$

and

$$\langle \mu_y, 1-[z]\rangle = 0 \quad (\mathrm{d}\mu_\Omega^s)\, a.e.\, y \in \Omega. \tag{3.6}$$

Since $0 \leq 1-[z] \leq 1$ on $\overline{\mathbb{B}}_V$, by (3.5),

$$p(y) = \langle \mu_y, 1-[z]\rangle^{-1} \geq 1 \quad (\mathrm{d}y)\, a.e.\, y \in \Omega.$$

On the other hand, since $1-[z] > 0$ on $\mathbb{B}_V$, by (3.4) and (3.6), μ_y is concentrated on $\mathbb{S}_V$ $(\mathrm{d}\mu_\Omega^s)\, a.e.\, y \in \Omega$, so

$$\mu_y(\mathbb{S}_V) = 1 \quad (\mathrm{d}\mu_\Omega^s)\, a.e.\, y \in \Omega. \tag{3.7}$$

Notice that the operator, which is defined $(\mathrm{d}y)\, a.e.\, y \in \Omega$,

$$\begin{aligned} \Gamma_y : C_0(V) &\to \mathbb{R} \\ \varphi &\mapsto p(y)\int_{\mathbb{B}_V} T_p\varphi(z)\,\mathrm{d}\mu_y(z) = p(y)\int_{\mathbb{B}_V}(1-[z])\varphi(\mathfrak{D}_p(z))\,\mathrm{d}\mu_y(z) \end{aligned}$$

is linear, positive and bounded

$$|\Gamma_y(\varphi)| \leq p(y)\langle \mu_y, 1-[z]\rangle \|\varphi\|_{L^\infty(V)} = \|\varphi\|_{L^\infty(V)}.$$

Indeed, for $\varphi \equiv 1$, we have $\Gamma_y(\varphi) = 1$, hence $\|\Gamma_y\| = 1$. By theorem [Riesz], there exists a unique probability $\nu_y \in \mathrm{Prob}(V)$ such that

$$p(y)\int_{\mathbb{B}_V} T_p\varphi(z)\,\mathrm{d}\mu_y(z) = \Gamma_y(\varphi) = \langle \nu_y, \varphi\rangle = \int_V \varphi(w)\,\mathrm{d}\nu_y(w)$$

for every $\varphi \in C_0(V)$. Define the Radon measure

$$\lambda = \mu_y(\mathbb{S}_V)\mu_\Omega \in \mathcal{M}^+(\Omega).$$

Hence, by the Lebesgue-Radon-Nikodým decomposition (3.3) and (3.7),

$$\mathrm{d}\lambda = p(y)\mu_y(\mathbb{S}_V)\,\mathrm{d}y + \mathrm{d}\mu_\Omega^s. \tag{3.8}$$

Notice that the operator, which is defined $(\mathrm{d}\lambda)\, a.e.\, y \in \Omega$,

$$\begin{aligned} \Gamma_y^\infty : C(\mathbb{S}_V) &\to \mathbb{R} \\ \varphi &\mapsto \frac{1}{\mu_y(\mathbb{S}_V)}\int_{\mathbb{S}_V} \varphi(z)\,\mathrm{d}\mu_y(z) \end{aligned}$$

is linear, positive and bounded with $\|\Gamma_y^\infty\| = 1$. By theorem [Riesz], there exists a unique probability $\nu_y^\infty \in \mathrm{Prob}(\mathbb{S}_V)$ such that

$$\frac{1}{\mu_y(\mathbb{S}_V)}\int_{\mathbb{S}_V} \varphi(z)\,\mathrm{d}\mu_y(z) = \Gamma_y^\infty(\varphi) = \langle \nu_y^\infty, \varphi\rangle = \int_{\mathbb{S}_V} \varphi(z)\,\mathrm{d}\nu_y^\infty(z)$$

for every $\varphi \in C(\mathbb{S}_V)$. Finally, all these observations yield that, for every $f \in \mathcal{F}_p(\mathscr{U}, V)$, since $T_p f = f^\infty$ on $\Omega \times \mathbb{S}_V$,

$$
\begin{aligned}
&\langle \nu_y, f(y,\cdot)\rangle \,\mathrm{d}y + \langle \nu_y^\infty, f^\infty(y,\cdot)\rangle \,\mathrm{d}\lambda(y) \\
&\quad = \Big(\langle \nu_y, f(y,\cdot)\rangle + p(y)\mu_y(\mathbb{S}_V)\langle \nu_y^\infty, f^\infty(y,\cdot)\rangle\Big)\,\mathrm{d}y + \langle \nu_y^\infty, f^\infty(y,\cdot)\rangle \,\mathrm{d}\mu_\Omega^s(y) \\
&\quad = \left(p(y)\int_{\mathbb{B}_V} T_p f(y,z)\,\mathrm{d}\mu_y(z) + p(y)\int_{\mathbb{S}_V} T_p f(y,z)\,\mathrm{d}\mu_y(z)\right)\mathrm{d}y \\
&\qquad + \left(\frac{1}{\mu_y(\mathbb{S}_V)}\int_{\overline{\mathbb{B}}_V} T_p f(y,z)\,\mathrm{d}\mu_y(z)\right)\mathrm{d}\mu_\Omega^s(y) \\
&\quad = \left(\int_{\overline{\mathbb{B}}_V} T_p f(y,z)\,\mathrm{d}\mu_y(z)\right)(p(y)\,\mathrm{d}y + \,\mathrm{d}\mu_\Omega^s(y)) \\
&\quad = \langle \mu_y, T_p f(y,\cdot)\rangle \,\mathrm{d}\mu_\Omega.
\end{aligned}
$$

Therefore, for every $f \in \mathcal{F}_p(\mathscr{U}, V)$,

$$
f(y, w^k(y))\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle \nu_y, f(y,\cdot)\rangle \,\mathrm{d}y + \langle \nu_y^\infty, f^\infty(y,\cdot)\rangle \,\mathrm{d}\lambda(y) \tag{3.9}
$$

(weak*) in $\mathcal{M}(\Omega)$.

If $\mathscr{U}$ is bounded the proof is done. If $\mathscr{U}$ is unbounded, take

$$
\Omega_N = \mathbb{B}_n(0,N) \cap \mathscr{U}.
$$

First, for Ω_1 we obtain a GYM $(\nu,\lambda,\nu^\infty)_1 \in \mathbb{GY}(\Omega_1; V)$ satisfying (3.9). After that, we can apply the result on Ω_2 for the subsequence of $(w^k)_{k\in\mathbb{N}}$ generated in the previous step obtaining a GYM $(\nu,\lambda,\nu^\infty)_2 \in \mathbb{GY}(\Omega_2; V)$ satisfying (3.9) which agrees with $(\nu,\lambda,\nu^\infty)_1$ (*a.e.*) on Ω_1 by construction. Iterating, we obtain a subsequence of $(w^k)_{k\in\mathbb{N}}$ which generates a GYM

$$
(\nu,\lambda,\nu^\infty) \in \mathbb{GY}(\mathscr{U}; V).
$$

Hence, for every $f \in \mathcal{F}_p(\mathscr{U}, V)$,

$$
\int_{\mathscr{U}} f(y, w^k(y))\phi(y)\,\mathrm{d}y \overset{*}{\rightharpoonup} \int_{\mathscr{U}} \langle \nu_y, f(y,\cdot)\rangle\phi(y)\,\mathrm{d}y + \int_{\mathscr{U}} \langle \nu_y^\infty, f^\infty(y,\cdot)\rangle\phi(y)\,\mathrm{d}\lambda(y)
$$

for every $\phi \in C_c(\mathscr{U})$, that is,

$$
f(y, w^k(y))\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle \nu_y, f(y,\cdot)\rangle \,\mathrm{d}y + \langle \nu_y^\infty, f^\infty(y,\cdot)\rangle \,\mathrm{d}\lambda(y)
$$

(weak*) in $\mathcal{R}(\mathscr{U})$. For $f(y,w) = [w]^p$, since $([\cdot]^p)^\infty \equiv 1$ by proposition 3.8,

$$
[w^k(y)]^p\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle \nu_y, [\cdot]^p\rangle \,\mathrm{d}y + \,\mathrm{d}\lambda(y)
$$

(weak*) in $\mathcal{R}(\mathscr{U})$. Hence, for every $\phi \in C_c(\mathscr{U})$ non negative,

$$\begin{aligned}\int_{\mathscr{U}} \langle \nu_y, [\cdot]^p\rangle \phi(y)\,\mathrm{d}y + \int_{\mathscr{U}} \phi(y)\,\mathrm{d}\lambda(y) &= \lim_{k\to\infty} \int_{\mathscr{U}} [w^k(y)]^p \phi(y)\,\mathrm{d}y \\ &\le \left(\sup_{k\in\mathbb{N}} \int_{\mathscr{U}} [w^k(y)]^p\,\mathrm{d}y\right) \|\phi\|_{L^\infty(\mathscr{U})}.\end{aligned}$$

Finally, denote

$$\omega = \langle \nu_y, [\cdot]^p\rangle \mathcal{L} + \lambda \in \mathcal{R}^+(\mathscr{U}).$$

Hence, since $\mathscr{U}$ is locally compact, by theorem [Riesz],

$$\omega(\mathscr{U}) = \sup\{\langle \omega, \phi\rangle \,:\, \phi \in C_c(\mathscr{U}),\, 0 \le \phi \le \mathbb{1}_{\mathscr{U}}\} \le \sup_{k\in\mathbb{N}} \int_{\mathscr{U}} [w^k(y)]^p\,\mathrm{d}y,$$

that is,

$$\int_{\mathscr{U}} \langle \nu_y, [\cdot]^p\rangle\,\mathrm{d}y + \lambda(\mathscr{U}) \le \sup_{k\in\mathbb{N}} \int_{\mathscr{U}} [w^k(y)]^p\,\mathrm{d}y.$$

□

To simplify we will use the notation

$$\langle \nu, \lambda, \nu^\infty; f, \phi\rangle = \int_{\mathscr{U}} \langle \nu_y, f(y,\cdot)\rangle \phi(y)\,\mathrm{d}y + \int_{\mathscr{U}} \langle \nu_y^\infty, f^\infty(y,\cdot)\rangle \phi(y)\,\mathrm{d}\lambda(y)$$

for every $f \in \mathcal{F}_p(\mathscr{U}, V)$ and every $\phi \in C_c(\mathscr{U})$.

Given $\nu \in \mathbb{Y}(\mathscr{U}; V)$ and $w \in L^p(\mathscr{U}; V)$, we define also the **shift** of the YM as

$$\mathcal{T}_w \nu = (T_w)_* \nu \in \mathbb{Y}(\mathscr{U}; V)$$

where T is the usual translation in V, that is,

$$\langle (\mathcal{T}_w \nu)_y, \varphi\rangle = \int_V \varphi(\omega + w(y))\,\mathrm{d}\nu_y(\omega)$$

for every $\varphi \in C_b(V)$.

Proposition 3.10. *Some properties of GYM.*

1) *There exists countable sets of functions $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_p(V)$ and $\{\phi_k\}_{k\in\mathbb{N}} \subset C_c(\mathscr{U})$ such that*

$$\langle \nu, \lambda, \nu^\infty; f_k, \phi_k\rangle = \langle \tilde\nu, \tilde\lambda, \tilde\nu^\infty; f_k, \phi_k\rangle \quad \forall k \in \mathbb{N} \quad \Rightarrow \quad (\nu, \lambda, \nu^\infty) = (\tilde\nu, \tilde\lambda, \tilde\nu^\infty).$$

2) *Let $w^k \to (\nu, \tilde\lambda, \nu^\infty)$ and $\tilde w^k \to (\tilde\nu, \tilde\lambda, \tilde\nu^\infty)$ in $\mathbb{GY}(\mathscr{U}; V)$. Then,*

$$w^k - \tilde w^k \to 0 \quad \textit{in } \mathcal{R}(\mathscr{U}) \quad \Rightarrow \quad \nu = \tilde\nu.$$

3) Let $w^k \to (\nu, \lambda, \nu^\infty)$ *in* $\mathbb{GY}(\mathscr{U}; V)$. *Then,*

$$w^k - \tilde{w}^k \to 0 \quad \textit{in } L^p_{loc}(\mathscr{U}; V) \quad \Rightarrow \quad \tilde{w}^k \to (\nu, \lambda, \nu^\infty).$$

4)

$$w^k \to w \quad \textit{in } L^p_{loc}(\mathscr{U}; V) \quad \Leftrightarrow \quad w^k \to (\delta_w, 0, 0) \quad \textit{in } \mathbb{GY}(\mathscr{U}; V).$$

5) Let $w^k \to (\nu, \lambda, \nu^\infty)$ *in* $\mathbb{GY}(\mathscr{U}; V)$ *and* $w \in L^p(\mathscr{U}; V)$. *Then*

$$w^k + w \to (\mathcal{T}_w \nu, \lambda, \nu^\infty) \quad \textit{in } \mathbb{GY}(\mathscr{U}; V).$$

Proof. Characterization of GYM... □

3.1.3 Lifted Generalized Young Measures

let $\vec{p} = (p_j)_{j=1}^N$ a collection of Lebesgue powers, that is, $\vec{p} \in [1, \infty)^N$, and $\vec{V} \equiv (V_1, \cdots, V_N)$ where each V_j is a d_j-dimensional real vector space equipped with some $[\cdot]_j$. Denote $V = V_1 \times \cdots \times V_N$. Suppose that we have a bounded sequence in

$$L^{\vec{p}}(\mathscr{U}; \vec{V}) \equiv L^{p_1}(\mathscr{U}; V_1) \times \cdots \times L^{p_N}(\mathscr{U}; V_N).$$

We can adapt the previous result to this situation. Denote $P = \max p_j$ and

$$\begin{array}{rcl} \mathfrak{d}_{\frac{\vec{p}}{P}} : V & \to & V \\ \vec{w} & \mapsto & (\mathfrak{d}_{\frac{p_j}{P}}(w_j))_{j=1}^d \end{array}$$

with correspondent inverse $\mathfrak{d}^{-1}_{\frac{P}{\vec{p}}} = \mathfrak{d}_{\frac{P}{\vec{p}}}$. Define

$$\begin{array}{rcl} [\cdot]_V : V & \to & \mathbb{R}^+ \\ \vec{w} & \mapsto & [([w_j]_j)_{j=1}^d] \end{array}$$

where $[\cdot]$ is a positive definite and homogeneous map in $\mathbb{R}^N$ (for example the Euclidean norm). It is clear that $\vec{V}$ is a $d_1 + \cdots d_N$-dimensional real vector space of equipped with $[\cdot]_V$ which is positive definite and homogeneous. Denote

$$\mathbb{S}^{\vec{p}}_{\vec{V}} = \{\vec{w} \in V \, : \, [\mathfrak{d}_{\frac{\vec{p}}{P}}(\vec{w})]_V = 1\} = \mathfrak{d}_{\frac{P}{\vec{p}}}(\mathbb{S}_V).$$

Definition 3.11. A triple $(\nu, \lambda, \nu^\infty)$ is called a $\mathrm{GYM}_{\vec{p}}$ from $\mathscr{U}$ to $\vec{V}$ if

$$\nu \in \mathbb{Y}(\mathscr{U}, \mathcal{L}; \vec{V}), \quad \lambda \in \mathcal{R}^+(\mathscr{U}), \quad \nu^\infty \in \mathbb{Y}(\mathscr{U}, \lambda; \mathbb{S}^{\vec{p}}_{\vec{V}}).$$

The space of such $\mathrm{GYM}_{\vec{p}}$ is denoted by

$$\mathbb{GY}_{\vec{p}}(\mathscr{U}; \vec{V}).$$

Definition 3.12. We define $\mathcal{F}_{\vec{p}}(\mathscr{U},\vec{V})$ as the class of continuous functions f on $\mathscr{U}\times V$ such that the function defined by

$$g(y,\vec{w}) = f(y,\mathfrak{d}_{\frac{P}{\vec{p}}}(\vec{w})), \quad (y,\vec{w})\in\mathscr{U}\times V$$

belongs to $\mathcal{F}_P(\mathscr{U},V)$.

For $f\in\mathcal{F}_{\vec{p}}(\mathscr{U},\vec{V})$ define its $\vec{p}$-recession function as

$$f^\infty(y,\vec{w}) = \lim_{\substack{\vec{w}'\to\vec{w}\\ s\to\infty}} \frac{f(y,s^{\frac{P}{\vec{p}}}\vec{w})}{s^P}, \quad (y,\vec{w})\in\mathscr{U}\times\mathbb{S}^{\vec{p}}_{\vec{V}}$$

(where $s^{\frac{P}{\vec{p}}}\vec{w} = (s^{\frac{P}{p_j}}w_j)_{j=1}^N$ is a pointwise multiplication) which is a well defined continuous function in $\mathscr{U}\times\mathbb{S}^{\vec{p}}_{\vec{V}}$. This is by definition of $\mathbb{S}^{\vec{p}}_{\vec{V}}$, observation

$$\mathfrak{d}_{\frac{p_j}{P}}(s^{\frac{P}{p_j}}w_j) = s\mathfrak{d}_{\frac{p_j}{P}}(w_j)$$

and

$$f^\infty(y,\vec{w}) = \lim_{\substack{\vec{w}'\to\vec{w}\\ s\to\infty}} \frac{f(y,s^{\frac{\vec{p}}{P}}\vec{w})}{s^P} = \lim_{\substack{\vec{w}'\to\vec{w}\\ s\to\infty}} \frac{g(y,s\mathfrak{d}_{\frac{\vec{p}}{P}}(\vec{w}))}{s^P} = g^\infty(y,\mathfrak{d}_{\frac{\vec{p}}{P}}(\vec{w}))$$

for every $(y,\vec{w})\in\mathscr{U}\times\mathbb{S}^{\vec{p}}_{\vec{V}}$.

Corollary 3.13 (Fundamental Theorem for $\mathrm{GYM}_{\vec{p}}$)**.** *Assume that $\mathscr{U}$ is open or closed. Let $(\vec{w}^k)_{k\in\mathbb{N}}$ a bounded sequence in $L^{\vec{p}}(\mathscr{U};\vec{V})$. Then, for a subsequence (not relabeled) there exists a GYM$_{\vec{p}}$*

$$(\nu,\lambda,\nu^\infty)\in\mathbb{GY}_{\vec{p}}(\mathscr{U};\vec{V})$$

such that, for every $f\in\mathcal{F}_{\vec{p}}(\mathscr{U},\vec{V})$,

$$f(y,\vec{w}^k(y))\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle\nu_y,f(y,\cdot)\rangle\,\mathrm{d}y + \langle\nu_y^\infty,f^\infty(y,\cdot)\rangle\,\mathrm{d}\lambda(y)$$

(weak) in $\mathcal{R}(\mathscr{U})$.*

Proof. Notice that the sequence $w^k = (\mathfrak{d}_{\frac{\vec{p}}{P}}(\vec{w}^k))$ is bounded in $L^P(\mathscr{U};V)$ because

$$|\mathfrak{d}_{\frac{p_j}{P}}(w_j^k)|^P \sim [\mathfrak{d}_{\frac{p_j}{P}}(w_j^k)]_j^P = [w_j^k]_j^{p_j} \sim |w_j^k|^{p_j}.$$

Applying theorem 3.9 to w^k we know that there exists a GYM $(\tilde{\nu},\tilde{\lambda},\tilde{\nu}^\infty)\in\mathbb{GY}(\mathscr{U};V)$ such that, for every $g\in\mathcal{F}_P(\mathscr{U},V)$,

$$g(y,w^k(y))\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle\nu_y,g(y,\cdot)\rangle\,\mathrm{d}y + \langle\nu_y^\infty,g^\infty(y,\cdot)\rangle\,\mathrm{d}\lambda(y)$$

(weak*) in $\mathcal{R}(\mathscr{U})$. Consider $\lambda = \tilde{\lambda} \in \mathcal{M}^+(\mathscr{U})$,

$$\nu = (\mathfrak{d}_{\frac{P}{\vec{p}}})_* \tilde{\nu} \in \mathbb{Y}(\mathscr{U}, \mathcal{L}; \vec{V}),$$
$$\nu^\infty = (\mathfrak{d}_{\frac{P}{\vec{p}}})_* \tilde{\nu}^\infty \in \mathbb{Y}(\mathscr{U}, \lambda; \mathbb{S}^{\vec{p}}_{\vec{V}}).$$

Hence, for every $f \in \mathcal{F}_{\vec{p}}(\mathscr{U}, \vec{V})$ we have

$$\begin{aligned} f(y, \vec{w}^k(y))\,\mathrm{d}y &= g(y, \mathfrak{d}_{\frac{\vec{p}}{P}}(\vec{w}^k(y)))\,\mathrm{d}y \\ &= g(y, w^k(y))\,\mathrm{d}y \\ &\overset{*}{\rightharpoonup} \langle \tilde{\nu}_y, g(y,\cdot)\rangle\,\mathrm{d}y + \langle \tilde{\nu}^\infty_y, g^\infty(y,\cdot)\rangle\,\mathrm{d}\lambda(y) \\ &= \langle \tilde{\nu}_y, f(y,\cdot)\circ \mathfrak{d}_{\frac{P}{\vec{p}}}\rangle\,\mathrm{d}y + \langle \tilde{\nu}^\infty_y, f^\infty(y,\cdot)\circ \mathfrak{d}_{\frac{P}{\vec{p}}}\rangle\,\mathrm{d}\lambda(y) \\ &= \langle \nu_y, f(y,\cdot)\rangle\,\mathrm{d}y + \langle \nu^\infty_y, g^\infty(y,\cdot)\rangle\,\mathrm{d}\lambda(y) \end{aligned}$$

□

3.2 Measure-valued solutions of the IEE

3.2.1 Leray solutions of the INSE

Theorem 3.14 (Leray ref 1934). *let $\mu > 0$ a fixed viscosity. For every $v_0 \in \mathcal{H}(\mathbb{R}^d)$ there exists a weak solution v^μ to the incompressible Navier-Stokes equations with initial velocity field v_0 satisfying the strong energy inequality*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |v^\mu(x,t)|^2\,\mathrm{d}x \le \int_{\mathbb{R}^d} |v_0(x)|^2\,\mathrm{d}x.$$

3.2.2 Measure-valued solutions of the IEE

Definition 3.15. A GYM $(\nu, \lambda, \nu^\infty) \in \mathbb{GY}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$ is called a **measure-valued solution** to the IEE if

$$\int_0^T \int_{\mathbb{R}^d} \Big(\partial_t \varphi \cdot \overline{v}(x,t) + \nabla\varphi : \langle \nu_{x,t}, \xi\otimes\xi\rangle\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d\times[0,T]} \nabla\varphi : \langle \nu^\infty_{x,t}, \theta\otimes\theta\rangle\,\mathrm{d}\lambda(x,t) = 0$$

for all $\varphi \in C^\infty_c(\mathbb{R}^d \times (0,T); \mathbb{R}^d) \cap \mathcal{H}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} \nabla\psi \cdot \overline{v}(x,t)\,\mathrm{d}x = 0$$

for all $\psi \in C^\infty_c(\mathbb{R}^d)$ for $a.e.\, t \in [0,T]$.

Theorem 3.16. *Let $v_0 \in \mathcal{H}(\mathbb{R}^d)$ and $(\mu_k)_{k\in\mathbb{N}}$ a vanishing sequence of positive viscosities, $\mu_k \downarrow 0$. Denote $v^k = v^{\mu_k}$ in the context of theorem 3.14 for the initial data v_0. Then, the sequence $(v^k)_{k\in\mathbb{N}}$ generates a GYM $(\nu,\lambda,\nu^\infty) \in \mathbb{GY}(\mathbb{R}^d\times[0,T];\mathbb{R}^d)$ which is a measure-valued solution of the IEE. Furthermore, such GYM satisfies:*

a) The concentration measure λ admits a desintegration of the form

$$\mathrm{d}\lambda(x,t) = \mathrm{d}\lambda_t(x) \otimes \mathrm{d}t$$

where the new λ is an uniformly bounded parametrized measure

$$\lambda \in L^\infty_{w^*}([0,T];\mathcal{M}^+(\mathbb{R}^d)).$$

b) For almost every time $t \in [0,T]$

$$\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle \,\mathrm{d}x + \lambda_t(\mathbb{R}^d) \leq \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x.$$

We call

$$E(t) = \frac{1}{2}\bigg(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle \,\mathrm{d}x + \lambda_t(\mathbb{R}^d) \bigg)$$

*the **energy of the measure-valued solution**.*

c) The barycenter belongs to $L^\infty([0,T];\mathcal{H}(\mathbb{R}^d))$. Hence, by theorem..., we can redefine it on a set of times of measure zero such that

$$\bar{v} \in C_b([0,T];\mathcal{H}(\mathbb{R}^d)).$$

Moreover,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |\bar{v}(x,t)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |v_0|^2 \,\mathrm{d}x.$$

d) The initial data is attained in the sense that

$$\bar{v}(t) \underset{t\to 0^+}{\longrightarrow} \bar{v}(0) = v_0 \quad \textit{(strong) in } L^2(\mathbb{R}^d;\mathbb{R}^d).$$

e) For all $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T);\mathbb{R}^d) \cap \mathcal{H}(\mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} \Big(\partial_t \varphi \cdot \bar{v}(x,t) + \nabla\varphi : \langle \nu_{x,t}, \xi \otimes \xi \rangle \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \nabla\varphi : \langle \nu^\infty_{x,t}, \theta\otimes\theta \rangle \,\mathrm{d}\lambda_t(x)\,\mathrm{d}t \\ = -\int_{\mathbb{R}^d} \varphi(x,0)\cdot \bar{v}(x,0)\,\mathrm{d}x$$

Proof. By theorem 3.14, for every finite time $s \in [0,T]$

$$\int_0^s \int_{\mathbb{R}^d} |v^k(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \int_0^s \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq s\int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x, \quad k \in \mathbb{N},$$

hence the sequence $(v^k)_{k\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^d \times [0,s])$. Hence, by the fundamental theorem of GYM 3.9, it generates a GYM $(\nu,\lambda,\nu^\infty) \in \mathbb{GY}(\mathbb{R}^d \times [0,s];\mathbb{R}^d)$ with $\lambda \in \mathcal{M}(\mathbb{R}^d \times [0,s])$. By disintegration theorem 3.4

$$\lambda = \tilde{\lambda}_t \otimes \lambda_{[0,s]}.$$

Therefore, for every $f \in \mathcal{F}_2(\mathbb{R}^d \times [0,T] \times \mathbb{R}^d)$,

$$f(x,t,v^k(x,t))\,\mathrm{d}x\,\mathrm{d}t \overset{*}{\rightharpoonup} \langle \nu_{x,t}, f(x,t,\cdot)\rangle\,\mathrm{d}x\,\mathrm{d}t + \langle \nu^\infty_{x,t}, f^\infty(x,t,\cdot)\rangle\,\mathrm{d}\tilde{\lambda}_t(x)\,\mathrm{d}\lambda_{(0,s)}(t)$$

(weak*) in $\mathcal{R}(\mathbb{R}^d \times [0,s])$. For $f(x,t,v) = |v|^2$, since $(|\cdot|^2)^\infty \equiv 1$ by proposition 3.8,

$$|v^k(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \overset{*}{\rightharpoonup} \langle \nu_{x,t}, |\cdot|^2\rangle\,\mathrm{d}x\,\mathrm{d}t + \,\mathrm{d}\tilde{\lambda}_t(x)\,\mathrm{d}\lambda_{[0,s]}(t)$$

(weak*) in $\mathcal{R}(\mathbb{R}^d \times [0,s])$. Hence, for every $\phi \in C_c(\mathbb{R}^d)$ and $\varphi \in C([0,s])$,

$$\begin{aligned}
&\int_0^s \varphi(t) \int_{\mathbb{R}^d} \phi(x)\langle \nu_{x,t}, |\cdot|^2\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_0^s \varphi(t)\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}\tilde{\lambda}_t(x)\,\mathrm{d}\lambda_{[0,s]}(t)\\
&\quad = \lim_{k\to\infty} \int_0^s \varphi(t) \int_{\mathbb{R}^d} \phi(x)|v^k(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad \le \|\varphi\|_{L^1([0,s])}\|\phi\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |v_0(x)|^2\,\mathrm{d}x.
\end{aligned}$$

For every $\varphi \in C([0,s])$ non negative, denote ω_φ by the positive Radon measure on $\mathbb{R}^d$ defined by

$$\int_0^s \varphi(t) \int_A \langle \nu_{x,t}, |\cdot|^2\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_0^s \varphi(t)\int_A \mathrm{d}\tilde{\lambda}_t(x)\,\mathrm{d}\lambda_{[0,s]}(t)$$

for every Borel subset A of $\mathbb{R}^d$. Analogously to the proof of theorem 3.9 we conclude that

$$\int_0^s \varphi(t) \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_0^s \varphi(t)\,\mathrm{d}\lambda_{(0,s)}(t) \le \|\varphi\|_{L^1([0,s])} \int_{\mathbb{R}^d} |v_0(x)|^2\,\mathrm{d}x. \tag{3.10}$$

for every $\varphi \in C([0,s])$ non negative. Observe that the second term implies that $\lambda_{[0,s]} << \mathrm{d}t$. This is because, since they are Radon measures, by theorem [Riesz], for every open subset U of $[0,s]$

$$\begin{aligned}
\lambda_{[0,s]}(U) &= \sup\{\langle \lambda_{[0,s]}, \varphi\rangle \,:\, \varphi \in C([0,s]),\, 0 \le \varphi \le \mathbb{1}_U\}\\
&\le \sup\{\langle\, \mathrm{d}t, \varphi\rangle \,:\, \varphi \in C([0,s]),\, 0 \le \varphi \le \mathbb{1}_U\} \int_{\mathbb{R}^d} |v_0(x)|^2\,\mathrm{d}x\\
&= \,\mathrm{d}t(U) \int_{\mathbb{R}^d} |v_0(x)|^2\,\mathrm{d}x,
\end{aligned}$$

and hence, for every Borel subset A of $[0,s]$, it is also

$$\lambda_{[0,s]}(A) \le \,\mathrm{d}t(A) \int_{\mathbb{R}^d} |v_0(x)|^2\,\mathrm{d}x.$$

Therefore, the Lebesgue-Radon-Nikodým decomposition of $\lambda_{[0,s]}$ with respect to $\mathrm{d}t$ is

$$\lambda_{[0,s]} = h(t)\,\mathrm{d}t$$

where h is the Lebesgue-Radon-Nikodým derivative $\frac{\mathrm{d}\lambda_{[0,s]}}{\mathrm{d}t}$. Now we can use Fatou's lemma to extend the inequality (3.10) to all non negative $\varphi \in L^1([0,s])$. Taking $\varphi_n \to \varphi$ in $L^1([0,s])$ (recall that it also converges $[\mathrm{d}t]$ pointwise)

$$\begin{aligned}
&\int_0^s \varphi(t) \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle \,\mathrm{d}x\,\mathrm{d}t + \int_0^s \varphi(t)h(t)\,\mathrm{d}t \\
&\quad \leq \liminf_{n\to\infty} \left(\int_0^s \varphi_n(t) \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle \,\mathrm{d}x\,\mathrm{d}t + \int_0^s \varphi_n(t)h(t)\,\mathrm{d}t \right) \\
&\quad \leq \liminf_{n\to\infty} \left(\|\varphi_n\|_{L^1([0,s])} \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x \right) \\
&\quad = \|\varphi\|_{L^1([0,s])} \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x.
\end{aligned}$$

Finally, for every $\varphi \in L^1([0,s])$ we deduce that

$$\int_0^s |\varphi(t)| \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle \,\mathrm{d}x\,\mathrm{d}t + \int_0^s |\varphi(t)|h(t)\,\mathrm{d}t \leq \|\varphi\|_{L^1([0,s])} \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x. \tag{3.11}$$

The second term implies that, since $L^1([0,s])^* \simeq L^\infty([0,s])$, $h \in L^\infty([0,s])$ with

$$\|h\|_{L^\infty([0,s])} \leq \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x.$$

We define

$$\lambda_t = h(t)\tilde{\lambda}_t \in \mathcal{M}^+(\mathbb{R}^d)$$

$[\mathrm{d}t]$ *a.e.* $t \in [0,s]$. Hence

$$\lambda = \lambda_t \otimes \mathrm{d}t \quad \text{and} \quad \lambda_t(\mathbb{R}^d) = h(t).$$

Again, since $L^1([0,s])^* \simeq L^\infty([0,s])$, (3.11) implies

$$\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle \,\mathrm{d}x + \lambda_t(\mathbb{R}^d) \leq \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x$$

for almost every time $t \in [0,s]$. We have just proved *a)* and *b)*.

c) Applying Jensen's inequality we deduce

$$\int_{\mathbb{R}^d} |\bar{v}(x,t)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} |\langle \nu_{x,t}, \xi \rangle|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\xi|^2 \rangle \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |v_0(x)|^2 \,\mathrm{d}x$$

$[\mathrm{d}t]$ *a.e.* $t \in [0,s]$. On the other hand, taking $f(x,t,\xi) = \xi \in \mathcal{F}_2(\mathbb{R}^d \times [0,T] \times \mathbb{R}^d)^d$, since $f^\infty \equiv 0$ by proposition 3.8,

$$v^k(x,t)\,\mathrm{d}x\,\mathrm{d}t \overset{*}{\rightharpoonup} \langle \nu_{x,t}, \xi \rangle \,\mathrm{d}x\,\mathrm{d}t = \bar{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{3.12}$$

(weak*) in $\mathcal{M}(\mathbb{R}^d \times [0,s];\mathbb{R}^d)$. Hence, for every $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$0 = \bigg(\int_{\mathbb{R}^d} \nabla\psi \cdot v^k(x,t)\,\mathrm{d}x\bigg)\,\mathrm{d}t \overset{*}{\rightharpoonup} \bigg(\int_{\mathbb{R}^d} \nabla\psi \cdot \bar{v}(x,t)\,\mathrm{d}x\bigg)\,\mathrm{d}t$$

(weak*) in $\mathcal{M}([0,s])$, that is,

$$\int_{\mathbb{R}^d} \nabla\psi \cdot \bar{v}(x,t)\,\mathrm{d}x = 0$$

$[\mathrm{d}t]$ *a.e.* $t \in [0,s]$.

d)...

e) First recall that, since v^k is a weak solution to the INSE,

$$\int_0^T \int_{\mathbb{R}^d} \Big((\partial_t\varphi + \mu_k\Delta\varphi)\cdot v^k + \nabla\varphi : v^k \otimes v^k\Big)(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^d} \varphi(x,0)\cdot v_0(x)\,\mathrm{d}x \quad (3.13)$$

for every $\varphi \in C_c(\mathbb{R}^d \times [0,s);\mathbb{R}^d) \cap \mathcal{H}(\mathbb{R}^d)$. For $f(x,t,\xi) = \xi\otimes\xi \in \mathcal{F}_2(\mathbb{R}^d \times [0,T] \times \mathbb{R}^d)^{d\times d}$, since

$$f^\infty(x,t,\theta) = \lim_{\substack{\theta'\to\theta \\ s\to\infty}} \frac{(s\theta')\otimes(s\theta')}{s^2} = \theta\otimes\theta, \quad \theta \in \mathbb{S}^{d-1}$$

we obtain

$$v^k \otimes v^k(x,t)\,\mathrm{d}x\,\mathrm{d}t \overset{*}{\rightharpoonup} \langle \nu_{x,t}, \xi\otimes\xi\rangle\,\mathrm{d}x\,\mathrm{d}t + \langle \nu_{x,t}^\infty, \theta\otimes\theta\rangle\,\mathrm{d}\lambda_t(x)\,\mathrm{d}t \quad (3.14)$$

(weak*) in $\mathcal{M}(\mathbb{R}^d \times [0,s];\mathbb{R}^{d\times d})$. Hence, by 3.12 and 3.14, when we make $k\uparrow\infty$ on 3.13 we obtain

$$\int_0^T \int_{\mathbb{R}^d} \Big(\partial_t\varphi\cdot\bar{v}(x,t) + \nabla\varphi : \langle \nu_{x,t}, \xi\otimes\xi\rangle\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \nabla\varphi : \langle \nu_{x,t}^\infty, \theta\otimes\theta\rangle\,\mathrm{d}\lambda_t(x)\,\mathrm{d}t$$
$$= -\int_{\mathbb{R}^d} \varphi(x,0)\cdot\bar{v}(x,0)\,\mathrm{d}x$$

for every $\varphi \in C_c(\mathbb{R}^d \times [0,s);\mathbb{R}^d) \cap \mathcal{H}(\mathbb{R}^d)$.

Finally, since this is true for all finite time $0 \le s \le T$, the results are extended naturally to the case $T = \infty$.

Notice that *d)* implies that it is a measure-valued solution when we test it with functions in $C_c(\mathbb{R}^d \times (0,T);\mathbb{R}^d)$. □

Definition 3.17. Given $v_0 \in \mathcal{H}(\mathbb{R}^d)$, a GYM $(\nu,\lambda,\nu^\infty) \in \mathbb{GY}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ is called an **admissible measure-valued solution** to the IEE with initial data v_0 if it satisfies *a)*, *b)*, *c)*, *d)* and *e)* of theorem 3.16.

3.2.3 Measure-valued subsolution of the IEE

In this part we equip, in the sense of section..., $V_1 = \mathbb{R}^d$ and $V_2 = S_0^d$ with

$$[\cdot]_1 = \frac{1}{\sqrt{d}}|\cdot| \quad \text{and} \quad [\cdot]_2 = \lambda_{\max}.$$

Define $[\cdot]_{\mathbb{R}^d \times S_0^d}$ via the Euclidean norm in $\mathbb{R}^2$. As we want to consider sequences in $L_t^\infty(L^2 \times L^1)$, we consider $\vec{p} = (2,1)$. The “sphere” is

$$\mathscr{S}_{2,1} \equiv \mathbb{S}^{2,1}_{\mathbb{R}^d, S_0^d} = \left\{ (v,u) \in \mathbb{R}^d \times S_0^d \, : \, \frac{1}{d}|v|^2 + \lambda_{\max}(u) = 1 \right\}$$

and the 2, 1-recession function of $f \in \mathcal{F}_{2,1}(..., \mathbb{R}^d, S_0^d)$ is

$$f^\infty(y,v,u) = \lim_{\substack{(v',u') \to (v,u) \\ s \to \infty}} \frac{f(y, sv', s^2 u')}{s^2}, \quad (y,v,u) \in ... \times \mathscr{S}_{2,1}.$$

The election of such $[\cdot]$ is due to the map

$$\begin{aligned} Q : \mathbb{R}^d &\to \mathbb{R}^d \times S_0^d \\ v &\mapsto (v, v \bigcirc v). \end{aligned}$$

satisfies $Q(\mathbb{S}^{d-1}) = \mathscr{S}_{2,1}$. It is clear from

$$\frac{1}{d}|v|^2 + \lambda_{\max}(v \bigcirc v) = \lambda_{\max}(v \otimes v) = \max_{\xi \in \mathbb{S}^{d-1}} \langle \xi, (v \otimes v)\xi \rangle = \max_{\xi \in \mathbb{S}^{d-1}} \langle v, \xi \rangle^2 = |v|^4, \quad v \in \mathbb{R}^d.$$

Theorem 3.18. *Let $(v^k, u^k)_{k \in \mathbb{N}}$ a bounded sequence in $L^\infty([0,T]; L^{2,1}(\mathbb{R}^d; \mathbb{R}^d, S_0^d))$. Then, for a subsequence (not relabeled) there exists a $GYM_{2,1}$*

$$(\nu, \lambda, \nu^\infty) \in \mathbb{GY}_{2,1}(\mathscr{D}_T, \mathcal{L}; \mathbb{R}^d, S_0^d)$$

such that, for every $f \in \mathcal{F}_{2,1}(\mathscr{D}_T, \mathbb{R}^d, S_0^d)$,

$$f(y, v^k(y), u^k(y)) \, \mathrm{d}y \overset{*}{\rightharpoonup} \langle \nu_y, f(y, \cdot) \rangle \, \mathrm{d}y + \langle \nu_y^\infty, f^\infty(y, \cdot) \rangle \, \mathrm{d}\lambda(y)$$

(weak) in $\mathcal{R}(\mathscr{D}_T)$. The concentration measure λ admits a desintegration of the form*

$$\mathrm{d}\lambda(x,t) = \mathrm{d}\lambda_t(x) \otimes \mathrm{d}t$$

where the new λ is an uniformly bounded parametrized measure

$$\lambda \in L^\infty_{w^*}([0,T]; \mathcal{M}^+(\mathbb{R}^d)).$$

Proof. Repeat the proof with lifted GYM. □

Definition 3.19. Let $(\nu, \lambda, \nu^\infty) \in \mathbb{GY}_{2,1}(\mathscr{D}_T; \mathbb{R}^d, S_0^d)$. Denote $\pi_1 : \mathbb{R}^d, S_0^d \to \mathbb{R}^d$ and $\pi_2 : \mathbb{R}^d, S_0^d \to S_0^d$ by the canonical projections. Notice that $\pi_1^\infty \equiv 0$ and $\pi_2^\infty = \pi_2$. Consider the **barycenter**

$$\begin{aligned}\bar{v}(x,t) &= \langle \nu_{x,t}, \pi_1 \rangle \\ \bar{u}(x,t) &= \langle \nu_{x,t}, \pi_2 \rangle \,\mathrm{d}x\,\mathrm{d}t + \langle \nu_{x,t}^\infty, \pi_2 \rangle \,\mathrm{d}\lambda(x,t)\end{aligned}$$

Note that $\bar{u}$ is only a measure. Such $\mathrm{GYM}_{2,1}$ is called a **measure-valued subsolution** if $(\bar{v}, \bar{u})$ is a subsolution in the sense that

$$\partial_t \bar{v} + \mathrm{div}\bar{u} = 0$$

in $(C_c^\infty(\mathscr{D}_T; \mathbb{R}^d) \cap C([0,T]; \mathcal{H}(\mathscr{D})))^*$

Observe that

$$\varrho^\infty(v,u) = \lim_{\substack{v' \to v \\ s \to \infty}} \frac{\varrho(sv', s^2 u')}{s^2} = \frac{d}{2} \lim_{\substack{v' \to v \\ s \to \infty}} \frac{\lambda_{\max}((sv') \otimes (sv') - s^2 u')}{s^2} = \varrho(v,u)$$

Definition 3.20. Given a measure-valued subsolution $(\nu, \lambda_t \otimes \mathrm{d}t, \nu^\infty) \in \mathbb{GY}_{2,1}(\mathscr{D}_T; \mathbb{R}^d, S_0^d)$ we will say that it is an **admissible measure-valued subsolution** is its energy

$$E(t) = \int_{\mathbb{R}^d} \langle \nu_{x,t}, \varrho \rangle \,\mathrm{d}x + \int_{\mathbb{R}^d} \langle \nu_{x,t}^\infty, \varrho \rangle \,\mathrm{d}\lambda_t(x)$$

satisfies

$$E(t) \leq \frac{1}{2} \int_{\mathbb{R}^d} |\bar{v}(x,0)| \,\mathrm{d}x$$

for almost every time.

Given $f \in \mathcal{F}_{2,1}(\mathscr{D}_T, \mathbb{R}^d, S_0^d)$, then $f \circ Q \in \mathcal{F}_2(\mathscr{D}_T, \mathbb{R}^d \times S_0^d)$ with

$$(f \circ Q)^\infty(v) = \lim_{\substack{v' \to v \\ s \to \infty}} \frac{f \circ Q(sv')}{s^2} = \lim_{\substack{v' \to v \\ s \to \infty}} \frac{f(sv', (sv') \bigcirc (sv'))}{s^2} = f^\infty \circ Q(v)$$

We define also

$$\tilde{\nu} = Q_* \nu \in \mathbb{Y}(\mathscr{D}_T, \mathcal{L}; \mathbb{R}^d \times S_0^d) \quad \text{and} \quad \tilde{\nu}^\infty = Q_* \nu^\infty \in \mathbb{Y}(\mathscr{D}_T, \lambda; \mathscr{S}_{2,1}).$$

Hence,

$$(\tilde{\nu}, \lambda, \tilde{\nu}^\infty) \in \mathbb{GY}_{2,1}(\mathscr{D}_T; \mathbb{R}^d, S_0^d)$$

Proposition 3.21. *Let $(\nu, \lambda, \nu^\infty)$ be a measure-valued solution with bounded energy and $(\tilde{\nu}, \lambda, \tilde{\nu}^\infty)$ be defined as above. Suppose (v^k, u^k) is bounded in $L^\infty([0,T]; L^{2,1}(\mathbb{R}^d; \mathbb{R}^d, S_0^d))$ such that*

$$(v^k, u^k) \to (\tilde{\nu}, \lambda, \tilde{\nu}^\infty) \qquad in \quad \mathbb{GY}_{2,1}(\mathscr{D}_T; \mathbb{R}^d, S_0^d).$$

Then

a) The $GYM_{2,1}$ $(\tilde{\nu}, \lambda, \tilde{\nu}^\infty)$ is a measure-valued subsolution.

b) If $\tilde{E}$ and E denote the energy of $(\tilde{\nu}, \lambda, \tilde{\nu}^\infty)$ and $(\nu, \lambda, \nu^\infty)$ respectively, then

$$\tilde{E}(t) = E(t)$$

for almost every time.

c)

$$v^k \xrightarrow{\mathbb{GY}_2} (\nu, \lambda, \nu^\infty)$$

d)

$$|u^k - v^k \bigcirc v^k| \to 0$$

in $L^1_{loc}(\mathscr{D}_T)$.

Proof. a) On the one hand

$$\begin{aligned}
0 &= \int_{\mathscr{D}_T} \Big(\partial_t \varphi \cdot \overline{v}(x,t) + \nabla \varphi : \langle \nu_{x,t}, \xi \otimes \xi \rangle \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathscr{D}_T} \nabla \varphi : \langle \nu^\infty_{x,t}, \theta \otimes \theta \rangle \,\mathrm{d}\lambda_t(x) \,\mathrm{d}t \\
&= \int_{\mathscr{D}_T} \Big(\partial_t \varphi \cdot \overline{v}(x,t) + \nabla \varphi : \langle \nu_{x,t}, \xi \bigcirc \xi \rangle \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathscr{D}_T} \nabla \varphi : \langle \nu^\infty_{x,t}, \theta \bigcirc \theta \rangle \,\mathrm{d}\lambda_t(x) \,\mathrm{d}t \\
&\quad + \frac{1}{d} \int_{\mathscr{D}_T} \mathrm{div} \varphi \langle \nu_{x,t}, |\xi|^2 \rangle \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{d} \int_{\mathscr{D}_T} \mathrm{div} \varphi \,\mathrm{d}\lambda_t(x) \,\mathrm{d}t \\
&= \int_{\mathscr{D}_T} \Big(\partial_t \varphi \cdot \overline{v}(x,t) + \nabla \varphi : \langle \nu_{x,t}, \pi_2 \circ Q \rangle \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathscr{D}_T} \nabla \varphi : \langle \nu^\infty_{x,t}, \pi_2 \circ Q(\theta) \rangle \,\mathrm{d}\lambda_t(x) \,\mathrm{d}t \\
&= \int_{\mathscr{D}_T} \partial_t \varphi \cdot \overline{v}(x,t) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathscr{D}_T} \nabla \varphi : \,\mathrm{d}\bar{u}(x,t)
\end{aligned}$$

for every $\varphi \in$ .... On the other hand

$$div...$$

b) It is clear by definition

$$\begin{aligned}
\tilde{E}(t) &= \int_{\mathbb{R}^d} \langle \tilde{\nu}_{x,t}, \varrho \rangle \,\mathrm{d}x + \int_{\mathbb{R}^d} \langle \tilde{\nu}^\infty_{x,t}, \varrho \rangle \,\mathrm{d}\lambda_t(x) \\
&= \int_{\mathbb{R}^d} \langle \nu_{x,t}, \varrho(\xi, \xi \bigcirc \xi) \rangle \,\mathrm{d}x + \int_{\mathbb{R}^d} \langle \nu^\infty_{x,t}, \varrho(\theta, \theta \bigcirc \theta) \rangle \,\mathrm{d}\lambda_t(x) \\
&= \frac{1}{2} \bigg(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\xi|^2 \rangle \,\mathrm{d}x + \lambda_t(\mathbb{R}^d) \bigg) = E(t).
\end{aligned}$$

c) Let $f \in \mathcal{F}_2(\mathscr{D}_T, \mathbb{R}^d)$. Take $g = f \circ \pi_1$. Notice that $g \in \mathcal{F}_{2,1}(\mathscr{D}_T, \mathbb{R}^d, S^d_0)$ and $f = g \circ Q$. Hence, by hypothesis,

$$\begin{aligned}
f(y, v^k(y)) \,\mathrm{d}y &= g(y, v^k(y), u^k(y)) \,\mathrm{d}y \\
&\overset{*}{\rightharpoonup} \langle \tilde{\nu}_y, g(y, \cdot) \rangle \,\mathrm{d}y + \langle \tilde{\nu}^\infty_y, g^\infty(y, \cdot) \rangle \,\mathrm{d}\lambda(y) \\
&= \langle \nu_y, f(y, \cdot) \rangle \,\mathrm{d}y + \langle \nu^\infty_y, f^\infty(y, \cdot) \rangle \,\mathrm{d}\lambda(y)
\end{aligned}$$

d) Note that the function

$$f(v,u) = |u - v \bigcirc v|, \quad (v,u) \in \mathbb{R}^d \times S_0^d$$

belongs to $\mathcal{F}_{2,1}(\mathscr{D}_T, \mathbb{R}^d, S_0^d)$ with

$$f^\infty(v,u) = \lim_{\substack{(v',u')\to(v,u)\\ s\to\infty}} \frac{|s^2 u' - (sv') \bigcirc (sv')|}{s^2} = |u - v \bigcirc v| = f(v,u).$$

Also $f \circ Q \equiv 0$. Therefore,

$$|u^k - v^k \bigcirc v^k|(y)\,\mathrm{d}y \overset{*}{\rightharpoonup} \langle \tilde{\nu}_y, f\rangle\,\mathrm{d}y + \langle \tilde{\nu}_y^\infty, f\rangle\,\mathrm{d}\lambda(y) = 0$$

(weak*) in $\mathcal{R}(\mathscr{D}_T)$. Density extends it to $L^1_{loc}(\mathscr{D}_T)$. $\square$

3.3 Density of wild initial datas

Theorem 3.22. *a) A GYM $(\nu, \lambda, \nu^\infty) \in \mathbb{GY}(\mathscr{D}_T; \mathbb{R}^d)$ is a measure-valued solution of the IEE with bounded energy if and only if there exists a sequence $(v^k)_{k\in\mathbb{N}}$ of weak solutions to the IEE bounded in $C_b([0,T]; \mathcal{H}_w(\mathbb{R}^d))$ which generates such GYM, that is,*

$$v^k \to (\nu, \lambda, \nu^\infty) \qquad \text{in } \mathbb{GY}(\mathscr{D}_T; \mathbb{R}^d).$$

b) If in addition $(\nu, \lambda, \nu^\infty)$ is admissible with initial data $v_0 \in \mathcal{H}(\mathbb{R}^d)$, then the generating sequence $(v^k)_{k\in\mathbb{N}}$ can be chosen such that they are all admissible and

$$v_0^k \to v_0 \qquad \text{in } L^2(\mathbb{R}^d; \mathbb{R}^d).$$

Instead of prove *b)*, first we are going to prove a weaker version *b')* which will allow us to prove *b)* after.

b') If in addition $(\nu, \lambda, \nu^\infty)$ is admissible with initial data $v_0 \in \mathcal{H}(\mathbb{R}^d)$, then the generating sequence $(v^k)_{k\in\mathbb{N}}$ can be chosen such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |v^k(x,t)|^2\,\mathrm{d}x \le \int_{\mathbb{R}^d} |v_0(x,t)|^2\,\mathrm{d}x + \frac{2}{k}.$$

and

$$v_0^k \to v_0 \qquad \text{in } L^2(\mathbb{R}^d; \mathbb{R}^d).$$

Corollary 3.23. *The set of wild initial datas is L^2-dense in the set of selenoidal initial datas $\mathcal{H}(\mathbb{R}^d)$.*

3.3.1 From subsolutions to exact solutions

Proposition 3.24. *a) We can generate* $(\nu, \lambda, \nu^\infty)$ *as required in theorem 3.22 a) provided we can generate the* $GYM_{2,1}$ $(\tilde{\nu}, \lambda, \tilde{\nu}^\infty)$ *by a sequence* (v^k, u^k) *bounded in* $L^\infty([0,T]; L^{2,1}(\mathbb{R}^d; \mathbb{R}^d \times S_0^d))$ *such that it is a smooth subsolution in* $\mathbb{R}^d \times [0,T]$.

b) If in addition $(\nu, \lambda, \nu^\infty)$ *is admissible with initial data* $v_0 \in \mathcal{H}(\mathbb{R}^d)$*, then we can generate it as required in theorem 3.22 b') if the sequence* (v^k, u^k) *additionally satisfies*

$$\limsup_{k\to\infty} \sup_{t\in[0,T]} \int_{\mathbb{R}^d} \varrho(v^k, u^k)\,\mathrm{d}x \le \operatorname*{esssup}_{t\in[0,T]} E(t)$$

and

$$v_0^k \to v_0 \qquad \text{in } L^2(\mathbb{R}^d; \mathbb{R}^d).$$

Proof. a) Since $(v^k, u^k) \in L^\infty([0,T]; L^{2,1}(\mathbb{R}^d; \mathbb{R}^d \times S_0^d))$ is smooth, the function $\varrho^k = \varrho(v^k, u^k)$ belongs to $C(\mathbb{R}^d \times (0,T)) \cap C_b([0,T], L^1(\mathbb{R}^d))$ by proposition 2.9. Hence, we can take $\alpha > d-2$ and $\varepsilon_k > 0$ small enough such that the function

$$e^k = \varrho^k + \varepsilon_k \min\{t, t^{-2}\} \min\{1, |x|^{-\alpha}\} \in C(\mathbb{R}^d \times (0,T)) \cap C_b([0,T], L^1(\mathbb{R}^d))$$

with

$$\varrho(v^k, u^k) < e^k \qquad \text{in} \quad \mathbb{R}^d \times (0,T)$$

satisfies

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} (e^k - \varrho^k)\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} (e^k - \varrho^k)\,\mathrm{d}x\,\mathrm{d}t < \frac{1}{k}.$$

Since (v^k, u^k) is a subsolution, the space $\mathcal{S}_{q^k}(\mathscr{D}_T, v_0^k, e^k)$ is non empty for some pressure q^k. Hence, by subsolution criterion 2.13, there exists a sequence $(v^{k,n})_{n\in\mathbb{N}} \subset C_b([0,T], \mathcal{H}_w(\mathbb{R}^d))$ of weak solutions of the IEE with initial data v_0^k such that

$$v^{k,n} \to v^k \qquad \text{in} \quad C_b([0,T], \mathcal{H}_w(\mathbb{R}^d)),$$

that is,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} (v^{k,n} - v^k) \cdot \varphi\,\mathrm{d}x \to 0 \qquad \forall \varphi \in L^2(\mathbb{R}^d; \mathbb{R}^d),$$

and

$$\frac{1}{2}|v^{k,n}|^2 = e^k \qquad \text{in} \quad C_b([0,T]; L^1(\mathbb{R}^d)).$$

Therefore (?????????? usar que $v^k \in C_b([0,T], L^2(\mathbb{R}^d; \mathbb{R}^d))$ y smooth), we can choose $n(k) \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} (v^{k,n} - v^k) \cdot v^k\,\mathrm{d}x \right| < \frac{1}{k}.$$

Now, for every finite time $s \in [0,T]$ and for every bounded subdomain $\Omega \subset \mathbb{R}^d$ we have

$$\begin{aligned}\int_\Omega |v^{k,n} - v^k|^2 \,\mathrm{d}x &= \int_\Omega |v^{k,n}|^2 \,\mathrm{d}x + \int_\Omega |v^k|^2 \,\mathrm{d}x - 2\int_\Omega v^{k,n} \cdot v^k \,\mathrm{d}x \\ &= \int_\Omega 2e^k \,\mathrm{d}x - \int_\Omega |v^k|^2 \,\mathrm{d}x - 2\int_\Omega (v^{k,n} - v^k) \cdot v^k \,\mathrm{d}x\end{aligned}$$

By prop...

$$\begin{aligned}\int_{\Omega_s} \Big(\varrho^k - \frac{1}{2}|v^k|^2\Big) \,\mathrm{d}x \,\mathrm{d}t &= \frac{d}{2}\int_{\Omega_s} \lambda_{\max}(v^k \bigcirc v^k - u^k) \,\mathrm{d}x \,\mathrm{d}t \\ &\leq \frac{d}{2}\int_{\Omega_s} |v^k \bigcirc v^k - u^k| \,\mathrm{d}x \,\mathrm{d}t \to 0\end{aligned}$$

when $k \to \infty$. Hence,

$$\int_{\Omega_s} 2e^k \,\mathrm{d}x \,\mathrm{d}t - \int_{\Omega_s} |v^k|^2 \,\mathrm{d}x \,\mathrm{d}t = 2\int_{\Omega_s} (e^k - \varrho^k) \,\mathrm{d}x \,\mathrm{d}t + 2\int_{\Omega_s} \Big(\varrho^k - \frac{1}{2}|v^k|^2\Big) \,\mathrm{d}x \,\mathrm{d}t$$

□

3.3.2 Approximation of Generalized Young Measures

3.3.3 Discrete Homogeneous Young Measures

References

[AB] J.J. Alibert and G. Bouchitté, *Non-uniform integrability and generalized Young measures*, J. Convex Anal. 4(1), 129-147 (1997).

[KCD] P. K. Kundu, I. M. Cohen and D. R. Dowling *Fluid Mechanics*, Fifth edition, Elsevier Inc., 2012.

[LS1] C. De Lellis and L. Székelyhidi Jr. *The Euler equations as a differential inclusion*, Annals of Mathematics (2), 170(3), 1417-1436 (2009).

[LS2] C. De Lellis and L. Székelyhidi Jr. *On admissibility criteria for weak solutions of the Euler equations*, Arch. Rational Mech. Anal. 195(1), 225-260 (2010).

[MB] A. Majda and A. Bertozzi, *Vorticity and incompressible flow*, Cambridge texts in Applied Mathematics, 2002.

[MD] R. DiPerna and A. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Phys., 108(4), 667-689 (1987).

[SW] L. Székelyhidi Jr. and E. Wiedemann, *Young measures generated by ideal incompressible flows*, Arch. Rational Mech. Anal. 206, 333-366 (2012).

[W] E. Wiedemann, *Existence of weak solutions for the incompressible Euler equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28(5), 727-730 (2011).