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Scattering Theory and Transmission Eigenvalues

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Declaración de originalidad

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En Murcia, a 27 de junio de 2022

Fdo.: Fernando Ballesta Yagüe

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Y a mis padres y a mi hermana, por ser mi luz.

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Introduction

This Master's Thesis (from now on denoted by TFM) is concerned with **Scattering Theory**. In particular, it is an introduction to a problem that has received much attention in the last two decades: the *transmission eigenvalue problem.*

What is Scattering Theory?

Sometimes, waves depart from their expected path and spread out in multiple directions. This phenomenon is known as *scattering.*

It is quite present in our daily lives. For example, it is the reason why the sky appears to be blue: the white light from the sun hits the molecules in the atmosphere, which causes the shorter blue wavelengths to scatter out in multiple directions (in fact, the violet wavelengths scatter more, but our eye is less sensitive to them; see [26]).

Another example is the scattering of the rays of light coming out from traffic lights or car lights when there is much humidity in the air (see Figure 1). This is due to the scattering of the light rays when they hit the water molecules in the air.

Figure 1: Scattering of traffic lights in presence of humidity.

For us, roughly speaking, scattering theory deals with how waves behave when they find an *scatterer* (object) or *inhomogeneous media* on their way (see Figure 2).

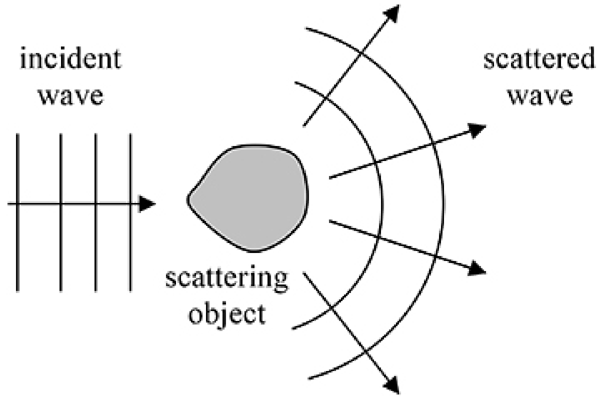

Figure 2: Diagram of the scattering phenomenon.

Topic of the TFM

There is a vast literature on scattering theory impossible to cover in an essay of this length. So we have decided to focus on **acoustic/Helmholtz scattering in the case of an isotropic inhomogeneous medium with an inhomogeneity of compact support**. The final **objectives** are

- To **introduce the transmission eigenvalue problem**.
- To **prove** (under certain conditions) the **existence of a discrete set of real transmission eigenvalues that accumulates at infinity**.

But before moving on to the structure of the TFM, let us give a general view of the subject that we are going to study.

Overview of the topic

Acoustic scattering in an inhomogeneous medium

We begin by describing what is acoustic scattering in an inhomogeneous medium of compact support.

Suppose we have an inhomogeneous bounded region $D \subset \mathbb{R}^3$ in an homogeneous space (i.e., $\mathbb{R}^3 \backslash D$ is an homogeneous region). We assume that D is connected with C^2 boundary, and denote by ν be the outward unit normal vector to ∂D defined on ∂D.

Let $c(x)$ be the speed of sound at a point $x \in \mathbb{R}^3$. Then, $c(x) = c_0$ is constant for $x \in \mathbb{R}^3 \backslash D$. We define the refractive index as

$$n(x) := \frac{c_0^2}{c(x)^2}.$$

Notice that $n \equiv 1$ on $\mathbb{R}^3 \backslash D$, and $n(x) > 0$ on $\mathbb{R}^3$. We assume that $n \in L^\infty(D)$.

The propagation of waves in a homogeneous space is modeled by the wave equation

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \Delta U$$

where U denotes the velocity potential, and $c = c_0$ is the sound speed (constant since we are in a homogeneous region). When we have a time-harmonic wave, i.e., a wave of the form

$$U(x,t) = \operatorname{Re}\left[u(x)e^{-i\omega t}\right]$$

for a frequency $\omega > 0$, then the wave equation can be reduced to the Helmholtz equation

$$\Delta u + k^2 u = 0.$$

where $k = \frac{\omega}{c} > 0$ is the wave number. Notice that it is proportional to the frequency ω.

In brief, Helmholtz equation models the propagation of time-harmonic waves in an homogeneous region,

Under the above assumptions, the *scattering problem for the inhomogeneous media* (D, n) that we are going to consider is the following.

Suppose we have an **incident field** u^i, that is, a solution of Helmholtz equation in all of[1] $\mathbb{R}^3$:

$$\Delta u^i + k^2 u^i = 0 \quad \text{on } \mathbb{R}^3.$$

This incident field scatters when it finds the inhomogeneity (D, n) on its path, i.e., it generates another wave u^s, and in physical reality we percieve the total field $u = u^i + u^s$ (u^s has no physical appearance; it is just defined for mathematical reasons). Its behavior can be modeled by the equation

$$\Delta u(x) + k^2 n(x) u(x) = 0 \quad \text{in } \mathbb{R}^3. \tag{1}$$

Substituting $u = u^i + u^s$ in the previous equation, and using that u^i is a solution on $\mathbb{R}^3$ of Helmholtz equation, we have that (1) is equivalent to the following equation in the unknown u^s:

$$\Delta u^s(x) + k^2 n(x) u^s(x) = k^2(1 - n(x)) u^i(x) \quad \text{in } \mathbb{R}^3. \tag{2}$$

To abbreviate, we sometime use the notation $m(x) := 1 - n(x)$. Notice that, since $n(x) \equiv 1$ on $\mathbb{R}^3 \setminus \overline{D}$, the previous equation on $\mathbb{R}^3 \setminus \overline{D}$ becomes $\Delta u^s + k^2 n u^s = 0$, which is Helmholtz equation for u^s and simply states that u^s is a time-harmonic wave traveling in the homogeneous region $\mathbb{R}^3 \setminus D$.

So the scattering problem we are going to consider is to find the total field u that satisfies (1) from a knowledge of the incident field u^i, the wave number k and the inhomogeneous medium (D, n). Notice that this is equivalent to solving (2) for u^s (a knowledge of one of them gives us the other).

The field u^s is called the **scattered field**. As we have said, you cannot perceive/measure it, but it is defined as $u^s = u - u^i$ and it is understood as the wave that generates u^i when it scatters. The supperposition of u^i and u^s (i.e. the sum) is the **total wave** that we perceive physically.

Radiation condition and far field patterns

Because of physical considerations skechted in the essay, and in order to to have uniqueness of the direct scattering problem, a radiation condition is imposed to u^s. In Physics, radiation conditions are imposed to select which kind of wave we want to obtain as a solution and to exclude the rest of them.

This condition, known as **Sommerfeld's radiation condition**, is the following:

$$\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \tag{3}$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $\frac{x}{|x|}$. The radial derivative is defined as $\frac{\partial u^s}{\partial r}(x) = \nabla u^s(x) \cdot \frac{x}{|x|}$.

[1] Except for possibly a subset of measure zero in the exterior of $\overline{D}$; this could be a single point, if we have a point source, or a surface.

Solutions to Helmholtz equation on an exterior domain (i.e., whose domain of definition contains the exterior of some sphere) that satisfy Sommerfeld Radiation Condition (3) are called **radiating solutions**.

It can be proved that every radiating solution to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave, i.e.,

$$u(x) = \frac{e^{ik|x|}}{|x|}\left[u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right], \quad |x| \to \infty$$

uniformly in all directions $\hat{x} = \frac{x}{|x|}$. The function u_∞, defined on the unit sphere $\mathbb{S}^2$, is also known as the **far field pattern** of u, and is given by

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\left[u(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)} - \frac{\partial u}{\partial\nu}(y)e^{-ik\hat{x}\cdot y}\right] ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{4}$$

In the case of the scattered field u^s, which is a radiating solution of Helmholtz equation, the far field pattern can also be expressed as

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y)u(y)\, dy \tag{5}$$

where the last integral can be taken over any measurable set containing $supp(m)$.

This function is truly important. For applications, it is used to recover radiating solutions of the Helmholtz equation from a knowledge of their far field patterns. In physical reality you can measure the asymptotic behavior of the wave, which is described by the far field pattern, and you would like to find out what is the scattered wave or what are the properties of the inhomogeneity (D, n). This kind of problems are known as **inverse problems**. A treatment of the subject can be found in Chapter 5 of [13] and Chapter 7 of [19].

Completeness of far field patterns

In applications, you send several different incident fields, usually plane waves of the form $u^i(x; d) = e^{ikx\cdot d_n}$ where $d \in \mathbb{S}^2$ is the direction of propagation of the wave, they scatter and you measure their respective far field patterns $u_\infty(\hat{x}; d)$. Now, from the knowledge of these collection of far field patterns $u_\infty(\cdot; d_n)$ for a dense collection of vectors $d_n \in \mathbb{S}^2$ (with $n \in \mathbb{N}$ for example, although it could be a finite collection; in reality, a non-countable one does not have much sense), you want to reconstruct properties of the inhomogeneity (D, n).

There are several reconstruction methods developed to this end. For instance, one of the most important is the *Factorization Method*. For a detailed treatment of it, see Section 7.5 of [19] or the book [20]. There are other methods (see Chapter 2 of [7]), but we will not go into further detail.

In order to apply some the aforementioned methods, it is important that the set of far field patterns $\mathcal{F} := \{u_\infty(\cdot, d_n) : n \in \mathbb{N}\}$ is complete in $L^2(\mathbb{S}^2)$. An important characterization of this property is the following, which we will prove in this TFM:

Theorem. *The orthogonal complement of $\mathcal{F}$ in $L^2(\mathbb{S}^2)$ consists of the conjugate of those functions $g \in L^2(\mathbb{S}^2)$ for which there exists $w \in H^2(D)$ and a Herglotz wave function*

$$v(x) = \int_{\mathbb{S}^2} e^{-ikx\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3,$$

such that the pair v, w is a solution to

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \text{ in } D \tag{6}$$

satisfying

$$w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \tag{7}$$

Transmission eigenvalues

In view of the previous Theorem, it seems important to study the problem

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } D$$

$$w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D,$$

which is known as the **interior transmission problem**. Values[2] of $k > 0$ for which this problem has non-trivial solutions are called (interior) **transmission eigenvalues**.

If we want $\mathcal{F}$ to be dense in $L^2(\mathbb{S}^2)$, its orthogonal complement has to be $\{0\}$. So a sufficient condition for this to occur is that k is not a transmission eigenvalue. Therefore, this sufficient condition is usually imposed in the reconstruction methods such as the factorization method (see, for example, Theorems 7.38 and 7.39 from Section 7.5 of [19]).

It seems that this problem had not received much attention until the last two decades. In fact, existence of real transmission eigenvalues for general media was an open problem for about 20 years, until in 2010 the question was quite completely answered by Cakoni, Gintides and Haddar in [8]. We will explain the proof of this theorem in Chapter 4.

But, according to [5], appart from that theorem there have not been new results on real transmission eigenvalues, although there has been intense research on complex ones; see chapter 10 of the book by Colton and Kress [13].

Non-Scattering Inhomogeneities

The transmission eigenvalue problem is closely related to another fundamental and perplexing problem in scattering theory: the problem of **non-scattering inhomogeneities**.

The problem is the following: given an inhomogeneity (D, n), does there exist a wave number $k > 0$ and an incident wave u^i such that the corresponding far field pattern u_∞ is identically zero?

Such an incident field is referred to as a **non-scattering incident wave** and the corresponding $k > 0$ as a **non-scattering wave number**. Again, we consider positive wave numbers $k > 0$ because they are the only ones that have physical meaning.

How can we formulate this problem mathematically? Notice that (because of a result that we will prove in the TFM, as a consequence of Rellich's lemma) a radiating solution to the Helmhotz Equation with identically zero far field pattern is identically zero as well. Therefore, since u^s is a radiating solution of Helmholtz equation outside D, an equivalent formulation of the problem is the following: an inhomogeneous medium (D, n) is non-scattering if there exists a wave number $k > 0$ and an incident field u^i such that the corresponding scattered field is zero outside the inhomogeneity.

For a given incident field u^i, the scattered field $u^s \in H^2_{loc}(\mathbb{R}^3)$ satisfies the equation

$$\Delta u^s + k^2 n u^s = k^2(1 - n) u^i \quad \text{on } \mathbb{R}^3,$$

[2]One can consider complex transmission eigenvalues, but the only one that have physical meaning and are therefore important for applications are real ones, so we will only deal with these ones.

together with Sommerfeld's radiation condition.

So, explicitly, an inhomogeneity (D, n) **does not scatter** if there exits a wave number $k > 0$ and an incident field u^i, i.e., a solution of

$$\Delta u^i + k^2 u^i = 0 \quad \text{on } \mathbb{R}^3,$$

such that

$$\Delta u^s + k^2 n u^s = k^2(1-n)u^i \quad \text{on } \mathbb{R}^3,$$

and

$$u^s \equiv 0 \quad \text{on } \mathbb{R}^3 \backslash D.$$

This is an overdetermined system of elliptic equations (it has more equations than unknowns). However, it can be proved that for the simplest case, i.e., spherically stratified media, using separation of variables, there exists non-scattering media (see minutes 13-18 of [5] for the sketch of a proof).

So it makes sense to consider this problem, i.e., in general we are not talking about the empty set.

Notice that this problem is physically astonishing: it considers incident fields of specific frequencies for which the inhomogeneity is *invisible*, in the sense that the wave does not change its behavior (there is no nonzero scattered wave). Apparently, this phenomenon is quite rare. See for example the article by Blasten, Paivarinta and Sylvester [3].

Relation between transmission eigenvalues and non-scattering inhomogeneities

Let us see what is the relationship between transmission eigenvalues and non-scattering inhomogeneities

Suppose that u^i and u^s satisfy the non-scattering condition for a wave number $k > 0$ (recall that $supp(n-1) = \overline{D}$). Restricting u^i to D, we have that it is a solution to Helmholtz equation on the domain D, that is,

$$\Delta u^i + k^2 u^i = 0 \quad \text{on } D.$$

Since $u^s \equiv 0$ on $\mathbb{R}^3 \backslash D$, we have that $u^s = 0$ and $\frac{\partial u^s}{\partial \nu} = 0$ on ∂D, Therefore, $u := u^s$ and $v := u^i\big|_D$ are solutions to the transmission eigenvalue problem, i.e., the problem of finding $u \in H_0^2(D)$ and $v \in L^2(D)$ such that

$$\Delta u + k^2 n u = k^2(1-n)v \quad \text{and} \quad \Delta v + k^2 v = 0 \text{ in D}$$

(where $u \in H_0^2(D)$ means $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on ∂D).

Remark. Notice that u and v are solutions to the transmission eigenvalue problem since taking $w = u + v$ (w would be the total field, since $u = u^s$, $v = u^i$ in the above discussion), then $v, w \in L^2(D)$ satisfy that $v - w \in H_0^2(D)$ and

$$\Delta v + k^2 v = 0 \quad \text{in } D,$$

$$\Delta w + k^2 n w = 0 \quad \text{in } D.$$

Therefore, solutions to the non-scattering problem are solutions to the transmission eigenvalue problem. Or, equivalently, **non-scattering wave numbers are a subset of (positive) transmission eigenvalues**.

So a neccesary condition for k being a non-scattering wave number is that k is a transmission eigenvalue.

It is natural then to ask another important question in scattering theory: when a (real) transmission eigenvalue is a non-scattering wave number? This is a partially open question yet. Many papers have been published lately regarding this topic.

To attack the problem, it has been related to regularity of the eigenfuctions of the laplacian (i.e. solutions of Helmholtz equation) and free boundary problems.

Let us give an overview here. Suppose that $k > 0$ is a transmission eigenvalue, that is, there exists $u \in H_0^2(D)$ and $v \in L^2(D)$ such that

$$\Delta u + k^2 nu = k^2(1-n)v \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D$$

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D.$$

We want to see if k is a non-scattering wave number, i.e., if there exists an incident wave v and a scattered wave v such that

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^3$$

$$\Delta u + k^2 nu = k^2(1-n)v \text{ in } D$$

$$u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D.$$

Notice that the only difference is that v must exist as a solution to Helmholtz equation in all of $\mathbb{R}^3$ and not only on D. Since H^1 solutions to Helmholtz equation are analytic (see [28] page 6), the problem is if it is possible to extend v outside of D in such a way that it is analytic. That is, under what assumption the function $v \in L^2(D)$ is a solution to Helmholtz equation in a region including D? In that case, the eigenfunction v has to be analytic, because H^1 solutions to Helmholtz equation are analytic. So it is a regularity issue of eigenfuctions up to the boundary.

There have been several approaches to this problems, such as the use of free boundary methods by Cakoni and Vogelius in [10] or by Salo and Shangolian in [28]. For an overview of the recent advances in this problem, see [5]. The most studied case is that of spherically symmetric media. In that case, the set of non-scattering wave numbers and the set of transmission eigenvalues coincide.

Motivation to study Transmission Eigenvalues

As explained in the previous section, transmission eigenvalues play a central role in applications: a sufficient condition to apply some of the reconstruction methods is that the wake number $k > 0$ is not a transmission eigenvalue.

Besides, being a transmission eigenvalue is a necessary condition for a wave number $k > 0$ to be a non-scattering wave number. And non-scattering inhomogeneities are a little known part of scattering theory and a truly recent area of research, where very modern techniques are being employed.

Therefore, the central motivation of this essay is to give an introduction to the topic, explaining acoustic scattering in an inhomogeneous medium so that we can define what is a transmission eigenvalue and give one of the first important results that was known for this problem: the existence of an infinite discrete set of transmission eigenvalues that accumulates at ∞ if $1 - n$ does not change sign in $\overline{D}$.

Structure of the TFM

This essay is divided in four chapters:

1. The first one is focused on Helmholtz Equation $\Delta u + k^2 u = 0$. We study several properties of this equation that will be fundamental for the remaining parts of the essay.

2. The second chapter is a short one. It gathers the theory of dual systems needed in the following chapter. Specifically, the aim of this chapter is to expose Fredholm's theorem and a theorem due to Lax that allows us to generalize mapping properties in a dual system to another *bigger* dual system (for example, to generalize properties proved for Hölder spaces to properties for Sobolev spaces).

3. The third chapter is devoted to the scattering problem in an inhomogeneous medium (2)-(3). First, we describe the model we are going to work with. Then, we reformulate the scattering problem as an integral equation, known as the *Lippman-Schwinger equation*. We prove existence and uniqueness of a solution via the mapping properties of the volume potential. In order to do this, we will apply Fredholm's theorem, so it is enough to prove uniqueness in order to have existence as well. Uniqueness will be proved via a Unique Continuation Principle.

   Then, we move on to the question of completeness of the far field patterns. As we have already explained, this question is related to the interior transmission problem: a couple of second order equations linked via their boundary conditions on a bounded domain. Values of the wave number for which there exist non-trivial solutions to this problem are called transmission eigenvalues. We state and prove this relation, and define the concept of transmission eigenvalues.

4. Transmission eigenvalues are the core of the last chapter. The objective of the chapter is to prove that, under certain conditions for the refractive index of the media, there exists a countable set of real transmission eigenvalues which accumulate at ∞. In order to do this, we reformulate the transmission eigenvalue problem as a classical eigenvalue problem $(K - \frac{1}{\tau}I)U = 0$ for $\tau > 0$. The operator K is not self-adjoint (although it is compact) and therefore non-standard methods must be used to prove existence of eigenvalues.

   The approach then is the following: we will reformulate the problem as finding the values of τ for which $N(A_\tau - \tau B) \neq \{0\}$ for $\{A_\tau\}_{\tau>0}$ a family of self-adjoint, compact and coercive (or stricly positive) operators and B a self-adjoint, compact and non-negative operator. This problem requires a theory to study the spectral decomposition of a compact, self-adjoint and strictly positive operator A with respect to another compact, self-adjoint, positive operator B, i.e., a generalization of the spectral theory for $A - \lambda I$ when I is substituted by a more general operator B. Then, we apply this theory to prove the existence of transmission eigenvalues aforementioned.

We include an Appendix with the needed background to understand the content of the essay.

References

The main references used for this TFM are the book of Cakoni, Colton and Haddar [7], the book of Colton and Kress [13], the book of Kirsch [19] and the book of Kirsch and Hettlich [21].

Notation

Given $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we denote by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j$$

the Euclidean scalar product of $\mathbb{R}^n$, and by

$$|x| = \sqrt{x \cdot x} = (x_1^2 + \ldots + x_n^2)^{1/2}$$

the Euclidean norm of $\mathbb{R}^n$.

Let f be a real or complex valued function f defined on an open subset of $\mathbb{R}^n$. We will denote its (first) partial derivative with respect to the j-th variable x_j evaluated at the point x by $\partial_j f(x)$, by $\partial_{x_j} f(x)$ or by $\frac{\partial f}{\partial x_j}(x)$. We will denote the m-th partial derivative of f with respect to the j-th variable by $\partial_j^m f$.

For a given multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, we denote by $\partial^{\boldsymbol{\alpha}} f$ the derivative $\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f$. The number $|\boldsymbol{\alpha}|$ indicates the total order of differentiation of $\partial^{\boldsymbol{\alpha}} f$. The space of functions defined on $\mathbb{R}^n$ such that all of its partial derivatives up to order $|\boldsymbol{\alpha}| \leqslant N$ are continuous is denoted by $C^N(\mathbb{R}^n)$ and the space of functions on $\mathbb{R}^n$ which are infinitely differentiable (that is, the functions in $C^N(\mathbb{R}^n)$ for all $N \in \mathbb{N}$) is denoted by $C^\infty(\mathbb{R}^n)$.

We denote $f \in C_0(\mathbb{R}^n)$ if f is continuous and $\lim_{|x| \to +\infty} f(x) = 0$.

We denote that a function f is in $C^N(\mathbb{R}^n)$ ($N \geqslant 0$) and has compact support by $f \in C_c^N(\mathbb{R}^n)$. Analogously, $f \in C_c^\infty$ means that $f \in C^\infty(\mathbb{R}^n)$ and has compact support.

Given $A \subseteq \mathbb{R}^n$, we denote by $\chi_A \colon \mathbb{R}^n \to \mathbb{R}$ the characteristic function of the set A:

$$\chi_A(x) = \begin{cases} 1, & \text{si } x \in A \\ 0, & \text{si } x \notin A \end{cases}.$$

Let $R > 0$, $p \in \mathbb{R}^n$. We denote:

- The open ball of radius R centered at p by

$$B(p, R) := \{x \in \mathbb{R}^3 : |x - p| < R\},$$

- The closed ball of radius R centered at p by

$$B[p, R] := \{x \in \mathbb{R}^3 : |x - p| \leqslant R\},$$

- The sphere of radius R centered at p by

$$S(p, R) := \{x \in \mathbb{R}^3 : |x - p| = R\}.$$

In particular, we denote the unit sphere of $\mathbb{R}^n$ by

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}.$$

Given a surface of class C^2, we will denote its surface measure by ds. In general, the surface will be the boundary of a domain or an sphere $S(p, R)$ for $p \in \mathbb{R}^n$, $R > 0$. In the case of a sphere, we may use $d\sigma$ instead of ds.

Given a linear a linear operator $A\colon X \to Y$ between vector spaces X and Y, we denote its kernel by $N(A)$.

Chapter 1

The Helmholtz equation

In this chapter, we study the Helmholtz equation, establishing some of its basic properties of this equation. This will be important for the study of scattering of acoustic waves in an inhomogeneous medium. We follow mainly Chapter 2 of [13] and Chapter 2 of [21]. We will work on $\mathbb{R}^3$. For a development of some parts of the chapter on arbitrary dimension, see [25], Section 7.6.

1.1 Helmholtz's Equation

The Helmholtz equation models the behavior of time-harmonic acoustic waves travelling in an homogeneous medium (we will see in a moment what does this means). For a careful deduction of this model, see [13], Section 2.1. We present here the basic ideas.

To model the propagation of sound waves in a homogeneous isotropic medium in $\mathbb{R}^3$, suppose we view them as an inviscid fluid. This fluid has velocity field $v = v(x,t)$, pressure $p = p(x,t)$, density $\rho = \rho(x,t)$ and specific entropy $S = S(x,t)$, where $x \in \mathbb{R}^3$, $t > 0$.

When you relate these magnitudes via Euler's equation, the equation of continuity, the state equation and the adiabatic hypothesis, you obtain a system of nonlinear partial differential equations. To simplify the model, it is linearized using small perturbations of the static state $v_0 = 0$, $p_0 =$ constant, $\rho_0 =$ constant and $S_0 =$ constant. After making computations with the linearized equations, eventually you arrive at the fact that there exists a *velocity potential* $U = U(x,t)$ such that

$$v = \frac{1}{\rho_0}\nabla_x U$$

and

$$p = -\frac{\partial U}{\partial t}.$$

This velocity potential satisfies the **wave equation**

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \Delta_x U.$$

We are going to consider solutions which are **time-harmonic acoustic waves**, i.e., functions of the form

$$U(x,t) = \mathrm{Re}\left[u(x)e^{-i\omega t}\right]$$

where $\omega > 0$ is the frequency of the wave, and c is the speed of sound (constant, since we are considering waves in a homogeneous medium). In order for this function to be a solution of the wave equation, the complex valued space dependent part u has to satisfy the **Helmholtz equation** (or reduced wave equation)

$$\Delta u + k^2 u = 0$$

where the **wave number** k is given by the constant $k := \frac{\omega}{c} > 0$ (notice that it is proportional to the frequency ω).

Therefore, to study the propagation of time-harmonic acoustic/sound waves in an homogeneous medium, we just need to study the equation that satisfies the space-dependent part u: the Helmholtz equation.

1.2 Fundamental solution

Although the physical interpretation of the Helmholtz equation only has sense for $k > 0$, many results are still true for $k \in \mathbb{C}$.

In this section, we introduce the most important function in acoustic scattering theory:

Lemma 1.2.1. *For $k \in \mathbb{C}$ the function $\Phi_k : \left\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \neq y\right\} \to \mathbb{C}$, defined by*

$$\Phi_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y,$$

is a solution to Helmholtz equation on the variable x, that is,

$$\Delta_x \Phi_k(x, y) + k^2 \Phi_k(x, y) = 0 \quad \text{for } x \neq y.$$

Remark 1.2.2. It is called the *fundamental solution of the Helmholtz equation*, because it is a fundamental solution to Helmholtz equation: see Theorem 7.33 of [25], pages 269-272 for a proof of this fact. However, we will not use this throughout the essay. We will just use Lemma 1.2.1.

Proof of Lemma 1.2.1. It is enough to show that $\Phi(x) := \frac{e^{ik|x|}}{4\pi|x|}$, $x \neq 0$, satisfies Helmholtz equation. We can write

$$\Phi(x) = \frac{1}{4\pi}\frac{e^{ik|x|}}{|x|} = \frac{1}{4\pi} f(r(x))$$

where $f(r) := \frac{e^{ikr}}{r}$ for $r > 0$ and $r(x) := |x|$, $x \in \mathbb{R}^3$. We then have

$$\begin{aligned}
\partial_i \Phi(x) &= \frac{1}{4\pi} f'(r(x)) \cdot \partial_i r(x) \\
&= \frac{e^{ikr} \cdot ik \cdot r - e^{ikr} \cdot 1}{4\pi r^2} \cdot \frac{x_i}{r} \\
&= \frac{ik|x| \cdot e^{ik|x|}}{4\pi|x|^2} \cdot \frac{x_i}{|x|} - \frac{e^{ik|x|}}{4\pi|x|^2} \cdot \frac{x_i}{|x|} \\
&= \Phi(x) \cdot \frac{ikx_i}{|x|} - \Phi(x) \cdot \frac{x_i}{|x|^2} \\
&= \Phi(x) \cdot \frac{x_i}{|x|} \cdot \left[ik - \frac{1}{|x|}\right]
\end{aligned}$$

So

$$\begin{aligned}
\partial_{ii}\Phi(x) &= \Phi(x)\cdot\frac{x_i}{|x|}\cdot\left[ik-\frac{1}{|x|}\right]\cdot\frac{x_i}{|x|}\cdot\left[ik-\frac{1}{|x|}\right] \\
&\quad+\Phi(x)\cdot\frac{1\cdot|x|-x_i\cdot\frac{x_i}{|x|}}{|x|^2}\cdot\left[ik-\frac{1}{|x|}\right] \\
&\quad+\Phi(x)\cdot\frac{x_i}{|x|}\cdot\left[0-\frac{0\cdot|x|-1\cdot\frac{x_i}{|x|}}{|x|^2}\right] \\
&=\Phi(x)\cdot\left[\frac{x_i^2}{|x|^2}\cdot\frac{[ik|x|-1]^2}{|x|^2}+\frac{|x|^2-x_i^2}{|x|^3}\cdot\frac{ik|x|-1}{|x|}+\frac{x_i^2}{|x|^4}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta\Phi(x) &= \sum_{i=1}^{3}\partial_{ii}\Phi(x) \\
&=\Phi(x)\left[\frac{|x|^2}{|x|^2}\cdot\frac{[ik|x|-1]^2}{|x|^2}+\frac{3|x|^2-|x|^2}{|x|^3}\cdot\frac{ik|x|-1}{|x|}+\frac{|x|^2}{|x|^4}\right] \\
&=\Phi(x)\left[\frac{-k^2|x|^2-2ik|x|+1}{|x|^2}+\frac{2ik|x|-2}{|x|^2}+\frac{1}{|x|^2}\right] \\
&=\Phi(x)\left[\frac{-k^2|x|^2}{|x|^2}\right] \\
&=-k^2\Phi(x).
\end{aligned}$$

□

Remark 1.2.3. In the following, we will not use the subindex k; that is, we will write Φ for Φ_k.

1.3 Representation formulas on bounded domains

In this section we develop a fundamental tool to treat solutions of the Helmholtz equation: Green's representation formula for solutions of the Helmholtz equation on bounded domains. This representation formula, when applied to a solution u of the Helmholtz equation, allows us to describe u in terms of its Dirichlet and Neumann data on the boundary, since the volume integral vanishes in that case. It is a consequence of the following general theorem, valid for every function u sufficiently regular.

Theorem 1.3.1. *[Green's Representation Theorem in the Interior of D] Let D be a bounded domain with C^1 boundary ∂D. For any $k \in \mathbb{C}$ and $u \in C^2(D) \cap C^1(\bar{D})$ we have the representation*

$$\int_{\partial D}\left[\frac{\partial u}{\partial\nu}(y)\Phi(x,y)-u(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right]ds(y)-\int_D\left[\Delta u(y)+k^2u(y)\right]\Phi(x,y)\,dy=u(x),\quad x\in D.$$

Proof. The idea of the proof is, given a point $x \in D$, to remove a neighbourhood of it from the domain so that $\Phi(x, y)$ is of class C^2 in that reduced domain and we can apply Green's Second Formula (5.5). Let us explain the details.

Fix $x \in D$ and, since D is a bounded domain, let $r > 0$ be such that $B[x, r] \subseteq D$. Let $D_r := D\backslash B[x, r]$. Then, $\partial D_r = \partial D \cup S(x, r)$, with exterior unit normal $\nu(y)$ for $y \in \partial D$ and $\frac{x-y}{|y-x|}$ for $y \in S(x, r)$ (since $\frac{y-x}{|y-x|}$ would be the exterior normal to the domain $B[x, r]$).

We apply Green's second identity (5.5) to u and $v(y) := \Phi(x,y)$ in the domain D_r and obtain

$$\begin{aligned}\int_{\partial D_r}\left(\Phi(x,y)\frac{\partial u}{\partial \nu}(y) - u(y)\cdot\frac{\partial \Phi(x,\cdot)}{\partial \nu}(y)\right) ds(y) &= \int_{D_r}\left[\Phi(x,y)\Delta u(y) - u(y)\Delta_y\Phi(x,y)\right] dy\\ &= \int_{D_r}\Phi(x,y)\left[\Delta u(y) + k^2 u(y)\right] dy,\end{aligned}$$

where we have used in the last step that $\Phi(x,\cdot)$ is a solution of the Helmholtz equation (Lemma 1.2.1).

Since $\partial D_r = \partial D \cup S(x,r)$, we can split the integral over ∂D_r in two integrals: one over ∂D and one over $S(x,r)$.

In order to study the integral over $S(x,r)$, we have calculate $\frac{\partial\Phi(x,\cdot)}{\partial\nu(y)}$ knowing that for the domain D_r, $\nu(y) = \frac{x-y}{|y-x|} = \frac{x-y}{r}$ for $y \in \partial S(x,r)$.

We can write $\Phi(x,y) = f(r(y))$ with

$$f(r) := \frac{e^{ikr}}{4\pi r} \quad \text{and} \quad r(y) = |x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}.$$

Therefore, we can apply the chain rule to obtain:

$$\begin{aligned}\nabla_y\Phi(x,y) &= f(r(y))\cdot\nabla_y r(y)\\ &= \frac{ik\cdot e^{ikr}4\pi r - e^{ikr}\cdot 4\pi}{(4\pi r)^2}\cdot\left(\frac{y_1-x_1}{r},\frac{y_2-x_2}{r},\frac{y_3-x_3}{r}\right)\\ &= e^{ikr}\left[\frac{ik\cdot 4\pi r - 4\pi}{(4\pi r)^2}\right]\cdot\frac{y-x}{r}\\ &= \frac{e^{ikr}}{4\pi r}\left[\frac{ik\cdot 4\pi r - 4\pi}{4\pi r}\right]\cdot\frac{y-x}{r}\\ &= \frac{e^{ik|x-y|}}{4\pi|x-y|}\cdot\left[ik - \frac{1}{|x-y|}\right]\cdot\frac{y-x}{|x-y|}\\ &= \frac{e^{ik|x-y|}}{4\pi|x-y|}\cdot\left[\frac{1}{|x-y|} - ik\right]\cdot\nu(y),\end{aligned}$$

(recall that $\nu(y) = \frac{x-y}{|y-x|}$) where we have used that

$$\frac{\partial r}{\partial y_j}(y) = \frac{1}{2\sqrt{(x_1-y_1)^2+\dots}}2(x_j-y_j)\cdot(-1) = \frac{y_j-x_j}{r},$$

and, in consequence, $\nabla_y r(y) = \frac{y-x}{r(y)}$. So, for $|y-x| = r$ we have:

$$\begin{aligned}\Phi(x,y) &= \frac{\exp(ikr)}{4\pi r}\\ \frac{\partial\Phi}{\partial\nu(y)}(x,y) &= \nu(y)\cdot\nabla_y\Phi(x,y)\\ &= \nu(y)\cdot\frac{e^{ik|x-y|}}{4\pi r}\left[\frac{1}{r} - ik\right]\cdot\nu(y)\\ &= \frac{e^{ik|x-y|}}{4\pi r}\left[\frac{1}{r} - ik\right]\end{aligned}$$

Therefore, we compute the integral over $S(x,r)$ as

$$\int_{S(x,r)} \left[\Phi(x,y)\frac{\partial u}{\partial \nu}(y) - u(y) \cdot \frac{\partial \Phi(x,\cdot)}{\partial \nu}(y) \right] ds(y) = \int_{S(x,r)} \left[\frac{e^{ikr}}{4\pi r}\frac{\partial u}{\partial \nu}(y) - u(y) \cdot \left(\frac{1}{r} - ik\right) \cdot \frac{e^{ikr}}{4\pi r} \right] ds(y).$$

We are going to split the last integral into two parts:

$$\int_{S(x,r)} \frac{e^{ikr}}{4\pi r}\frac{\partial u}{\partial \nu}(y)\, ds(y) \quad \text{and} \quad \int_{S(x,r)} u(y) \cdot \left(\frac{1}{r} - ik\right) \cdot \frac{e^{ikr}}{4\pi r}\, ds(y).$$

The first one tends to 0 as $r \to 0^+$, since

$$\begin{aligned} \int_{S(x,r)} \left| \frac{\partial u}{\partial \nu}(y) \cdot \frac{e^{ikr}}{4\pi r} \right| ds(y) &= \frac{1}{4\pi r} \int_{S(x,r)} |\nabla u(y) \cdot \nu(y)|\, ds(y) \\ &\leqslant \frac{1}{4\pi r} \cdot \|\nabla u\|_{L^\infty(\overline{D})} \cdot \sigma(S(x,r)) \\ &= \frac{4\pi r^2}{4\pi r} \cdot \|\nabla u\|_{L^\infty(D)} \xrightarrow{r \to 0} 0, \end{aligned} \tag{1.1}$$

having used that $\sigma(S(x,r)) = 4\pi r^2$.

The second one tends to $u(x)$ as $r \to 0^+$. Let us see why. We have

$$-\frac{e^{ikr}}{4\pi r}\left(\frac{1}{r} - ik\right) \int_{S(x,r)} u(y)\, ds(y) = -\frac{e^{ikr}}{4\pi r^2} \int_{S(x,r)} u(y)\, ds(y) + \frac{e^{ikr}}{4\pi r} \int_{S(x,r)} iku(y)\, ds(y). \tag{1.2}$$

The first term in the right-hand side of (1.2) tends to $u(x)$ as $r \to 0^+$ because, as $\sigma(S(x,r)) = 4\pi r^2$,

$$\begin{aligned} \left| \frac{1}{4\pi r^2} \int_{S(x,r)} u(y)\, ds(y) - u(x) \right| &= \left| \frac{1}{4\pi r^2} \int_{S(x,r)} [u(y) - u(x)]\, ds(y) \right| \\ &\leqslant \frac{1}{4\pi r^2} \int_{S(x,r)} |u(y) - u(x)|\, ds(y) \\ &\leqslant \frac{1}{4\pi r^2} \sigma(S(x,r)) \cdot \sup_{y \in S(x,r)} |u(y) - u(x)| \\ &= \sup_{y \in S(x,r)} |u(y) - u(x)| \xrightarrow{r \to 0^+} 0. \end{aligned}$$

Therefore, since $\frac{1}{4\pi r^2} \int_{S(x,r)} u(y)\, ds(y) \to u(x)$ and $-e^{ikr} \to -1$ when $r \to 0^+$, the product tends to $-u(x)$.

For the second term of (1.2), we have

$$\begin{aligned} \left| \frac{e^{ikr}}{4\pi r} \int_{S(x,r)} iku(y)\, ds(y) \right| &\leqslant \frac{k}{4\pi r} \int_{S(x,r)} |u(y)|\, ds(y) \\ &\leqslant \frac{k}{4\pi r} \sigma(S(x,r)) \cdot \|u\|_{L^\infty(\overline{D})} \\ &= \frac{k}{4\pi r} \cdot 4\pi r^2 \cdot \|u\|_{L^\infty(\overline{D})} \\ &= kr \cdot \|u\|_{L^\infty(\overline{D})} \xrightarrow{r \to 0^+} 0 \end{aligned}$$

where we have used that u is bounded on $\overline{D}$ (since D is a bounded set, so $\overline{D}$ is compact, and u is continuous on $\overline{D}$ by hypothesis, therefore bounded).

So

$$\begin{aligned}
&\int_D \Phi(x,y)[\Delta u(y) + k^2 u(y)]\,dy = \lim_{r\to 0^+} \int_{D_r} \Phi(x,y)[\Delta u(y) + k^2 u(y)]\,dy \\
&= \int_{\partial D} \left(\Phi(x,y)\frac{\partial u(y)}{\partial \nu(y)} - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right) ds(y) + \lim_{r\to 0^+} \int_{S(x,r)} \left(\Phi(x,y)\frac{\partial u(y)}{\partial \nu(y)} - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right) \\
&= \int_{\partial D} \left(\Phi(x,y)\frac{\partial u(y)}{\partial \nu(y)} - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right) ds(y) - u(x),
\end{aligned}$$

obtaining the desired conclusion. □

Remark 1.3.2. - This theorem tells us that, for $x \in D$, any function u can be expressed as a sum of three potentials:

$$\begin{aligned}
(\tilde{\mathcal{S}}\varphi)(x) &= \int_{\partial D} \varphi(y)\Phi(x,y)ds(y), \quad x \notin \partial D \\
(\tilde{\mathcal{D}}\varphi)(x) &= \int_{\partial D} \varphi(y)\frac{\partial \Phi}{\partial \nu(y)}(x,y)ds(y), \quad x \notin \partial D \\
(\mathcal{V}\varphi)(x) &= \int_{D} \varphi(y)\Phi(x,y)dy, \quad x \in \mathbb{R}^3
\end{aligned}$$

which are called **single layer potential, double layer potential**, and **volume potential**, respectively, with density φ.

We note that the volume integral of Green's representation formula vanishes if u is a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ in D. In this case the function u can be expressed solely as a combination of a single and a double layer surface potential. This observation is useful to solve boundary value problems of Helmholtz equation using integral equation methods. We refer the reader to Chapter 3 of [13], Chapters 2 and 3 of [12], and Section 3.1 of [21]. We state this fact in the following theorem.

Corollary 1.3.3. *For any $k \in \mathbb{C}$ and any solution $u \in C^2(D) \cap C^1(\bar{D})$ of the Helmholtz equation $\Delta u + k^2 u = 0$, we have the representation*

$$u(x) = \int_{\partial D} \left[\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y), \quad x \in D. \tag{1.3}$$

1.4 Representation formula for unbounded domains

For solutions of the Helmholtz equation on exterior domains, the following condition is important.

Definition 1.4.1. A solution u to the Helmholtz equation whose domain of definition contains the complement of some ball is called **radiating** if it satisfies the **Sommerfeld radiation condition**

$$\lim_{r\to\infty} r(x)\left(\frac{\partial u}{\partial r}(x) - iku(x)\right) = 0. \tag{1.4}$$

where $r(x) = |x|$ and the limit is assumed to hold uniformly in all directions $x/|x|$.

Remark 1.4.2. Before going into technical details, let us give some motivation about why is this radiation condition asked for solutions of Helmholtz equation in exterior domains.

Consider the spherically symmetric solutions

$$u_1(x) = \frac{e^{ik|x|}}{|x|} \quad \text{and} \quad u_2(x) = \frac{e^{-ik|x|}}{|x|}.$$

Both satisfy the decay estimate $u(x) = O(\frac{1}{|x|})$ when $|x| \to \infty$ (condition known as Sommerfeld's finiteness condition: see [13], page 23). Therefore, this decay condition is not enough to have uniqueness of solutions to the Helmholtz equation on exterior domains.

However, it can be checked that the only of these two functions that satisfies the radiation condition (1.4) is $\frac{e^{ik|x|}}{|x|}$. The time-harmonic wave corresponding to this space-dependent part is

$$\mathrm{Re}\left(\frac{e^{ik|x|-i\omega t}}{|x|}\right) = \frac{\cos(k|x| - \omega t)}{|x|},$$

which physically corresponds to an outgoing wave (see [13], page 18) This is what we want, since the Helmholz equation is related to scattering of electromagnetic waves and, in that context, the scattered waves have to be outgoing (see [21], Section 1.4).

So Sommerfeld's radiation condition is a way to characterize outgoing waves and exclude the ingoing ones. In Physics, radiation conditions are usually imposed to select which type of solution we want (see again [21], page 14).

Although this is the phsyical interpretation, from a mathematical point of view Sommerfeld's radiation condition is a sufficient condition to prove existence and uniqueness of many scattering problems.

We would like to prove a representation formula for radiating solutions on exterior domains similar to the one we proved for interior domains (formula (1.3)). In order to do this, we need to study the asymptotic behavior of the fundamental solution.

Lemma 1.4.3. *For any compact set $K \subseteq \mathbb{R}^3$ the fundamental solution Φ satisfies the Sommerfeld radiation condition* (1.4) *uniformly with respect to $y \in K$. More precisely,*

$$\Phi(x, y) = \frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x}\cdot y} \left[1 + O\left(\frac{1}{|x|}\right)\right], \quad |x| \to \infty \tag{1.5}$$

and

$$\nabla_x \Phi(x, y) = ik\hat{x}\frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x}\cdot y} \left[1 + O\left(\frac{1}{|x|}\right)\right], \quad |x| \to \infty \tag{1.6}$$

uniformly with respect to $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^2$ and $y \in K$.

Proof. Recall from the proof of Lemma 1.2.1 that

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x - y|}$$

and

$$\nabla_x \Phi(x, y) = \Phi(x, y) \cdot \frac{x - y}{|x - y|} \cdot \left[ik - \frac{1}{|x - y|}\right].$$

We are going to need the asymptotic formula

$$|x-y| = |x| - y\cdot x/|x| + O(|x|^{-1}), \quad |x|\to\infty. \tag{1.7}$$

uniformly for y on compact sets. This formula is true because

$$\begin{aligned}
|x-y| &= \sqrt{(x-y)(x-y)}\\
&= \sqrt{|x|^2+|y|^2-2x\cdot y}\\
&= |x|\cdot\sqrt{1+\left(\frac{|y|^2}{|x|^2}-\frac{2x\cdot y}{|x|^2}\right)}\\
&= |x|\cdot\left(1+\frac{\left(\frac{|y|^2}{|x|^2}-\frac{2x\cdot y}{|x|^2}\right)}{2}+O\left(\left(\frac{|y|^2}{|x|^2}-\frac{2x\cdot y}{|x|^2}\right)^2\right)\right)\\
&= |x| - \frac{x\cdot y}{|x|}+\frac{|y|^2}{|x|}+|x|\cdot O\left(\frac{(|y|^2-2x\cdot y)^2}{|x|^4}\right)\\
&= |x| - \frac{x\cdot y}{|x|}+O(|x|^{-1})+O\left(\frac{|y|^2-4xy|y|^2+4(xy)^2}{|x|^3}\right)\\
&= |x| - \frac{x\cdot y}{|x|}+O(|x|^{-1})+O(O(|x|^{-3})+O(|x|^{-2})+O(|x|^{-1}))\\
&= |x| - \frac{x\cdot y}{|x|}+O(|x|^{-1})+O(|x|^{-1})\\
&= |x| - \frac{x\cdot y}{|x|}+O(|x|^{-1})
\end{aligned}$$

where we have used that $O(O(h)) = O(h)$ for a function h and the Taylor's series $\sqrt{1+t} = 1+\frac{t}{2}+O(t^2)$ when $t\to 0$. Notice that we can apply this series on the fourth equality since, as we are considering y on a compact set (therefore bounded), $\frac{|y|^2}{|x|^2}-\frac{2x\cdot y}{|x|^2}\to 0$ when $x\to\infty$.

Now, we are ready to prove (1.5) and (1.6).

Proof of (1.5)
We have, using (1.7), that

$$\begin{aligned}
4\pi\cdot\frac{e^{ik|x-y|}}{4\pi|x-y|}\cdot|x|\cdot e^{-ik|x|}-e^{-ik\hat{x}\cdot y} &= \frac{e^{ik[|x-y|-|x|]}}{|x-y|}\cdot|x|-e^{-ik\hat{x}\cdot y}\\
&= \frac{e^{ik[-\hat{x}\cdot y+O(\frac{1}{|x|})]}}{|x|-\hat{x}\cdot y+O(\frac{1}{|x|})}\cdot|x|-e^{ik\hat{x}\cdot y}\\
&= e^{-ik\hat{x}\cdot y}\cdot\left[\frac{e^{ik\cdot O(\frac{1}{|x|})}}{1-\frac{\hat{x}\cdot y}{|x|}+O(\frac{1}{|x|^2})}-1\right]\\
&= e^{-ik\hat{x}\cdot y}\cdot\left[\frac{e^{ik\cdot O(\frac{1}{|x|})}-\left[1-\frac{\hat{x}\cdot y}{|x|}+O(\frac{1}{|x|})\right]}{1-\frac{\hat{x}\cdot y}{|x|}+O(\frac{1}{|x|^2})}\right]\leqslant C\cdot\frac{1}{|x|}
\end{aligned}$$

when $|x|\to\infty$ because, as $y\in K$ with K compact, the denominator tends to 1 as $|x|\to\infty$, and in the numerator $e^{ikO(\frac{1}{|x|})}\to 1$ when $|x|\to\infty$, so the numerator is bounded by $\leqslant C\cdot\frac{1}{|x|}$.

Proof of (1.6)

Equation (1.6) is equivalent to

$$\nabla_x \Phi(x,y) \cdot \frac{4\pi|x|}{ik} e^{-ik|x|+ik\hat{x}\cdot y} = \hat{x} \cdot \left[1 + O\left(\frac{1}{|x|}\right)\right], \quad |x| \to \infty.$$

So we only need to prove this. Using the value of $\nabla_x \Phi(x,y)$, we have

$$\begin{aligned}
\nabla_x \Phi(x,y) \cdot \frac{4\pi|x|}{ik} \cdot e^{-ik|x|+ik\hat{x}\cdot y} &= \frac{e^{ik|x-y|}}{4\pi|x-y|} \cdot \left[ik - \frac{1}{|x-y|}\right] \cdot \frac{x-y}{|x-y|} \cdot \frac{4\pi|x|}{ik} \cdot e^{-ik|x|+ik\hat{x}\cdot y} \\
&= e^{ik[|x-y|-|x|+\hat{x}\cdot y]} \cdot \left[1 - \frac{1}{|x-y|ik}\right] \cdot \frac{x-y}{|x-y|} \cdot \frac{|x|}{|x-y|} \\
&= e^{ikO(\frac{1}{|x|})} \cdot \frac{x-y}{|x-y|} \cdot \frac{1}{|\hat{x} - \frac{y}{|x|}|} \cdot \left[1 - \frac{1}{|x-y|ik}\right] \\
&= \hat{x} \cdot \frac{e^{ikO(\frac{1}{|x|})}}{|\hat{x} - \frac{y}{|x|}|^2} - \hat{x} \cdot \frac{1}{ik|x-y|} \cdot \frac{e^{ikO(\frac{1}{|x|})}}{|\hat{x} - \frac{y}{|x|}|^2} - y \cdot \frac{1}{|x-y|} \frac{e^{ikO(\frac{1}{|x|})}}{|\hat{x} - \frac{y}{|x|}|} + y \cdot \frac{1}{ik|x-y|^2} \frac{e^{ikO(\frac{1}{|x|})}}{|\hat{x} - \frac{y}{|x|}|} \\
&= \hat{x} - (\hat{x} + y) \cdot \frac{1}{|x-y|} \frac{e^{ikO(\frac{1}{|x|})}}{|\hat{x} - \frac{y}{|x|}|} \cdot [\frac{1}{|\hat{x} - \frac{y}{|x|}|} + 1] + y \cdot O\left(\frac{1}{|x|^2}\right) \\
&= \hat{x} - (\hat{x} + y) O\left(\frac{1}{|x|}\right) + y \cdot O\left(\frac{1}{|x|^2}\right).
\end{aligned}$$

□

Corollary 1.4.4. *The fundamental solution $\Phi(x,y)$, seen as a function of x, satisfies Sommerfeld Radiation Condition* (1.4)*.*

Proof. Using (1.5) and (1.6), we have that (recall that $\hat{x} = \frac{x}{|x|}$

$$\begin{aligned}
r\left(\frac{\partial \Phi(x,y)}{\partial r(x)} - ik\Phi(x,y)\right) &= |x| \left(\hat{x} \cdot \nabla_x \Phi(x,y) - ik\Phi(x,y)\right) \\
&= |x| \left(\hat{x} \cdot \nabla_x \Phi(x,y) - ik\Phi(x,y)\right) \\
&= |x| \left(\hat{x} ik \cdot \hat{x} \frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x}\cdot y} [1 + O(|x|^{-1})] - ik \frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x}\cdot y} [1 + O(|x|^{-1})]\right) \\
&= |x| \left(ik \frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x}\cdot y} O(|x|^{-1}) - ik \frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x}\cdot y} O(|x|^{-1})\right) \\
&= |x| \cdot O(|x|^{-2}) = O(|x|^{-1}), \quad |x| \to \infty.
\end{aligned}$$

□

Corollary 1.4.5. *Let $D \subset \mathbb{R}^3$ be a bounded domain of class C^2, with exterior unit normal ν. Then $\frac{\partial \Phi(x,y)}{\partial \nu(y)}$, as a function of x, is a radiating solution of the Helmholtz equation.*

Proof. We have that

$$\begin{aligned}
\Delta_x \frac{\partial \Phi(x,y)}{\partial \nu(y)} &= \sum_{i=1}^{3} \partial_{x_i x_i} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \\
&= \sum_{i=1}^{3} \partial_{x_i x_i} \left(\sum_{j=1}^{3} \partial_{y_j} \Phi(x,y) \cdot \nu^j(y) \right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \partial_{y_j} (\partial_{x_i x_i} \Phi(x,y)) \nu^j(y) \\
&= \sum_{j=1}^{3} \partial_{y_j} \left(\sum_{i=1}^{3} \partial_{x_i x_i} \Phi(x,y) \right) \nu^j(y) \\
&= \sum_{j=1}^{3} \partial_{y_j} (\Delta_x \Phi(x,y)) \nu^j(y) \\
&= \sum_{j=1}^{3} \partial_{y_j} (-k^2 \Phi(x,y)) \nu^j(y) \\
&= -k^2 \sum_{j=1}^{3} \partial_{y_j} \Phi(x,y) \nu^j(y) \\
&= -k^2 \frac{\partial \Phi(x,y)}{\partial \nu(y)}.
\end{aligned}$$

Therefore, $\tilde{u}(x) = \frac{\partial \Phi(x,y)}{\partial \nu(y)}$ is a solution to Helmholtz equation.
Now, we want to see that $\tilde{u}$ satisfies (1.4). We have that

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial r}(x) &= \sum_{j=1}^{3} \frac{x_j}{|x|} \partial_{x_j} \tilde{u}(x) \\
&= \sum_{j=1}^{3} \frac{x_j}{|x|} \partial_{x_j} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \\
&= \sum_{j=1}^{3} \frac{x_j}{|x|} \partial_{x_j} \left(\sum_{i=1}^{3} \partial_{y_i} \Phi(x,y) \nu^i(y) \right) \\
&= \sum_{j=1}^{3} \sum_{i=1}^{3} \partial_{y_i} \left(\frac{x_j}{|x|} \partial_{x_j} \Phi(x,y) \right) \nu^i(y) \\
&= \sum_{i=1}^{3} \partial_{y_i} \left(\sum_{j=1}^{3} \frac{x_j}{|x|} \partial_{x_j} \Phi(x,y) \right) \nu^i(y) \\
&= \sum_{i=1}^{3} \partial_{y_i} \frac{\partial \Phi(x,y)}{\partial r(x)} \nu^i(y) \\
&= \frac{\partial}{\partial \nu(y)} \frac{\partial \Phi(x,y)}{\partial r(x)}.
\end{aligned}$$

Therefore,

$$\begin{aligned} r\left(\frac{\partial \tilde{u}}{\partial r}(x) - ik\tilde{u}(x)\right) &= r\left(\frac{\partial}{\partial \nu(y)}\frac{\partial \Phi(x,y)}{\partial r(x)} - ik\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right) \\ &= r\frac{\partial}{\partial \nu(y)}\left(\frac{\partial \Phi(x,y)}{\partial r(x)} - ik\Phi(x,y)\right) \\ &= \frac{\partial}{\partial \nu(y)}\left(r\left(\frac{\partial \Phi(x,y)}{\partial r(x)} - ik\Phi(x,y)\right)\right). \end{aligned}$$

Since, by the previous corollary, Φ satisfies Sommerfeld Radiation Condition (1.4), we have that

$$r \cdot \left(\frac{\partial \Phi(x,y)}{\partial r(x)} - ik\Phi(x,y)\right) \xrightarrow{r\to\infty} 0$$

uniformly for $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^2$. Thus, the radiation condition is also true for $\tilde{u}$, as we wanted to prove. □

Now, we can prove the following theorem, which states that the radiation condition is equivalent to an integral radiation condition and is also equivalent to Green's representation formula for exterior domains.

Theorem 1.4.6. *Let Ω be an exterior domain, i.e., $\Omega = \mathbb{R}^3 \backslash \overline{D}$ with D a bounded domain of class C^2. Let u be a solution of Helmholtz equation in Ω such that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.*

Then, the following conditions are equivalent:

1. *(Uniform Sommerfeld Radiation Condition)*

$$\frac{\partial u}{\partial r}(x) - ik \cdot u(x) = o(r^{-1})$$

when $r = |x| \to \infty$ uniformly on $\hat{x} = \frac{x}{r} \in \mathbb{S}^2$.

2. *(Sommerfeld Radiation Condition on L^2)*

$$\lim_{R\to\infty} \int_{S(0,R)} \left|\frac{\partial u}{\partial r} - iku\right|^2 ds_R = 0$$

3. *(Green's representation formula for unbounded domains)*

$$u(x) = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}(y) \cdot \Phi(x,y) - u(y) \cdot \frac{\partial \Phi(x,\cdot)}{\partial \nu}(y)\right) ds(y) \tag{1.8}$$

for all $x \in \Omega$, where ν is the interior unit normal of $\partial\Omega$.

Remark 1.4.7. One needs to be careful with Green's representation formula on exterior domains. If we take ν to be the interior unit normal of the exterior domain, then the representation formula is the same as for interior domains. However, if we take ν to be the exterior unit normal of $\partial\Omega$ (as is usually the case), we have to change the sign of the right-hand side of (1.8), and therefore the representation formula is different to that of the interior domain.

Proof of Theorem 1.4.6. $\boxed{1 \implies 2}$

Suppose that

$$\frac{\partial u}{\partial r}(x) - iku(x) = o(r^{-1})$$

when $r = |x| \to \infty$ uniformly in $\hat{x} = \frac{x}{r} \in \mathbb{S}^2$.

That is,

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}(x) - ik\cdot u(x)\right) = 0.$$

So, for every $\varepsilon > 0$ there exists $R_0 > 0$ such that if $r \geqslant R_0$, then

$$r\left|\frac{\partial u}{\partial r}(x) - ik\cdot u(x)\right| < \sqrt{\varepsilon}$$

for every x with $|x| = r$.

Therefore

$$\begin{aligned}\int_{S(0,r)} \left|\frac{\partial u}{\partial r} - iku\right|^2 ds_r &\leqslant \int_{S(0,r)} \frac{\varepsilon}{r^2}\, ds_r \\ &= \varepsilon\frac{1}{r^2}\sigma(S(0,r)) \\ &= \varepsilon\frac{1}{r^2}r^2\sigma(\mathbb{S}^2) \\ &= \varepsilon\sigma(\mathbb{S}^2)\end{aligned}$$

having used that, if we make the change of variables $y = \frac{x}{r} \implies r^2 dy = dx$, we have that

$$\sigma(S(0,r)) = \int_{S(0,r)} 1\, ds_r(x) = \int_{S(0,1)} r^2\, ds_1(y) = r^2\sigma(S(0,1))$$

being $S(0,1) = \mathbb{S}^{n-1}$.

$\boxed{2 \implies 3}$

Let $R > 0$ be sufficiently big so that $D \subset B(0,R)$ and consider

$$\Omega_R := \Omega \cap B(0,R).$$

Fix $x \in \Omega$.

Since Ω_R is a bounded domain on which u is a solution to Helmholtz equation (by hypothesis of the theorem), we can apply Theorem 1.4.3 which gives us Green's representation formula for solutions of Helmholtz equation on bounded domains

$$u(x) = \left(\int_{\partial\Omega} + \int_{S(0,R)}\right)\left(\frac{\partial u}{\partial \nu}(y)\cdot\Phi(x,y) - u(y)\cdot\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right) ds(y),$$

where ν is the exterior unit normal of the interior domain Ω_R (and therefore, on $\partial\Omega$ is the interior unit normal with respect to Ω).

To obtain condition 3., it is enough to prove that the integral over the sphere tends to 0 when $R \to \infty$. The radiation condition on L^2 given by 2. can be written as

$$\int_{S(0,R)}\left(\left|\frac{\partial u}{\partial\nu}\right|^2 + k^2|u|^2\right) ds + 2k\int_{S(0,R)} \mathrm{Im}\left(u\cdot\frac{\partial\overline{u}}{\partial\nu}\right) ds \xrightarrow{R\to\infty} 0, \tag{1.9}$$

since

$$\begin{aligned}\left|\frac{\partial u}{\partial r} - iku\right|^2 &= \left|\frac{\partial u}{\partial \nu}\right|^2 + k^2|u|^2 - 2\operatorname{Re}\left(iku \cdot \overline{\frac{\partial u}{\partial r}}\right)\\ &= \left|\frac{\partial u}{\partial \nu}\right|^2 + k^2|u|^2 + 2\operatorname{Im}\left(ku \cdot \overline{\frac{\partial u}{\partial r}}\right)\\ &= \left|\frac{\partial u}{\partial \nu}\right|^2 + k^2|u|^2 + 2k\operatorname{Im}\left(u \cdot \frac{\partial \overline{u}}{\partial r}\right).\end{aligned}$$

having used:

- On the first step, that

$$\begin{aligned}|z-w|^2 &= (z-w)\overline{z-w} = (z-w)(\overline{z}-\overline{w}) = z\cdot\overline{z} - z\cdot\overline{w} - w\cdot\overline{z} + w\cdot\overline{w}\\ &= |z|^2 + |w|^2 - z\overline{w} - w\overline{z} = |z|^2+|w|^2 - (z\overline{w}+\overline{z}w) = |z|^2+|w|^2 - 2\operatorname{Re}(z\cdot\overline{w}).\end{aligned}$$

- On the second step, that given $z \in \mathbb{C}$, $z = a + bi$ with $a, b \in \mathbb{R}$, then

$$iz = i(a+bi) = -b + ia \implies \operatorname{Re}(iz) = -b = -\operatorname{Im}(z).$$

- On the third step, that k is real, so

$$\overline{\frac{\partial u}{\partial r}} = k \cdot \operatorname{Im}\left(u \cdot \frac{\partial \overline{u}}{\partial r}\right).$$

We have that

$$\begin{aligned}\int_{S(0,R)} \operatorname{Im}\left(u\cdot\frac{\partial\overline{u}}{\partial\nu}\right) ds &= \operatorname{Im}\left(\int_{S(0,R)} u\cdot\frac{\partial\overline{u}}{\partial\nu}\, ds\right)\\ &= \operatorname{Im}\left(-\int_{\partial\Omega} u\cdot\frac{\partial\overline{u}}{\partial\nu}\, ds + \int_{\Omega_R} u\cdot\Delta\overline{u} + \nabla u\cdot\nabla\overline{u}\right)\\ &= \operatorname{Im}\left(-\int_{\partial\Omega} u\cdot\frac{\partial\overline{u}}{\partial\nu}\, ds + \int_{\Omega_R} u\cdot(-k^2\overline{u}) + |\nabla u|^2\right)\\ &= \operatorname{Im}\left(-\int_{\partial\Omega} u\cdot\frac{\partial\overline{u}}{\partial\nu}\, ds - k^2\int_{\Omega_R}[|u|^2+|\nabla u|^2]\,dx\right)\\ &= \operatorname{Im}\left(-\int_{\partial\Omega} u\cdot\frac{\partial\overline{u}}{\partial\nu}\, ds\right)\\ &= -\int_{\partial\Omega} \operatorname{Im}\left(\operatorname{Im}\left(u\cdot\frac{\partial\overline{u}}{\partial\nu}\right)\right) ds\end{aligned}$$

That is,

$$\int_{S(0,R)} \operatorname{Im}\left(u\cdot\frac{\partial\overline{u}}{\partial\nu}\right) ds = -\int_{\partial\Omega} \operatorname{Im}\left(u\cdot\frac{\partial\overline{u}}{\partial\nu}\right) ds.$$

That is, $\int_{S(0,R)} \operatorname{Im}\left(u\cdot\frac{\partial\overline{u}}{\partial\nu}\right) ds$ does not depend on R (for R sufficiently big so that $D \subset B(0,R)$).

So (1.9) can be written as

$$\lim_{R\to\infty}\int_{S(0,R)}\left(\left|\frac{\partial u}{\partial\nu}\right|^2+k^2\cdot|u|^2\right)ds=2k\cdot\int_{\partial\Omega}\operatorname{Im}\left(u\cdot\frac{\partial\overline{u}}{\partial\nu}\right)ds. \tag{1.10}$$

Since $\left|\frac{\partial u}{\partial\nu}\right|^2\geqslant 0$ y $k^2\geqslant 0$, the fact that the limit exists and is finite means that $\lim_{R\to+\infty}\int_{S(0,R)}k^2\cdot|u|^2\,ds$ is bounded, that is,

$$\int_{S(0,R)}|u|^2\,ds=O(1)\quad\text{ when } R\to\infty.$$

Recall that we wanted to prove that

$$\lim_{R\to\infty}\int_{S(0,R)}\left(\frac{\partial u}{\partial\nu}(y)\cdot\Phi(x,y)-u(y)\cdot\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right)ds(y)=0.$$

Let

$$I_R:=\int_{S(0,R)}\left(\frac{\partial u}{\partial\nu}(y)\cdot\Phi(x,y)-u(y)\cdot\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right)ds(y).$$

Let us write I_R as $I_R=I_1(R)-I_2(R)$ where

$$I_1(R):=\int_{S(0,R)}\left(\frac{\partial u}{\partial\nu}(y)-ik\cdot u(y)\right)\cdot\Phi(x,y)\,ds(y)$$

and

$$I_2(R):=\int_{S(0,R)}\left(\frac{\partial\Phi(x,y)}{\partial\nu(y)}-ik\cdot\Phi(x,y)\right)\cdot u(y)\,ds(y).$$

We are going to check that both integrals tend to 0 as $R\to\infty$.

1. $$(I_1)^2\leqslant\int_{S(0,R)}\left|\frac{\partial u}{\partial r}-iku\right|^2 ds\cdot\int_{S(0,R)}|\Phi(x,y)|^2\,ds(y)=o(1)O(1)\text{ when } R\to\infty.$$

   So $I_1\to 0$ when $R\to\infty$.

2. $$(I_2)^2\leqslant\int_{S(0,R)}|u(y)|^2\,ds\cdot\int_{S(0,R)}\left|\frac{\partial\Phi(x,y)}{\partial r}-ik\cdot\Phi(x,y)\right|^2 ds(y)=O(1)\cdot o(1).$$

   So $I_2\to 0$ when $R\to\infty$.

Therefore, $I_R=I_1(R)-I_2(R)\to 0$ when $R\to\infty$.
So

$$u(x)=u(x)=\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}(y)\cdot\Phi(x,y)-u(y)\cdot\frac{\partial\Phi(x,\cdot)}{\partial\nu}(y)\right)ds(y),\quad x\in\Omega.$$

$\boxed{3\implies 1}$
Suppose that

$$u(x)=\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}(y)\cdot\Phi(x,y)-u(y)\cdot\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right)ds(y).$$

Differentiating under the integral sign we have that

$$\begin{aligned}
&r\left(\frac{\partial u}{\partial r}(x) - iku(x)\right) \\
&= r\Bigg[\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}(y)\cdot\frac{\partial \Phi}{\partial r(x)}(x,y) - u(y)\frac{\partial}{\partial r(x)}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right)\,ds(y) \\
&\quad - ik\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}(y)\Phi(x,y) - u(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right)\,ds(y)\Bigg] \\
&= \int_{\partial\Omega}\left[\frac{\partial u}{\partial\nu}(y)\cdot r\left[\frac{\partial\Phi}{\partial r(x)}(x,y) - ik\Phi(x,y)\right] - u(y)\cdot r\left[\frac{\partial}{\partial r(x)}\frac{\partial\Phi(x,y)}{\partial\nu(y)} - ik\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right]\right]\,ds(y)
\end{aligned}$$

We are going to check that

$$\lim_{r\to\infty} r\left(\frac{\partial\Phi}{\partial r(x)}(x,y) - ik\Phi(x,y)\right) = 0 \tag{1.11}$$

and

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r(x)}\frac{\partial\Phi(x,y)}{\partial\nu(y)} - ik\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right] = 0 \tag{1.12}$$

uniformly in $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^2$ for y on compact sets. By the Dominated Convergence Theorem, this will end the proof.

But (1.11) is true by Corollary 1.4.4, because it states that Φ is a radiating solution and, therefore, it satisfies Sommerfeld's Radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial\Phi}{\partial r(x)}(x,y) - ik\Phi(x,y)\right) = 0$$

uniformly in $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^2$ for y on compact sets.

And (1.12) is true by Corollary 1.4.5, because it asserts that $\frac{\partial\Phi(x,y)}{\partial\nu(y)}$ is a radiating solution to Helmholtz equation, so it satisfies Sommerfeld's radiation condition, which is exactly (1.12).

So by the Dominated Convergence Theorem,

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}(x) - ik\cdot u(x)\right) = 0$$

uniformly in $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^2$ for y on compact sets. □

Definition 1.4.8. A solution u to Helmholtz equation defined in all of $\mathbb{R}^3$ is called an *entire solution*.

Notice that an entire solution to the Helmholtz equation of class $C^2(\mathbb{R}^3)$ which satisfies the radiation condition must vanish identically, because considering it as a solution on $\mathbb{R}^3\backslash B(0,r)$ we have, by (1.8) (which we can apply because the solution is radiating) and (5.5) (taking ν to be the unit normal vector to the boundary $S(0,r)$ directed to the exterior of $\mathbb{R}^3\backslash B(0,r)$ and therefore to the interior of $B(0,r)$), that

$$\begin{aligned}
u(x) &= \int_{S(0,r)}\left[u(y)\cdot\frac{\partial\Phi(x,y)}{\partial\nu(y)} - \frac{\partial u}{\partial\nu}(y)\Phi(x,y)\right]\,ds(y) \\
&= -\int_{B(0,r)}(u(y)\Delta_y\Phi(x,y) - \Phi(x,y)\Delta u(y))\,dy \\
&= \int_{B(0,r)}[u(y)(-k^2\Phi(x,y)) - \Phi(x,y)(-k^2u(y))]\,dy = 0
\end{aligned}$$

for all $x \in \mathbb{R}^3 \backslash B[0, R]$, where the minus sign on the penultimate expression is due to the fact that ν is directed into the interior of $B(0, r)$ (instead of the exterior). Notice that we can apply (5.5) because $u \in C^2(\mathbb{R}^3)$ and $\Phi(x, y)$ as a function of y is in $C^2(\mathbb{R}^3 \backslash \{x\})$, in particular, it is in $C^2(B[0, R])$ which is what we need to apply (5.5).

Since we have this for all $R > 0$, we have that $u \equiv 0$ for all $x \in \mathbb{R}^3 \backslash \{0\}$. The continuity of u implies that $u \equiv 0$ on $\mathbb{R}^3$.

Remark 1.4.9. To make the previous reasoning, we had to apply Green's representation formula for exterior domains. If we try to apply Green's representation formula for interior domains instead, let us say for $x \in B(0, r)$, the function Φ does not satisfy the necessary hypothesis to apply Green's Second Formula (5.5), since $\Phi(x, y)$ as a function of y is not even continuous at $y = x$, so it is not $C^2(B(0, r))$.

1.5 Separation of variables in Helmholtz equation

In this section, we study the series expansion of solutions of the Helmholtz equation $\Delta u + k^2 u = 0$ inside of balls (i.e., for spherically symmetric media). For a comprehensive and clear treatment of this subject, we refer to Chapter 2 of [21], where a comparative study of the Laplace equation and the Helmholtz equation is made.

We are more interested in the ideas used in the proof of the following result than in the statement itself, because we will use them later (specifically, in Rellich's Lemma 1.7.1).

Theorem 1.5.1. *Let $k \in \mathbb{C} \backslash \{0\}$ with $\mathrm{Im}(k) \geqslant 0$ and $R > 0$ and $u \in C^2(B(0, R))$ solve the Helmholtz equation $\Delta u + k^2 u = 0$ in $B(0, R)$. Then there exist unique $\alpha_n^m \in \mathbb{C}$, $|m| \leqslant n$, $n = 0, 1, 2, \ldots$ such that*

$$u(x) = u(r\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_n^m j_n(kr) Y_n^m(\hat{x}), \qquad 0 \leqslant r < R, \hat{x} \in \mathbb{S}^2.$$

The series converges uniformly with all of its derivatives in every closed ball $B[0, R']$ with $R' < R$.

Proof. We have that $u_r(\hat{x}) := u(r\hat{x}) \in L^2(\mathbb{S}^2)$ since $u \in C^2(B(0, R))$, so it is bounded. Therefore, we can use Theorem 5.3.7 to expand u_r in a series of spherical harmonics

$$u_r(\hat{x}) = u(r\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_n^m(r) Y_n^m(\hat{x}), \quad \hat{x} \in \mathbb{S}^2.$$

where the coefficients $u_n^m(r)$ are given by

$$u_n^m(r) = (u_r, Y_n^m)_{L^2(\mathbb{S}^2)}.$$

Claim: the coefficients u_n^m satisfy the spherical Bessel differential equation. To see this, we use the Helmholtz equation for u in spherical polar coordinates, and that the functions Y_n^m are eigenfunctions of the self-adjoint Laplace-Beltrami operator with corresponding eigenvalue $n(n+1)$.

Recall that the laplacian in polar coordinates takes the following expression

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \Delta_{\mathbb{S}^2} =: \Delta_r + \frac{1}{r^2} \cdot \Delta_{\mathbb{S}^2}.$$

Since u is solution of the Helmholtz equation, $\Delta u = \Delta_r u + \frac{1}{r^2}\Delta_{\mathbb{S}^2} u = -k^2 u \iff \Delta_r u = -\frac{1}{r^2}\Delta_{\mathbb{S}^2} u - k^2 u$. Therefore, differentiating under the integral, we have

$$\begin{aligned}
\Delta_r u_n^m(r) = \Delta_r \int_{\mathbb{S}^2} u(r\hat{x}) \cdot \overline{Y_n^m(\hat{x})}\, ds(\hat{x}) = \\
&= \int_{\mathbb{S}^2} \Delta_r u(r\hat{x}) \cdot \overline{Y_n^m(\hat{x})}\, ds = \\
&= -\frac{1}{r^2}\int_{\mathbb{S}^2} \Delta_{\mathbb{S}} u(r\hat{x}) \cdot \overline{Y_n^m(\hat{x})}\, ds - k^2 \int_{\mathbb{S}^2} u(r\hat{x}) \cdot \overline{Y_n^m(\hat{x})}\, ds = \\
&= -\frac{1}{r^2}\int_{\mathbb{S}^2} u(r\hat{x}) \cdot \overline{\Delta_{\mathbb{S}^2} Y_n^m(\hat{x})}\, ds - k^2 u_n^m(r) = \\
&= \frac{1}{r^2}\int_{\mathbb{S}^2} n(n+1) \cdot u(r\hat{x}) \cdot \overline{Y_n^m(\hat{x})}\, ds - k^2 u_n^m(r) = \\
&= u_n^m(r) \cdot \left[\frac{1}{r^2} n(n+1) - k^2\right]
\end{aligned}$$

having used in that $\Delta_{\mathbb{S}^2}$ is self-adjoint in $L^2(\mathbb{S}^2)$ and that $\Delta_S Y_n^m = -n(n+1)Y_n^m$ (see Theorem 5.3.8 of the Appendix).

That is, u_n^m satisfies Bessel's spherical differential equation (5.12). Therefore, the general solution is given by

$$u_n^m(r) = \alpha_n^m j_n(kr) + \beta_n^m y_n(kr)$$

for certain α_n^m, β_n^m. Therefore, u has the form

$$u(r\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[\alpha_n^m j_n(kr) + \beta_n^m y_n(kr)\right] Y_n^m(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, r > 0. \tag{1.13}$$

The convergence of the series (1.13) is in $L^2(\mathbb{S}^2)$ for each $r > 0$ fixed.

We are interested in smooth solutions in the ball $B(0, R)$. Therefore, $\beta_n^m = 0$ (because y_n has a pole in 0).

The proof of the uniform convergence requires the use of some properties of the Legendre polynomials. Check Theorem 2.33 of [21], pages 62-64, for a proof. □

1.6 The far field pattern

We are now in a position to introduce the definition of the far field pattern (or scattering amplitude) of a radiating solution. It is a function defined on the unit sphere $\mathbb{S}^2$ that gathers the asymptotical information of the corresponding radiating solution, and it just depends on the values of the solution on the boundary of the exterior domain, and on the shape of the boundary (because it depends on the unit normal ν).

The *far field pattern*, as we will explain in Chapter 3, is quite fundamental in Scattering Theory.

Theorem 1.6.1. *Let D de a bounded domain of class C^2 and let u be a radiating solution of Helmholtz equation in the exterior domain $\mathbb{R}^3 \backslash \overline{D}$, with $u \in C^2(\mathbb{R}^3 \backslash \overline{D}) \cap C^1(\mathbb{R}^3 \backslash D)$. Then u has the asymptotic behavior of an outgoing spherical wave*

$$u(x) = \frac{e^{ik|x|}}{|x|}\left[u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right], \quad |x| \to \infty \tag{1.14}$$

uniformly in all directions $\hat{x} = x/|x|$ where the function u_∞ defined on the unit sphere $\mathbb{S}^2$ is known as the far field pattern of u, and is given by

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\left[u(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial v(y)} - \frac{\partial u}{\partial v}(y)e^{-ik\hat{x}\cdot y}\right] ds(y), \quad \hat{x}\in\mathbb{S}^2, \tag{1.15}$$

where ν denotes the unit normal vector to the boundary ∂D directed into the exterior of D.

Proof. From (1.5), we have

$$\Phi(x,y) = \frac{e^{ik|x|}}{4\pi|x|}\left[e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right] \tag{1.16}$$

and from (1.6), we have

$$\begin{aligned}\frac{\partial\Phi(x,y)}{\partial\nu(y)} &= -[-ike^{-ik\hat{x}\cdot y}\hat{x}]\frac{e^{ik|x|}}{4\pi|x|}\left[1+O\left(\frac{1}{|x|}\right)\right]\cdot\nu(y)\\ &= \nabla_y e^{-ik\hat{x}\cdot y}\cdot\nu(y)\cdot\frac{e^{ik|x|}}{4\pi|x|}\left[1+O\left(\frac{1}{|x|}\right)\right]\\ &= \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)}\cdot\frac{e^{ik|x|}}{4\pi|x|}\left[1+O\left(\frac{1}{|x|}\right)\right]\end{aligned} \tag{1.17}$$

uniformly for all $y\in\partial D$ (since ∂D is closed and bounded, therefore compact). Inserting (1.16)-(1.17) into Green's representation formula for unbounded domains (1.8) (taking ν to be the unit normal vector to the boundary ∂D directed into the exterior of D), we obtain

$$\begin{aligned}u(x) &= \int_{\partial D}\left[u(y)\cdot\frac{\partial\Phi(x,y)}{\partial\nu(y)} - \frac{\partial u}{\partial\nu}(y)\Phi(x,y)\right] ds(y)\\ &= \int_{\partial D}\left[u(y)\cdot\frac{e^{ik|x|}}{4\pi|x|}\left[\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)} + O\left(\frac{1}{|x|}\right)\right] - \frac{\partial u}{\partial\nu}(y)\frac{e^{ik|x|}}{4\pi|x|}e^{-ik\hat{x}\cdot y}\left[1+O\left(\frac{1}{|x|}\right)\right]\right] ds(y)\\ &= \frac{e^{ik|x|}}{|x|}\cdot\frac{1}{4\pi}\int_{\partial D}\left[u(y)\cdot\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)} - \frac{\partial u}{\partial\nu}(y)\cdot e^{-ik\hat{x}\cdot y}\right] ds(y)\\ &\quad+ \frac{e^{ik|x|}}{|x|}\cdot\frac{1}{4\pi}\int_{\partial D}\left[u(y)\cdot\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)} - \frac{\partial u}{\partial\nu}(y)\cdot e^{-ik\hat{x}\cdot y}\right] ds(y)\cdot O\left(\frac{1}{|x|}\right)\\ &= \frac{e^{ik|x|}}{|x|}\left[u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right],\end{aligned}$$

as we wanted to prove. □

Remark 1.6.2. One of the main problems in scattering theory is to recover radiating solutions of the Helmholtz equation from a knowledge of their far field patterns. Given the mapping $A\colon u\mapsto u_\infty$ transferring a radiating solution u into its far field pattern u_∞, an important question is to solve the equation $Au = u_\infty$ for a given u_∞. The operator A is smoothing, since we can pass as many derivatives as we want under the integral sign in expression (1.15).

We will not deal with this problem. See Section 7.5 of [19] or the book [20] for a treatment of the subject.

1.7 Rellich's Uniqueness Theorem

In this section we are going to state and prove Rellich's Uniqueness Lemma and some of its direct consequences. The most important one of them is that a radiating solution to the Helmholtz equation is uniquely determined by its far field pattern.

The proof of Rellich's Lemma uses arguments similar to those in the proof of Theorem 1.5.1 plus some asymptotic properties of the spherical Bessel and Hankel functions which are presented in section 5.4 of the appendix.

Lemma 1.7.1. *[Rellich's Lemma] Let $u \in C^2\left(\mathbb{R}^3 \backslash B\left[0, R_0\right]\right)$ be a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $|x| > R_0$ and wave number $k \in \mathbb{R}_{>0}$ such that*

$$\lim_{R \to \infty} \int_{|x|=R} |u|^2 ds = 0.$$

Then u vanishes for $|x| > R_0$.

Proof. The general solution of the Helmholtz equation in the exterior of $B\left(0, R_0\right)$ is given by (1.13) (instead of the pair of linearly independent solutions j_n and y_n, we consider the pair j_n and $h_n^{(1)}$); that is,

$$u(r\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[a_n^m h_n^{(1)}(kr) + b_n^m j_n(kr)\right] Y_n^m(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, r > R_0,$$

for some $a_n^m, b_n^m \in \mathbb{C}$. The spherical harmonics $\left\{Y_n^m : |m| \leqslant n, n \in \mathbb{N}_0\right\}$ form an orthogonal system of $L^2(\mathbb{S}^2)$. Therefore, Parseval's theorem yields, defining $u_r(\hat{x}) := u(r\hat{x})$ for $\hat{x} \in \mathbb{S}^2$,

$$\|u_r\|_{L^2(\mathbb{S}^{n-1})} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a_n^m h_n^{(1)}(kr) + b_n^m j_n(kr)|^2$$

where

$$\|u_r\|_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} |u_r(\hat{x})|^2 \, ds(\hat{x}) = \int_{\mathbb{S}^2} |u(r\hat{x})|^2 \, ds(\hat{x}) = \int_{S(0,r)} |u(y)|^2 \cdot \frac{1}{r^2} \, ds(y)$$

having done in the last step the change of variables $y = r\hat{x}$ that takes $\hat{x} \in S(0,1)$ to $y \in S(0,r)$, being the determinant of the jacobian of the inverse mapping $\frac{1}{r^2}$.

That is,

$$r^2 \cdot \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m h_n^{(1)}(kr) + b_n^m j_n(kr)\right|^2 = \int_{S(0,r)} |u(y)|^2 \, ds(y).$$

The assumption of the theorem says precisely that the right-hand side tends to 0 when $r \to \infty$. Therefore, the left-hand side tends to 0 as well when $r \to \infty$. In particular, for every fixed $n \in \mathbb{N}_0$ and m with $|m| \leqslant n$, we have that

$$r^2 \left|a_n^m h_n^{(1)}(kr) + b_n^m j_n(kr)\right|^2 \xrightarrow{r \to \infty} 0.$$

Defining $c_n^m = a_n^m + b_n^m$ and using that $h_n^{(1)} = j_n + i \cdot y_n$, we can write it as

$$r \cdot \left[i a_n^m y_n(kr) + c_n^m j_n(kr)\right] \xrightarrow{r \to \infty} 0.$$

Now we use the asymptotic behavior of $j_n(kr)$ and $y_n(kr)$ as r tends to infinity. By Theorem 5.4.4,

$$h_n^{(1)}(z) = \frac{\exp\left(i(z - \frac{\pi}{2}(n+1))\right)}{z} \cdot \left[1 + O\left(\frac{1}{|z|}\right)\right] \quad |z| \to \infty,$$

uniformly with respect to $\frac{z}{|z|}$.

Since $j_n = \mathrm{Re}(h_n^{(1)})$ and $y_n = \mathrm{Im}(h_n^{(1)})$, we have that

$$kr \cdot y_n(kr) = kr \cdot \mathrm{Im}(h_n^{(1)}(kr))$$

and

$$kr \cdot j_n(kr) = kr \cdot \mathrm{Re}(h_n^{(1)}(kr)).$$

Therefore, using the above asymptotic expression from Theorem 5.4.4, we have that

$$\begin{aligned} 0 &= \lim_{r\to\infty}[i \cdot a_n^m \cdot kry_n(kr) + c_n^m \cdot kr \cdot j_n(kr)] \\ &= i \cdot a_n^m \cdot \mathrm{Im}\left[\exp\left(i\left(kr - \frac{\pi}{2}(n+1)\right)\right)\right] + c_n^m \cdot \mathrm{Re}\left[\exp\left(i\left(kr - \frac{\pi}{2}(n+1)\right)\right)\right] + O\left(\frac{1}{r}\right). \end{aligned}$$

Therefore, the only possibility is that

$$\lim_{r\to\infty}\left[ia_n^m \cdot \mathrm{Im}(\exp(i \cdot \ldots)) + c_n^m \cdot \mathrm{Re}(i \cdot \ldots)\right] = 0.$$

Notice that

$$\exp\left(ikr - i\frac{\pi}{2}(n+1)\right) = e^{ikr} \cdot e^{-i\frac{\pi}{2}(n+1)} = e^{ikr} \cdot (-i)^{n+1}.$$

The values of $(-i)^{n+1}$ alternate between ± 1 and $\pm i$. Therefore, we have to consider two cases.

- If $(-i)^{n+1} = \pm i$, then $\mathrm{Im}(e^{ikr} \cdot (\pm i)) = \pm\cos(kr)$ and $\mathrm{Re}(e^{ikr} \cdot (\pm i)) = \mp\sin(kr)$ (notice that they have opposite signs). Therefore, in this case we have

$$\pm[ia_n^m \cos(kr) - c_n^m \sin(kr)] \xrightarrow{r\to\infty} 0.$$

- If $(-i)^{n+1} = \pm 1$, then $\mathrm{Im}(e^{ikr} \cdot (\pm i)) = \pm\sin(kr)$ and $\mathrm{Re}(e^{ikr} \cdot (\pm i)) = \pm\cos(kr)$ (notice that they have the same sign). Therefore, in this case we have

$$\pm[ia_n^m \sin(kr) + c_n^m \cos(kr)] \xrightarrow{r\to\infty} 0.$$

In any of the two cases, we can choose sequences $\{r_j\}_{j\in\mathbb{N}}$ such that $r_j \to \infty$ when $j \to \infty$ in such a way that $\sin(kr_j) = 1$ and $\cos(kr_j) = 0$ and in such a way that $\sin(kr_j) = 0$ and $\cos(kr_j) = 1$. Therefore, we conclude that $a_n^m = 0 = c_n^m$. So we also have $b_n^m = 0$.

This reasoning is valid for all n, m. Therefore, $u \equiv 0$, as we wanted to prove. □

Theorem 1.7.2. *Let D be a bounded domain with boundary ∂D of class C^2. Let v be the unit normal directed into the exterior of D, and assume $u \in C^2\left(\mathbb{R}^3\backslash\bar{D}\right) \cap C\left(\mathbb{R}^3\backslash D\right)$ is a radiating solution to the Helmholtz equation with wave number $k > 0$ for which*

$$\mathrm{Im}\int_{\partial D} u\frac{\partial\bar{u}}{\partial v}\,ds \geqslant 0.$$

Then $u = 0$ in $\mathbb{R}^3\backslash\bar{D}$

Proof. From the identity (1.10) and the assumption of the theorem, we conclude that the hypothesis of Rellich's Lemma 1.7.1 is satisfied. Hence, the theorem follows. □

Applying Rellich's lemma we can now prove that the far field pattern uniquely determines a radiating solution (the converse is also true because the far field pattern is defined for a given radiating solution, so Rellich's lemma establishes the one-to-one correspondence between radiating waves and their far field patterns).

Theorem 1.7.3. *Let D be as in Theorem 1.7.2 and let $u \in C^2\left(\mathbb{R}^3 \backslash \bar{D}\right)$ be a radiating solution to the Helmholtz equation for which the far field pattern vanishes identically. Then $u = 0$ in $\mathbb{R}^3 \backslash \bar{D}$.*

Proof. Since from (1.14) we deduce

$$\int_{|x|=r} |u(x)|^2 ds = \int_{\mathbb{S}^2} \left|u_\infty(\hat{x})\right|^2 ds + O\left(\frac{1}{r}\right), \quad r \to \infty$$

the assumption $u_\infty = 0$ on $\mathbb{S}^2$ implies that the assumption of Rellich's lemma is satisfied, because

$$\int_{|x|=r} |u(x)|^2 \, ds = O\left(\frac{1}{R}\right).$$

Therefore, the theorem follows. □

Remark 1.7.4. Rellich's Lemma is also important to prove uniqueness of many scattering problems. For an application of such kind, see for example Theorem 3.23 of [21], pages 131-132.

1.8 Remarks to Chapter 1

Weaker hypothesis

To begin with, we can weaken the hypothesis made in some of the theorems. In the theorems about representations formulas, we have asked that u is in $C^2(D) \cap C^1(\overline{D})$ or in $C^2(\mathbb{R}^3 \backslash \overline{D}) \cap C^1(\mathbb{R}^3 \backslash D)$, where D is an interior domain. This hypothesis can be relaxed, taking u to be continuous up to the boundary instead of C^1. But then, we need to understand the normal derivative in the sense of uniform convergence:

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h \to 0^+} \nu(x) \cdot \nabla u(x - h\nu(x)), \quad x \in \partial D,$$

so to make the reasonings of the proofs of the representation theorems one has to integrate over paralel surfaces instead of directly on the boundary. For a description of this concept, see [22], pages 84-85, or [12], page 37.

Regularity of solutions to Helmholtz equation

It can be proved that any C^2 solution to Helmholtz equation is in fact analytic. For a proof, see Corollary 3.4 of [21].

Chapter 2

Dual Systems, Fredholm Alternative and Lax's Theorem

In this chapter we develop some of the tools that we will need later. More specifically, we give a brief survey/overview of the theory of dual systems, which tries to generalize Hilbert Space Theory (Riesz Representation Theorem, Fredholm alternative...). We skip all of the proof except the one of Lax's Theorem. Our reference is Chapter 4 of the book of Kress [22].

In this chapter we assume that all vector spaces are complex (the real case can be treated analogously). Unless stated otherwise, in this chapter $G \subset \mathbb{R}^n$ denotes a nonempty compact set.

2.1 Dual systems via bilinear forms

Definition 2.1.1. Let X, Y be vector spaces. A mapping

$$\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{C}$$

is called a *bilinear form* if

$$\begin{aligned} \langle \alpha_1\varphi_1 + \alpha_2\varphi_2, \psi \rangle &= \alpha_1 \langle \varphi_1, \psi \rangle + \alpha_2 \langle \varphi_2, \psi \rangle, \\ \langle \varphi, \beta_1\psi_1 + \beta_2\psi_2 \rangle &= \beta_1 \langle \varphi, \psi_1 \rangle + \beta_2 \langle \varphi, \psi_2 \rangle \end{aligned}$$

for all $\varphi_1, \varphi_2, \varphi \in X, \psi_1, \psi_2, \psi \in Y$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. The bilinear form is called *non-degenerate* if for every $\varphi \in X$ with $\varphi \neq 0$ there exists $\psi \in Y$ such that $\langle \varphi, \psi \rangle \neq 0$; and for every $\psi \in Y$ with $\psi \neq 0$ there exists $\varphi \in X$ such that $\langle \varphi, \psi \rangle \neq 0$.

Definition 2.1.2. Two normed spaces X and Y equipped with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{C}$ are called a *dual system* and it is denoted by $\langle X, Y \rangle$.

A classical example of a dual system is the following.

Theorem 2.1.3. *Each normed space X together with its dual space X^* forms the canonical dual system $\langle X, X^* \rangle$ with the bilinear form*

$$\langle \varphi, F \rangle := F(\varphi), \quad \varphi \in X, F \in X^*.$$

Proof. See Theorem 4.3 of [22], page 46. □

Another example is the classical integral of a product $\langle f, g \rangle = \int_G fg$, which can be considered over a vector space such as $C(G)$ (the space of continuous functions defined on G) instead of $L^2(G)$.

Theorem 2.1.4. *Let $G \subset \mathbb{R}^n$ be a nonempty compact set. Then $\langle C(G), C(G)\rangle$ is a dual system with the bilinear form*

$$\langle \varphi, \psi \rangle := \int_G \varphi(x)\psi(x)dx, \quad \varphi, \psi \in C(G).$$

Proof. See Theorem 4.4 of [22], page 46. □

Adjoint of an operator in a dual system

Definition 2.1.5. Let $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$ be two dual systems. Then two operators $A : X_1 \to X_2, B : Y_2 \to Y_1$ are called *adjoint* (with respect to these dual systems) if

$$\langle A\varphi, \psi\rangle_2 = \langle \varphi, B\psi\rangle_1$$

for all $\varphi \in X_1, \psi \in Y_2$. (where $\langle \cdot, \cdot\rangle_1$ denotes the bilinear form of $\langle X_1, Y_1\rangle$ and $\langle \cdot, \cdot\rangle_2$ denotes the bilinear form of $\langle X_2, Y_2\rangle$).

Theorem 2.1.6. *Let $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$ be two dual systems. If an operator $A : X_1 \to X_2$ has an adjoint $B : Y_2 \to Y_1$, then B is uniquely determined, and A and B are linear.*

Proof. See Theorem 4.6 of [22], page 46. □

The typical example is an operator given by a continuous or weakly singular kernel (see Definition 5.1.18 of the Appendix).

Theorem 2.1.7. *Let K be a continuous or a weakly singular kernel. Then in the dual system $\langle C(G), C(G)\rangle$ the (compact) integral operators defined by*

$$(A\varphi)(x) := \int_G K(x, y)\varphi(y)dy, \quad x \in G,$$

and

$$(B\psi)(x) := \int_G K(y, x)\psi(y)dy, \quad x \in G,$$

are adjoint.

Proof. See Theorem 4.7 of [22], page 47. □

Remark 2.1.8. Note that not every operator has an adjoint. For example, the operator $A : C[0, 1] \to C[0, 1]$ defined by $A\varphi := \varphi(1)$ is a compact operator that does not have an adjoint operator with respect to the dual system $\langle C[0, 1], C[0, 1]\rangle$ of Theorem 2.1.4. We can prove it by *reductio ad absurdum*: suppose $B : C[0, 1] \to C[0, 1]$ was an adjoint of A. We can choose a function $\psi \in C[0, 1]$ with $\int_0^1 \psi(x)\, dx = 1$. By the Cauchy-Schwarz inequality,

$$1 = |\varphi(1)| = |\langle A\varphi, \psi\rangle| = |\langle \varphi, B\psi\rangle| \leqslant \|\varphi\|_2 \|B\psi\|_2$$

for all $\varphi \in C[0, 1]$ with $\varphi(1) = 1$. Considering this inequality for the sequence (φ_n) with $\varphi_n(x) := x^n$ we arrive at a contradiction, since the right-hand side tends to zero as $n \to \infty$.

The following is a classical result.

Theorem 2.1.9. *Let $A : X \to Y$ be a bounded linear operator mapping a normed space X into a normed space Y. Then the adjoint operator $A^* : Y^* \to X^*$ with respect to the canonical dual systems $\langle X, X^* \rangle$ and $\langle Y, Y^* \rangle$ exists. It is given by*

$$A^* F := FA, \quad F \in Y^*,$$

and is called the dual operator of A. It is bounded with norm $\|A\| = \|A^*\|$.

Proof. See Theorem 4.8 of [22], pages 47-48.

□

Remark 2.1.10. It can be shown that under the assumptions of Theorem 2.1.9 and assuming that Y is a Banach space, the adjoint operator A^* is compact if and only if A is compact.

2.2 Dual Systems via Sesquilinear Forms

We can develop a similar theory for sesquilinear forms instead of bilinear ones.

Definition 2.2.1. Let X, Y be linear spaces. A mapping $(\cdot, \cdot) : X \times Y \to \mathbb{C}$ is called a *sesquilinear form* if

$$(\alpha_1 \varphi_1 + \alpha_2 \varphi_2, \psi) = \alpha_1 (\varphi_1, \psi) + \alpha_2 (\varphi_2, \psi),$$

$$(\varphi, \beta_1 \psi_1 + \beta_2 \psi_2) = \bar{\beta}_1 (\varphi, \psi_1) + \bar{\beta}_2 (\varphi, \psi_2)$$

for all $\varphi_1, \varphi_2, \varphi \in X, \psi_1, \psi_2, \psi \in Y$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. (Here, the bar indicates the complex conjugate.)

The definitions and theorems formulated for non-degenerate bilinear forms can be adapted to non-degenerate sesquilinear forms and are still true.

Remark 2.2.2. Sesquilinear forms differ from bilinear forms by their anti-linearity with respect to the second space. However, they are closely related: assume there exists a mapping $* : Y \to Y$ with the properties $(\beta_1 \psi_1 + \beta_2 \psi_2)^* = \bar{\beta}_1 \psi_1^* + \bar{\beta}_2 \psi_2^*$ and $(\psi^*)^* = \psi$ for all $\psi_1, \psi_2, \psi \in Y$ and $\beta_1, \beta_2 \in \mathbb{C}$. Such a mapping (called an *involution*) provides a one-to-one correspondence between bilinear and sesquilinear forms by $(\varphi, \psi) = \langle \varphi, \psi^* \rangle$. For example, in the space $C(G)$ the natural involution is given by $\psi^*(x) := \overline{\psi(x)}$ for all $x \in G$ and all $\psi \in C(G)$.

Hilbert Spaces

As we have already said in the introduction to this chapter, (pre-)Hilbert spaces are a particular case of dual systems. Indeed, any scalar product on a linear space X can be seen as a non-degenerate sesquilinear form that is symmetric, that is, $(\varphi, \psi) = \overline{(\psi, \varphi)}$ for all $\varphi, \psi \in X$, and positive definite, that is, $(\varphi, \varphi) > 0$ for all $\varphi \in X$ with $\varphi \neq 0$. Thus **each pre-Hilbert space is a dual system canonically**.

For this special case, we recall some well-known theorems that we will use later:

Theorem 2.2.3 (Riesz Representation Theorem)**.** *Let X be a Hilbert space. Then for each bounded linear functional $F : X \to \mathbb{C}$ there exists a unique element $f \in X$ such that*

$$F(\varphi) = (\varphi, f)$$

for all $\varphi \in X$. The norms of the element f and the linear function F coincide, i.e.,

$$\|f\| = \|F\|.$$

Proof. See Theorem 4.10 of [22], page 49. □

Remark 2.2.4. The Riesz Representation theorem establishes the existence of a bijective anti-linear mapping $J : X \to X^*$ between a Hilbert space and its dual space, given by

$$(J(f))(\varphi) = (\varphi, f)$$

for all $\varphi, f \in X$, which is isometric in the sense that it preserves the norms.

As opposed to what happened with general dual systems (even for compact operators, as seen in an Remark 2.1.8), any bounded linear operator on a Hilbert space has an adjoint. This is thanks to the Riesz Representation Theorem, which is not true in general for dual systems.

Theorem 2.2.5. *Let X and Y be Hilbert spaces, and let $A : X \to Y$ be a bounded linear operator. Then there exists a uniquely determined linear operator $A^* : Y \to X$ with the property*

$$(A\varphi, \psi) = (\varphi, A^*\psi)$$

for all $\varphi \in X$ and $\psi \in Y$, i.e., A and A^* are adjoint with respect to the dual systems (X, X) and (Y, Y) generated by the scalar products on X and Y. The operator A^* is bounded and $\|A^*\| = \|A\|$. (Again we use the same symbol $(\cdot, \cdot)$ for the scalar products on X and Y.)

Proof. See Theorem 4.11 of [22], page 50.

□

Theorem 2.2.6. *Let X and Y be Hilbert spaces and let $A : X \to Y$ be a compact linear operator. Then the adjoint operator $A^* : Y \to X$ is also compact.*

Proof. See Theorem 4.12 of [22], page 50.

□

2.2.1 Lax's Theorem

The following theorem is due to Lax and provides a usefool tool to extend results on the boundedness of linear operators from a given norm to a weaker scalar product norm.

Theorem 2.2.7 (Lax)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces equipped with scalar products*[1] *$(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$ respectively and suppose that there exists a positive constant $C > 0$ such that*

$$|(\varphi, \psi)_X| \leqslant C\|\varphi\|_X \cdot \|\psi\|_X \tag{2.1}$$

for all $\varphi, \psi \in X$. Let $U \subset X$ be a subspace and let $A\colon U \to Y$ y $B\colon Y \to X$ be bounded linear operators that satisfy

$$(A\varphi, \psi)_Y = (\varphi, B\psi)_X \tag{2.2}$$

for all $\varphi \in U$, $\psi \in Y$.

Then $A\colon U \to Y$ is bounded with respect to the norms induced by the scalar products. We will denoted these norms by $\|\cdot\|_{s,X}$ and $\|\cdot\|_{s,Y}$.

[1] These scalar products do not need to induce the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Proof. Let $M := B \circ A \colon U \to X$. Then M is bounded because it is a composition of bounded operators, and we have $\|M\|_{U\to X} \leqslant \|B\|_{Y\to X} \cdot \|A\|_{U\to Y}$.

M is self-adjoint, that is:

$$(M\varphi, \psi)_X = (\varphi, M\psi)_X \tag{2.3}$$

for all $\varphi, \psi \in U$, because when we apply (2.2) we obtain:

$$(\varphi, M\psi)_X = (\varphi, BA\psi)_X \underset{\substack{\uparrow\\(2.2)}}{=} (A\varphi, A\psi)_Y = \overline{(A\psi, A\varphi)_Y} \underset{\substack{\uparrow\\(2.2)}}{=} \overline{(\psi, BA\varphi)_X} = \overline{(\psi, M\varphi)_X} = (M\varphi, \psi)_X.$$

By Cauchy-Schwarz inequality,

$$\|M^n\varphi\|_{s,X}^2 = (M^n\varphi, M^n\varphi)_X = (\varphi, M^{2n}\varphi)_X \leqslant \|\varphi\|_{s,X} \cdot \|M^{2n}\varphi\|_{s,X} \leqslant \|M^{2n}\varphi\|_{s,X}$$

for every $\varphi \in U$ with $\|\varphi\|_s \leqslant 1$, and every $n \in \mathbb{N}$.

From this, by induction, it follows that

$$\|M\varphi\|_{s,X} \leqslant \|M^{2^n}\varphi\|_{s,X}^{2^{-n}}, \tag{2.4}$$

for all $\varphi \in U$ with $\|\varphi\|_{s,X} \leqslant 1$, because

$$\|M\varphi\|_{s,X} \leqslant \|M^2\varphi\|_{s,X}^{1/2} \leqslant (\|M^4\varphi\|_{s,X}^{1/2})^{1/2} \leqslant \ldots \leqslant \|M^{2^n}\varphi\|_{s,X}^{2^{-n}}.$$

From (2.1) we have that

$$\|\varphi\|_{s,X} = (\varphi, \varphi)_X^{1/2} \underset{\substack{\uparrow\\(2.1)}}{\leqslant} (C\|\varphi\|_X\|\varphi\|_X)^{1/2} = \sqrt{C} \cdot \|\varphi\|_X.$$

That is,

$$\|\varphi\|_{s,X} \leqslant \sqrt{C} \cdot \|\varphi\|_X. \tag{2.5}$$

Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|M\varphi\|_{s,X} \underset{\substack{\uparrow\\(2.4)}}{\leqslant} \|M^{2^n}\varphi\|_{s,X}^{2^{-n}} &\underset{\substack{\uparrow\\(2.5)}}{\leqslant} [\sqrt{C} \cdot \|M^{2^n}\varphi\|_X]^{2^{-n}} \\ &\underset{\substack{\uparrow\\M \text{ bounded}}}{\leqslant} [\sqrt{C} \cdot \|M\|_{U\to X}^{2^n} \cdot \|\varphi\|_X]^{2^{-n}} \\ &\leqslant [\sqrt{C} \cdot \|\varphi\|_X]^{2^{-n}} \cdot \|M\|_{U\to X}.\end{aligned}$$

Taking $\lim_{n\to\infty}$, we obtain

$$\|M\varphi\|_{s,X} \leqslant \|M\|_{U\to X} < \infty \tag{2.6}$$

for all $\varphi \in U$ with $\|\varphi\|_{s,X} \leqslant 1$. By Cauchy-Schwarz inequality, we have that for every $\varphi \in U$ with $\|\varphi\|_{s,X} \leqslant 1$:

$$\|A\varphi\|_{s,Y}^2 = (A\varphi, A\varphi)_Y \underset{\substack{\uparrow\\(2.3)}}{=} (\varphi, M\varphi)_X \underset{\substack{\uparrow\\\text{C-S}}}{\leqslant} \|\varphi\|_{s,X}\|M\varphi\|_{s,X} \underset{\substack{\uparrow\\(2.6) \text{ and } \|\varphi\|_{s,X}\leqslant 1}}{\leqslant} \|M\|_{U\to X}.$$

Therefore, A is a bounded operator from U to Y with the norms induced by the scalar product. □

2.2.2 Fredholm's Alternative

The following theorem is Fredholm Alternative for a dual system generated by a bilinear form. The case of a sesquilinear form is analogous.

Theorem 2.2.8. *Let $A\colon X \to X$, $B\colon Y \to Y$ be compact adjoint operators in a dual system $\langle X, Y\rangle$. Then*

1. *Either $I - A$ and $I - B$ are bijective,*
2. *Or $I - A$ and $I - B$ have nontrivial nullspaces with finite dimension*

$$\dim N(I - A) = \dim N(I - B) \in \mathbb{N}$$

and the ranges are given by

$$(I - A)(X) = \{f \in X : \langle f, \psi\rangle = 0, \psi \in N(I - B)\}$$

and

$$(I - B)(Y) = \{g \in Y : \langle \varphi, g\rangle = 0, \varphi \in N(I - A)\}.$$

Proof. See Theorem 4.17 of [22], page 55. □

Remark 2.2.9. Note that this theorem implies that one of the four properties $I - A$ is injective, $I - A$ is surjective, $I - B$ is injective and $I - B$ is surjective implies the three other ones.

Usually, this theorem is applied in the following way: we have a partial differential equation expressed in terms of a compact operator. Then, Fredholm's Theorem guarantees that if we have uniqueness of the solution (i.e. injectivity with respect to the initial data), which usually is easier to prove, then we have existence as well.

Chapter 3

Scattering in an Inhomogeneous medium

For the remaining chapters of this essay, we are going to consider the *direct scattering problem of acoustic waves by an inhomogeneous medium of compact support*.

To study the problem, we are going to use the method of integral equations. Since there will not be boundary conditions, we will only use volume potentials (instead of surface potentials such as the single- or double-layer potentials).

The structure of the chapter is the following:

- We begin the chapter by deriving the linearized equations governing the propagation of small amplitude sound waves in an inhomogeneous medium.

- We then reformulate the direct scattering problem for such a medium as an integral equation known as the *Lippmann-Schwinger equation*.

- Then, in order to apply Fredholm's theorem (that is, in order to prove uniqueness of the homogeneous equation), we prove a Unique Continuation Principle. This allows us to prove the existence of a unique solution to the Lippmann-Schwinger equation, and therefore the existence and uniqueness of a solution to the scattering problem.

- Finally, we investigate the set $\mathcal{F}$ of far field patterns of the scattered fields corresponding to incident time-harmonic plane waves moving in arbitrary directions (i.e., waves whose space dependent part is of the form $e^{-ikx\cdot d}$ for $d \in \mathbb{S}^2$ arbitrary). By proving a reciprocity relation for far field patterns, we show that the completeness of the set $\mathcal{F}$ is equivalent to the nonexistence of an specific type of eigenfunctions (called Herglotz wave functions) for a new type of boundary value problem for the Helmholtz equation called the *interior transmission problem*.

- Lastly, we define the concept of *non-scattering inhomogeneity* and relate it to the transmission eigenvalue problem.

For this chapter, we follow [13], sections 8.1-8.4, and [19], sections 7.1-7.3.

3.1 Physical model

In this section, we give a brief description of the physical model. For a detailed deduction of the model, see [19], pages 239-241, or [13], pages 304-306.

By a deduction analogous to that needed in Chapter 1 to deduce Helmholtz equation, making several assumptions to simplify the model, it can be shown that, if we consider acoustic waves travelling in a medium such as a fluid and suppose that the linear part of the pressure of the fluid $p(x,t)$ is time-harmonic, i.e.,

$$p(x,t) = p_0 + \varepsilon p_1(x,t) + O(\varepsilon^2),$$

with

$$p_1(x,t) = \mathrm{Re}[u(x)e^{-i\omega t}]$$

for some frequency $\omega > 0$ and a complex-valued function u which only depends on the spatial variable, then the wave behavior can be modeled by the following equation for u:

$$\Delta u(x) + \frac{\omega^2}{c(x)^2}\left(1 + i\frac{\gamma}{\omega}\right) u = 0. \tag{3.1}$$

In free space (that is, outside the inhomogeneity), $c = c_0$ is constant and $\gamma = 0$. We define the *wave number* and the *index of refraction* by

$$k := \frac{\omega}{c_0} > 0 \quad \text{and} \quad n(x) := \frac{c_0^2}{c(x)^2}\left(1 + i\frac{\gamma}{\omega}\right)$$

respectively. Notice that n is a complex-valued function with $\mathrm{Re}\, n(x) \geqslant 0$ and $\mathrm{Im}\, n(x) \geqslant 0$. Notice that outside the inhomogeneity, since $c = c_0$ and $\gamma = 0$, then $n(x) \equiv 1$. So we assume that the inhomogeneous region, let us called it $D \subset \mathbb{R}^3$, is given by

$$D = supp(m)$$

where m is defined as

$$m(x) := 1 - n(x), \quad x \in \mathbb{R}^3. \tag{3.2}$$

Since we are considering inhomogeneities of compact support, $D = supp(m)$ is assumed to be compact.

With these definitions, equation (3.1) takes the form

$$\Delta u + k^2 n u = 0.$$

We assume that there exists a source (outside the inhomogeneity) which generates an *incident* field u^i that satisfies the unperturbed Helmholtz equation $\Delta u^i + k^2 u^i = 0$ outside the sources (for example, if a whistle is taking place at a point z in space, it satisfies Helmholtz equation outside this point, i.e., in $\mathbb{R}^3\backslash\{z\}$). We typically imagine that u^i is generated by a *point source* or that it is a *plane wave*; that is, the time-dependent incident fields have the form

$$p_1^i(x,t) = \frac{1}{|x-z|}\,\mathrm{Re}\left[e^{ik|x-z|-i\omega t}\right]; \quad \text{that is,} \quad u^i(x) = \frac{e^{ik|x-z|}}{|x-z|},$$

for a *point source* at $z \in \mathbb{R}^3$, or

$$p_1^i(x,t) = \mathrm{Re}\left[e^{ik\hat{\theta}\cdot x - i\omega t}\right]; \quad \text{that is,} \quad u^i(x) = e^{ik\hat{\theta}\cdot x},$$

for *plane wave* travelling in the direction of a unit vector $\hat{\theta} \in \mathbb{R}^3$.

In both cases, u^i is a solution of the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3\backslash\{z\}$ or $\mathbb{R}^3$ respectively. In the first case, the pressure p_1^i describes a *spherical wave* that travels away from the source with velocity $c_0 = \frac{\omega}{k}$. In the second case, p_1^i describes a plane wave that travels in the direction $\hat{\theta}$ with velocity c_0.

The incident field is disturbed by the inhomogeneous medium (D, n) described by the index of refraction n and produces a *scattered wave* u^s. The total field $u = u^i + u^s$ satisfies the equation $\Delta u + k^2 nu = 0$ outside the sources; that is, the scattered field u^s satisfies the inhomogeneous equation

$$\Delta u^s + k^2 nu^s = k^2(1-n)u^i$$

where the right-hand side is a function of compact support in D. Furthermore, we assume the scattered field u^s to behave as a spherical wave far from the medium (see description (1.14) to understand what we mean). We describe this, as in Chapter 1, by Sommerfeld Radiation Condition

$$\frac{\partial u^s(x)}{\partial r} - iku^s(x) = O\left(\frac{1}{r^2}\right) \qquad \text{as } r = |x| \to \infty,$$

uniformly in $\frac{x}{|x|} \in \mathbb{S}^2$. The smoothness of the solution u^s depends on the smoothness of the refractive index n. We have now described a quite complete model of the direct scattering problem.

Direct Scattering Problem

Let the wave number $k > 0$, the index of refraction $n \in L^\infty(\mathbb{R}^3)$ with $n \equiv 1$ on $\mathbb{R}^3 \backslash D$, and the incident field u^i be given. The scattering problem is to determine the scattered field u^s that satisfies the equation

$$\Delta u^s + k^2 nu^s = k^2(1-n)u^i$$

and the radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

uniformly for $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^2$.

3.1.1 Summary of the model

To fix ideas, equation $\Delta u + k^2 n(x)u = 0$ governs the propagation of time-harmonic acoustic waves of small amplitude in an inhomogeneous medium. We shall only consider the simplest case:

- The inhomogeneity (D, n) is determined by the inhomogeneous region $D \subset \mathbb{R}^3$ and the refractive index n, with $D = supp(1-n)$. We further assume that:
  - The inhomogeneous medium is *slowly varying*. This translates in the fact that c and γ, and therefore n, are assumed to be *piecewise continuous* at least.
  - That the inhomogeneity is of compact support, which means that D is bounded or, equivalently, that the support of $1-n$ is compact.

  Later, we will specify some regularity of the boundary ∂D.
- The region of propagation under consideration is all of $\mathbb{R}^3$.
- The wave motion is caused by an incident field u^i satisfying Helmholtz equation, which is scattered by the inhomogeneous medium.

Under these conditions, the **scattering problem** under consideration is to find u (or, equivalently, u^s) such that

$$\Delta u + k^2 n(x)u = 0 \text{ in } \mathbb{R}^3 \tag{3.3}$$

$$u = u^i + u^s, \tag{3.4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{3.5}$$

Remark 3.1.1. In general, we will assume that the refractive index n is real and positive. If we want to include the possibility that the medium is absorbing, then the refractive index has an imaginary component (see [13], page 306):

$$n(x) = n_1(x) + i\frac{n_2(x)}{k}, \tag{3.6}$$

where $n_1(x) = \mathrm{Re}(n(x)) \geqslant 0$ and $n_2(x) = \mathrm{Im}(n(x)) \geqslant 0$.

3.2 The Lippmann-Schwinger Equation

The aim of this section is to derive an integral equation that is equivalent to the scattering problem (3.3)-(3.5) where we assume the refractive index n of the general form (3.6) to be piecewise continuous in $\mathbb{R}^3$ such that

$$m := 1 - n$$

has compact support and

$$n_1(x) > 0 \quad \text{and} \quad n_2(x) \geqslant 0$$

for all $x \in \mathbb{R}^3$. Throughout this chapter, we shall always assume that these assumptions are valid and let $D := supp(m) = \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ (the last equality being true because m is piecewise continuous).

To derive an integral equation equivalent to (3.3)-(3.5), we shall need to consider the *volume potential*

$$u(x) := \int_{\mathbb{R}^3} \Phi(x, y)\varphi(y)\, dy, \quad x \in \mathbb{R}^3, \tag{3.7}$$

where

$$\Phi(x, y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x - y|}, \quad x \neq y,$$

is the fundamental solution to the Helmholtz equation and φ is a continuous function in $\mathbb{R}^3$ with compact support, i.e., $\varphi \in C_c(\mathbb{R}^3)$. We do not consider surface potentials (like the single or double layer) since we do not have boundary conditions.

3.2.1 Mapping properties of the volume potential

The main result about the volume potential is the following:

Theorem 3.2.1. *The volume potential u given by (3.7) exists as an improper integral for all $x \in \mathbb{R}^3$ and has the following properties. If $\varphi \in C_c(\mathbb{R}^3)$ then $u \in C^{1,\alpha}(\mathbb{R}^3)$ and the orders of differentiation and integration can be interchanged. If $\varphi \in C_c(\mathbb{R}^3) \cap C^{0,\alpha}(\mathbb{R}^3)$ then $u \in C^{2,\alpha}(\mathbb{R}^3)$ and*

$$\Delta u + k^2 u = -\varphi \text{ in } \mathbb{R}^3. \tag{3.8}$$

In addition, we have

$$\|u\|_{2,\alpha,\mathbb{R}^3} \leqslant C\|\varphi\|_{\alpha,\mathbb{R}^3}$$

for some positive constant C depending only on the support of φ. Furthermore, if $\varphi \in C_C(\mathbb{R}^3) \cap C^{1,\alpha}(\mathbb{R}^3)$, then $u \in C^{3,\alpha}(\mathbb{R}^3)$.

Proof. The theorem follows from a slight modification of Theorem 3.9 and Corollary 3.10 of [21], pages 107-110. We skip the proof since it is quite technical and needs a couple of lemmas. The idea to prove the Hölder estimates is to bound Φ and its partial derivatives so that, differentiating under the integral sign, we can bound the volume potential and its partial derivatives. $\square$

Generalizations for H^2 functions

Since for piecewise continuous n we cannot expect C^2 solutions of (3.4) (because if $u \in C^2$ and $\Delta u + k^2 n u = 0$, every term is continuous except for n) we require the solutions to belong to the Sobolev space $H^2_{loc}(\mathbb{R}^3)$ of functions with locally square integrable weak derivatives. So we want to generalize the results of Theorem 3.2.1 to Sobolev spaces.

First, we use Lax's Theorem 2.2.7 to extend the mapping properties for the volume potential for Hölder spaces given in the previous theorem to the more general setting of Sobolev spaces.

Theorem 3.2.2. *Given two bounded domains D and G, the volume potential*

$$(V\varphi)(x) := \int_D \Phi(x,y)\varphi(y)\, dy, \qquad x \in \mathbb{R}^3,$$

defines a bounded operator $V\colon L^2(D) \to H^2(G)$.

Proof. Let B be an open ball such that $\overline{G} \subset B$ (there exists such a ball since G is bounded) and let $\gamma \in C^2_c(B)$ be such that

$$\begin{cases} \gamma(x) \geqslant 0 & \forall x \in B \\ \gamma(x) = 1 & \forall x \in G \end{cases}$$

We want to use Lax's Theorem 2.2.7 to generalize the mapping property of the previous theorem from Hölder spaces to Sobolev spaces. The previous theorem gives a mapping property for the volume potential from $C^{0,\alpha}(D)$ to $C^{2,\alpha}(D)$. Therefore, following the notation of Lax's theorem, the known spaces we need to consider are $X := C^{0,\alpha}(D)$ e $Y := C^{2,\alpha}(B)$ equipped with their respective usual Hölder norms. Now, we need to introduce a scalar product on each of them:

1. On X, we define the scalar product of $L^2(D)$:

$$(u,v)_X := \int_D u \cdot \overline{v}.$$

2. On Y, we define the following weighted Sobolev scalar product:

$$(u,v)_Y := \int_B \gamma \left[u \cdot \overline{v} + \sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial \overline{v}}{\partial x_i} + \sum_{i,j=1}^{3} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 \overline{v}}{\partial x_i \partial x_j} \right] dx.$$

Notice that condition (2.1) of Lax's theorem is satisfied, because

$$\begin{aligned} |(u,v)_X| &= \left| \int_D u\overline{v} \right| \leqslant \int_D |u| \cdot |v| \leqslant \int_D \sup_{x\in D} |u(x)| \cdot \sup_{x\in D} |v(x)| \\ &= \operatorname{vol}(D) \sup_{x\in D} |u(x)| \cdot \sup_{x\in D} |v(x)| \leqslant C\|u\|_{C^{0,\alpha}(D)} \|v\|_{C^{0,\alpha}(D)} = C\|u\|_X \|v\|_X, \end{aligned}$$

where $C := \operatorname{vol}(D) > 0$ is the measure of D with respect to the n-dimensional Lebesgue measure.

Now, let

$$(V^*\psi)(x) := \int_B \psi(y) \cdot \Phi(x,y)\, dy, \qquad \forall \psi \in Y$$

(it is the same definition as for V, but we integrate over B instead of over V).

To be able to apply Lax's theorem, we want to choose a vector subspace $U \subset X$ and to find an operator $W : Y \to X$ which is the adjoint of $V : U \to Y$, that is, an operator such that

$$(V\varphi, \psi)_Y = (\varphi, W\psi)_X \quad \forall \varphi \in U, \psi \in Y.$$

In order to do so, we need to integrate by parts on the expression

$$(V\varphi, \psi)_Y = \int_B \gamma \left[(V\varphi)\overline{v} + \sum_{i=1}^{3} (\partial_i (V\varphi))(\partial_i \overline{v}) + \sum_{i,j=1}^{3} (\partial_i \partial_j (V\varphi))(\partial_i \partial_j \overline{v}) \right] dx.$$

We study this expression term by term.

First term. We begin by studying expressions of the form $\int_B \gamma \cdot (V\varphi) \cdot \psi \, dx$. Recall that γ is a (bump) function (not a scalar).

We have

$$\nabla_x \Phi(x, y) = -\nabla_y \Phi(x, y), \tag{3.9}$$

since

$$\nabla_x \Phi(x, y) = f'(r_y(x)) \cdot \nabla r_y(x) = f'(r_x(y)) \cdot (-\nabla r_x(y)) = -\nabla_y \Phi(x, y)$$

where $f(r) := \frac{e^{ikr}}{4\pi r}$, $r_y(x) := |x - y|$ for $x \in \mathbb{R}^3$, and $r_x(y) := |x - y|$ for $x \in \mathbb{R}^3$, having used that $\nabla r_x(y) = -\nabla r_y(x)$.

Using this and Fubini's theorem we have that, for $\varphi \in X$ and $\psi \in Y$,

$$\int_B \gamma \cdot (V\varphi) \cdot \psi \, dx = \int_D \varphi \cdot V^*(\gamma \cdot \psi) \, dx, \tag{3.10}$$

since

$$\begin{aligned}
\int_B \gamma(V\varphi) \cdot \psi \, dx &= \int_B \gamma(x) \cdot \left(\int_D \Phi(x, y)\varphi(y) \, dy \right) \cdot \psi(x) \, dx \\
&= \int_D \left(\int_B \gamma(x)\Phi(x, y)\varphi(y)\psi(x) \, dx \right) dy \\
&= \int_D \left(\int_B \gamma(x)\Phi(x, y)\psi(x) \, dx \right) \varphi(y) \, dy \\
&= \int_D V^*(\gamma \cdot \psi)(y) \cdot \varphi(y) \, dy.
\end{aligned}$$

Second term. Now we move on to the second term, which is formed by integrals of the form

$$\int_B \gamma \cdot \partial_i (V\varphi) \cdot \partial_i \overline{v} \, dx.$$

For $\varphi \in X$ y $\psi \in Y$ we have that

$$\int_B \gamma \cdot \frac{\partial}{\partial x_i} V\varphi \cdot \frac{\partial \psi}{\partial x_i} \, dx = -\int_D \varphi \cdot \frac{\partial}{\partial x_i} V^* \left(\gamma \cdot \frac{\partial \psi}{\partial x_i} \right) dx. \tag{3.11}$$

because using again (3.9) and Fubini's theorem we obtain

$$
\begin{aligned}
\int_B \gamma \cdot \frac{\partial}{\partial x_i}(V\varphi) \cdot \frac{\partial \psi}{\partial x_i}\, dx &= \int_B \gamma(x) \cdot \frac{\partial}{\partial x_i} \left(\int_D \Phi(x,y)\varphi(y)\, dy \right) \cdot \frac{\partial \psi}{\partial x_i}(x)\, dx = \\
&= \int_B \gamma(x) \cdot \left(\int_D \frac{\partial \Phi}{\partial x_i}(x,y)\varphi(y)\, dy \right) \frac{\partial \psi}{\partial x_i}(x)\, dx = \\
&= \int_B \gamma(x) \cdot \left(\int_D -\frac{\partial \Phi}{\partial y_i}(x,y)\varphi(y)\, dy \right) \frac{\partial \psi}{\partial x_i}(x)\, dx = \\
&= \int_D \int_B \gamma(x) \cdot \left(-\frac{\partial \Phi}{\partial y_i}(x,y) \right) \cdot \varphi(y) \cdot \frac{\partial \psi}{\partial x_i}(x)\, dx\, dy = \\
&= -\int_D \varphi(y) \cdot \left(\int_B \frac{\partial \Phi}{\partial y_i}(x,y) \cdot \gamma(x) \cdot \frac{\partial \psi}{\partial x_i}(x)\, dx \right) dy = \\
&= -\int_D \varphi(y) \cdot \frac{\partial}{\partial y_i} \left(\int_B \Phi(x,y) \cdot \gamma(x) \cdot \frac{\partial \psi}{\partial x_i}(x)\, dx \right) dy = \\
&= -\int_D \varphi(y) \cdot \frac{\partial}{\partial y_i} V^* \left(\gamma \cdot \frac{\partial \psi}{\partial x_i} \right)(y)\, dy
\end{aligned}
$$

Third term. Now, we study the third term, which involves integrals of the form

$$
\int_B (\partial_i \partial_j (V\varphi)) \cdot (\partial_i \partial_j \overline{v})\, dx.
$$

For $\varphi \in C_c^1(D)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial x_i} \int_D \Phi(x,y) \cdot \varphi(y)\, dy &= \int_D \frac{\partial \Phi}{\partial x_i}(x,y) \cdot \varphi(y)\, dy \\
&= \int_D -\frac{\partial \Phi}{\partial y_i}(x,y) \cdot \varphi(y)\, dy \\
&= \int_D \Phi(x,y) \cdot \frac{\partial \varphi}{\partial y_i}(y)\, dy - \int_{\partial D} \Phi(x,y) \cdot \varphi(y) \cdot \nu^i(y)\, ds(y) \\
&= \int_D \Phi(x,y) \cdot \frac{\partial \varphi}{\partial y_i}(y)\, dy,
\end{aligned}
$$

having used in the last step that φ has compact support. That is the reason to choose $\varphi \in C_c^1(D)$: to eliminate the boundary terms after integrating by parts. So,

$$
\frac{\partial}{\partial x_i} V\varphi = V\left(\frac{\partial \varphi}{\partial x_i} \right). \tag{3.12}
$$

By (3.11), for $\varphi \in C_c^1(D)$ and $\psi \in Y$, we have that

$$\begin{aligned}
\int_B \gamma \left(\frac{\partial^2}{\partial x_i \partial x_j}(V\varphi)\right) \cdot \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right) dx &\underset{(3.12)}{\overset{\uparrow}{=}} \int_B \gamma \cdot \frac{\partial}{\partial x_i} V(\partial_j \varphi) \cdot \frac{\partial^2 \psi}{\partial x_i \partial x_j}\, dx \\
&\underset{(3.11)}{\overset{\uparrow}{=}} -\int_D (\partial j \varphi) \cdot \frac{\partial}{\partial x_i} V^* \left(\gamma \cdot \frac{\partial}{\partial x_i}(\partial_j \psi)\right) dx \\
&\underset{\text{Parts}}{\overset{\uparrow}{=}} \int_D \varphi \cdot \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} V^*(\gamma \cdot \partial_i \partial_j \psi)\, dx - \int_{\partial D} \varphi \cdot \frac{\partial}{\partial x_i} V^*(\gamma \cdot \partial_i \partial_j \psi) \cdot \nu^j\, ds \\
&\underset{\varphi \in C_c^1(D)}{\overset{\uparrow}{=}} \int_D \varphi \cdot \partial_j \partial_i (V^*(\gamma \cdot \partial_i \partial_j \psi))\, dx.
\end{aligned}$$

That is,

$$\int_B \gamma(\partial_i \partial_j (V\varphi)) \cdot (\partial_i \partial_j \psi)\, dx = \int_D \varphi \cdot \partial_j \partial_i (V^*(\gamma \cdot \partial_i \partial_j \psi))\, dx \quad \forall \varphi \in C_c^1(D), \psi \in Y = C^{2,\alpha}(D). \tag{3.13}$$

Therefore, the choice of the subspace U must be

$$U := C_c^1(D) \subset X.$$

Using (3.10), (3.11) and (3.13), we want to find $W : Y \to X$ the adjoint of $V : U \to Y$, that is, the operator that satisfies

$$(V\varphi, \psi)_Y = (\varphi, W\psi)_X \quad \forall \varphi \in U, \psi \in Y.$$

Let us find it.

$$\begin{aligned}
(V\varphi, \psi)_Y &= \int_B \gamma \left[(V\varphi)\overline{v} + \sum_{i=1}^3 (\partial_i (V\varphi))(\partial_i \overline{v}) + \sum_{i,j=1}^3 (\partial_i \partial_j (V\varphi))(\partial_i \partial_j \overline{v})\right] dx \\
&= \int_B \gamma V\varphi \cdot \overline{v}\, dx + \sum_{i=1}^3 \int_B \gamma \cdot \partial_i V\varphi \cdot \partial_i \overline{v}\, dx + \sum_{i,j=1}^3 \int_B \gamma \cdot (\partial_i \partial_j V\varphi) \cdot (\partial i \partial_j \psi)\, dx
\end{aligned}$$

Applying (3.10) to the first term, (3.11) to the second, and (3.13) to the third, we obtain:

$$\begin{aligned}
(V\varphi, \psi)_Y &= \int_D \varphi \cdot V^*(\gamma \cdot \overline{v})\, dx + \sum_{i=1}^3 - \int_D \varphi \cdot \partial_i V^*(\gamma \cdot \partial_i \overline{v}) + \sum_{i,j=1}^3 \int_D \varphi \cdot \partial_i \partial_j V^*(\gamma \cdot \partial_i \partial_j \overline{v}) \\
&= \int_D \varphi \cdot \left[V^*(\gamma \cdot \overline{v}) - \sum_{i=1}^3 \partial_i V^*(\gamma \cdot \partial_i \overline{v}) + \sum_{i,j=1}^3 \partial_i \partial_j (V^*(\gamma \cdot \partial_i \partial_j \overline{v}))\right] \\
&= \int_D \varphi \overline{W\psi}\, dx \\
&= (\varphi, W\psi)_{L^2(D)} \\
&= (\varphi, W\psi)_X,
\end{aligned}$$

having defined W as the operator

$$W\psi := \overline{V^*(\gamma \cdot \overline{v}) - \sum_{i=1}^3 \partial_i V^*(\gamma \cdot \partial_i \overline{v}) + \sum_{i,j=1}^3 \partial_i \partial_j (V^*(\gamma \cdot \partial_i \partial_j \overline{v}))}$$

That is, the adjoint of $V: U \to Y$ is the operator $W: Y \to X$ given by

$$W\psi = \overline{V^*\gamma \cdot \overline{v}} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \overline{V^*\left(\gamma \cdot \frac{\partial \overline{v}}{\partial x_i}\right)} + \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} \overline{V^*\left(\gamma \cdot \frac{\partial^2 \overline{v}}{\partial x_i \partial x_j}\right)}.$$

(Notice that since γ takes only real values, its conjugate is itself).

By Theorem 3.2.1, V and W are bounded with respect to the Hölder norms of their spaces of definition. Therefore, because of Lax's Theorem 2.2.7), $V: U \to Y$ is **bounded** with respect to the norms induced by the scalar products, i.e., there exists a constant $C > 0$ such that

$$\|V\varphi\|_{s,Y} \leqslant C \cdot \|\varphi\|_{s,X} \quad \forall \varphi \in U.$$

Using that $\|\cdot\|_{s,X} = \|\cdot\|_{L^2(D)}$, that $U = C^1_c(D)$ and that the norm of Y (because of how we have defined γ) dominates the norm of $H^2(G)$ (i.e., $\|u\|_{H^2(G)} \leqslant \|u\|_{s,Y}$), we have that there exists a constant $C > 0$ such that

$$\|V\varphi\|_{H^2(G)} \leqslant C \cdot \|\varphi\|_{L^2(D)} \quad \forall \varphi \in C^1_c(D).$$

The proof is concluded if we observe that $C^1_c(D)$ is dense in $L^2(D)$. □

Using this mapping property, we can extend (3.8) to H^2 functions.

Proposition 3.2.3. *Let $\varphi \in L^2_c(D)$. Then $u := V\varphi \in H^2_{loc}(\mathbb{R}^3)$ satisfies (3.8) in the H^2 sense (i.e, almost everywhere in $\mathbb{R}^3$).*

Proof. Let $\varphi \in L^2_c(\mathbb{R}^3)$. The density of $C^{0,\alpha}_c(\mathbb{R}^n)$ in $L^2_c(\mathbb{R}^3)$ implies that there exists a sequence $\{\varphi_n\} \subset C^{0,\alpha}_c(\mathbb{R}^n)$ such that $\varphi_n \xrightarrow{L^2} \varphi$. We can take this sequence with supports contained in $B(0, R)$ for some $R > 0$ such that $supp(\varphi) \subset B(0, R)$.

By Theorem 3.2.2, V is continuous from $L^2(B(0, R))$ to $H^2(D)$ for any bounded $D \subset \mathbb{R}^3$, so $V\varphi_n \xrightarrow{H^2(D)} V\varphi$ for every bounded D.

Therefore, fix a bounded domain D. Let $u_n := V\varphi_n$ and $u := V\varphi$. By Theorem 3.2.1,

$$\Delta(V\varphi_n) + k^2 V\varphi_n = -\varphi_n \quad \text{in } \mathbb{R}^3 \quad \forall n \in \mathbb{N}.$$

That is,

$$\Delta u_n + k^2 u_n = -\varphi_n \quad \text{in } \mathbb{R}^3 \quad n \in \mathbb{N}.$$

Since u_n are bounded in $H^2(D)$ (because, since they are a convergent sequence, $\|u_n\|_{H^2(D)} \leqslant C\|u\|_{H^2(D)}$), Theorem 5.1.23 from the Appendix (which we can apply since $H^2(D)$ is a separable Hilbert space, therefore reflexive) gives us that $u_n \rightharpoonup u$ in $H^2(D)$. Besides, by Rellich-Kondrachov's Compactness Theorem (see Theorem 5.1.12 of the Appendix), $u_n \xrightarrow{L^2(D)} u$.

Since $\|u_n\|_{H^2(D)} \leqslant C\|u\|_{H^2(D)}$, then $\|\Delta u_n\|_{L^2(D)} \leqslant C\|u\|_{H^2(D)}$, so Δu_n is a bounded sequence in $H^2(D)$. Therefore, again by Theorem 5.1.23, $\Delta u_n \rightharpoonup f$ in $L^2(D)$. By theorem 5.1.8, we have that $\Delta u_n \rightharpoonup \Delta u$ in $L^2(D)$.

Given $\eta \in C^\infty_c$, $f_\eta(\psi) := \int \psi \cdot \eta$ for $\psi \in L^2$ is a bounded linear functional in L^2. Therefore, we have that

$$\lim_n \int (\Delta u_n + k^2 u_n + \varphi_n) \cdot \eta = \int (\Delta u + k^2 u + \varphi_n) \cdot \eta.$$

The integrand of the left-hand side is 0 for every $n \in \mathbb{N}$. Therefore

$$\int (\Delta u + k^2 u + \varphi_n) \cdot \eta = 0$$

for every $\eta \in C_c^\infty(\mathbb{R}^3)$.

So by the theorem of convergence of Approximations of the Identity (see Theorem 5.1.3 from the Appendix), taking η so that the product $(\Delta u + k^2 u + \varphi_n) \cdot \eta$ is the convolution with an Aproximation of the Identity, we have that $\Delta u + k^2 u = -\varphi$ almost everywhere in D. Since this can be done for every bounded domain D in $\mathbb{R}^3$, we have that $\Delta u + k^2 u = -\varphi$ almost everywhere in $\mathbb{R}^3$, as we wanted to prove. ☐

For a bounded domain $D \subset \mathbb{R}^3$ with C^2 boundary, the Sobolev embedding theorem states that $H^2(D)$ functions are continuous. Moreover, Green's integral theorem remains valid for functions $u \in H^2(D)$ (see Lemma 5.1.7 of the Appendix).

Therefore, the proof of Green's representation formula for bounded domains (Theorem 1.3.1) can be carried over to H^2 functions, since the only tools used in it are the continuity of the function u and Green's second formula (5.5), and we can use them for $u \in H^2$ as we have just reasoned. Well, to be precise, we used in (1.1) one more condition: that the L^∞ norm of the gradient was bounded. This can be avoided if we use instead the Cauchy-Schwarz inequality, obtaining

$$\int_{S(x,r)} |\nabla u(y) \cdot \nu(y)| \, ds(y) \leqslant \|\nabla u\|_{L^2(S(x,r))} \sigma(S(x,r)) \leqslant C \|\nabla u\|_{L^2(B(x,r))} r^2 \xrightarrow{r \to 0^+} 0.$$

having used the trace inequality (Theorem 5.1.13) in the penultimate step because $u \in H^2(D)$ so $\nabla u|_{\partial D} \in H^{1/2}(\partial D)$, and on the last step that $\nabla u \in H^1(D)$ and therefore $\|\nabla u\|_{L^2(B(x,r))} \leqslant \|\nabla u\|_{L^2(D)}$ for r sufficiently small.

In particular, the formula for a solution of Helmholtz equation (1.3) remains valid for H^2 solutions to the Helmholtz equation. This implies that H^2 solutions to the Helmholtz equation automatically are C^2 solutions, since we can differentiate under the integral sign in expression (1.3) using Theorem 5.1.2 of the Appendix. Therfore, the Sommerfeld Radiation Condition is well defined for H^2 solutions.

3.2.2 Reformulation of the scattering problem

We now show that the scattering problem (3.3)-(3.5) is equivalent to the problem of solving the integral equation

$$u(x) = u^i(x) - k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y) \, dy, \quad x \in \mathbb{R}^3, \tag{3.14}$$

for u which is known as the *Lippmann-Schwinger equation.*

Theorem 3.2.4. *If $u \in H^2_{loc}(\mathbb{R}^3)$ is a solution of (3.3)-(3.5), then u is a solution of (3.14). Conversely, if $u \in C(\mathbb{R}^3)$ is a solution of (3.14) then $u \in H^2_{loc}(\mathbb{R}^3)$ and u is a solution of (3.3)-(3.5).*

Proof. Let $u \in H^2_{loc}(\mathbb{R}^3)$ be a solution of (3.3)-(3.5). Let $x \in \mathbb{R}^3$ be an arbitrary point and choose an open ball B (with unitary exterior normal ν) that contains the support of m and the point x (i.e., $supp(m) \subset B$ and $x \in B$).

We can apply Green's representation formula (Theorem 1.3.1) to u on B (since $\overline{B}$ is compact, so $u \in H^2(B)$ and therefore, by the discussion previous to the theorem, Green's representation formula can be applied) to the first integral of the following expression, obtaining that for $x \in B$

$$\begin{aligned}
&\int_{\partial B}\left[\frac{\partial u(y)}{\partial \nu(y)}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) - k^2\int_B \Phi(x,y)m(y)u(y)\,dy \\
&= u(x) + \int_B [\Delta u(y) + k^2 u(y)]\Phi(x,y)\,dy - k^2\int_B \Phi(x,y)m(y)u(y)\,dy \\
&\underset{\substack{\uparrow \\ (3.3)}}{=} u(x) + \int_B [-k^2 n(y)u(y) + k^2 u(y)]\Phi(x,y)\,dy - k^2\int_B \Phi(x,y)\cdot m(y)u(y)\,dy \\
&= u(x) + \int_B \Phi(x,y)k^2 u(y)[-n(y) + 1 - m(y)]\,dy \\
&\underset{\substack{\uparrow \\ m=1-n}}{=} u(x).
\end{aligned}$$

That is,

$$\int_{\partial B}\left[\frac{\partial u(y)}{\partial \nu(y)}\Phi(x,y) - u(y)\cdot\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) - k^2\int_B \Phi(x,y)\cdot m(y)\cdot u(y)\,dy = u(x), \quad x \in B. \tag{3.15}$$

Since u^i is an entire solution of Helmholtz equation (i.e., a solution on $\mathbb{R}^3$), in particular it is a solution on B, so we can apply to it Green's representation formula (1.3) obtaining

$$u^i(x) = \int_{\partial B}\left[\frac{\partial u^i(y)}{\partial \nu(y)}\Phi(x,y) - u^i(y)\cdot\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y), \quad x \in B. \tag{3.16}$$

Since u^s satisfies the radiation condition (3.5), Green's formula (5.5) and the *Sommerfeld Radiation Condition* (3.5) give us

$$\int_{\partial B}\left[\frac{\partial u^s}{\partial \nu}\Phi(x,\cdot) - u^s\cdot\frac{\partial \Phi(x,\cdot)}{\partial \nu}\right] ds = 0, \quad x \in B. \tag{3.17}$$

This is because, if $R > 0$ is such that $B \subsetneq B[0,R]$, then

$$\begin{aligned}
&-\int_{\partial B}\left[\frac{\partial u^s(y)}{\partial \nu(y)}\cdot\Phi(x,y) - u^s(y)\cdot\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) + \int_{S_R}\left[\frac{\partial u^s(y)}{\partial \nu(y)}\cdot\Phi(x,y) - u^s(y)\cdot\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) \\
&\underset{\substack{\uparrow \\ (5.5)}}{=} \int_{B_R\setminus B}[\Phi(x,y)\cdot\Delta u^s(y) - u^s\cdot\Delta\Phi(x,y)]\,dy \\
&= \int_{B_R\setminus B}[\Phi(x,y)\cdot(-k^2u^s) - u^s(y)\cdot(-k^2)\cdot\Phi(x,y)]\,dy = 0
\end{aligned}$$

for all R sufficiently big, having used in the second-to-last step that Φ is a solution of Helmholtz equation (that is, $\Delta\Phi = -k^2\Phi$) and that u^s is a radiating solution of Helmholtz equation on the exterior domain[1] $\mathbb{R}^3\setminus B$ since $supp(m) \subset B$. Therefore, we can proceed in exactly the same way as in Theorem 1.4.6 (because u^s satisfies the exact same hypothesis) in such a way that when we take $\lim_{R\to\infty}$ we have

$$0 = \lim_{R\to\infty}\int_{S_R}\left[\frac{\partial u^s(y)}{\partial \nu(y)}\Phi(x,y) - u^s(y)\cdot\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right] ds(y) - \int_{\partial B}(\text{same}) = -\int_{\partial B}(\text{same})$$

[1]Recall that we already noticed this: when we wrote (3.3) as $\Delta u^s + k^2u^s = k^2(1-n)u^i$, it was clear that u^s is a solution to the Helmholtz equation outside the support of $m = 1-n$.

as we wanted to prove.

We combine equations (3.15), (3.16) and (3.17) with (3.4) (i.e., with $u = u^i + u^s$) to obtain that

$$\begin{aligned} u(x) &\underset{(3.15)}{\overset{\uparrow}{=}} \int_{\partial B} \left[\frac{\partial u(y)}{\partial \nu(y)} \cdot \Phi(x,y) - u(y) \cdot \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right] ds(y) - k^2 \int_B \Phi(x,y) m(y) u(y)\, dy \\ &\underset{(3.4)}{\overset{\uparrow}{=}} \int_{\partial B} \left[\frac{\partial u^i(y)}{\partial \nu(y)} \Phi(x,y) - u^i(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right] ds(y) + \int_{\partial B} \left[\frac{\partial u^s(y)}{\partial \nu(y)} \Phi(x,y) - u^s(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right] ds(y) \\ &\quad - k^2 \int_B \Phi(x,y) m(y) u(y)\, dy \underset{(3.16) \text{ and } (3.17)}{\overset{\uparrow}{=}} u^i(x) - k^2 \int_B \Phi(x,y) m(y) u(y)\, dy, \end{aligned}$$

which is (3.14), as we wanted to prove.

Let us now prove the **converse**. Let $u \in C(\mathbb{R}^3)$ be a solution of (3.14). Let define u^s as

$$u^s(x) := -k^2 \int_{\mathbb{R}^3} \Phi(x,y) \cdot m(y) \cdot u(y)\, dy, \qquad x \in \mathbb{R}^3$$

(it's the logical way to define it so that $u = u^i + u^s$)

Since Φ satisfies the S.R.C. (3.5) uniformly with respect to y on compact sets (see Lemma 1.4.3) and m has compact support, then u^s also satisfies (3.5), since

$$\begin{aligned} &\lim_{r\to\infty} r \cdot \left(\frac{\partial u^s}{\partial r}(x) - iku^s(x) \right) \\ &= \lim_{r\to\infty} |x| \cdot \left(\nabla_x \left(\int_{\mathbb{R}^3} -k^2 \Phi(x,y) m(y) u(y)\, dy \right) \cdot \frac{x}{|x|} - ik \int_{\mathbb{R}^3} -k^2 \Phi(x,y) \cdot m(y) u(y)\, dy \right) \\ &= \lim_{r\to\infty} |x| \cdot \left(\int_{\mathbb{R}^3} -k^2 \nabla_x \Phi(x,y) m(y) u(y)\, dy \cdot \frac{x}{|x|} - ik \cdot \int_{\mathbb{R}^3} -k^2 \Phi(x,y) \cdot m(y) u(y)\, dy \right) \\ &= \lim_{r\to\infty} |x| \cdot \left(\int_{\mathbb{R}^3} -k^2 \nabla_x \Phi(x,y) \cdot \frac{x}{|x|} m(y) u(y)\, dy - ik \int_{\mathbb{R}^3} -k^2 \Phi(x,y) m(y) u(y)\, dy \right) \\ &= \lim_{r\to\infty} |x| \cdot \int_{supp(m)} -k^2 m(y) u(y) \left(\frac{\partial \Phi(x,y)}{\partial r(x)} - ik\Phi(x,y) \right) dy \\ &= -k^2 \int_{supp(m)} -k^2 m(y) u(y) \cdot \lim_{r\to\infty} |x| \cdot \left(\frac{\partial \Phi}{\partial r(x)}(x,y) - ik\Phi(x,y) \right) dy = 0, \end{aligned}$$

where we have used in the last step that $m(y)u(y)$ is bounded on the support of m (u because it is continuous, and m because it is piecewise continuous), and the limit inside the integral is 0 because of Corollary 1.4.4, because we are considering $y \in supp(m)$, which is a compact set.

Since m is piecewise continuous with compact support, then $\varphi := -k^2 mu \in L^2(D)$ with $D = supp(m)$ (since $u \in C(\mathbb{R}^3)$, so it is bounded on $supp(m)$, thus $\varphi \in L^\infty(D)$, and since D has finite measure because it is bounded, $\varphi \in L^2(D)$). So by Theorem 3.2.2, $V\varphi \in H^2(G)$ for every bounded domain G. That is, $u^s = V\varphi \in H^2_{loc}(\mathbb{R}^3)$.

Besides, because of (3.8) from Theorem 3.2.1, $u^s = V\varphi$ satisfies

$$\Delta u^s + k^2 u = k^2 mu = -\varphi \text{ on } \mathbb{R}^3.$$

Since u^i is an entire solution to the Helmholtz equation, i.e., $\Delta u^i + k^2 u^i = 0$ on $\mathbb{R}^3$, then

$$\Delta u + k^2 u = (\Delta u^i + k^2 u^i) + (\Delta u^s + k^2 u^s) = k^2 mu,$$

that is,

$$\Delta u + k^2(1 - m)u = \Delta u + k^2 nu = 0$$

on $\mathbb{R}^3$, as we wanted to prove. $\square$

Remark 3.2.5. Notice that in (3.14) we can replace the region of integration by any domain G such that the support of m is contained in $\overline{G}$ and look for solutions in $C(\overline{G})$. So if we have a solution u to (3.14) defined on a domain $\overline{G}$ that contains the support of m, we can extend it for $x \in \mathbb{R}^3 \backslash \overline{G}$ defining $u(x)$ by the right-hand side of (3.14), and we obtain in this way a continuous solution u to the Lippmann-Schwinger equation (3.14) in all of $\mathbb{R}^3$.

3.3 The Unique Continuation Principle

In order to establish the existence of a unique solution to the scattering problem (3.3)-(3.5) for all positives values of the wave number k, we see from Theorem 3.2.4 that it is sufficient to establish the existence of a unique solution to the Lippmann-Schwinger equation (3.14). To this end, we would like to apply Fredholm's Theorem.

Define the integral operator[2] $T\colon (C(\overline{B}), \|\cdot\|_\infty) \to (C(\overline{B}), \|\cdot\|_\infty)$ by

$$(Tu)(x) := -\int_B \Phi(x,y)m(y)u(y)\,dy, \quad x \in \overline{B} \tag{3.18}$$

where B is a ball such that $supp(m) \subset B$ (notice that, by Remark 3.2.5, it does not matter which region of integration we choose as long as it contains the support of m). This integral operator has a weakly singular kernel $\Phi(x,y)m(y)$, since $m(y)$ is bounded (it is piecewise continuous and of compact support) and $|\Phi(x,y)| \leqslant \frac{1}{|x-y|}$, and therefore, they satisfy Definition 5.1.18 from the Appendix for $\alpha = 2$, $n = 3$. Hence, by Theorem 5.1.19 from the Appendix, $T\colon C(\overline{B}) \to C(\overline{B})$ is a **compact** operator.

In terms of this operator T, the Lippman-Schwinger equation (3.14) is equivalent to the following: given an incident field u^i, find $u \in C(\overline{B})$ such that

$$(I - T)u = u^i. \tag{3.19}$$

Since T is compact, Fredholm's theorem 2.2.8 asserts that it is enough to prove injectivity[3] of $I - T$ in order to prove bijectivity (i.e. existence and uniqueness). Therefore, we must show that the homogeneous equation has only the trivial solution, which is equivalent to[4] prove that the only solution of

$$\Delta u + k^2 n(x) u = 0 \text{ in } \mathbb{R}^3, \tag{3.20}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{3.21}$$

is $u \equiv 0$

To prove this, we need a **unique continuation principle**. The tough part is to prove the following lemma:

Lemma 3.3.1. *Let G be a domain in $\mathbb{R}^3$ and let $u_1, \dots, u_P \in H^2(G)$ be real valued functions satisfying*

$$|\Delta u_p| \leqslant c \sum_{q=1}^{P} [|u_q| + |\nabla u_q|] \text{ in } G \tag{3.22}$$

for $p = 1, \dots, P$ and some constant c. Assume that u_p vanishes in a neighborhood of some $x_0 \in G$ for $p = 1, \dots, P$. Then u_p is identically zero in G for $p = 1, \dots, P$.

Proof. See Section 3.6 for a detailed proof of this lemma. □

[2]Notice that $C(\overline{B})$ with the maximum norm $\|\cdot\|_\infty$ (maximum, since they are continuous functions on a compact set) is a Banach Space because it is a closed subspace of $(L^\infty(\overline{B}), \|\cdot\|_\infty)$ (closed because the uniform limit of continuous functions on a compact set is continuous).

[3]That is, the homogeneous equation $(I-T)u = 0$ has zero as its only solution.

[4]This is true because the Lippmann-Schwinger equation $(I-T)u = u^i$ on $C(\overline{B})$ is equivalent by Theorem 3.2.4 to (3.3)-(3.5). Since the homogeneous equation is the one with $u^i \equiv 0$, equations (3.3)-(3.5) are equivalent to (3.20)-(3.21), since $u^i = 0$ if and only if $u = u^s$, and therefore the radiation condition for u^s is equivalent to the radiation condition for u.

Theorem 3.3.2 (Unique continuation principle). *Let G be a domain in $\mathbb{R}^3$ and suppose $u \in H^2(G)$ is a solution of*

$$\Delta u + k^2 n(x) u = 0$$

in G such that n is piecewise continuous in G and u vanishes in a neighborhood of some $x_0 \in G$. Then u is identically zero in G.

Proof. We can apply Lemma 3.3.1 to $u_1 := \operatorname{Re} u$ and $u_2 := \operatorname{Im} u$. This is because, by definition of u_1 and u_2, $u = u_1 + iu_2$ and $\Delta u = \Delta u_1 + i\Delta u_2$, and because, by hypothesis, $\Delta u = -k^2 nu$. These equations imply that

$$|\Delta u_1| \leqslant |\Delta u| \leqslant k^2 \|n\|_\infty |u| \leqslant k^2 \|u\|_\infty [|u_1| + |u_2|].$$

and the same applies to u_2. So u_1 and u_2 are real valued functions that satisfy (3.22).

If u vanishes in a neighbourhood of some point $x_0 \in G$, then $u_1 = \operatorname{Re} u$ and $u_2 = \operatorname{Im} u$ do vanish as well in that same neighbourhood. Therefore, by Lemma 3.3.1, u_1 and u_2 vanish identically on G, so $u = u_1 + iu2$ as well. □

We can now show that for all $k > 0$ there exists a unique solution to the scattering problem (3.3) - (3.5).

Theorem 3.3.3. *For each $k > 0$ there exists a unique solution $u \in H^2_{loc}(\mathbb{R}^3)$ to (3.3)-(3.5) and u depends continuously with respect to the maximum norm on the incident field u^i.*

Proof. By Theorem 3.2.4, the existence and uniquenes of a solution $u \in H^2_{loc}(\mathbb{R}^3)$ to (3.3)-(3.5) is equivalent to existence and uniqueness of a solution $u \in C(\mathbb{R}^3)$ to the Lippman-Schwinger equation (3.14).

Notice that this is equivalent to proving existence and uniqueness of a solution $u \in C(\overline{B})$ to (3.14) for B a ball or any bounded region that contains *supp*(m), as reasoned in Remark 3.2.5.

In brief, we just need to prove existence and uniqueness of the Lippmann-Schwinger equation (3.14) in $C(\overline{B})$ for a ball B that contains *supp*(m). As argued at the beginning of this section, the Lippmann-Schwinger equation (3.14) can be expressed as: given u^i an entire solution to Helmholtz equation, find $u \in C(\overline{B})$ such that $(I - T)u = u^i$ where T is defined in (3.18). We deduced that T is a compact operator. Hence, we can apply Fredholm's theorem, which stablishes that in order to prove existence and uniqueness, it is enough to prove that the only solution of $(I - T)u = 0$ is $u = 0$. This, because of the equivalences we have just explained, is equivalent to prove that the only solution to (3.3)-(3.5) for $u^i = 0$ is $u = 0$, i.e., that if u is a solution of (3.20)-(3.21), then $u \equiv 0$.

So suppose u is a solution to (3.20)-(3.21). Let $B := B(0, r)$ be a ball of radius r centered at the origin such that m vanishes outside of B, i.e., *supp*$(m) \subset B$. Let ν denote the exterior unit normal to ∂B. From Green's theorem (5.4) and from $\Delta u + k^2 n(x) u = 0$ in $\mathbb{R}^3$ (which is true because u is a solution of (3.20)-(3.21)) we have that

$$\int_{|x|=r} u \frac{\partial \overline{u}}{\partial \nu}\, ds = \int_{|x|<r} [u \cdot \Delta \overline{u} + \nabla u \cdot \nabla \overline{u}] = \int_{|x|<r} [-k^2 \overline{n} |u|^2 + |\nabla u|^2]\, dx.$$

Since $\operatorname{Im}(n) \geqslant 0$, $\operatorname{Im}(\overline{n}) = -\operatorname{Im}(n)$, and $\operatorname{Im}(\lambda z) = \lambda \operatorname{Im}(z)$ for $\lambda \in \mathbb{R}$, we have

$$\operatorname{Im}\left(\int_{|x|=r} u \frac{\partial \overline{u}}{\partial \nu}\, ds \right) = \int_{|x|<r} \operatorname{Im}\left(-k^2 \overline{n} |u|^2 + |\nabla u|^2\right) dx = k^2 \int_{|x|\leqslant r} \operatorname{Im}(n) \cdot |u|^2\, dx \geqslant 0. \tag{3.23}$$

Therefore, Theorem 1.7.2 shows that $u(x) = 0$ for $|x| \geqslant r$. So, by the Unique Continuation Principle 3.3.2,

$$u(x) = 0 \quad \text{for all } x \in \mathbb{R}^3.$$

Therfore, by Fredholm's Theorem 2.2.8, the operator $I - T$ is bijective. Since it is linear and bounded, Theorem 5.1.22 from the Appendix gives us that its inverse is bounded as well. That is, the integral equation (3.14) can be inverted in $C(\overline{B})$ and the inverse operator is bounded.

From this, the boundedness of $(I - T)^{-1}$ implies that $u = (I - T)^{-1}u^i$ depends continuously on the incident field u^i with respect to the maximum norm (i.e., $\|\cdot\|_\infty$). $\square$

3.4 The Far Field Pattern

From (3.14) and Theorem 3.2.4 we see that the scattered field is given by

$$u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)\,dy, \quad x \in \mathbb{R}^3.$$

Remark 3.4.1. In fact, since m has compact support $D = supp(m)$, the above integral can be taken over D (or any measurable set that contains it). See Remark 3.2.5.

Hence, letting $|x|$ tend to infinity, we can apply formula (1.5) to obtain

$$\begin{aligned} u^s(x) &= -k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)\,dy \\ &\underset{(1.16)}{\overset{}{\underset{\uparrow}{=}}} -k^2 \int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{e^{ik|x|}}{|x|}\left[e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right] m(y)u(y)\,dy \\ &= \frac{e^{ik|x|}}{|x|}\left(-\frac{k^2}{4\pi}\right)\int_{\mathbb{R}^3}\left[e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right] m(y)u(y)\,dy. \end{aligned}$$

Defining

$$u_\infty(\hat{x}) := -\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y)u(y)\,dy \tag{3.24}$$

for $\hat{x} = x/|x|$ on the unit sphere $\mathbb{S}^2$, we have that

$$\begin{aligned} u^s(x) &= \frac{e^{ik|x|}}{|x|}\cdot u_\infty(\hat{x}) + \frac{e^{ik|x|}}{|x|}\left(-\frac{k^2}{4\pi}\right)\int_{\mathbb{R}^3} O\left(\frac{1}{|x|}\right) m(y)u(y)\,dy \\ &= \frac{e^{ik|x|}}{|x|}\cdot u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right) \end{aligned}$$

being the last equality true because $\frac{e^{ik|x|}}{|x|} = O\left(\frac{1}{|x|}\right)$ and

$$\left|\int_{\mathbb{R}^3} O\left(\frac{1}{|x|}\right) m(y)u(y)\,dy\right| \leqslant \int_{\mathbb{R}^3}\left|O\left(\frac{1}{|x|}\right) m(y)u(y)\right| dy \leqslant \int_{\mathbb{R}^3}\frac{C}{|x|}|m(y)u(y)|\,dy \leqslant \frac{C'}{|x|}$$

where the last step is true because $supp(m)$ is compact, m is bounded and $u \in C(\mathbb{R}^3)$, so it is bounded on compact sets. In brief,

$$u^s(x) = \frac{e^{ik|x|}}{|x|}u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

where u_∞ is called the *far field pattern* and is given by (3.24) for $\hat{x} = x/|x|$ on the unit sphere $\mathbb{S}^2$.

Remark 3.4.2. Up to now, we have given two definitions of far field pattern: one in Theorem 1.6.1 for arbitrary radiating solutions of Helmholtz, and the one we have just given for the scattered field of problem (3.3)-(3.5). However, this scattered field that satisfies (3.3)-(3.5) is a radiating solution to Helmholtz equation on $\mathbb{R}^3\backslash D$: radiating because of (3.5); solution to Helmholtz equation because (3.3)-(3.4) imply that $u^s + k^2nu^s = k^2(n-1)u^i$ on $\mathbb{R}^3$, with $n \equiv 1$ on $\mathbb{R}^3\backslash D$.

So for this scattered field we have two definitions of far field pattern: the one in (3.24) and the one in (1.15).

Let us see that these two definitions coincide. Let ν be the unit normal of ∂D directed into the exterior of $\mathbb{R}^3 \backslash D$ (i.e., the interior of D). Then, by definition (1.15), we have

$$\begin{aligned} u_\infty(\hat{x}) &= \frac{1}{4\pi}\int_{\partial D}\left[u^s(y)\cdot\frac{\partial}{\partial\nu(y)}e^{-ik\hat{x}\cdot y} - e^{ik\hat{x}\cdot y}\cdot\frac{\partial}{\partial\nu}u^s(y)\right]ds(y) \\ &= \frac{1}{4\pi}\int_{\partial D}\Bigg[-k^2\int_{\mathbb{R}^3}m(y)\Phi(y,z)u(z)\,dz\cdot\frac{\partial}{\partial\nu(y)}e^{-ik\hat{x}\cdot y} \\ &\quad - e^{ik\hat{x}\cdot y}\cdot(-k^2)\int_{\mathbb{R}^3}m(z)\frac{\partial}{\partial\nu(y)}\Phi(y,z)u(z)\,dz\Bigg]ds(y) \\ &= -\frac{k^2}{4\pi}\left[\int_{\mathbb{R}^3}m(z)u(z)\cdot\left[\int_{\partial D}\Phi(y,z)\frac{\partial}{\partial\nu(y)}e^{-ik\hat{x}\cdot y} - e^{ik\hat{x}\cdot y}\frac{\partial}{\partial\nu(y)}\Phi(y,z)\,ds(y)\right]dz\right] \\ &= -\frac{k^2}{4\pi}\int_{\mathbb{R}^3}m(z)u(z)(-e^{-ik\hat{x}\cdot z})\,dz \end{aligned}$$

where in the last step we have used Green's representation formula on the bounded domain $D = supp(m)$ applied to the entire solution of Helmholtz equation $e^{-ik\hat{x}\cdot y}$ as a function of y. Notice that, as in Remark 3.4.1, the volume integrals can be taken over $D = supp(m)$, and therefore it is enough to consider $z \in D$, with D an interior domain of class C^2 by hypothesis. Then, for $z \in D$, it is legitimate to apply Theorem 1.3.1, which is what we have done.

Notice as well that the minus sign obtained in the last step is because ν is the unit normal on ∂D directed into the interior of D, so we have to change the sign of the representation formula of Theorem 1.3.1.

3.4.1 Reciprocity relation

Let us consider the case when the incident field u^i is a plane wave, i.e.,

$$u^i(x) = e^{ikx\cdot d}$$

where $d \in \mathbb{S}^2$ is a unit vector giving the direction of propagation.

Notation: we denote the dependence of the far field pattern u_∞ on d by writing $u_\infty(\hat{x}) = u_\infty(\hat{x}; d)$. Similarly, we write $u^s(x) = u^s(x; d)$ and $u(x) = u(x; d)$ for the scattered field an the total field respectively.

First, we prove a reciprocity principle for u_∞. $u_\infty(\hat{x}; d)$ represents the observation in the direction $\hat{x}$ of the scattered wave produced by a plane wave travelling in the direction d. Therefore, the reciprocity principle states the (physically reasonable) fact that the observation made in the direction $\hat{x}$ of an object illuminated from the direction $\hat{\theta}$ is the same as the obervation made in the direction $-\hat{\theta}$ of the same object but illuminated from $-\hat{x}$. Since we are dealing with acoustic waves instead of electromagnetic ones, change the word "illumination" by "sound emited" (but light is easier to imagine).

Theorem 3.4.3. *The far field pattern satisfies the reciprocity relation*

$$u_\infty(\hat{x}; \hat{\theta}) = u_\infty(-\hat{\theta}; -\hat{x})$$

for all $\hat{x}$, $\hat{\theta}$ on the unit sphere $\mathbb{S}^2$.

Proof. First, notice that if $u \in H^2_{loc}(\mathbb{R}^3)$ is a solution of (3.3)-(3.5) for the incident field $u^i(x;d) := e^{ikx\cdot d}$, then $u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)\,dy$ is a function of class[5] $C^2(\mathbb{R}^3)$, and therefore $u = u^i + u^s$ also is $C^2(\mathbb{R}^3)$ (since u^i is analytic as well).

Therefore, we can apply Green's second formula (5.5): since u^i is an entire solution of the Helmholtz equation, applying it on the interior domain $B(0,a)$ we obtain

$$\begin{aligned}
&\int_{|y|=a} \left[u^i(y;\hat{\theta}) \frac{\partial}{\partial \nu} u^i(y;-\hat{x}) - u^i(y;-\hat{x}) \frac{\partial}{\partial \nu} u^i(y;\hat{\theta}) \right] ds(y) \\
&= \int_{B(0,a)} [u^i(y,\hat{\theta})\Delta u^i(y;-\hat{x}) - u^i(y;-\hat{x})\Delta u^i(y;\hat{\theta})]\,dy \\
&= \int_{B(0,a)} \left(u^i(y;\hat{\theta}) \left(-k^2 u^i(y;-\hat{x})\right) - u^i(y;-\hat{x}) \left(-k^2 u^i(y;\hat{\theta})\right) \right) dy = 0.
\end{aligned}$$

That is,

$$\int_{|y|=a} \left[u^i(y;\hat{\theta}) \frac{\partial}{\partial \nu} u^i(y;-\hat{x}) - u^i(y;-\hat{x}) \frac{\partial}{\partial \nu} u^i(y;\hat{\theta}) \right] ds(y) = 0. \tag{3.25}$$

On the other hand, u^s is a solution of the Helmholtz equation on the exterior domain $\mathbb{R}^3 \backslash B(0,a)$. Therefore, applying Green's second formula on (5.5) on $D_R := B(0,R)\backslash B(0,a)$ for $R > a$, and denoting by ν the unit exterior normal to D_R, we obtain

$$\begin{aligned}
&\int_{\partial D_R} \left[u^s(y;\hat{\theta}) \frac{\partial}{\partial \nu} u^s(y;-\hat{x}) - u^s(y;-\hat{x}) \frac{\partial}{\partial \nu} u^s(y;\hat{\theta}) \right] ds(y) \\
&= \int_{B(0,R)\backslash B(0,a)} [u^s(y;\hat{\theta})\Delta u^s(y;-\hat{x}) - u^s(y;-\hat{x})\Delta u^s(y;\hat{\theta})]\,dy \\
&= \int_{B(0,R)\backslash B(0,a)} \left(u^s(y;\hat{\theta}) \left(-k^2 u^s(y;-\hat{x})\right) - u^s(y;-\hat{x})(-k^2 u^s(y;\hat{\theta})) \right) dy = 0.
\end{aligned}$$

Since $\partial D_R = S(0,a) \cup S(0,R)$, we have that

$$\begin{aligned}
0 = &\int_{|y|=a} \left[u^s(y;\hat{\theta}) \frac{\partial}{\partial \nu} u^s(y;-\hat{x}) - u^s(y;-\hat{x}) \frac{\partial}{\partial \nu} u^s(y;\hat{\theta}) \right] ds(y) \\
&+ \int_{|y|=R} \left[u^s(y;\hat{\theta}) \frac{\partial}{\partial \nu} u^s(y;-\hat{x}) - u^s(y;-\hat{x}) \frac{\partial}{\partial \nu} u^s(y;\hat{\theta}) \right] ds(y).
\end{aligned}$$

Since u^s satisfies the same hypothesis as in Theorem 1.4.6, following its proof we see that the limit of the second integral as $R \to \infty$ is 0. Therefore,

$$0 = \int_{|y|=a} \left[u^s(y;\hat{\theta}) \frac{\partial}{\partial \nu} u^s(y;-\hat{x}) - u^s(y;-\hat{x}) \frac{\partial}{\partial \nu} u^s(y;\hat{\theta}) \right] ds(y). \tag{3.26}$$

Of the two expressions that we have for the far field pattern, we now use the one given in (1.15), replacing the exponentials of that formula by u^i, because we are considering $u^i(y,-\hat{x}) = e^{iky\cdot(-\hat{x})}$. We apply it to the far field patterns $u_\infty(\hat{x};\hat{\theta})$ and $u_\infty(-\hat{\theta},-\hat{x})$, obtaining that

$$4\pi u_\infty(\hat{x};\hat{\theta}) = \int_{|y|=a} \left[u^s(y;\hat{\theta}) \frac{\partial}{\partial \nu} u^i(y;-\hat{x}) - u^i(y;-\hat{x}) \frac{\partial}{\partial \nu} u^s(y;\hat{\theta}) \right] ds(y) \tag{3.27}$$

[5] In fact, $C^\infty(\mathbb{R}^3)$; in fact, it is analytic outside of the ball $B(0,a)$, but we will just need it to be $C^2(\mathbb{R}^3)$.

and

$$4\pi u_\infty(-\hat{\theta}, -\hat{x}) = \int_{|y|=a} \left[u^s(y; -\hat{x}) \frac{\partial}{\partial \nu} u^i(y; \hat{\theta}) - u^i(y; \hat{\theta}) \frac{\partial}{\partial \nu} u^s(y; -\hat{x}) \right] ds(y). \tag{3.28}$$

We substract the (3.28) from the sum of (3.25)-(3.27). This yields

$$4\pi[u_\infty(\hat{x}; \hat{\theta}) - u_\infty(-\hat{\theta}, -\hat{x})] = \int_{|y|=a} \left[u(y; \hat{\theta}) \frac{\partial}{\partial \nu} u(y; -\hat{x}) - u(y; -\hat{x}) \frac{\partial}{\partial \nu} u(y; \hat{\theta}) \right] ds(y).$$

To obtain the statement of the theorem, it suffices to show that this expression vanishes. But this follows directly from Green's first formula (5.4), since

$$\int_{|y|=a} u(y; \hat{\theta}) \frac{\partial}{\partial \nu} u(y; -\hat{x})\, ds(y) = \int_{B(0,a)} [u(y; \hat{\theta}) \Delta u(y; -\hat{x}) + \nabla u(y; \hat{\theta}) \cdot \nabla u(y; -\hat{x})]\, dy$$

and

$$\int_{|y|=a} u(y; -\hat{x}) \frac{\partial}{\partial \nu} u(y; \hat{\theta})\, ds(y) = \int_{B(0,a)} [u(y; -\hat{x}) \Delta u(y; \hat{\theta}) + \nabla u(y; -\hat{x}) \cdot \nabla u(y; \hat{\theta})]\, dy,$$

so substracting the first expression from the second we get

$$\begin{aligned} &\int_{|y|=a} \left[u(y; \hat{\theta}) \frac{\partial}{\partial \nu} u(y; -\hat{x}) - u(y; -\hat{x}) \frac{\partial}{\partial \nu} u(y; \hat{\theta}) \right] ds(y) \\ &= \int_{B(0,a)} \left[u(y; \hat{\theta}) \Delta u(y; -\hat{x}) - u(y; -\hat{x}) \Delta u(y; \hat{\theta}) \right] dy = 0 \end{aligned}$$

having used in the last step that u is solution of (3.3) and therefore $\Delta u(y; -\hat{x}) = -k^2 n(y) u(y; -\hat{x})$ and $\Delta u(y; \hat{\theta}) = -k^2 n(y) u(y; \hat{\theta})$.

Hence $u_\infty(\hat{x}; \hat{\theta}) - u_\infty(-\hat{\theta}; -\hat{x}) = 0$, i.e.,

$$u_\infty(\hat{x}; \hat{\theta}) = u_\infty(-\hat{\theta}; -\hat{x}),$$

as desired. □

3.4.2 Completeness of the far field patterns

For applications, it is important to study if the far field patterns corresponding to all incident plane waves are complete in $L^2(\mathbb{S}^2)$. We are going to see that the far field patterns are complete provided k^2 is not a transmission eigenvalue having an eigenfunction that is a Herglotz wave function. We will explain what are these concepts in a moment.

Definition 3.4.4. A *Herglotz wave function* is a function of the form

$$v(x) = \int_{\mathbb{S}^2} e^{ikx \cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3 \tag{3.29}$$

where $g \in L^2(\mathbb{S}^2)$. The function g is called the Herglotz kernel of v.

Herglotz wave functions are understood as a superposition of plane waves, i.e., a superposition of waves of the form $e^{ikx \cdot d}$ with $d \in \mathbb{S}^2$. Because we can differentiate under the integral sign and $e^{ikx \cdot d}$ is a solution to Helmholtz equation as a function of x, Herglotz wave functions are entire solutions to the Helmholtz equation.

Remark 3.4.5. In this TFM, we have only considered scattering by an inhomogeneous medium, but we have not said anything about the case of obstacle scattering (see Chapter 3 of [13] for a treatment of this case). However, in the case of obstacle scattering, there is an analogous result to the one we are going to study here for scattering by an inhomogeneous medium. In obstacle scattering, the set of far field patterns for this problem is complete in $L^2(\mathbb{S}^2)$ provided that k^2 is not a a Dirichlet eigenvalue having an eigenfunction that is a Herglotz wave function. That is, provided that $\Delta u = -k^2 u$ on D, $u = 0$ on ∂D does not have as a solution a Herglotz wave function. Herglotz wave functions are already solutions to $\Delta u = -k^2 u$, the set of far field patterns is complete provided that no Herglotz wave function vanishes on ∂D. For a complete discussion see [13], pages 75-76, specifically Theorem 3.29.

In the present case of scattering by an inhomogeneous medium we have a similar result except that the Dirichlet problem is replaced by a new type of boundary value problem: the *interior transmission problem*. This name is motivated by the fact that we have two partial differential equations defined on the same interior domain and linked together by their Cauchy data on the boundary.

We assume that $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ is connected with a connected C^2 boundary ∂D and D contains the origin.

Theorem 3.4.6. *Let $\{d_n : n \in \mathbb{N}\}$ be a countable dense set of vectors on the unit sphere $\mathbb{S}^2$ and define the class $\mathcal{F}$ of far field patterns by*

$$\mathcal{F} := \{u_\infty(\cdot, d_n) : n \in \mathbb{N}\}.$$

The orthogonal complement of $\mathcal{F}$ in $L^2(\mathbb{S}^2)$ consists of ***the conjugate*** *of those functions $g \in L^2(\mathbb{S}^2)$ for which there exists $w \in H^2(D)$ and a Herglotz wave function*

$$v(x) = \int_{\mathbb{S}^2} e^{-ikx\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3,$$

such that the pair v, w is a solution to

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } D \tag{3.30}$$

satisfying

$$w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \tag{3.31}$$

Proof. Let $\mathcal{F}^\perp$ denote the orthogonal complement to $\mathcal{F}$. We will show that $\overline{g} \in \mathcal{F}^\perp$ if and only if g satisfies the assumptions stated in the theorem.

$$\begin{aligned}
\overline{g} \in \mathcal{F}^\perp &\iff \langle u_\infty(\cdot, d_n), \overline{g} \rangle_{L^2(\mathbb{S}^2)} = 0 \quad \forall n \in \mathbb{N} \\
&\iff \int_{\mathbb{S}^2} u_\infty(\hat{x}; d_n) \cdot g(\hat{x})\, ds(\hat{x}) = 0 \quad \forall n \in \mathbb{N} \\
&\underset{\substack{\uparrow \\ \text{Thm.3.4.3}}}{\iff} \int_{\mathbb{S}^2} u_\infty(-d_n; -\hat{x}) \cdot g(\hat{x})\, ds(\hat{x}) = 0 \quad \forall n \in \mathbb{N}
\end{aligned}$$

From the continuity of u_∞ as a function of d (due to the continuity of u_∞ with respect to $\hat{x}$ and the reciprocity principle) and Theorem 3.4.3, we obtain

$$\begin{aligned}
\int_{\mathbb{S}^2} u_\infty(-d_n; -\hat{x}) \cdot g(\hat{x})\, ds(\hat{x}) = 0 \quad \forall n \in \mathbb{N} &\iff \int_{\mathbb{S}^2} u_\infty(-d; -\hat{x}) \cdot g(\hat{x})\, ds(\hat{x}) = 0 \quad \forall d \in \mathbb{S}^2 \\
&\iff \int_{\mathbb{S}^2} u_\infty(\hat{x}; d) \cdot g(-d)\, ds(d) = 0 \quad \forall \hat{x} \in \mathbb{S}^2,
\end{aligned}$$

having renamed in the last step $-d \mapsto \hat{x}$ and $-\hat{x} \mapsto d$.

In brief,

$$\overline{g} \in \mathcal{F}^{\perp} \iff \int_{\mathbb{S}^2} u_\infty(\hat{x}; d) \cdot g(-d)\, ds(d) = 0 \quad \forall \hat{x} \in \mathbb{S}^2. \tag{3.32}$$

Remark 3.4.7. What the deduction of (3.32) is really telling us is that the density of the directions $\{d_n\}_{n\in\mathbb{N}}$ and the continuity of $u_\infty(\hat{x}, d)$ with respect to the direction of propagation d allow us to pass from

the orthogonality of $\overline{g}$ in $L^2(\mathbb{S}^2)$ to $\{u_\infty(\hat{x}; d_n)\}_n$ with $\{d_n\}_n$ a dense set of directions in $\mathbb{S}^2$

to

the orthogonality of $\overline{g}$ in $L^2(\mathbb{S}^2)$ to the collection $\{u_\infty(\hat{x}; d) : d \in \mathbb{S}^2\}$ for every direction $d \in \mathbb{S}^2$.

That is why in the statement of this theorem we stablish the equality of the orthogonal complement of $\mathcal{F}$ (which depends *a priori* on the directions d_n) and a set of functions where the directions d_n play no role (that is, the conjugate of those functions for which there exist solutions of a specific kind to the problem (3.30) and (3.31)). So we could substitute the countable dense set $\{d_n\}_{n\in\mathbb{N}}$ by any dense set of $\mathbb{S}^2$ (including $\mathbb{S}^2$).

We now continue with the proof. From Theorem 3.2.4, we have that

$$u(x; d) = u^i(x; d) + u^s(x; d) = e^{ikxd} + u^s(x; d) \tag{3.33}$$

where

$$u^s(x; d) = -k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) u(y; d)\, dy.$$

Let us prove the equivalence stated in the Theorem.

To begin with, suppose that $\overline{g} \in \mathcal{F}$. As a consequence of (3.32), this is equivalent to

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(-d)\, ds(d) = 0 \quad \forall \hat{x} \in \mathbb{S}^2.$$

We need to find w and v satisfying the conditions of the Theorem. The function v is already given by the theorem: it has to be

$$v(x) := \int_{\mathbb{S}^2} e^{-ikx\cdot d} g(d)\, ds(d), \qquad x \in \mathbb{R}^3,$$

which already satisfies $\Delta v + k^2 v = 0$ in D.

Let us see which function must be w. Multiplying (3.33) by $g(-d)$ and integrating over $d \in \mathbb{S}^2$ we have that

$$\int_{\mathbb{S}^2} u(x; d) g(-d)\, ds(d) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(-d)\, ds(d) + \int_{\mathbb{S}^2} u^s(x; d) g(-d)\, ds(d). \tag{3.34}$$

Notice that

$$w(x) := \int_{\mathbb{S}^2} u(x, d) g(-d)\, ds(d), \quad x \in \mathbb{R}^3,$$

$$w^i(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(-d)\, ds(d), \quad x \in \mathbb{R}^3$$

and

$$w^s(x) = \int_{\mathbb{S}^2} u^s(x, d) g(-d)\, ds(d)$$

satisfy (3.3)-(3.5), because

- Since we can differentiate under the integral sign and e^{ikxd} is an entire solution of Helmholtz's equation, w^i is an entire solution of Helmholtz's equation.
- Since we can differentiate under the integral sign and $u(\cdot, d)$ is a solution of $\Delta u + k^2 nu = 0$ in $\mathbb{R}^3$, we have that w is as well a solution of (3.3).
- Since u^s satisfies the radiation condition (3.5), we can derivate under the integral sign and apply the Dominated Convergence Theorem to obtain that w^s also satisfies the radiation condition (3.5).

In brief, if we define an incident field as

$$w^i(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(-d)\, ds(d),$$

its scattered field is

$$w^s(x) = \int_{\mathbb{S}^2} u^s(x,d) g(-d)\, ds(d).$$

Let us see what is the far field pattern w_∞ associated to w.

$$\begin{aligned}
w_\infty(\hat{x}) &= -\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) w(y)\, dy \\
&= -\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) \left(\int_{\mathbb{S}^2} u(x,d) g(-d)\, ds(d)\right) dy \\
&= \int_{\mathbb{S}^2}\left(-\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot d} m(y) u(y)\, dy\right) g(-d)\, ds(d) \\
&= \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(-d)\, ds(d).
\end{aligned}$$

That is,

$$w_\infty(\hat{x}) = \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(-d)\, ds(d),$$

Therefore, since we are assuming that $\overline{g} \in \mathcal{F}$, then by (3.32) we have that

$$w_\infty(\hat{x}) \equiv 0 \quad \text{in } \mathbb{S}^2.$$

To summarize, the far field pattern of the scattered field w^s corresponding to the incident field

$$w^i(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(-d)\, ds(d) = \int_{\mathbb{S}^2} e^{-ikx\cdot d} g(d)\, ds(d).$$

is $\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(-d)\, ds(d)$, which is zero by assumption.

Therefore, by Theorem 1.7.3, $w^s = 0$ in all of $\mathbb{R}^3 \backslash D$. So, if we define

$$v := w^i \quad \text{and} \quad w := w^i + w^s$$

then (3.30) is satisfied (as we have already discussed). Moreover, they satisfy (3.31) as well because

$$w|_{\partial D} = w^i\Big|_{\partial D} + w^s|_{\partial D} = w^i\Big|_{\partial D} = v|_{\partial D}$$

and

$$\left.\frac{\partial w}{\partial \nu}\right|_{\partial D} = \left.\frac{\partial w^i}{\partial \nu}\right|_{\partial D} + \left.\frac{\partial w^s}{\partial \nu}\right|_{\partial D} = \left.\frac{\partial w^i}{\partial \nu}\right|_{\partial D} = \left.\frac{\partial v}{\partial \nu}\right|_{\partial D}.$$

To prove the **converse**, suppose $g \in L^2(\mathbb{S}^2)$ is such that there exists $w \in H^2(D)$ and a Herglotz wave function

$$v(x) = \int_{\mathbb{S}^2} e^{-ikx\cdot d} g(d)\, ds(d) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(-d)\, ds(d), \quad x \in \mathbb{R}^3$$

satisfying (3.30) and (3.31).

The idea is to show that there exists w such that its far field pattern is $\int_{\mathbb{S}^2} u_\infty(\hat{x}, d)\cdot g(-d)\, ds(d)$ and that its far field pattern is identically zero, therefore proving (3.32) and, consequently, that $\overline{g} \in \mathcal{F}$.

In order to do so, we extend w, initially defined by hypothesis on D, to all of $\mathbb{R}^3$ by setting $w := v$ in $\mathbb{R}^3\backslash D$. Notice that $v \in H^2_{loc}(\mathbb{R}^3)$ (since $v \in C^\infty(\mathbb{R}^3)$, because we can derivate under the integral sign as many times as we want by Theorem 5.1.2 of the Appendix), and $w \in H^2(D)$ and $v = w$ and $\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu}$ on ∂D by hypothesis. So the boundary values of w from the interior and the exterior of ∂D coincide. Thus, the extension satisfies $w \in H^2_{loc}(\mathbb{R}^3)$.

Besides, we have that, for any test function $\phi \in C^2_c(\mathbb{R}^3)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^3} (\Delta w + k^2 n w)\phi\, dx &= \int_{\mathbb{R}^3} w(\Delta\phi + k^2 n\phi)\, dx && \text{Int. by parts}\\
&= \int_D w(\Delta\phi + k^2 n\phi)\, dx + \int_{\mathbb{R}^3\backslash D} w(\Delta\phi + k^2\phi)\, dx && n \equiv 1 \text{ in } \mathbb{R}^3\backslash D\\
&= \int_D w(\Delta\phi + k^2 n\phi)\, dx + \int_{\mathbb{R}^3\backslash D} v(\Delta\phi + k^2\phi)\, dx. && w \equiv v \text{ in } \mathbb{R}^3\backslash D
\end{aligned}$$

Applying Green's formula (5.5) to the first integral, we have

$$\begin{aligned}
\int_D w(\Delta\phi + k^2 n\phi)\, dx &= \int_D [w\Delta\phi - (-k^2 n w)\phi]\, dx\\
&= \int_D [w\Delta\phi - \Delta w\phi]\, dx && \Delta w + k^2 n w = 0 \text{ on } D\\
&= \int_{\partial D} \left[w\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial w}{\partial\nu}\right] ds && (5.5)\\
&= \int_{\partial D} \left[v\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial v}{\partial\nu}\right] ds && v = w,\ \tfrac{\partial v}{\partial\nu} = \tfrac{\partial w}{\partial\nu} \text{ on } \partial D\\
&= \int_D [v\Delta\phi - \Delta v\phi]\, dx && (5.5)\\
&= \int_D v(\Delta\phi + k^2\phi)\, dx. && \Delta v + k^2 v = 0 \text{ on } D
\end{aligned}$$

So

$$\begin{aligned}
\int_{\mathbb{R}^3} (\Delta w + k^2 n w)\phi\, dx &= \int_D v(\Delta\phi + k^2\phi)\, dx + \int_{\mathbb{R}^3\backslash D} v(\Delta\phi + k^2\phi)\, dx\\
&= \int_{\mathbb{R}^3} v(\Delta\phi + k^2\phi)\, dx\\
&= \int_{\mathbb{R}^3} (\Delta v + k^2 v)\phi\, dx = 0,
\end{aligned}$$

where we have used on the last step that v is an entire solution of Helmholtz's equation, and on the penultimate step we have integrated by parts over a ball $B(0,R)$ that contains the support of ϕ (and, therefore, with boundary terms that are 0).

Therefore, taking ϕ in such a way that $\int_{\mathbb{R}^3}(\Delta w + k^2 nw)\cdot \phi\, dx$ is a convolution with an approximation of the identity (see Theorem 5.1.3 of the Appendix), we obtain that $\Delta w + k^2 nw = 0$ a.e in $\mathbb{R}^3$. That is, it is a solution in $H^2_{loc}(\mathbb{R}^3)$ of $\Delta w + k^2 nw = 0$.

Since $w = v$ on $\mathbb{R}^3\backslash D$, the difference $w - v$ vanishes in the exterior of D and therefore it trivially satisfies the radiation condition. Therefore, if we take as incident field $w^i = v$ (it is an entire solution of Helmholtz's equation, so it can be an incident field), and define

$$w^s := \begin{cases} w & \text{in } D, \\ 0 & \text{in } \mathbb{R}^3\backslash D, \end{cases}$$

we have that $w := w^i + w^s = v$ is the unique total field corresponding to the incident field $w^i = v$; i.e., it is the unique solution solution of (3.3)-(3.5) for $w^i = v$ (uniqueness follows from Theorem 3.3.3).

Since

$$w^s(x) = \frac{e^{ik|x|}}{|x|} w_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

the fact that $w^s(x) \equiv 0$ on $\mathbb{R}^3\backslash D$ implies that $w_\infty \equiv 0$ on $\mathbb{S}^2$.

But, at the same time, from (3.34) we see that

$$\int_{\mathbb{S}^2} u(x,d) g(-d)\, ds(d)$$

is also a solution of (3.3)-(3.5) for the incident field $w^i = v$. Therefore, uniqueness gives us that

$$w(x) = \int_{\mathbb{S}^2} u(x,d) g(-d)\, ds(d) \quad \forall x \in \mathbb{R}^3.$$

Since the far field pattern of the right-hand side is, as we have already computed,

$$\int_{\mathbb{S}^2} u_\infty(x,d) g(-d)\, ds(d),$$

we have that

$$\int_{\mathbb{S}^2} u_\infty(x,d) g(-d)\, ds(d), = w_\infty(\hat{x}) = 0.$$

□

3.4.3 The interior transmission problem

Motivated by Theorem 3.4.6 we now define the interior transmission problem.

Interior Transmision Problem Given $f \in H^{3/2}(\partial D)$ and $g \in H^{1/2}(\partial D)$ find two functions $v, w \in L^2(D)$ with $w - v \in H^2(D)$ such that

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \text{ in } D \tag{3.35}$$

and

$$w - v = f, \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = g \quad \text{on } \partial D \tag{3.36}$$

where the differential equations for w and v are understood in the distributional sense and the boundary conditions are well defined for the difference $w - v$.

Remark 3.4.8. When we say that the equations in (3.35) are understood in the distributional sense, we mean that

$$\int_D w \cdot (\Delta\phi + k^2 n\phi)\, dx = 0$$

and

$$\int_D v \cdot (\Delta\phi + k^2\phi)\, dx = 0$$

for every $\phi \in C_c^\infty(\mathbb{R}^3)$.

Notice that the boundary conditions (3.36) are not written precisely, because $w|_{\partial D}$, $v|_{\partial D}$, $\frac{\partial w}{\partial \nu}$ and are not necessarily defined for arbitrary functions of $L^2(D)$ such as v and w. What we mean is that $(w - v)\big|_{\partial D} = f$ and $\frac{\partial}{\partial \nu}(w - v) = g$ on ∂D, and in order for this conditions to be well defined we impose the necessary condition $w - v \in H^2(D)$ (so that $w - v|_{\partial D} \in H^{3/2}(\partial D)$ and $\frac{\partial}{\partial \nu}(w - v) \in H^{1/2}(\partial D)$ and therefore the boundary conditions (3.36) make sense).

In this TFM, we will only study the homogeneous interior transmission problem, i.e., for $f \equiv 0$ and $g \equiv 0$. For information on the inhomogeneous interior transmission problem, we refer the reader to section 3.1 of [7].

In particular, our main concern is going to be the existence of positive values of the wave number k such that nontrivial solutions exist to the homogeneous interior transmission problem since, according to Theorem 3.4.6, this is the only case for which there is a possibility that $\mathcal{F}$ is not complete in $L^2(\mathbb{S}^2)$. Because of this, we make the following definition.

Definition 3.4.9. If $k > 0$ is such that the homogeneous interior transmission problem, i.e., the problem (3.35) and (3.36) with $f = g = 0$ has a nontrivial solution, then k is called a ***transmission eigenvalue***.

Remark 3.4.10. Complex transmission eigenvalues can also exists, but we will only deal with real, positive transmission eigenvalues, because they correspond to a positive wave number k, the only case that has physical meaning.

Remark 3.4.11. We have studied the completeness of the set $\mathcal{F}$ of far field patterns relating it to the existence of a nontrivial solution to the homogeneous interior transmission problem. However, there is a different approach: one can study the completeness of $\mathcal{F}$ by analyzing the *far field operator* $F\colon L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2. \tag{3.37}$$

We refer to [13], pages 323-340 for an introduction to this approach in the case of scattering by an inhomogeneous media, and pages 76-87 for the case of obstacle scattering.

3.5 Introduction to Non-Scattering Inhomogeneities

The transmission eigenvalue problem is closely related to a perplexing question in scattering theory: the problem of *non-scattering inhomogeneities*. Given an inhomogeneity (D, n), does there exist a wave number $k > 0$ and an incident wave u^i such that the corresponding far field pattern u_∞ is identically zero?

Such an incident field is referred to as a *non-scattering incident wave* and the corresponding $k > 0$ as a *non-scattering wave number*. Again, we consider positive wave numbers $k > 0$ because they are the only ones that have physical meaning.

Notice that, by Theorem 1.7.3, a radiating solution of the Helmholtz equation with identically zero far field pattern is identically zero as well. Therefore, since u^s is a radiating solution of the Helmholtz equation outside D, an inhomogeneous media (D, n) is non-scattering if and only if there exists a wave number $k > 0$ and an incident field u^i such that the corresponding scattered field is zero outside the inhomogeneity.

For a given incident field u^i, we know that the scattered field $u^s \in H^2_{loc}(\mathbb{R}^3)$ satisfies the equation

$$\Delta u^s + k^2 n u^s = k^2(1-n)u^i \quad \text{ on } \mathbb{R}^3,$$

together with Sommerfeld's radiation condition.

So, explicitly, an inhomogeneity (D, n) **does not scatter** if there exits a wave number $k > 0$ and an incident field u^i, i.e., a solution of

$$\Delta u^i + k^2 u^i = 0 \quad \text{ on } \mathbb{R}^3,$$

such that

$$\Delta u^s + k^2 n u^s = k^2(1-n)u^i \quad \text{ on } \mathbb{R}^3,$$

and

$$u^s \equiv 0 \quad \text{ on } \mathbb{R}^3 \backslash D.$$

This is an overdetermined system of elliptic equations (it has more equations than unknowns), so it could have no solution. However, it can be proved that for the simplest case, i.e., spherically stratified media (to which we can apply separation of variables), there exist non-scattering media (see minutes 13-18 of [5] for the sketch of a proof). So it makes sense to consider this problem, because in general we are not talking about the empty set.

Notice that this problem is physically astonishing: it considers incident fields of specific frequencies for which the inhomogeneity is *invisible*, in the sense that the wave does not change its behavior (there is no nonzero scattered wave).

Relation between transmission eigenvalues and non-scattering inhomogeneities

Let us see how transmission eigenvalues and non-scattering inhomogeneities relate.

Suppose that u^i and u^s satisfy the non-scattering condition for a wave number $k > 0$ (recall that $supp(n-1) = \overline{D}$). Restricting u^i to D, we have that it is a solution to Helmholtz equation on the domain D, that is,

$$\Delta u^i + k^2 u^i = 0 \quad \text{ on } D.$$

Since $u^s \equiv 0$ on $\mathbb{R}^3$, we have that $u^s = 0$ and $\frac{\partial u^s}{\partial \nu} = 0$ on ∂D, Therefore, $u := u^s$ and $v := u^i\big|_D$ are solutions to the transmission eigenvalue problem, i.e., the problem of finding $u \in H^2_0(D)$ and $v \in L^2(D)$ such that

$$\Delta u + k^2 n u = k^2(1-n)v \quad \text{ and } \quad \Delta v + k^2 v = 0 \text{ in D}$$

where $u \in H^2_0(D)$ means $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on ∂D.

Remark. Notice that u and v are solutions to the transmission eigenvalue problem since taking $w = u + v$ (w would be the total field, since $u = u^s$, $v = u^i$ in the above discussion), then $v, w \in L^2(D)$ satisfy that $v - w \in H^2_0(D)$ and

$$\Delta v + k^2 v = 0 \text{ in } D,$$

$$\Delta w + k^2 n w = 0 \text{ in } D.$$

Therefore, solutions to the non-scattering problem are solutions to the transmission eigenvalue problem. That is,

Theorem 3.5.1. *Given an inhomogeneous medium (D, n), non-scattering wave numbers are a subset of transmission eigenvalues.*

So a neccesary condition for k being a non-scattering wave number is that k is a transmission eigenvalue. It is natural then to ask another important question in scattering theory: when a (real) transmission eigenvalue is a non-scattering wave number? This is a partially open question yet. Many papers have been published lately regarding this topic.

This problem is beyond the scope of this TFM. To attack it, it has been related to regularity of the eigenfuctions of the laplacian (i.e. solutions of Helmholtz equation) and free boundary problems.

Let us give an overview here. Suppose that $k > 0$ is a transmission eigenvalue, that is, there exists $u \in H_0^2(D)$ and $v \in L^2(D)$ such that

$$\Delta u + k^2 nu = k^2(1-n)v \text{ and } \Delta v + k^2 v = 0 \text{ in } D$$

$$u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D.$$

We want to see if k is a non-scattering wave number, i.e., if there exists an incident wave v and a scattered wave v such that

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^3$$

$$\Delta u + k^2 nu = k^2(1-n)v \text{ in } D$$

$$u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D.$$

Notice that the only difference is that v must exist as a solution to Helmholtz equation in all of $\mathbb{R}^3$ and not only on D. Since H^1 solutions to Helmholtz equation are analytic (see [28], page 6), the problem is if it is possible to extend v outside of D in such a way that it is analytic. That is, under what assumptions the function $v \in L^2(D)$ is a H^1 solution to Helmholtz equation in a region including D?. In that case, the eigenfunction v has to be analytic, because H^1 solutions to Helmholtz equation are analytic. So it is a regularity issue of eigenfuctions up to the boundary.

There have been several approaches to this problems, such as the use of free boundary methods by Cakoni and Vogelius [10] and by Salo and Shangolian in [28]. For an overview of the recent advances in this problem, see [5].

3.6 Appendix to Chapter 3

In this appendix, we prove Lemma 3.3.1.

Proof of Lemma 3.3.1. For $0 < R \leqslant 1$, let $B[x_0; R]$ be the closed ball of radius R centered at x_0. Choose R such that $B[x_0; R] \subset G$. We shall show that $u_p(x) = 0$ for $x \in B[x_0; R/2]$ and $p = 1, \ldots, P$. The theorem follows from this fact since any other point $x_1 \in G$ can be connected to x_0 by a finite number of overlapping balls. Without loss of generality, we shall assume that $x_0 = 0$ and for convenience we temporarily write $u = u_p$.

For $r = |x|$ and n an arbitrary positive integer (the refractive index n does not appear in this proof, so there is no risk of confussion), we define $v \in H^2(G)$ by

$$v(x) := \begin{cases} e^{r^{-n}} u(x), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{3.38}$$

Then:

Lemma 3.6.1. *Given u and v as above, we have*

$$\Delta u = e^{-r^{-n}} \left[\Delta v + \frac{2n}{r^{n+1}} \frac{\partial v}{\partial r} + \frac{n}{r^{n+2}} \left(\frac{n}{r^n} - n + 1 \right) v \right]. \tag{3.39}$$

Proof of Lemma 3.6.1. Let us check this. Since

$$\frac{\partial v}{\partial r}(x) = \nabla v \cdot \frac{x}{|x|} = \nabla(e^{r^{-n}} u(x)) \cdot \frac{x}{|x|},$$

we need to compute the partial derivatives of $e^{r^{-n}} u(x)$.

As used repeatedly in Chapter 1, we have that $\partial_i r = \frac{x_i}{r}$ for $i = 1, 2, 3$. Therefore,

$$\partial_i v(x) = \partial_i(e^{r^{-n}} \cdot u(x)) = e^{r^{-n}} \partial_i u(x) + e^{r^{-n}} (-n) r^{-n-2} x_i u(x). \tag{3.40}$$

So

$$\nabla(e^{r^{-n}} u(x)) = e^{r^{-n}} \nabla u(x) + e^{r^{-n}} r^{-n-2} u(x)(-n)\vec{x}$$

Hence,

$$\begin{aligned}\frac{\partial v}{\partial r}(x) = \nabla v(x) \cdot \frac{x}{|x|} &= e^{r^{-n}} \nabla u(x) \cdot \frac{x}{|x|} + e^{r^{-n}} r^{-(n+2)} u(x)(-n)\vec{x} \cdot \frac{\vec{x}}{|x|} \\ &= e^{r^{-n}} \nabla u(x) \cdot \frac{x}{r} + v(x) r^{-(n+2)}(-nr).\end{aligned}$$

Since $\Delta v = \partial_1^2 v + \partial_2^2 v + \partial_3^2 v$, we have to compute $\partial_i^2 v$ for $i = 1, 2, 3$. By (3.40), we have

$$\begin{aligned}\partial_i^2 v(x) =& \partial_i(e^{r^{-n}} \cdot \partial_i u(x)) + \partial_i(e^{r^{-n}} \cdot (-n) \cdot r^{-n-2} \cdot x_i \cdot u(x)) + \\ &+ (-n) \cdot [e^{r^{-n}} \cdot (-n) \cdot r^{-n-1} \cdot \partial_i r \cdot r^{-n-2} \cdot x_i \cdot u(x) + e^{r^{-n}} \cdot \partial_i(r^{-n-2} \cdot x_i \cdot u(x))]\end{aligned}$$

The last expression is

$$\partial_i(r^{-n-2} x_i u(x)) = (-n-2) r^{-n-3} \frac{x_i}{r} x_i \cdot u(x) + r^{-n-2} \partial_i(x_i u(x))$$

where $\partial_i(x_i \cdot u(x)) = 1 \cdot u(x) + x_i \cdot \partial_i u(x)$. Therefore, the above expression results in

$$\begin{aligned}\partial_i^2 v(x) = e^{r^{-n}} &\cdot \partial_i^2 u(x) \\ &+ e^{r^{-n}} \cdot (-n) \cdot r^{-n-2} \cdot x_i \cdot \partial_i u(x) \\ &+ n^2 \cdot e^{r^{-n}} \cdot r^{-2n-3} \cdot \partial_i r \cdot x_i \cdot u(x) \\ &+ (-n) \cdot e^{r^{-n}} \cdot (-n-2) \cdot r^{-n-4} \cdot x_i^2 \cdot u(x) \\ &+ (-n) \cdot e^{r^{-n}} \cdot r^{-n-2} \cdot u(x) + (-n) \cdot e^{r^{-n}} \cdot r^{-n-2} \cdot x_i \cdot \partial_i u(x).\end{aligned}$$

where, as we have already commented, $\partial_i r = \frac{x_i}{r}$. Therefore, we obtain

$$
\begin{aligned}
&e^{-r^{-n}} \cdot \left[\Delta v + \frac{2n}{r^{n+1}} \cdot \frac{\partial v}{\partial r} + \frac{n}{r^{n+2} \cdot \left(\frac{n}{r^n} - n + 1 \right) \cdot v} \right] \\
&\quad = \Delta u(x) + (-n) \cdot r^{-n-2} \nabla u(x) \cdot x + n^2 \cdot r^{-2n-3} u(x) \cdot r + (-n) \cdot (-n-2) \cdot r^{-n-4} \cdot u(x) \cdot r^2 \\
&\quad + 3(-n) \cdot r^{-n-2} \cdot u(x) + (-n) \cdot r^{-n-2} \cdot \nabla u(x) \cdot x + \frac{2n}{r^{n+1}} \cdot \nabla u(x) \cdot \frac{x}{r} \\
&\quad + \frac{2n}{r^{n+1}} \cdot r^{-n-2} \cdot u(x) \cdot (-n) \cdot r + \frac{n}{r^{n+2}} \cdot \frac{n}{r^n} \cdot u(x) - \frac{n}{r^{n+2}} \cdot n \cdot u(x) + \frac{n}{r^{n+2}} \cdot u(x) \\
&\quad = \Delta u(x) + \nabla u(x) \cdot x \cdot \left[(-n) \cdot r^{-n-2} + (-n) \cdot r^{-n-2} + \frac{2n}{r^{n+2}} \right] \\
&\quad + u(x) \cdot \left[n^2 \cdot r^{-2n-2} + (-n)(-n-2) \cdot r^{-n-2} \right. \\
&\quad \left. + 3(-n) \cdot r^{-n-2} + \frac{2n}{r^{n+1}} \cdot r^{-n-2} \cdot (-n) \cdot r + \frac{n^2}{r^{2n+2}} - \frac{n^2}{r^{n+2}} + \frac{n}{r^{n+2}} \right] \\
&\quad = \Delta u(x) + u(x) \cdot \left[r^{-2n-2} \cdot (n^2 - 2n^2 + n^2) + r^{-n-2} \cdot (n^2 + 2n - 3n - n^2 + n) \right] \\
&\quad = \Delta u(x).
\end{aligned}
$$

This ends the verification of (3.39). □

Using the inequality $(a+b)^2 \geqslant 2ab$ and calling $b := \frac{2n}{r^{n+1}} \cdot \frac{\partial v}{\partial r}$ and a to the rest of the expression in brackets of (3.39), we see that

$$
\begin{aligned}
(\Delta u)^2 &= (e^{-r^{-n}}) \left[\Delta v + \frac{n}{r^{n+2}} \left(\frac{n}{r^n} - n + 1 \right) v + \frac{2r}{r^{n+1}} \cdot \frac{\partial v}{\partial r} \right]^2 \\
&= e^{-r^{-n}} (a+b)^2 \\
&\geqslant e^{-r^{-n}} 2ba \\
&= \frac{4ne^{-2r^{-n}}}{r^{n+1}} \cdot \frac{\partial v}{\partial r} \left[\Delta v + \frac{n}{r^{n+2}} \left(\frac{n}{r^n} - n + 1 \right) v \right].
\end{aligned}
$$

Now, let $\varphi \in C^2(\mathbb{R}^3)$ be such that $\begin{cases} \varphi(x) = 1 & \text{for } |x| \leqslant R/2 \\ \varphi(x) = 0 & \text{for } |x| \geqslant R \end{cases}$ and is decreasing with respect to $r = |x|$. Then if we define $\hat{u} := \varphi \cdot u$ and $\hat{v} := \varphi \cdot v$, it can be seen that the above inequality is also valid for u and v replaced by $\hat{u}$ and $\hat{v}$ respectively.

In particular, multiplying by r on both sides, we have the inequality

$$
r^{n+2} e^{2r^{-n}} (\Delta \hat{u})^2 \geqslant 4nr \cdot \frac{\partial \hat{v}}{\partial r} \left[\Delta \hat{v} + \frac{n}{r^{n+2}} \left(\frac{n}{r^n} - n + 1 \right) v \right].
$$

Integrating over G:

$$
\int_G r^{n+2} e^{2r^{-n}} (\Delta \hat{u})^2 \, dx \geqslant 4n \int_G r \frac{\partial \hat{v}}{\partial r} \left[\Delta \hat{v} + \frac{n}{r^{n+2}} \left(\frac{n}{r^n} - n + 1 \right) \hat{v} \right] dx. \tag{3.41}
$$

Next, we are going to integrate by parts in (3.41).

First notice that, since $supp(\varphi) \subset B[x_0, R] \subset G$ with $x_0 = 0$ (the inclusion due to our choice of R), we have that all the boundary terms vanish (if it was necessary, we could choose a bit smaller R), because $\hat{u} = \varphi \cdot u$ and $\hat{v} = \varphi \cdot v$ have compact support contained in G.

We are going to use as well the following vector identity:

$$2\nabla(x \cdot \nabla\hat{v}) \cdot \nabla\hat{v} = \operatorname{div}[x|\nabla\hat{v}|^2] - |\nabla\hat{v}|^2. \tag{3.42}$$

Let us check this: to begin with, recall that if $F = (F_x, F_y, F_z)$ is a differentiable vector field, then

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

First, we compute $\nabla(x \cdot \nabla\hat{v})$ (the left-hand side of (3.42)) :

$$\begin{aligned}\partial_i(x \cdot \nabla\hat{v}) \cdot \partial_i\hat{v} &= \partial_i(x_1\partial_1\hat{v} + x_2\partial_2\hat{v} + x_3\partial_3\hat{v}) \cdot \partial_i\hat{v} \\ &= \big(x_1 \cdot \partial^2_{1i}\hat{v} + x_2 \cdot \partial^2_{2i}\hat{v} + x_3 \cdot \partial^2_{3i}\hat{v} + 1 \cdot \partial_i\hat{v}\big) \cdot \partial_i\hat{v} \\ &= x_1 \cdot \partial_i\hat{v} \cdot \partial^2_{1i}\hat{v} + x_2 \cdot \partial_i\hat{v} \cdot \partial^2_{2i}\hat{v} + x_3 \cdot \partial_i\hat{v} \cdot \partial^2_{3i}\hat{v} + (\partial_i\hat{v})^2.\end{aligned}$$

Then,

$$\begin{aligned}2\nabla(x \cdot \nabla\hat{v}) \cdot \nabla\hat{v} &= 2\sum_{i=1}^{3} \partial_i(x \cdot \nabla\hat{v}) \cdot \partial_i\hat{v}) \\ &= 2\sum_{i=1}^{3}[x_1 \cdot \partial_i\hat{v} \cdot \partial^2_{1i}\hat{v} + x_2 \cdot \partial_i\hat{v} \cdot \partial^2_{2i}\hat{v} + x_3 \cdot \partial_i\hat{v} \cdot \partial^2_{3i}\hat{v} + (\partial_i\hat{v})^2] \\ &= 2\sum_{i=1}^{3}[x_1 \cdot \partial_i\hat{v} \cdot \partial^2_{1i}\hat{v} + x_2 \cdot \partial_i\hat{v} \cdot \partial^2_{2i}\hat{v} + x_3 \cdot \partial_i\hat{v} \cdot \partial^2_{3i}\hat{v}] + 2 \cdot |\nabla\hat{v}|^2.\end{aligned}$$

We now compute the right-hand side of (3.3.1), which is $\operatorname{div}[x \cdot |\nabla\hat{v}|^2] - |\nabla\hat{v}|^2$. The divergence is

$$\begin{aligned}&\partial_i(x_i \cdot [(\partial_1\hat{v})^2 + (\partial_2\hat{v})^2 + (\partial_3\hat{v})^2]) \\ &= 1 \cdot [(\partial_1\hat{v})^2 + (\partial_2\hat{v})^2 + (\partial_3\hat{v})^2] + x_i \cdot 2\partial_1\hat{v} \cdot \partial^2_{1i}\hat{v} + x_i \cdot 2\partial_2\hat{v} \cdot \partial^2_{2i}\hat{v} + x_i \cdot 2\partial_3\hat{v} \cdot \partial^2_{3i}\hat{v} \\ &= |\nabla\hat{v}|^2 + 2x_i(\nabla\hat{v} \cdot \nabla(\partial_i\hat{v}))\end{aligned}$$

So

$$\begin{aligned}\operatorname{div}[x \cdot |\nabla\hat{v}|^2] - |\nabla\hat{v}|^2 &= 3|\nabla\hat{v}|^2 + 2[x_1 \cdot (\nabla\hat{v} \cdot \nabla(\partial_1\hat{v}) + x_2 \cdot (\nabla\hat{v} \cdot \nabla(\partial_2\hat{v}) + x_3 \cdot (\nabla\hat{v} \cdot \nabla(\partial_3\hat{v})] - |\nabla\hat{v}|^2 \\ &= 2|\nabla\hat{v}|^2 + 2[x_1 \cdot (\nabla\hat{v} \cdot \nabla(\partial_1\hat{v}) + x_2 \cdot (\nabla\hat{v} \cdot \nabla(\partial_2\hat{v}) + x_3 \cdot (\nabla\hat{v} \cdot \nabla(\partial_3\hat{v})].\end{aligned}$$

So $2\nabla(x \cdot \nabla\hat{v}) \cdot \nabla\hat{v} = \operatorname{div}[x|\nabla\hat{v}|^2] - |\nabla\hat{v}|^2$, which ends the verification of (3.42).

From (5.4), we have that:

$$\int_G r \cdot \frac{\partial\hat{v}}{\partial r}\Delta\hat{v}\,dx = -\int_G \nabla(r\frac{\partial\hat{v}}{\partial r}) \cdot \nabla\hat{v}\,dx + \int_{\partial G} \frac{\partial\hat{v}}{\partial\nu} \cdot \left(r\frac{\partial\hat{v}}{\partial r}\right)\,ds.$$

The boundary integral is 0 since $\hat{v}$ has compact support contained in G. Using that

$$r\frac{\partial\hat{v}}{\partial r} = r\nabla\hat{v} \cdot \frac{x}{r} = \nabla\hat{v} \cdot x,$$

the volume integral gives

$$
\begin{aligned}
-\int_G \nabla(r\cdot\frac{\partial\hat{v}}{\partial r})\cdot\nabla\hat{v}\,dx &= -\int_G \nabla(x\cdot\nabla\hat{v})\cdot\nabla\hat{v}\,dx \\
&= -\int_G \frac{1}{2}[\mathrm{div}[x\cdot|\nabla\hat{v}|^2]-|\nabla\hat{v}|^2]\,dx \\
&= -\int_{\partial G}\frac{1}{2}x\cdot|\nabla\hat{v}|^2\cdot\nu\,ds+\frac{1}{2}\int_G|\nabla\hat{v}|^2\,dx \qquad \text{Divergence thm.} \\
&= \frac{1}{2}\int_G|\nabla\hat{v}|^2\,dx.
\end{aligned}
$$

Therefore, we obtain

$$
\int_G r\frac{\partial\hat{v}}{\partial r}\Delta\hat{v}\,dx=\frac{1}{2}\int_G|\nabla\hat{v}|^2\,dx. \tag{3.43}
$$

For m an integer, we have that:

$$
\begin{aligned}
\int_G\frac{1}{r^m}\cdot\hat{v}\cdot\frac{\partial\hat{v}}{\partial r}\,dx &= \int_G\frac{1}{r^m}\cdot\hat{v}\cdot\nabla\hat{v}\frac{x}{r}\,dx \\
&= \int_G\frac{1}{r^{m+1}}\cdot\hat{v}\cdot(\partial_1\hat{v}\cdot x_1+\partial_2\hat{v}\cdot x_2+\partial_3\hat{v}\cdot x_3)\,dx \\
&= -\int_G\left[\hat{v}\partial_1\left(\frac{x_1\cdot\hat{v}}{r^{m+1}}\right)+\hat{v}\partial_2\left(\frac{x_2\cdot\hat{v}}{r^{m+1}}\right)+\hat{v}\partial_3\left(\frac{x_3\cdot\hat{v}}{r^{m+1}}\right)\right]dx. \qquad supp(\hat{x})\subset G
\end{aligned}
$$

Let us compute first the integrand:

$$
\partial_i\left(\frac{x_i\hat{v}}{r^{m+1}}\right)=\partial_i\left(\hat{v}\frac{x_i}{r^{m+1}}\right)=\partial_i\hat{v}\frac{x_i}{r^{m+1}}+\hat{v}\partial_i\left(\frac{x_i}{r^{m+1}}\right). \tag{3.44}
$$

We have that

$$
\begin{aligned}
\partial_i\left(\frac{x_i}{r^{m+1}}\right) &= \frac{\partial_i(x_i)\cdot r^{m+1}-x_i\cdot\partial_i(r^{m+1})}{r^{2m+2}}= \\
&= \frac{r^{m+1}-x_i(m+1)\cdot r^m\cdot\partial_i(r)}{r^{2m+2}}= \\
&= r^{-(m+1)}-r^{(m-1)-(2m+2)}\cdot x_i^2\cdot(m+1)= \\
&= r^{-(m+1)}-r^{-(m+3)}\cdot x_i^2\cdot(m+1).
\end{aligned}
$$

Therefore, (3.44) turns into

$$
(3.44)=\hat{v}\cdot r^{-(m+1)}+\partial_i\hat{v}\cdot\frac{x_i}{r^{m+1}}-(m+1)\cdot\hat{v}\cdot x_i^2\cdot r^{-(m+3)}
$$

Hence,

$$
\begin{aligned}
\sum_{i=1}^3\partial_i\left(\frac{x_i\hat{v}}{r^{m+1}}\right) &= 3\hat{v}r^{-(m+1)}+\sum_{i=1}^3\partial_i\hat{v}\cdot\frac{x_i}{r^{m+1}}-\sum_{i=1}^3(m+1)\hat{v}\cdot x_i^2r^{-(m+3)} \\
&= 3\hat{v}\cdot r^{-(m+1)}+\frac{1}{r^m}\nabla\hat{v}\cdot\frac{x}{r}-(m+1)\hat{v}\cdot r^{-(m+3)}\sum_{i=1}^3 x_i^2 \\
&= 3\hat{v}r^{-(m+1)}+\frac{1}{r^m}\frac{\partial\hat{v}}{\partial r}-(m+1)\hat{v}r^{-(m+1)} \\
&= \hat{v}r^{-(m+1)}(3-(m+1))+\frac{1}{r^m}\cdot\frac{\partial\hat{v}}{\partial r}.
\end{aligned}
$$

So
$$\sum_{i=1}^{3} \hat{v}\left(\frac{x_i \hat{v}}{r^{m+1}}\right) = \frac{\hat{v}^2}{r^{m+1}} \cdot (2-m) + \frac{1}{r^m}\hat{v}\frac{\partial \hat{v}}{\partial r}.$$
Therefore,
$$\int_G \frac{1}{r^m}\hat{v} \cdot \frac{\partial \hat{v}}{\partial r}\, dx = -\int_G \left[\frac{\hat{v}^2}{r^{m+1}}(2-m) + \frac{1}{r^m}\hat{v} \cdot \frac{\partial \hat{v}}{\partial r}\right] dx = -\int_G \frac{1}{r^m}\hat{v} \cdot \frac{\partial \hat{v}}{\partial r}\, dx + (m-2) \cdot \int_G \frac{\hat{v}}{r^{m+1}}\, dx.$$
That is,
$$\int_G \frac{1}{r^m} \cdot \hat{v} \cdot \frac{\partial \hat{v}}{\partial r}\, dx = \frac{1}{2}(m-2) \cdot \int_G \frac{\hat{v}^2}{r^{m+1}}\, dx. \tag{3.45}$$
Since
$$\int_G \frac{\hat{v}^2}{r^{2n+2}}\, dx \geqslant \int_G \frac{\hat{v}^2}{r^{n+2}}\, dx \tag{3.46}$$
(which we will justify in a moment) we obtain
$$\begin{aligned}
&\int_G r^{n+2} e^{2r^{-n}} (\Delta \hat{u})^2\, dx \\
&\underset{(3.41)}{\overset{}{\geqslant}}\; 4n \int_G r\frac{\partial \hat{v}}{\partial r}\Delta \hat{v}\, dx + 4n \int_G r\frac{\partial \hat{v}}{\partial r} \cdot \frac{n}{r^{n+2}}\frac{n}{r^n}\hat{v}\, dx + 4n \int_G r\frac{\partial \hat{v}}{\partial r} \cdot \frac{n}{r^{n+2}}(-n+1)\hat{v}\, dx.
\end{aligned}$$

Using (3.43) on the first integral, (3.45) with $m = 2n+1$ on the second integral and (3.45) with $m = n+1$ on the third one, we obtain
$$\begin{aligned}
&4n \cdot \frac{1}{2}\int_G |\nabla \hat{v}|^2\, dx + 4n \cdot \frac{1}{2}(2n+1-2) \cdot \int_G \frac{\hat{v}^2}{r^{2n+1+1}}\, dx \cdot n^2 + 4n \cdot \frac{1}{2}(n+1-2) \cdot \int_G \frac{\hat{v}^2}{r^{n+1+1}}\, dx \cdot n(-n+1) \\
&= 2n \int_G |\nabla \hat{v}|^2\, dx + 2n(2n-1) \cdot n^2 \int_G \frac{\hat{v}^2}{r^{2n+2}}\, dx + 2n(n-1)n(1-n) \int_G \frac{\hat{v}^2}{r^{n+2}}\, dx \\
&\geqslant 2n \int_G |\nabla \hat{v}|^2\, dx + 2n^2(2n^2-n) \int_G \frac{\hat{v}^2}{r^{2n+2}}\, dx + (-2n^2) \cdot (n^2-2n+1) \int_G \frac{\hat{v}^2}{r^{2n+2}}\, dx \\
&= 2n \int_G |\nabla \hat{v}|^2\, dx + \int_G \frac{\hat{v}^2}{r^{2n+2}}\, dx \cdot 2n^2(n^2+n-1).
\end{aligned}$$

That is,
$$\int_G r^{n+2} e^{2r^{-n}} (\Delta \hat{u})^2\, dx \geqslant 2n \int_G |\nabla \hat{v}|^2\, dx + 2n^2(n^2+n-1) \int_G \frac{\hat{v}^2}{r^{2n+2}}\, dx. \tag{3.47}$$
The justification of (3.46) is the following:

- If $r \geqslant 1$, then $r^{2n+2} \geqslant r^{n+2} \frac{1}{r^{2n+2}} \leqslant \frac{1}{r^{n+2}}$.
- If $r \leqslant 1$, $r^{2n+2} \leqslant r^{n+2} \implies \frac{1}{r^{2n+2}} \geqslant \frac{1}{r^{n+2}}$.

Since $\hat{v}(x) = 0$ for $r = |x| \geqslant R$ and $0 < R \leqslant 1$, we just need to consider the case $\hat{v}(x) \neq 0$, i.e., $r = |x| < R$. In this case, $r < R \leqslant 1$, so $\frac{1}{r^{2n+2}} \geqslant \frac{1}{r^{n+2}}$.

Now, from (3.47), we have

$$
\begin{aligned}
&\int_G r^{n+2} e^{2r^{-n}} (\Delta \hat{u})^2 \, dx \\
&\geqslant n \left[\int_G 2|\nabla \hat{v}|^2 \, dx + \int_G \frac{2n^2}{r^{2n+2}} \cdot \hat{v}^2 \, dx - \int \frac{2n^2}{r^{2n+2}} \cdot \hat{v}^2 \, dx \right] + 2n^2(n^2+n-1) \int_G \frac{\hat{v}^2}{r^{2n+2}} \\
&= n \cdot \left[\int_G 2|\nabla \hat{v}|^2 \, dx + \int_G \frac{2n^2}{r^{2n+2}} \cdot \hat{v}^2 \, dx \right] - 2n^3 \cdot \int_G \frac{\hat{v}^2}{r^{2n+2}} \, dx + 2n^2(n^2+n-1) \int_G \frac{\hat{v}^2}{r^{2n+2}} \, dx \\
&= n \cdot \left[\int_G (2|\nabla \hat{v}|^2 + \frac{2n^2}{r^{2n+2}} \cdot \hat{v}^2) \, dx \right] + (2n^4 - 2n^2) \int_G \frac{\hat{v}^2}{r^{2n+2}} \, dx \\
&\geqslant n \int_G e^{2r^{-n}} |\nabla \hat{u}|^2 \, dx + n^4 \int_G \frac{\hat{v}^2}{r^{2n+2}} \, dx \\
&= n \int_G e^{2r^{-n}} |\nabla \hat{u}|^2 \, dx + n^4 \int_G \frac{e^{2r^{-n}}}{r^{2n+2}} \hat{u}^2 \, dx.
\end{aligned}
$$

That is,

$$
\int_G r^{n+2} e^{2r^{-n}} (\Delta \hat{u})^2 \, dx \geqslant n \int_G e^{2r^{-n}} |\nabla \hat{u}|^2 \, dx + n^4 \int_G \frac{e^{2r^{-n}}}{r^{2n+2}} \hat{u}^2 \, dx, \tag{3.48}
$$

where in the last inequality we have used that

$$
e^{2r^{-n}} |\nabla \hat{u}|^2 \leqslant 2|\nabla \hat{v}|^2 + \frac{2n^2}{r^{2n+2}} |\hat{v}|^2. \tag{3.49}
$$

Let us see why this is true. To begin with,

$$
\nabla \hat{u} = e^{-r^{-n}} \left[\nabla \hat{v} + \frac{n}{r^{n+1}} \cdot \frac{x}{r} \cdot \hat{v} \right],
$$

since

$$
\nabla \hat{v}(x) = \nabla(e^{r^{-n}} \cdot \hat{u})(x)
$$

and

$$
\begin{aligned}
\partial_i (e^{r^{-n}} \cdot \hat{u})(x) &= e^{r^{-n}} \cdot (-n) \cdot r^{-(n+1)} \cdot \partial_i(r) \cdot \hat{u}(x) + e^{r^{-n}} \cdot \partial_i \hat{u}(x) \\
&= (-n) \cdot r^{-(n+2)} \cdot e^{r^{-n}} \cdot \hat{u}(x) \cdot x_i + e^{r^{-n}} \cdot \partial_i \hat{u}(x),
\end{aligned}
$$

so

$$
\begin{aligned}
&\nabla \hat{v}(x) = \nabla(e^{r^{-n}} \hat{u})(x) = (-n) \cdot r^{-(n+2)} \cdot e^{r^{-n}} \cdot \hat{u}(x) \cdot x + e^{r^{-n}} \cdot \nabla \hat{u}(x) \implies \\
&e^{-r^{-n}} \cdot \nabla \hat{v}(x) = (-n) \cdot r^{-(n+2)} \cdot \hat{u}(x) \cdot x + \nabla \hat{u}(x) \implies \\
&\nabla \hat{u}(x) = e^{-r^{-n}} \cdot \nabla \hat{v}(x) + n \cdot r^{-(n+2)} \cdot \hat{u}(x) \cdot x = e^{-r^{-n}} \cdot \left[\nabla \hat{v}(x) + \frac{n}{r^{n+1}} \cdot \frac{x}{r} \cdot \hat{v}(x) \right].
\end{aligned}
$$

Therefore,

$$
|\nabla \hat{u}|^2 = |e^{-r^{-n}}|^2 \cdot |\nabla \hat{v} + \frac{n}{r^{n+1}} \frac{x}{r} \cdot \hat{v}|^2 \leqslant e^{-2r^{-n}} \cdot \left[|\nabla \hat{v}| + \left(\frac{n}{r^{n+1}} |\hat{v}| \right) \right]^2 \leqslant e^{-2r^{-n}} \cdot 2 \left[|\nabla \hat{v}|^2 + \frac{n^2}{r^{2n+2}} |\hat{v}|^2 \right]
$$

From here we obtain (3.49), as desired. Now we can continue the proof of the Lemma. We have not used hypothesis (3.22) yet, which established that

$$|\Delta u_p| \leqslant C \sum_{q=1}^{P}][|u_q| + |\nabla u_q|] \quad \text{ on } G$$

for every $p = 1, \ldots, P$ and some constant $C > 0$.

Therefore, by (3.22),

$$|\Delta u_p(x)|^2 \leqslant C^2 \left(\sum_{q=1}^{P} [|u_q| + |\nabla u_q|] \right)^2$$

Cauchy-Schwarz inequality gives us that $\sum a_i b_i \leqslant (\sum a_i^2)^{1/2} (\sum b_i^2)^{1/2}$. Using it with $a_i = |u_q| + |\nabla u_q|$ and $b_i = 1$ we obtain

$$\begin{aligned} C^2 \left(\sum_{q=1}^{P} [|u_q| + |\nabla u_q|] \right)^2 &\leqslant C^2 \cdot \left(\sum_{q=1}^{P} [|u_q| + |\nabla u_q|]^2 \right) \cdot \left(\sum_{1}^{P} 1^2 \right) \\ &\leqslant P \cdot C^2 \sum_{q=1}^{P} 2[|u_q| + |\nabla u_q|^2] \qquad\qquad (a+b)^2 \leqslant 2(a^2+b^2) \\ &= 2PC^2 \sum_{q=1}^{P} [|u_q|^2 + |\nabla u_q|^2] \\ &\leqslant 2PC^2 \sum_{q=1}^{P} \left[\frac{|u_q|^2}{r^{3n+4}} + \frac{|\nabla u_q|^2}{r^{n+2}} \right], \end{aligned}$$

having used on the last step that since $R \leqslant 1$, we have

- If $|x| = r \leqslant \frac{R}{2}$ it holds
$$\frac{1}{r} \geqslant \frac{2}{R} \geqslant 2 > 1 \implies \frac{1}{r^{n+2}} \geqslant 2^{n+2} > 1 \quad \text{y} \quad \frac{1}{r^{3n+4}} \geqslant 2^{3n+4} > 1.$$
- If $r = |x| \in [\frac{R}{2}, R]$, then $\frac{1}{r} \in [\frac{1}{R}, \frac{2}{R}]$ with $\frac{1}{R} \geqslant 1$ because $R \leqslant 1$. So
$$\frac{1}{r^{3n+4}} \in \left[\frac{1}{R^{3n+4}}, \frac{2^{3n+4}}{R^{3n+4}} \right]$$
with $\frac{1}{R^{3n+4}} > 1$ since $R \leqslant 1$.

Thus, in both cases we have

$$|\Delta \hat{u}_p(x)|^2 \leqslant \frac{|\Delta \hat{u}_p(x)|^2}{r^{3n+4}}.$$

Since $u_p(x) = \hat{u}_p(x)$ for all x satisfying $|x| \leqslant \frac{R}{2}$, using (3.48) we obtain

$$n \int_{|x| \leqslant \frac{R}{2}} e^{2r^{-n}} |\nabla \hat{u}_p|^2 \, dx + n^4 \int_{|x| \leqslant \frac{R}{2}} \frac{e^{2r^{-n}}}{r^{2n+2}} \hat{u}_p^2 \, dx \underset{\substack{\uparrow \\ (3.48)}}{\leqslant} \int_G r^{n+2} \cdot e^{2r^{-n}} |\Delta \hat{u}_p|^2 \, dx$$

$$\leqslant 2PC^2 \sum_{q=1}^{P} \left[\int_{|x| \leqslant R/2} e^{2r^{-n}} |\nabla \hat{u}_q|^2 \, dx + \int_{|x| \leqslant R/2} \frac{e^{2r^{-n}}}{r^{2n+2}} \hat{u}_q^2 \, dx \right] + \int_{R/2 \leqslant |x| \leqslant R} \frac{e^{2r^{-n}} |\Delta \hat{u}_p(x)|^2}{r^{2n+2}} \, dx,$$

having separated on the last step the integral in two domains: $|x| \leqslant \frac{R}{2}$ and $|x| \in [R/2, R]$:

- On $|x| \leqslant R/2$ we use that $|\Delta u_p(x)|^2 \leqslant 2PC^2 \cdot \sum_{q=1}^{P} \dots$ and the linearity of the integral.
- On $|x| \in [R/2, R]$, we use that $|\Delta \hat{u}_p(x)|^2 \leqslant \frac{|\Delta \hat{u}_p(x)|^2}{r^{3n+4}}$.

Substracting the first two terms of the last expression on both sides, we obtain

$$\begin{aligned} & n \int_{|x| \leqslant R/2} e^{2r^{-n}} \cdot |\nabla \hat{u}_p|^2 \, dx - 2PC^2 \sum_{q=1}^{P} \int_{|x| \leqslant R/2} e^{2r^{-n}} \cdot |\nabla \hat{u}_q|^2 \, dx \\ & + n^4 \int_{|x| \leqslant R/2} \frac{e^{2r^{-n}}}{r^{2n+2}} \cdot \hat{u}_p^2 \, dx - 2PC^2 \sum_{q=1}^{P} \int_{|x| \leqslant R/2} \frac{e^{2r^{-n}}}{r^{2n+2}} \hat{u}_q^2 \, dx \\ & \leqslant \int_{R/2 \leqslant |x| \leqslant R} \frac{e^{2r^{-n}} |\Delta \hat{u}_p(x)|^2}{r^{2n+2}} \, dx, \quad n \in \mathbb{N}. \end{aligned}$$

Taking n sufficiently large, the only non-negligible term of the left-hand side is

$$n^4 \int_{|x| \leqslant R/2} \frac{e^{2r^{-n}}}{r^{2n+2}} u_p^2 \, dx$$

so

$$n^4 \int_{|x| \leqslant R/2} \frac{e^{2r^{-n}}}{r^{2n+2}} u_p^2 \, dx \leqslant C \int_{R/2 \leqslant |x| \leqslant R} \frac{e^{2r^{-n}} |\Delta \hat{u}_p(x)|^2}{r^{2n+2}} \, dx$$

for every $p = 1, \dots, P$, with n sufficiently big and C some positive constant.

Since $f(r) := \frac{e^{2r^{-n}}}{r^{2n+2}}$ $(r > 0)$ is increasing, we have

- If $r \in [0, R/2]$, then
$$\frac{e^{2r^{-n}}}{r^{2n+2}} \geqslant \frac{e^{2(R/2)^{-n}}}{(R/2)^{2n+2}}. \tag{3.50}$$
- If $r \in [R/2, R]$, then
$$\frac{e^{2r^{-n}}}{r^{2n+2}} \leqslant \frac{e^{2(R/2)^{-n}}}{(R/2)^{2n+2}}. \tag{3.51}$$

Therefore,

$$\begin{aligned} n^4 \int_{|x| \leqslant R/2} \frac{e^{2(R/2)^{-n}}}{(R/2)^{2n+2}} \cdot u_p^2 \, dx & \underset{(3.50)}{\overset{\uparrow}{\leqslant}} n^4 \int_{|x| \leqslant R} \frac{e^{2r^{-n}}}{r^{2n+2}} \cdot u_p^2 \, dx \\ & \leqslant C \int_{R/2 \leqslant |x| \leqslant R} \frac{e^{2r^{-n}}}{r^{2n+2}} \cdot |\Delta \hat{u}_p(x)|^2 \, dx \\ & \underset{(3.51)}{\overset{\uparrow}{\leqslant}} C \cdot \int_{R/2 \leqslant |x| \leqslant R} \frac{e^{2(R/2)^{-n}}}{(R/2)^{2n+2}} |\Delta \hat{u}_p(x)|^2 \, dx. \end{aligned}$$

Cancelling out the constants $\frac{e^{2(R/2)^{-n}}}{(R/2)^{2n+2}}|$ on both sides, the remaining expression is

$$n^4 \int_{|x| \leqslant R/2} u_p^2 \, dx \leqslant C \int_{R/2 \leqslant |x| \leqslant R} |\Delta \hat{u}_p(x)|^2 \, dx \quad \forall p = 1, \dots, P.$$

Since the right-hand side is constant with respect to n, taking $n \to \infty$ we obtain that the only possibility is that

$$\int_{|x| \leqslant R/2} u_p^2 \, dx = 0$$

which implies that

$$u_p = 0 \text{ en } |x| \leqslant R/2.$$

and this completes the proof of the Lemma. $\square$

Chapter 4

Existence of Transmission Eigenvalues

4.1 Introduction

In the previous chapter we have seen that the transmission eigenvalue problem is important for establishing the completeness of a set of far field patterns in $L^2(\mathbb{S}^2)$. In this chapter we study the question of discreteness and existence of transmission eigenvalues.

Particular cases can be considered. For example, there exists a rather complete knowledge of the radially symetric case. One known result[1] is the following:

Theorem 4.1.1. *Suppose that* $n \in C^2[0, R]$, $\operatorname{Im}(n(r)) = 0$, *and either*

$$n(R) \neq 1$$

or

$$n(R) = 1 \text{ and } \frac{1}{R}\int_0^R \sqrt{n(\rho)}\, d\rho \neq 1.$$

Then, there exists an infinite discrete set of transmission eigenvales with spherically symmetric eigenfunctions.

Proof. See [7], Theorem 4.7, pages 130-131. □

However, we are interested in more general results that do not assume any kind of symmetry. This was an open problem for about twenty years, until in 2008 Päivärinta and Sylvester gave a proof in [27] for large enough index of refraction. Briefly after that, Cakoni, Gintides and Haddar gave a quite complete answer in [8] to the question of existence of transmission eigenvalues, under the only assumption that $m = 1 - n$ (where n is the refractive index) does not change sign in the inhomogeneity. The **objective** of this chapter is to state and prove this last result.

We follow sections 4.1-4.2 of [7] and Section 10.1 of [13].

Preliminary concepts

Throughout this chapter, we use the following concepts.

Definition 4.1.2. Let A be a bounded linear operator on a Hilbert space X. A is said to be

- **Non-negative** if $(Au, u) \geqslant 0$ for every $u \in X$.
- **Coercive** or **strictly positive** if there exists a constant $\beta > 0$ such that $(Au, u) \geqslant \beta\|u\|^2$.

[1] See also [13], Theorem 8.13, pages 321-323.

Structure of the chapter

The structure of the chapter is the following.

First, we reformulate the interior transmission problem as a fourth order equation. Then, we reformulate the transmission eigenvalue problem as a classical eigenvalue problem $(K - \frac{1}{\tau}I)U = 0$ for $\tau > 0$. The operator K is not self-adjoint (although it is compact) and therefore non-standard methods must be used to prove existence of eigenvalues. This is the reason why the problem was open so many years. The approach to overcome this difficulty is the following:

- We will reformulate the problem as finding the values of τ for which $N(A_\tau - \tau B) \neq \{0\}$ for $\{A_\tau\}_{\tau>0}$ a family of self-adjoint, compact and coercive (or stricly positive) operators and B a self-adjoint, compact and non-negative operator.

- This problem requires a functional analytic theory to study the spectral decomposition of a compact, self-adjoint and strictly positive operator A with respect to another compact, self-adjoint, positive operator B, i.e., a generalization of the spectral theory for $A - \lambda I$ when I is substituted by a more general operator B. This generalization requires several theorems that are quite technical to prove.

- Lastly, we apply this theory to prove the existence of transmission eigenvalues aforementioned.

4.2 The Transmission Eigenvalue Problem

In this section, we formulate the transmission eigenvalue problem for isotropic non-homogeneous medium in several ways.

Let $D \subset \mathbb{R}^3$ be the support of an isotropic inhomogeneous media with refractive index $n \in L^\infty(D)$ such that $\mathrm{Re}(n) \geqslant n_0 > 0$ and $\mathrm{Im}(n) \geqslant 0$. Recall the interior transmission problem corresponding to the scattering problem for this isotropic inhomogeneous media:

Interior transmission problem
Given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$, find $w \in L^2(D), v \in L^2(D)$ with $w - v \in H^2(D)$ such that

$$\begin{array}{ll} \Delta w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w - v = f & \text{on } \partial D \\ \dfrac{\partial w}{\partial v} - \dfrac{\partial v}{\partial v} = h & \text{on } \partial D \end{array} \tag{4.1}$$

where the equations for w and v are understood in the distributional sense and the boundary conditions are well defined for the difference $w - v$.

We were particularly interested in the homogeneous case:

Definition 4.2.1. Values of $k \in \mathbb{C}$ for which the homogeneous interior transmission problem

$$\begin{array}{ll} \Delta w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \partial D \\ \frac{\partial w}{\partial v} = \frac{\partial v}{\partial v} & \text{on } \partial D \end{array} \tag{4.2}$$

has nontrivial solutions $w \in L^2(D)$ and $v \in L^2(D)$, such that $w - v \in H_0^2(D)$, are called **transmission eigenvalues**.

Recall that we are going to be interested only in positive transmission eigenvalues: $k \in \mathbb{R}$, $k > 0$.

Reformulation of the Interior Transmission Problem as a fourth order equation

One appealing approach, given the structure of the boundary conditions in (4.1), is to take the difference $u := w - v$ as a new unknown and try to obtain an equivalent equation for u. Let us see how to do this.

Subtracting the second equation from the first, we have

$$\Delta u + k^2 nu = -k^2(n-1)v \quad \text{in } D, \tag{4.3}$$

together with the other equation

$$\Delta v + k^2 v = 0 \quad \text{in } D \tag{4.4}$$

and the boundary conditions

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial v} = h \quad \text{on } \partial D.$$

To eliminate v we need to divide by $n-1$, and therefore we have to assume that $n-1$ is bounded away from zero. This justifies the following assumption.

Assumption: the real part of the constrast $n-1$ does not change sign in D. More specifically, we assume that there exists $\alpha > 0$ such that

$$\text{either } \operatorname{Re}(n(x)) - 1 \geqslant \alpha > 0 \text{ or } 1 - \operatorname{Re}(n(x)) \geqslant \alpha > 0 \text{ for almost all } x \in D.$$

. Letting

$$n_{min} := \inf_D \operatorname{Re}(n) \quad \text{and} \quad n_{max} := \sup_D \operatorname{Re}(n), \tag{4.5}$$

the above assumption means that

$$\text{either } n_{min} > 1 \text{ or } 0 < n_{max} < 1.$$

Under this assumption, we can divide (4.3) by $k^2(n-1)$ to obtain

$$v = -\frac{1}{k^2(n-1)}[\Delta u + k^2 nu]$$

and substituting this into equation (4.4) we obtain

$$\begin{aligned}
&\Delta\left[-\frac{1}{k^2(n-1)}[\Delta u + k^2 nu]\right] + k^2\left[-\frac{1}{k^2(n-1)}[\Delta u + k^2 nu]\right] = 0\\
\iff\ & -\frac{1}{k^2}(\Delta + k^2)\frac{1}{n-1}[\Delta u + k^2 nu] = 0\\
\iff\ & (\Delta + k^2)\frac{1}{n-1}[\Delta u + k^2 nu] = 0.
\end{aligned}$$

Using that $\frac{n}{n-1} = 1 + \frac{1}{n-1}$, the left-hand side of the last expression can be rewritten as

$$\begin{aligned}
(\Delta + k^2)\frac{1}{n-1}[\Delta u + k^2 nu] &= \Delta\left[\frac{1}{n-1}\Delta u\right] + \Delta\left[\frac{1}{n-1}k^2 nu\right] + k^2\frac{1}{n-1}\Delta u + \frac{k^2}{n-1}k^2 nu\\
&= \Delta\left[\frac{1}{n-1}\Delta u\right] + \Delta[k^2 u] + \Delta\left[\frac{k^2 u}{n-1}\right] + k^2\frac{1}{n-1}\Delta u + \frac{k^2}{n-1}k^2 nu\\
&= \Delta\left[\frac{1}{n-1}\Delta u\right] + k^2\frac{n}{n-1}\Delta u + \Delta\left[\frac{k^2 u}{n-1}\right] + \frac{k^2}{n-1}k^2 nu\\
&= (\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u.
\end{aligned}$$

That is, we have obtained an equivalent formulation of equation (4.1) as a boundary value problem for a fourth order equation:

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0 \quad \text{in } D \tag{4.6}$$

$$u = f \quad \text{and } \frac{\partial u}{\partial \nu} = h \quad \text{on } \partial D \tag{4.7}$$

where we assume $u := w - v \in H^2(D)$. The functions v and w are related to u through

$$v = -\frac{1}{k^2(n-1)}(\Delta u + k^2 n u) \tag{4.8}$$

$$w = u + v = -\frac{1}{k^2(n-1)}(\Delta u + k^2 u) \tag{4.9}$$

Weak formulation of the Interior Transmission Problem

In this section we are concerned with proving the existence of real transmission eigenvalues, i.e., the values of $k > 0$ for which

$$\begin{aligned} \Delta w + k^2 n w &= 0 \text{ in } D, \\ \Delta v + k^2 v &= 0 \text{ in } D, \\ w - v &= 0 \text{ on } \partial D, \\ \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} &= 0 \text{ on } \partial D \end{aligned} \tag{4.10}$$

has non-trivial solutions $w \in L^2(D)$ and $v \in L^2(D)$ such that $w - v \in H^2(D)$; that is, the values of $k > 0$ for which

$$\begin{aligned} \Delta w + k^2 n w &= 0 \text{ in } D, \\ \Delta v + k^2 v &= 0 \text{ in } D, \end{aligned}$$

has non-trivial solutions $w \in L^2(D)$ and $v \in L^2(D)$ such that $w - v \in H_0^2(D)$, where

$$H_0^2(D) = \left\{ u \in H^2(D) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \right\}.$$

We have been able to write (4.10) as a fourth order equation

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0 \tag{4.11}$$

for $u = w - v$ in $H_0^2(D)$.

The weak formulation of this fourth order equation becomes, after integrating by parts, the following: find a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\, dx = 0 \text{ for all } v \in H_0^2(D). \tag{4.12}$$

A consequence of this formulation is the following:

Theorem 4.2.2. *Suppose $n \in L^\infty(D)$ with $\operatorname{Im} n > 0$ almost everywhere in a region $D_0 \subset D$ of positive measure. Then, there are no transmission eigenvalues $k > 0$ for the inhomogeneity (D, n). In particular, the set $\mathcal{F}$ of far field patterns is complete in $L^2(\mathbb{S}^2)$ for each $k > 0$.*

Proof. Let v, w be a solution to (3.35) and (3.36) for $f = g = 0$. Then, taking $u := w - v$, applying (4.12) with $v = u$ we obtain

$$\int_D \frac{1}{n-1}(\Delta\overline{u} + k^2 n\overline{u})(\Delta u + k^2 u)\, dx = 0.$$

We have that

$$\begin{aligned}(\Delta\overline{u} + k^2 n\overline{u})(\Delta u + k^2 u) &= |\Delta u + k^2 u|^2 + k^2 n\overline{u}\Delta u + k^4 n\overline{u}u - k^2\overline{u}\Delta u - k^2\overline{u}k^2 u \\ &= |\Delta u + k^2 u|^2 + k^2(n-1)\overline{u}(\Delta u + k^2 u)\end{aligned}$$

Therefore, after integrating by parts, we see that

$$\int_D \frac{1}{n-1}(\Delta\overline{u} + k^2 n\overline{u})(\Delta u + k^2 u)\, dx = \int_D \frac{1}{n-1}|\Delta u + k^2 u|^2\, dx + k^4 \int_D |u|^2\, dx - k^2 \int_D |\nabla u|^2\, dx = 0.$$

Since $\operatorname{Im} n > 0$ on a region D_0 of positive measure, then $\operatorname{Im}\left(\frac{1}{n-1}\right) \neq 0$ on this region, and the second integral is a real number, taking imaginary parts we have that

$$0 = \operatorname{Im}\left(\int_D \frac{1}{n-1}|\Delta u + k^2 u|^2\, dx\right) = \int_D \operatorname{Im}\left(\frac{1}{n-1}\right)|\Delta u + k^2 u|^2\, dx.$$

Therefore, $|\Delta u + k^2 u|^2 \equiv 0$ on D_0, a region of positive measure. Since, by (3.35), $\Delta u + k^2 u = k^2(1-n)w$, we have that $w \equiv 0$ on D_0. Therefore, by the Unique Continuation Principle (Theorem 3.3.2) we have that $w \equiv 0$ on D. Therefore, its Cauchy data are zero, i.e., $w = 0$ and $\frac{\partial w}{\partial \nu} = 0$ on ∂D. Thus, the Cauchy data of v are 0 as well, which implies[2] that $v \equiv 0$ on D. So for the inhomogeneous medium (D, n) there does not exist a transmission eigenvalue $k > 0$, since every solution to the interior transmission problem is the trivial one. □

However, when $\operatorname{Im}(n) \equiv 0$, there may exist values of k that are transmission eigenvalues. We investigate this case in the rest of the TFM.

4.2.1 Non self-adjointness of the transmission eigenvalue problem

So, from now on, we consider the case where n is **real** and

$$\boxed{n_{min} > 1}.$$

The case $0 < n_{max} < 1$ is completely analogous and is treated in [7].

We would like to write the transmission eigenvalue problem as an eigenvalue problem.

First, take $\tau := k^2$ in (4.12) to obtain

$$\int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta\overline{v} + \tau n\overline{v})\, dx = 0 \qquad \text{for all } v \in H_0^2(D) \tag{4.13}$$

Computing the product of the integrand, this is equivalent to

$$\int_D \frac{1}{n-1}\Delta u \cdot \Delta\overline{v}\, dx + \int_D \frac{1}{n-1}(\Delta u \cdot \tau n\overline{v} + \Delta\overline{v} \cdot \tau u)\, dx + \int_D \frac{1}{n-1}\tau^2 \cdot n u\overline{v}\, dx = 0 \tag{4.14}$$

for all $v \in H_0^2(D)$. We can express this in terms of three operators. Let us see how.

[2] For example, by Green's Representation Formula on bounded domains (which, as we reasoned in the previous chapter, is valid for $H^2(D)$ functions).

1. Consider the bounded antilinear functional that, for a fixed $u \in H_0^2(D)$, takes

$$\begin{array}{ccc} H_0^2(D) & \to & \mathbb{C} \\ v & \mapsto & \int_D \frac{1}{n-1} \Delta u \cdot \Delta \overline{v}\, dx. \end{array}$$

Therefore, by Riesz Representation Theorem for Hilbert Spaces, there exists a unique $Tu \in H_0^2(D)$ such that

$$(Tu, v)_{H^2(D)} = \int_D \frac{1}{n-1} \Delta u \cdot \Delta \overline{v}\, dx \qquad \forall v \in H_0^2(D).$$

Therefore, we have an operator $T\colon H_0^2(D) \to H_0^2(D)$ bounded, strictly positive and self-adjoint because

(a) Linear: because

$$\begin{aligned} (T(u + \lambda v), w)_{H^2(D)} &= \int_D \frac{1}{n-1} \Delta (u + \lambda v) \cdot \Delta \overline{w}\, dx \\ &= (Tu, w) + \lambda (Tv, w) = (Tu + \lambda Tv, w) \quad \forall w \in H_0^2(D). \end{aligned}$$

(b) Bounded: it is a consequence of Riesz Representation Theorem; or simply because

$$\begin{aligned} \|Tu\|^2_{H^2(D)} = (Tu, Tu)_{H^2(D)} = \int_D \frac{1}{n-1} \Delta u \cdot \Delta \overline{Tu}\, dx &\underset{\text{C-S}}{\underset{\uparrow}{\leqslant}} \left\| \frac{1}{n-1} \right\|_\infty \|\Delta u\|_{L^2(D)} \|\Delta \overline{Tu}\|_{L^2(D)} \\ \leqslant \left\| \frac{1}{n-1} \right\|_\infty c^2 \|u\|_{H^2(D)} \|\overline{Tu}\|_{H^2(D)} &= \left\| \frac{1}{n-1} \right\|_\infty c^2 \|u\|_{H^2(D)} \|Tu\|_{H^2(D)}, \end{aligned}$$

being $\|\frac{1}{n-1}\|_\infty < \infty$ since $n_{min} > 1$ and having used that, by Theorem 5.1.10, on $H_0^2(D)$ the $H^2(D)$-norm of a function is equivalent to the $L^2(D)$-norm of its laplacian, so there exists $c > 0$ such that

$$\|\Delta u\|_{L^2(D)} \leqslant c \|u\|_{H^2(D)}. \tag{4.15}$$

for all $u \in H_0^2(D)$.
Therefore, by cancelling the term $\|Tu\|_{H^2(D)}$ on both sides, we have that

$$\|Tu\|_{H^2(D)} \leqslant \left\| \frac{1}{n-1} \right\|_\infty c^2 \cdot \|u\|_{H^2(D)}.$$

We conclude then that T is bounded on $H_0^2(D)$.

(c) Strictly positive (coercive):

$$\begin{aligned} (Tu, u)_{H^2(D)} &= \int_D \frac{1}{n-1} \Delta u \cdot \Delta \overline{u}\, dx \\ &= \int_D \frac{1}{n-1} |\Delta u|^2\, dx \\ &\geqslant \int_D \frac{1}{n_{max} - 1} |\Delta u|^2\, dx \\ &= C \cdot \|\Delta u\|^2_{L^2(D)} \\ &\geqslant C' \cdot \|u\|^2_{H^2(D)} \end{aligned}$$

having used the hypothesis that $n(x) \leqslant n_{max} < \infty$ and, in the last step, the H^2-norm equivalence of Theorem 5.1.10 of the Appendix as in (4.15).

(d) Self-adjoint:

$$(u, Tv)_{H^2(D)} = \overline{(Tv, u)}_{H^2(D)} = \overline{\int_D \frac{1}{n-1} \Delta v \cdot \Delta \overline{u}\, dx} = \int_D \frac{1}{n-1} \cdot \Delta \overline{v} \cdot \overline{\Delta \overline{u}}\, dx = (Tu, v)_{H^2(D)},$$

having used that n is real-valued.

2. Consider now the bounded antilinear functional that, for a fixed $u \in H_0^2(D)$, takes

$$\begin{array}{ccc} H_0^2(D) & \to & \mathbb{C} \\ v & \mapsto & -\int_D \frac{1}{n-1}(n\overline{v}\Delta u + u\Delta\overline{v})\, dx. \end{array}$$

By Riesz Representation Theorem, there exists an operator $T_1 \colon H_0^2(D) \to H_0^2(D)$ such that

$$\begin{aligned} (T_1 u, v)_{H^2(D)} &= -\int_D \frac{1}{n-1}(n\overline{v}\Delta u + u\Delta\overline{v})\, dx \\ &= -\int_D \frac{1}{n-1}(\Delta u \cdot \overline{v} + u \cdot \Delta\overline{v})\, dx + \int_D \nabla u \cdot \nabla\overline{v}. \end{aligned} \tag{4.16}$$

This operator is

(a) Self-adjoint:

$$\begin{aligned} (T_1 u, v)_{H^2(D)} &= -\int_D \frac{1}{n-1}(n\overline{v}\Delta u + u\Delta\overline{v})\, dx \\ &= -\int_D \frac{1}{n-1}(\overline{v}\Delta u + nu\Delta\overline{v})\, dx \\ &= \overline{(T_1 v, u)}_{H^2(D)} \\ &= (u, T_1 v)_{H^2(D)}. \end{aligned}$$

(b) Compact: let $T_1^{(1)}$ be the part of the operator T_1 given by the second integral of (4.16). We have that

$$\|T_1^{(1)} u\|_{H^2} = \sup_{0 \neq v \in H^2} \frac{1}{\|v\|_{H^2}} \left| \int_D \frac{1}{n-1} u \Delta\overline{v}\, dx \right| \leqslant C\|u\|_{L^2}.$$

Therefore, for a sequence $\{u_n\}$ bounded on $H^2(D)$, thanks to the compact embedding of $H_0^2(D)$ into $L^2(D)$, we obtain that a subsequence of $\{T_1^{(1)} u_n\}$ converges strongly on $H^2(D)$. The second integral of (4.16) gives us the same result (considering the adjoint). So we can conclude that T_1 is compact.

3. Lastly, consider the bounded linear functional that, for a fixed $u \in H_0^2(D)$, takes

$$\begin{array}{ccc} H_0^2(D) & \to & \mathbb{C} \\ v & \mapsto & \int_D \frac{n}{n-1} u \cdot \overline{v}\, dx. \end{array}$$

Again, Riesz Representation Theorem gives us the existence of an operator $T_2 \colon H_0^2(D) \to H_0^2(D)$ that satisfies

$$(T_2 u, v)_{H^2(D)} = \int_D \frac{n}{n-1} u \cdot \overline{v}\, dx$$

and is

(a) Self-adjoint:

$$(T_2u, v)_{H^2(D)} = \int_D \frac{n}{n-1} u \cdot \overline{v}\, dx = \overline{\int_D \frac{n}{n-1} \overline{u} \cdot v} = \overline{(T_2v, u)}_{H^2(D)} = (u, T_2v)_{H^2(D)}.$$

(b) Non-negative:

$$(T_2u, u)_{H^2(D)} = \int_D \frac{n}{n-1} u \cdot \overline{u}\, dx = \int_D \frac{n}{n-1} |u|^2\, dx \geqslant 0$$

(c) Compact: the reasoning is analogous to that of T_1.

With these operators, (4.14) can be written as

$$(Tu, v)_{H^2(D)} - \tau(T_1u, v)_{H^2(D)} + \tau^2(T_2u, v)_{H^2(D)} = 0 \quad v \in H_0^2(D).$$

Since the scalar product is non-degenerate, the above expression means that

$$Tu - \tau T_1u + \tau^2 T_2u = 0.$$

Therefore, we conclude that the transmission eigenvalue problem is equivalent to finding a non-zero function $u \in H_0^2(D)$ such that

$$(T - k^2T_1u + k^4T_2)u = 0.$$

Theorem 4.2.3. *Assume that $n \in L^\infty(D)$ takes real values and $n_{min} > 1$, where n_{min} is given by (4.5). Then the set of transmission eigenvalues $k \in \mathbb{C}$ is discrete (possibly empty) with $+\infty$ as the only possible accumulation point. The multiplicity of the eigenvalues is finite with finite dimensional eigenspaces.*

Proof. As discussed above, $k \in \mathbb{C}$ is a transmission eigenvalue if and only if there exists a nonzero solution $u \in H_0^2(D)$ of

$$Tu - k^2T_1u + k^4T_2u = 0,$$

which, applying T^{-1}, is equivalent to the existence of a nonzero function $u \in H_0^2(D)$ such that

$$\left(I - k^2T^{-1}T_1 + k^4T^{-1}T_2\right) u = 0,$$

where I denotes de identity operator. Letting $\tau := k^2$ and setting $U := \left(u, \tau T^{-1}T_2u\right)$, the interior transmission eigenvalue problem becomes the eigenvalue problem

$$\left(K - \frac{1}{\tau} I\right) U = 0 \quad U \in H_0^2(D) \times H_0^2(D)$$

for the operator $K : H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$K := \begin{pmatrix} T^{-1}T_1 & -I \\ T^{-1}T_2 & 0 \end{pmatrix},$$

because

$$0 = \left(K - \frac{1}{\tau} I\right) U = \begin{pmatrix} T^{-1}T_1 - \frac{1}{\tau}I & -I \\ T^{-1}T_2 & -\frac{1}{\tau}I \end{pmatrix} \begin{pmatrix} u \\ \tau T^{-1}T_2u \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau}(I + \tau \cdot T^{-1}T_1 - \tau^2T^{-1}T_2)u \\ 0 \end{pmatrix}$$
$$\iff (I + \tau T^{-1}T_1 - \tau^2T^{-1}T_2)u = 0.$$

Notice that K is compact because each entry $K_{i,j}$ is compact. Therefore, each entry is the limit of a sequence of finite-rank operators $\{K_{i,j}(n)\}_{n\in\mathbb{N}}$. So K is the limit of the sequence of finite-rank operators

$$K(n) := \begin{pmatrix} K_{1,1}(n) & K_{1,2}(n) \\ K_{2,1}(n) & K_{2,2}(n) \end{pmatrix}.$$

Therefore, we can apply Theorem 5.1.16 from the appendix to conclude that its eigenvalues are either at most, a countable set that accumulates at 0 (including 0), and Theorem 5.1.14 to deduce that the multiplicity is finite with finite dimensional generalized eigenspaces. $\square$

4.3 Existence of Transmission Eigenvalues

The fact that the eigenvalue problem associated to the transmission eigenvalue problem is not self-adjoint makes it harder to prove the existence of transmission eigenvalues for media that is not spherically stratified (that is, radially symmetric). In fact, this problem was open until 2008, when Päivärinta and Sylvester proved in their article [27] that, for large enough index of refraction n, there exsits at least one transmission eigenvalue. Later, the existence of transmission eigenvalues was completely solved in [8], where they proved the existence of an infinite set of transmission eigenvalues with ∞ as the only accumulation point by just assuming that either $n_{min} > 1$ or $0 < n_{min} \leqslant n_{max} < 1$. We explain this proof in detail.

To clarify the exposition, we are just going to explain the case $n_{min} > 1$ (see [7], Section 4.2, pages 130-136 for the other one).

We have seen in the previous section that the transmission eigenvalue problem is equivalent to, given $k > 0$, finding a not identically zero function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta \overline{v} + \tau n \overline{v})\, dx = 0 \qquad \forall v \in H_0^2(D), \tag{4.17}$$

where $\tau = k^2$.

To formulate it in terms of operators, we give the following definition.

Definition 4.3.1. Let $\mathcal{A}_\tau, \mathcal{B}\colon H_0^2(D) \times H_0^2(D) \to \mathbb{C}$ with $\tau > 0$ the operators given by

$$\begin{aligned} \mathcal{A}_\tau(u,v) &:= \left(\frac{1}{n-1}(\Delta u + \tau u), (\Delta v + \tau v)\right)_D + \tau^2 (u,v)_D \\ &= \int_D \frac{1}{n(x)-1}(\Delta u(x) + \tau u(x)) \cdot (\Delta \overline{v}(x) + \tau \overline{v}(x))\, dx + \tau^2 \int_D u(x)\cdot \overline{v}(x)\, dx, \end{aligned}$$

and

$$\mathcal{B}(u,v) := (\nabla u, \nabla v)_D = \int_D \nabla u(x) \cdot \nabla \overline{v}(x)\, dx,$$

where $(\cdot,\cdot)_D$ denotes the scalar product of $L^2(D)$.

Proposition 4.3.2. *The operators $\mathcal{A}_\tau$ ($\tau > 0$) y $\mathcal{B}$ are symmetric sesquilinear operators (symmetric in the sense that $\mathcal{A}_\tau(u,v) = \overline{\mathcal{A}_\tau(v,u)}$ and the same for $\mathcal{B}$).*

Proof. It is a consequence of the fact that we consider $n(x)$ real-valued and $\tau \in \mathbb{R}$, and therefore $\overline{n} = n$ and $\overline{\tau} = \tau$. $\square$

Applying the Riesz Representation Theorem as we did in the previous section for T, T_1 and T_2, we can define the bounded linear operators $A_\tau\colon H_0^2(D) \to H_0^2(D)$ and $B\colon H_0^2(D) \to H_0^2(D)$ as the only ones that satisfy

$$(A_\tau u, v)_{H^2(D)} = \mathcal{A}_\tau(u,v) \quad \forall u, v \in H_0^2(D)$$

$$(Bu, v)_{H^2(D)} = \mathcal{B}(u,v) \quad \forall u, v \in H_0^2(D).$$

In terms of these operators, we can rewrite (4.17) in the following way. The integrand is

$$(\Delta u + \tau u)(\Delta \overline{v} + \tau n \overline{v}) = \Delta u \cdot \Delta \overline{v} + \Delta u \cdot \tau n \overline{v} + \tau u \cdot \Delta \overline{v} + \tau u \cdot \tau n \overline{v}.$$

Since

$$(\Delta u + \tau u)(\Delta \overline{v} + \tau \overline{v}) = \Delta u \cdot \Delta \overline{v} + \Delta u \cdot \tau \overline{v} + \tau u \cdot \Delta \overline{v} + \tau u \cdot \tau \overline{v},$$

we have that

$$\begin{aligned}(\Delta u + \tau u)(\Delta \overline{v} + \tau n \overline{v}) &= (\Delta u + \tau u)(\Delta \overline{v} + \tau \overline{v}) - \Delta u \cdot \tau \overline{v} + \Delta u \cdot \tau n \overline{v} - \tau u \cdot \tau \overline{v} + \tau u \cdot \tau n \overline{v} = \\ &= (\Delta u + \tau u)(\Delta \overline{v} + \tau \overline{v}) + \tau \Delta u \cdot \overline{v}(n-1) + \tau^2 \cdot u \cdot \overline{v}(n-1).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta \overline{v} + \tau n \overline{v})\, dx \\ &= \int_D \frac{1}{n-1}\left[(\Delta u + \tau u)(\Delta \overline{v} + \tau \overline{v}) + \tau \Delta u \cdot \overline{v}(n-1) + \tau^2 u \overline{v}(n-1)\right] dx \\ &= \int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta \overline{v} + \tau \overline{v})\, dx + \tau^2 \int u\overline{v} + \tau \int \Delta u \overline{v} \\ &= \int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta \overline{v} + \tau \overline{v})\, dx + \tau^2 \int_D u\overline{v} - \tau \int_D \nabla u \cdot \nabla \overline{v} \\ &= \mathcal{A}_\tau(u,v) - \tau \mathcal{B}(u,v) = (A_\tau u, v)_{H^2(D)} - \tau (Bu, v)_{H^2(D)} = (A_\tau u - \tau B u, v)_{H^2(D)}.\end{aligned}$$

Since k is a transmission eigenvalue if and only if for $\tau := k^2$ there exists $u \in H_0^2(D)$, $u \not\equiv 0$ such that

$$\int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta \overline{v} + \tau n \overline{v})\, dx = 0 \quad \forall v \in H_0^2(D),$$

we have that k is a transmission eigenvalue if and only if $\tau := k^2$ is such that

$$(A_\tau u - \tau B u, v)_{H^2(D)} = 0 \quad \text{for all } v \in H_0^2(D)$$

has a non-trivial solution u. That is, k is a transmission eigenvalue if and only if

$$N(A_\tau - \tau B) \neq \{0\}.$$

We are going to study some properties of the operators A_τ and B. In order to do so, let $\lambda_1(D)$ be the first Dirichlet eigenvalue of $-\Delta$ in D (that is, the smallest $\lambda > 0$ such that the problem $-\Delta u = \lambda u$ on D, $u = 0$ on ∂D has a non-trivial solution).

Lemma 4.3.3. *The operators $A_\tau, B\colon H_0^2(D) \to H_0^2(D)$ $(\tau > 0)$ are self-adjoint, and B is non-negative and compact. Besides, if $n_{min} > 1$, then A_τ is strictly positive (i.e. coercive).*

Proof. The self-adjointness is a consequence of Proposition 4.3.2. Indeed,

$$(A_\tau u, v)_{H^2(D)} = \mathcal{A}_\tau(u,v) = \overline{\mathcal{A}_\tau(v,u)} = \overline{(A_\tau v, u)_{H^2(D)}} = (u, A_\tau v)_{H^2(D)}.$$

and

$$(Bu, v)_{H^2(D)} = \mathcal{B}(u,v) = (\nabla u, \nabla v)_{L^2(D)} = \overline{(\nabla v, \nabla u)_{L^2(D)}} = \overline{\mathcal{B}(v,u)} = \overline{(Bv, u)_{H^2(D)}} = (u, Bv)_{H^2(D)}.$$

Now, we check that A_τ is strictly positive if $n_{min} > 1$. To begin with,

$$\frac{1}{n(x)-1} \geqslant \frac{1}{n_{max}-1} =: \gamma > 0$$

almost everywhere $x \in D$. Therefore

$$\begin{aligned}
(A_\tau u, u)_{H^2(D)} = \mathcal{A}_\tau(u,u) &= \left(\frac{1}{n-1}(\Delta u + \tau u), (\Delta u + \tau u)\right)_{L^2(D)} + \tau^2 (u,u)_{L^2(D)} \\
&= \int_D \frac{1}{n-1} |\Delta u + \tau u|^2 \, dx + \tau^2 \int_D |u|^2 \\
&\geqslant \gamma \|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2 \|u\|^2_{L^2(D)} \\
&= \gamma (\Delta u + \tau u, \Delta u + \tau u)_{L^2(D)} + \tau^2 \|u\|^2_{L^2(D)} \\
&= \gamma \left[\|\Delta u\|^2_{L^2(D)} + \tau^2 \|u\|^2_{L^2(D)} + 2\tau [(u, \Delta u) + (\Delta u, u)] \right] + \tau^2 \|u\|^2_{L^2(D)} \\
&\geqslant \gamma \|\Delta u\|^2_{L^2(D)} + \tau^2 (\gamma + 1) \|u\|^2_{L^2(D)} - 2\gamma\tau \|u\|_{L^2(D)} \cdot \|\Delta u\|_{L^2(D)} \\
&= \left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \|\Delta u\|^2_{L^2(D)} + \frac{\gamma^2}{\varepsilon} \|\Delta u\|^2_{L^2(D)} - 2\gamma\tau \|\Delta u\|_{L^2(D)} \|u\|_{L^2(D)} \\
&\quad + \varepsilon \tau^2 \|u\|^2_{L^2(D)} + (1 + \gamma - \varepsilon) \tau^2 \|u\|^2_{L^2(D)} \\
&= \left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \|\Delta u\|^2_{L^2(D)} + \varepsilon \left(\tau \|u\|_{L^2(D)} - \frac{\gamma}{\varepsilon} \|\Delta u\|_{L^2(D)}\right)^2 + (1 + \gamma - \varepsilon) \tau^2 \|u\|^2_{L^2(D)} \\
&\geqslant \left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \|\Delta u\|^2_{L^2(D)} + (1 + \gamma - \varepsilon) \tau^2 \|u\|^2_{L^2(D)} \qquad \varepsilon \in (\gamma, \gamma+1).
\end{aligned}$$

In brief, for all $\varepsilon \in (\gamma, \gamma+1)$ with $\gamma := \frac{1}{n_{max}-1}$ (positive because $n_{min} > 1$), we have

$$(A_\tau u, u)_{H^2(D)} = \mathcal{A}_\tau(u,u) = \left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \cdot \|\Delta u\|^2_{L^2(D)} + (1 + \gamma - \varepsilon) \tau^2 \cdot \|u\|^2_{L^2(D)} \tag{4.18}$$

Notice that we choose $\varepsilon \in (\gamma, \gamma+1)$ so that $\gamma - \frac{\gamma^2}{\varepsilon} > 0$ and $1 + \gamma - \varepsilon > 0$. So we can fix any $\varepsilon \in (\gamma, \gamma+1)$ (for example, $\varepsilon = \gamma + \frac{1}{2}$).

Besides, as $u \in H^2_0(D)$, then $\nabla u \in H^1_0(D)$, so by Poincaré's inequality (Theorem 5.1.9 of the Appendix)

$$\|\nabla u\|^2_{L^2(D)} \leqslant \frac{1}{\lambda_1(D)} \|\Delta u\|^2_{L^2(D)}.$$

Therefore

$$\begin{aligned}
(A_\tau u, u)_{H^2(D)} &\geqslant \left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \|\Delta u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2 \|u\|^2_{L^2} \\
&= \frac{1}{2}\left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \|\Delta u\|^2_{L^2(D)} + \frac{1}{2}\left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \|\Delta u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2 \|u\|^2_{L^2} \\
&\geqslant \frac{1}{2}\left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \|\Delta u\|^2_{L^2(D)} + \frac{1}{2}\left(\gamma - \frac{\gamma^2}{\varepsilon}\right) \lambda_1(D)\|\nabla u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2 \|u\|^2_{L^2} \\
&\geqslant C_\tau \cdot [\|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)} + \|\Delta u\|^2_{L^2(D)}]
\end{aligned}$$

with

$$C_\tau := \min\left\{ \frac{1}{2}\left(\gamma - \frac{\gamma^2}{\varepsilon}\right), \frac{1}{2}\left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\lambda_1(D), 1+\gamma-\varepsilon \right\}.$$

Since $u \in H^2_0(D)$, its H^2-norm is equivalent to the L^2-norm of its laplacian (see Proposition 5.1.10 of the Appendix). Therefore

$$C_\tau \cdot [\|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)} + \|\Delta u\|^2_{L^2(D)}] \geqslant C_\tau \|\Delta u\|^2_{L^2(D)} \geqslant C_\tau \|u\|^2_{H^2(D)}.$$

So

$$(A_\tau u, u)_{H^2(D)} \geqslant C_\tau \cdot \|u\|^2_{H^2(D)}$$

with $C_\tau > 0$ constant.

Finally, B is

- Non-negative:

$$(Bu, u)_{H^2(D)} = \mathcal{B}(u,u) = (\nabla u, \nabla u)_{L^2(D)} = \|\nabla u\|^2_{L^2(D)} \geqslant 0.$$

- Compact: let (u_n) be a bounded sequence in $H^2_0(D)$. We want to see that $(Bu_n)_n$ has a convergent subsequence in $H^2_0(D)$.

  Since (u_n) is bouned in $H^2_0(D)$, then (∇u_n) is bounded in $H^1_0(D)$, so by the compact embedding of $H^1_0(D)$ into $L^2(D)$ (Theorem 5.1.12) , there exists a subsequence of (∇u_n) that converges in $L^2(D)$, say (∇u_{n_k}). That is, there exists $u \in H^1_0(D)$ such that $\nabla u_{n_k} \to \nabla u$ in $L^2(D)$.

  Let us see that $Bu_{n_k} \to Bu$ in $H^2_0(D)$. We have

$$\begin{aligned}
\|Bu_{n_k} - Bu\|_{H^2_0(D)} &= \sup_{\varphi \in H^2_0, \|\varphi\|_{H^2_0(D)}=1} (B(u_{n_k} - u), \varphi)_{H^2_0(D)} \\
&= \int_D \nabla(u_{n_k} - u)\cdot \nabla\varphi \, dx && \text{Definition of } B \\
&\leqslant \left(\int_D |\nabla u_{n_k} - \nabla u|^2 \, dx\right)^{1/2} \left(\int_D |\nabla \varphi|^2\right)^{1/2} && \text{C-S} \\
&= \|\nabla u_{n_k} - \nabla u\|_{L^2(D)} \cdot \|\nabla\varphi\|_{L^2(D)} \\
&\leqslant \|\nabla u_{n_k} - \nabla u\|_{L^2(D)}, && \|\varphi\|_{H^2_0(D)} = 1.
\end{aligned}$$

  which tends to zero when $k \to \infty$ because $\nabla u_{n_k} \to \nabla u$ in $L^2(D)$. Notice that the first step of the above deduction follows from the characterization of a norm in a Hilbert Space in terms of the scalar product. In brief, $Bu_{n_k} \to Bu$ in $H^2_0(D)$, completing the proof. □

Remark 4.3.4. Notice that the condition $n_{min} > 1$ is used in this Lemma and in the following one to prove the coercivity of certain operators. More specifically, this condition is asked so that the bounds $\frac{1}{n-1} \geqslant \frac{1}{n_{max}-1} > 0$ are true.

Theorem 4.3.5. *If $n_{min} > 1$, then $A_\tau - \tau B$ is coercive for $0 < \tau < \frac{\lambda_1(D)}{n_{max}}$. That is, for every $0 < \tau < \frac{\lambda_1(D)}{n_{max}}$, there exists a constant $\alpha > 0$ (that only depends on τ and n_{max}) such that*

$$(A_\tau u - \tau B u, u)_{H^2} \geqslant \alpha \|u\|^2_{H^2} > 0 \quad \text{for all } u \in H^2_0(D),$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue for the negative Laplacian.

Proof. Let $\gamma := \frac{1}{n_{max}-1} > 0$ as in the proof of Lemma 4.3.3. We then have

$$\begin{aligned}(A_\tau u - \tau B u, u)_{H^2_0} &= \mathcal{A}_\tau(u,u) - \tau\|\nabla u\|^2_{L^2} \\ &\geqslant \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\|\Delta u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2\|u\|^2_{L^2(D)} - \tau\|\nabla u\|^2_{L^2}\end{aligned} \tag{4.19}$$

for $\gamma < \varepsilon < \gamma + 1$. Since $\nabla u \in H^1_0(D)$, the Poincaré inequality gives us

$$\|\nabla u\|^2_{L^2(D)} \leqslant \frac{1}{\lambda_1(D)}\|\Delta u\|^2_{L^2(D)}.$$

Therefore, apllying it to the above expresison we obtain

$$\begin{aligned}(A_\tau u - \tau B u, u) &\geqslant \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\cdot\|\Delta u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2\cdot\|u\|^2_{L^2(D)} - \tau\cdot\frac{1}{\lambda_1(D)}\|\Delta u\|^2_{L^2(D)} \\ &= \left(\gamma - \frac{\gamma^2}{\varepsilon} - \frac{\tau}{\lambda_1(D)}\right)\cdot\|\Delta u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2\cdot\|u\|^2_{L^2(D)}.\end{aligned}$$

Hence, since the norm of $H^2(D)$ is equivalent to the $L^2(D)$-norm of the Laplacian for functions in $H^2_0(D)$ (see Section 5.1.10 of the Appendix), $A_\tau - \tau B$ is strictly positive if

$$\tau < \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\lambda_1(D).$$

In particular, taking ε arbitrary closed to $\gamma + 1$, the latter becomes

$$\tau < \lim_{\varepsilon\to(\gamma+1)^-}\left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\lambda_1(D) = \gamma - \frac{\gamma^2}{\gamma+1}\lambda_1(D) = \frac{\gamma(\gamma+1)-\gamma^2}{\gamma+1}\lambda_1(D) = \frac{\gamma}{\gamma+1}\lambda_1(D).$$

Since $\gamma = n_{\frac{1}{n_{max}-1}}$, then $\frac{\gamma}{\gamma+1} = \frac{1}{n_{max}}$ and the condition becomes

$$\tau < \frac{\lambda_1(D)}{n_{max}}.$$

□

As a corollary, we have:

Corollary 4.3.6. *Assume that $n_{min} > 1$. Then*

$$k_0^2 \geqslant \frac{\lambda_1(D)}{n_{max}}, \tag{4.20}$$

where k_0 is the smallest transmission eigenvalue and $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

Proof. For $\tau = k^2 < \frac{\lambda_1(D)}{n_{max}}$, $A_\tau - \tau B$ is strictly positive, and therefore its kernel is $\{0\}$. Therefore, k cannot be a transmission eigenvalue. □

4.3.1 Spectral decomposition with respect to an operator

Remember that we are interested in studying the kernel of $A_\tau - \tau B$. That is, we are concerned with the spectral properties of A_τ with respect to B. The following step is to establish some general spectral properties for operators in the conditions of Theorem 4.3.3.

Throughout this section, let X be a Hilbert space with scalar product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$.

Recall that if $A\colon X \to X$ is a bounded, self-adjoint, and stricly positive definite operator, then we can define the operators $A^{\pm 1/2}$ by

$$A^{\pm 1/2} = \int_0^\infty \lambda^{\pm 1/2}\, dE_\lambda$$

where dE_λ is the spectral measure associated with A. In particular, $A^{\pm 1/2}$ are **also bounded, self-adjoint, and stricly positive definite** operators on X satisfying

$$A^{-1/2}A^{1/2} = I \quad \text{and} \quad A^{1/2}A^{1/2} = A$$

What we are going to do now is to describe the **spectral decomposition of the operator** A **with respect to self-adjoint nonnegative compact operators** B. That is, we are going to generalize the known properties of the eigenvalue problem

$$A - \lambda I$$

when the identity operator I is replaced by an arbitrary operator B (which has to satisfy some properties).

Specifically, we want to generalize the spectral decomposition and the min-max principle.

Generalized Spectral Decomposition

Theorem 4.3.7. *Let $A\colon X \to X$ be a bounded, self-adjoint, and stricly positive definite operator on a Hilbert space X and let $B\colon X \to X$ be a nonnegative, self-adjoint, and compact linear operator with null space $N(B)$. There exists an increasing sequence of positive real numbers (λ_j) and a sequence (u_j) of elements of X satisfying*

$$Au_j = \lambda_j B u_j \tag{4.21}$$

and

$$(Bu_j, u_l) = \delta_{jl} \tag{4.22}$$

such that each $u \in [A(N(B))]^\perp$ can be expanded in a series

$$u = \sum_{j=1}^{\infty} \gamma_j u_j. \tag{4.23}$$

If $N(B)^\perp$ has infinite dimension then $\lambda_j \to \infty$ as $j \to \infty$.

Remark 4.3.8. Notice that this theorem gives us a spectral decomposition because the λ_j are the eigenvalues of the operator A with respect to the operator B, with the eigenfunctions u_j being orthonormal with respect to the scalar product $[u,v]_B := (Bu, v)$ defined for $u, v \in X\backslash N(B)$ (see the end of Remark 4.3.14). In the case where B is the identity, we have the classical result.

Proof of Theorem 4.3.7. The idea of the proof is to apply the well-known spectral theorem for compact self-adjoint operators, and then derive the stated properties from it.

The operator $\tilde{B} = A^{-1/2}BA^{-1/2}$ is

- Nonnegative, because $A^{1/2}, A^{-1/2}$ are positive, and B is non-negative.
- Self-adjoint: in general if we denote the adjoint of an operator A by A^*, then $(AB)^* = B^*A^*$. So
$$(A^{-1/2}BA^{-1/2})^* = (A^{-1/2})^*B^*(A^{-1/2})^* = A^{-1/2}BA^{-1/2},$$
being the last step true because $A^{-1/2}$ and B are self-adjoint.
- Compact, because it is a composition of B, which is compact, with bounded operators, namely $A^{1/2}$ and $A^{-1/2}$.

Therefore, we can apply the spectral decomposition (5.10) and (5.11). Let (μ_j) be the decreasing sequence of positive eigenvalues of $\tilde{B}$ and (v_j) the corresponding orthonormal eigenelements of $\tilde{B}$ that are complete in $\overline{A^{-1/2}BA^{-1/2}(X)}$. Then

$$v = \sum_{j=1}^{\infty}(v, v_j)v_j \tag{4.24}$$

for all $v \in \overline{A^{-1/2}BA^{-1/2}(X)}$. Note that, because of the spectral theorem, zero is the only possible accumulation point for the sequence (μ_j).

We have that

$$\mu_j v_j = \tilde{B}v_j \iff \mu_j v_j = A^{-1/2}BA^{-1/2}v_j$$

Proof of (4.21) and (4.22).

To obtain (4.21), we want an equation with the operator A on the left-hand side acting on a vector, and the operator B on the right-hand side acting on that same vector and multiplied by an scalar. Therefore, we divide by μ_j (which is positive, thus non-zero) and apply on both sides the operator $A^{1/2} = (A^{-1/2})^{-1}$, thus obtaining

$$A^{1/2}v_j = \frac{1}{\mu_j}BA^{-1/2}v_j$$

Since we want to have operator A on the left-hand side acting on the same vector as B on the right-hand side, we express $A^{1/2} = AA^{-1/2}$, getting

$$AA^{-1/2}v_j = \frac{1}{\mu_j}BA^{-1/2}v_j$$

Therefore, our candidate for λ_j is $\frac{1}{\mu_j}$ and our candidate for u_j is $A^{-1/2}v_j$. However, notice that

$$\begin{aligned}(B(A^{-1/2}v_j), A^{-1/2}v_l) &\underset{\substack{\uparrow \\ A^{-1/2}\text{ self-adjoint}}}{=} (A^{-1/2}BA^{-1/2}v_j, v_l) = (\tilde{B}v_j, v_l) \underset{\substack{\uparrow \\ \tilde{B}v_j=\mu_j v_j}}{=} \mu_j(v_j, v_l) \\ &\underset{\substack{\uparrow \\ \{v_j\}_j\text{ orthonormal}}}{=} \mu_j \cdot \delta_{j,l} = \begin{cases} \mu_j & \text{if } j = l \\ 0 & \text{if } j \neq l. \end{cases}\end{aligned}$$

So, in order to have $(Bu_j, u_l) = \delta_{j,l}$, the right choice for u_j is not $A^{-1/2}v_j$ but $\frac{1}{\sqrt{\mu_j}}A^{-1/2}v_j$.

Therefore, what we have proved is that if we define

$$\boxed{\lambda_j := \frac{1}{\mu_j}} \quad \text{and} \quad \boxed{u_j := \sqrt{\lambda_j}A^{-1/2}v_j},$$

for $j = 1, 2, \ldots$, then they satisfy (4.21) and (4.22).

Proof of (4.23)

Let us check (4.23). The only tool we have to obtain it is equation (4.24) (i.e., the fact that $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\overline{A^{-1/2}BA^{-1/2}(X)}$), so we are going to apply it.

Let

$$u \in \overline{A^{-1}BA^{-1/2}(X)}.$$

Then, there exists a sequence $\{w_n\} \subset A^{-1}BA^{-1/2}(X)$ such that $u = \lim_n w_n$. Therefore, since $A^{+1/2}$ is bounded (i.e. continuous) we have

$$A^{1/2}u = A^{1/2}(\lim_n w_n) = \lim_n A^{1/2}w_n,$$

where $A^{1/2}w_n \in A^{1/2}A^{-1}BA^{-1/2}(X) = A^{-1/2}BA^{-1/2}(X)$ for all $n \in \mathbb{N}$. So

$$A^{+1/2}u \in \overline{A^{-1/2}BA^{-1/2}(X)}.$$

Therefore, we can write $A^{+1/2}u$ using the orthonormal basis $\{v_j\}_{j\in\mathbb{N}}$ (that is, we can apply (4.24)), obtaining

$$A^{+1/2}u = \sum_{j=1}^{\infty}(A^{+1/2}u, v_j)v_j.$$

Applying $A^{-1/2} = (A^{1/2})^{-1}$ and using its continuity and linearity, we have

$$\begin{aligned} u = A^{-1/2}\left(\sum_{j=1}^{\infty}(A^{1/2}u, v_j)v_j\right) & \underset{\substack{\uparrow \\ A^{-1/2} \text{ lin. and cont.}}}{=} \sum_{j=1}^{\infty} A^{-1/2}((A^{1/2}u, v_j)v_j) \\ \underset{\substack{\uparrow \\ A^{-1/2} \text{ linear}}}{=} \sum_{j=1}^{\infty}(A^{1/2}u, v_j)\cdot A^{-1/2}v_j & \underset{\substack{\uparrow \\ \text{Multiply by } 1=\frac{\sqrt{\lambda_j}}{\sqrt{\lambda_j}}}}{=} \sum_{j=1}^{\infty}\frac{1}{\sqrt{\lambda_j}}(A^{1/2}u, v_j)\cdot\sqrt{\lambda_j}A^{-1/2}v_j = \sum_{j=1}^{\infty}\gamma_j u_j, \end{aligned}$$

where

$$\gamma_j := \frac{1}{\sqrt{\lambda_j}}(A^{1/2}u, v_j).$$

We have thus obtained (4.23) for $u \in \overline{A^{-1}BA^{-1/2}(X)}$. Nevertheless, the statement asserts that (4.23) is true for all $u \in [A(N(B))]^{\perp}$. If we prove that

$$\overline{A^{-1}BA^{-1/2}(X)} = [A(N(B))]^{\perp},$$

then we will have finished. This is true since

$$\overline{A^{-1}BA^{-1/2}(X)} = \overline{A^{-1}B(X)} = [N(BA^{-1}]^{\perp} = [A(N(B))]^{\perp}$$

having used

- On the first equality, that $A^{-1/2}$ is bijective from X to X (since it is invertible being $(A^{-1/2})^{-1} = A^{1/2}$), so $A^{-1/2}(X) = X$. Therefore, $A^{-1}BA^{-1/2}(X) = A^{-1}B(X)$.
- On the second equality, we use Theorem 5.1.20, which gives us $N(A^*)^\perp = \overline{A(X)}$ for A an operator defined on X. We use that $(A^{-1}B)^* = B^*(A^{-1})^* = BA^{-1}$ where the last step is true because A (and therefore A^{-1}) and B are self-adjoint.
- On the third equality, we have used that $N(AB^{-1}) = A(N(B))$.

Proof of the last statement

The only thing left to prove is that if $N(B)^\perp$ has infinite dimension, then $\lambda_j \to \infty$ when $j \to \infty$.

If $N(B)^\perp$ has infinite dimension, then, since $A^{\pm 1/2}$ are bijective and striclty positive and $\tilde{B} = A^{-1/2}BA^{-1/2}$, then $N(\tilde{B})^\perp$ has infinite dimension as well. Since $N(\tilde{B})^\perp$ is an (at most countable) orthogonal direct sum of eigenspaces associated to different non-zero eigenvalues, and (because of the spectral theorem) the eigenspace associated to one eigenvalue is finite-dimensional, then $N(\tilde{B})^\perp$ being infinite-dimensional implies that **there must be an infinite number of distinct eigenvalues**. Therefore the eigenvalues form a non-negative decreasing sequence, with 0 as the only accumulation point: $\{\mu_j\}_{j=1}^\infty$ with $\mu_j \searrow 0$. Therefore, $\lambda_j = \frac{1}{\mu_j} \to \infty$ when $j \to \infty$, as we wanted to prove. $\square$

Generalized Min-Max Principle

The next important result is a generalized min-max principle. Before proceeding with it, we make some remarks that will be useful in the following.

Lemma 4.3.9. *Let X be a Hilbert space with scalar product $(\cdot,\cdot)_X$, and let $A\colon X \to X$ be a bounded linear operator, which is self-adjoint and stricly positive. Then the mapping $[\cdot,\cdot]_A\colon X \times X \to K$ given by*

$$[u, v]_A := (Au, v)_X$$

is an scalar product.

Proof. For simplicity, we denote by $(\cdot,\cdot)$ the scalar product of X. Let $u, v, w \in X$, $\alpha, \beta \in K$. We have that:

1. *Conjugate symmetry*:

$$[u, v]_A = (Au, v) \underset{\substack{\uparrow \\ A \text{ self-adjoint}}}{=} (u, Av) = \overline{(Av, u)} = \overline{[v, u]_A}.$$

2. *Sesquilinearity*:

$$[\alpha u + \beta v, w]_A = (A(\alpha u + \beta v), w) = \alpha(Au, w) + \beta(Av, w) = \alpha[u, w]_A + \beta[v, w]_A$$

3. *Positive-definiteness*: $[u, u]_A = (Au, u) \geqslant c \cdot \|u\|_X^2$ for some constant $c > 0$ because A is strictly positive. So $[u, u]_A \geqslant 0$ and the equality is reached if and only if $u = 0$. $\square$

The fact that $[\cdot,\cdot]_A$ is an scalar product is useful to prove that the vectors $\{u_j\}$ given in the previous theorem are linearly independent.

Corollary 4.3.10. *Let A and B be as in Theorem 4.3.7. Then, the vectors $\{u_j\}_{j\in\mathbb{N}}$ are orthogonal with respect to the scalar product $[\cdot,\cdot]_A$ and, therefore, they are linearly independent.*

Proof. We have that

$$[u_j, u_k]_A = (Au_j, u_k) \underset{\underset{(4.21)}{\uparrow}}{=} (\lambda_j Bu_j, u_k) = \lambda_j (Bu_j, u_k) \underset{\underset{(4.22)}{\uparrow}}{=} \lambda_j \cdot \delta_{j,k}.$$

Since they are orthogonal with respect to $[\cdot,\cdot]_A$, they are linearly independent. □

Corollary 4.3.11. *Let A and B be as before, Then, X can be decomposed in the following direct sum:*

$$X = N(B) \oplus [A(N(B))]^{\perp}$$

Proof. Since $N(B)$ is a closed subspace of X, we can decompose X as an orthogonal direct sum of $N(B)$ and its orthogonal complement with respecto to the scalar product $[\cdot,\cdot]_A$. This gives us

$$X = N(B) \oplus [N(B)]^{\perp_A}.$$

We have that

$$[N(B)]^{\perp_A} = [A(N(B))]^{\perp},$$

since

$$u \in [A(N(B))]^{\perp} \iff (u,v) = 0 \;\; \forall v \in A(N(B)) \iff (u, Aw) = 0 \;\; \forall w \in N(B) \underset{\underset{A \text{ self-adjoint}}{\uparrow}}{\iff}$$

$$\iff (Au, w) = 0 \;\; \forall w \in N(B) \underset{\underset{\text{Def. of } [\cdot,\cdot]_A}{\uparrow}}{\iff} [u,w]_A = 0 \;\; \forall w \in N(B) \iff u \in [N(B)]^{\perp_A}.$$

Therefore, we have the following decomposition of X as a direct sum:

$$X = N(B) \oplus [A(N(B))]^{\perp}.$$

□

Remark 4.3.12. Because of the direct sum given by Lemma 4.3.11, $u \notin N(B)$ implies that $u = b + v$ with $v \in N(B)$ and $v \in [A(N(B))]^{\perp} \backslash \{0\}$.

Theorem 4.3.13. *Let A, B, and (λ_j) be as in Theorem 4.3.7 and define the Rayleigh quotient as*

$$R(u) = \frac{(Au,u)}{(Bu,u)}$$

for $u \notin N(B)$, where $(\cdot,\cdot)$ is the scalar product in X. Then the following min-max principles hold

$$\lambda_j = \min_{W \in \mathcal{U}_j^A} \left(\max_{u \in W \backslash \{0\}} R(u) \right) = \max_{W \in \mathcal{U}_{j-1}^A} \left(\min_{u \in [A(W + N(B))]^{\perp} \backslash \{0\}} R(u) \right), \tag{4.25}$$

where $\mathcal{U}_j^A$ denotes the set of all j-dimensional subspaces of $[A(N(B))]^{\perp}$.

Remark 4.3.14. $R(u) = \frac{(Au,u)}{(Bu,u)}$ is **well defined** for $u \notin N(B)$. To check this, it is enough to see that $(Bu, u) \neq 0$ for $u \notin N(B)$.

By Theorem 4.3.7, for every $u \in [A(N(B))]^{\perp}$, there exist scalars γ_j such that

$$u = \sum_{j=1}^{\infty} \gamma_j u_j$$

with $(Bu_j, u_l) = \delta_{j,l}$. So, if $\tilde{u} \notin N(B)$, i.e., $\tilde{u}) = b + u$ with $b \in N(B)$, $u \in [A(N(B))]^\perp \backslash \{0\}$ (see Remark 4.3.12), then

$$\begin{aligned}(B\tilde{u}, \tilde{u}) &= (B(b+u), b+u) = (Bu, b+u) = (Bu, u) + (Bu, b) = (Bu, u) + (u, Bb) = (Bu, u) \\ &= \left(B\left(\sum_{j=1}^{\infty} \gamma_j u_j\right), \sum_{l=1}^{\infty} \gamma_l u_l \right) = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} \gamma_j \overline{\gamma_l} (Bu_j, u_l) = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} \gamma_j \overline{\gamma_l} \delta_{j,l} = \sum |\gamma_j|^2 \neq 0,\end{aligned}$$

since $u \notin N(B)$, so in particular $u \neq 0$, so $\gamma_j \neq 0$ for some j. In the above formula we have used that B is continuous and that the scalar product is continuous in each of its variables.

Notice as well that a consequence of the fact that $(Bu, u) \neq 0$ for $u \notin N(B)$ is that $[u, v]_B := (Bu, v)$ defines a **scalar product** on $X \backslash N(B)$ (as can be checked in a similar way to what we did with $[\cdot, \cdot]_A$ in Lemma 4.3.9).

Proof of Theorem 4.3.13. The proof is based on the fact that if $u \in [A(N(B))]^\perp$, then from Theorem 4.3.7 we can write

$$u = \sum_{j=k}^{\infty} \gamma_j u_j \tag{4.26}$$

for some coefficients γ_j, where the u_j are defined in Theorem 4.3.7. Using this expression of u, we can calculate $R(u)$ in the following way[3]:

$$R(u) = \frac{1}{\sum_{j=1}^{\infty} |\gamma_j|^2} \sum_{j=1}^{\infty} \lambda_j |\gamma_j|^2. \tag{4.27}$$

This is because

$$\begin{aligned}
R(u) &= \frac{(Au, u)}{(Bu, u)} \\
&= \frac{\left(A\left(\sum_1^\infty \gamma_j u_j\right), \sum_1^\infty \gamma_k u_k\right)}{\left(B\left(\sum_1^\infty \gamma_j u_j\right), \sum_1^\infty \gamma_k u_k\right)} \\
&= \frac{\left(\sum_1^\infty \gamma_j A u_j, \sum_1^\infty \gamma_k u_k\right)}{\left(\sum_1^\infty \gamma_j B u_j, \sum_1^\infty \gamma_k u_k\right)} && (A, B \text{ lin. and cont.}) \\
&= \frac{\sum_{j,k=1}^\infty \gamma_j \overline{\gamma_k} (Au_j, u_k)}{\sum_{j,k=1}^\infty \gamma_j \overline{\gamma_k} (Bu_j, u_k)} && (\cdot,\cdot) \text{ lin. and cont. in 1st var.} \\
&= \frac{\sum_{j,k=1}^\infty \gamma_j \overline{\gamma_k} \lambda_j (Bu_j, u_k)}{\sum_{j,k=1}^\infty \gamma_j \overline{\gamma_k} (Bu_j, u_k)} && (4.21)\text{: } Au_j = \lambda_j Bu_j \\
&= \frac{\sum_{j,k=1}^\infty \gamma_j \overline{\gamma_k} \lambda_j \delta_{j,k}}{\sum_{j,k=1}^\infty \gamma_j \overline{\gamma_k} \delta_{j,k}} && (4.22)\text{: } (Bu_j, u_l) = \delta_{jl} \\
&= \frac{\sum_{j=1}^\infty |\gamma_j|^2 \cdot \lambda_j}{\sum_{j=1}^\infty |\gamma_j|^2}.
\end{aligned}$$

[3]Notice that, given $u \in [A(N(B))]^\perp$, *a priori* the expression (4.26) may not be unique. But this is not important: (4.27) is the formula for $R(u)$ for every expression of the form (4.26). That is, given $u \in [A(N(B))]^\perp$ in the form (4.26), $R(u)$ can be computed with formula (4.27).

We have to prove (4.25).

Proof of the first equality of (4.25) Let $W_j = span\{u_1, \ldots, u_j\}$. We have that $W_j \in \mathcal{U}_j^A$ because

- $W_j \subset [A(N(B))]^\perp$ since $u_k \in [A(N(B))]^\perp$ for every $k = 1, \ldots, j$. This is because given $v \in A(N(B))$, that is, $v = Aw$ with $w \in X$ such that $Bw = 0$, we have that
$$(u_k, v) = (u_k, Aw) \underset{\substack{\uparrow \\ A \text{ self-adjoint}}}{=} (Au_k, w) = (\lambda_k Bu_k, w) \underset{\substack{\uparrow \\ B \text{ self-adjoint}}}{=} \lambda_k(u_k, Bw) = \lambda_k(u_k, 0) = 0.$$

- $W_j = span\{u_1, \ldots, u_j\}$ has dimension j because $\{u_j\}_{j\in\mathbb{N}}$ are linearly independent because of Lemma 4.3.10.

First step: Let us prove that

$$\lambda_j = \max_{u \in W_j\backslash\{0\}} R(u) = \min_{u\in[A(W_{j-1}+N(B))]^\perp\backslash\{0\}} R(u). \tag{4.28}$$

1. *First equality of* (4.28): if $u \in W_j\backslash\{0\}$, i.e., $u \in span\{u_1, \ldots, u_j\}$, then $u = \sum_{k=1}^j \gamma_k u_k$, which is an expression of the form (4.26). So by (4.27),
$$R(u) \underset{\substack{\uparrow \\ \gamma_k=0 \text{ for } k\geqslant j+1 \text{ in } (4.26)}}{=} \frac{\sum_{k=1}^j |\gamma_k|^2\lambda_k}{\sum_{k=1}^j |\gamma_k|^2} \underset{\substack{\uparrow \\ \lambda_j>0, \{\lambda_j\} \text{ increasing}}}{\leqslant} \frac{\sum_{k=1}^j |\gamma_k|^2 \cdot \lambda_j}{\sum_{k=1}^j |\gamma_k|^2} = \lambda_j \cdot 1 = \lambda_j.$$

   So $\lambda_j \geqslant \max_{u_j\in W_j\backslash\{0\}} R(u)$. To see the equality, it is enough that the bound is reached for some vector. That is, it is enough that $R(u) = \lambda_j$ for some $u \in W_j\backslash\{0\}$. Let us see that $u_j \in W_j\backslash\{0\}$ satisfies that. To begin with, $u_j \in W_j\backslash\{0\}$ since $u_j \neq 0$ because of condition (4.22) and because $u_j \in W_j$ by definition W_j. Since $u_j = 1 \cdot u_j$ is an expression of the form (4.26), by (4.27) we have that $R(u_j) = \frac{1^2\cdot\lambda_j}{1^2} = \lambda_j$.

   Therefore $\lambda_j = \max_{u\in W_j\backslash\{0\}} R(u)$, thus obtaining the first equality.

2. *Second equality of* (4.28): let $u \in A(W_{j-1} + N(B))^\perp\backslash\{0\}$.

   Notice that $A(W_{j-1} + N(B))^\perp \subset [A(N(B))]^\perp$ because $A(N(B)) \subset A(W_{j-1} + N(B))$, so it makes sense to apply formula (4.26) to elements of $A(W_{j-1} + N(B))^\perp$.

   That is, since $u \in [A(N(B))]^\perp$, Theorem 4.3.7 gives us that u can be expressed in the form (4.26):
$$u = \sum_{k=1}^\infty \gamma_k \cdot u_k.$$

Let $j \in \{1, \ldots, j-1\}$. Then

$$\begin{aligned}
0 &= (u, Au_{j_0}) = \\
&= \left(\sum_{k=1}^{\infty} \gamma_k u_k, Au_{j_0}\right) \\
&= \sum_{k=1}^{\infty} \gamma_k (u_k, Au_{j_0}) && (\cdot,\cdot) \text{ lin. and cont. on 1st var.} \\
&= \sum_{k=1}^{\infty} \gamma_k (u_k, \lambda_{j_0} Bu_{j_0}) && (4.21) \\
&= \sum_{k=1}^{\infty} \gamma_k \cdot \lambda_{j_0} (u_k, Bu_{j_0}) && (\cdot,\cdot) \text{ sesquilinear; } \lambda_j \in \mathbb{R} \\
&= \sum_{k=1}^{\infty} \gamma_k \lambda_{j_0} (Bu_k, u_{j_0}) && B \text{ self-adjoint} \\
&= \lambda_{j_0} \sum_{k=1}^{\infty} \gamma_k \delta_{k,j_0} = \gamma_{j_0} \lambda_{j_0}. && (4.22)
\end{aligned}$$

Then, since $\lambda_{j_0} > 0$ (by Theorem 4.3.7), it has to be $\gamma_{j_0} = 0$.

Since this is valid for all $j_0 = 1, \ldots, j-1$, we have that $u = \sum_{k=j}^{\infty} \gamma_k u_k$, so applying (4.27) we have that

$$R(u) = \frac{\sum_{k=j}^{\infty} |\gamma_k|^2 \cdot \lambda_k}{\sum_{k=j}^{\infty} |\gamma_k|^2} \geqslant \frac{\sum_{k=j}^{\infty} |\gamma_k|^2 \cdot \lambda_j}{\sum_{k=j}^{\infty} |\gamma_k|^2} = \lambda_j \cdot 1 = \lambda_j.$$

Therefore

$$\lambda_j \leqslant \min_{u \in [A(W_{j-1}+N(B))]^{\perp} \setminus \{0\}} R(u).$$

To see the equality, it is enough to find $u \in [A(W_{j-1} + N(B))]^{\perp} \setminus \{0\}$ such that $R(u) = \lambda_j$. We have seen when checking the first equality that $R(u_j) = \lambda_j$ (and that $u_j \neq 0$). So it is enough to see that $u_j \in [A(W_{j-1} + N(B))]^{\perp} \setminus \{0\}$.

Lemma 4.3.15. *Under the conditions of the theorem, if $W_{j-1} := span\{u_1, \ldots, u_{j-1}\}$, then*

$$u_l \in [A(W_{j-1} + N(B))]^{\perp} \setminus \{0\} \quad \forall l \geqslant j.$$

Proof. Let $v \in A(W_{j-1} + N(B))$, i.e., $v = Aw$ with $w \in W_{j-1} + N(B)$. Then

$$(u_l, v) = (u_l, Aw) \underset{\substack{\uparrow \\ A \text{ self-adjoint}}}{=} (Au_l, w) \underset{\substack{\uparrow \\ (4.21)}}{=} (\lambda_l Bu_l, w)$$

Since $w \in W_{j-1} + N(B)$, $w = \sum_{k=1}^{j-1} a_k u_k + b$ for some $a_k \in \mathbb{C}$, $b \in N(B)$. So

$$\lambda_l (Bu_l, w) = \lambda_l \left(Bu_l, \sum_{k=1}^{j-1} a_k u_k + b \right) = \lambda_j \cdot \left[\sum_{k=1}^{j-1} \overline{a_k} (Bu_l, u_k) + (Bu_l, b) \right].$$

The sum is 0 because, by (4.22), $(Bu_l, u_k) = \delta_{l,k} = 0$ for $k = 1, \ldots, j-1$ and in this case $l \geqslant j$. So what remains is

$$(u_l, v) = \lambda_l (Bu_l, b) \underset{\substack{\uparrow \\ B \text{ self-adjoint}}}{=} \lambda_l (u_l, Bb) \underset{\substack{\uparrow \\ b \in N(B)}}{=} \lambda_l (u_l, 0) = 0.$$

Then, since this is true for all $v \in A(W_{j-1} + N(B))$, we have that

$$u_l \in [A(W_{j-1} + N(B))]^{\perp} \backslash \{0\} \quad \forall l \geqslant j. \qquad \square$$

So, applying Lemma 4.3.15, we end the proof of the second equality of (4.28).

Next, let W be any element of $\mathcal{U}_j^A$, that is, W has dimension j and $W \subset [A(N(B))]^{\perp}$. Therefore,

$$W \cap [AW_{j-1} + A(N(B))]^{\perp} \neq \{0\}. \tag{4.29}$$

By Lemma 4.3.15 we have that

$$\overline{span}\{u_k\}_{k \geqslant j} \subset [AW_{j-1} + A(N(B))]^{\perp},$$

so to prove (4.29) it is enough to see that $W \cap \overline{span}\{u_k\}_{k \geqslant j} \neq \{0\}$. Let us check this.

Since W is a subspace of dimension j of $[A(N(B))]^{\perp}$, there exists a basis $\{v_1, \ldots, v_j\}$. As $[A(N(B))]^{\perp} = \overline{span}\{u_k\}_{k=1}^{\infty}$, there exist scalars $\gamma_l^{(k)}$, $l \in \mathbb{N}$, $k = 1, \ldots, n$ such that

$$v_k = \sum_{l=1}^{\infty} \gamma_l^{(k)} u_l.$$

We can thus consider the matrix of dimension $j \times (j-1)$ whose k-th row is composed of the coefficients $\gamma_1^{(k)}, \ldots, \gamma_{j-1}^{(k)}$. We can then make Gaussian elimination (i.e., row reduction) to obtain matrix whose last row is composed by zeros.

This means that, from the basis $\{v_1, \ldots, v_j\}$ of W, we can obtain another basis $\{w_1, \ldots, w_j\}$ such that $w_j \in \overline{span}\{u_k\}_{k \geqslant j}$. In particular, since it is a basis, we have that $w_j \neq 0$ and $W_j \in W$, so

$$W \cap \overline{span}\{u_k\}_{k \geqslant j} \neq 0.$$

Therefore, we obtain (4.29).

So

$$\max_{u \in W \backslash \{0\}} R(u) \geqslant \min_{u \in W \cap [A(W_{j-1}+N(B))]^{\perp} \backslash \{0\}} R(u) \geqslant \min_{u \in [A(W_{j-1}+N(B))]^{\perp} \backslash \{0\}} R(u) = \lambda_j$$

where

- The first inequality is true because that maximum over $W \backslash \{0\}$ is $\geqslant$ that the minimum over any subset of $W \backslash W \backslash \{0\}$ and $W \cap [A(W_{j-1} + N(B))]^{\perp} \backslash \{0\}$ is a subset of $W \backslash \{0\}$.

- The second inequality is true because that set on the left-hand side is smaller, so the minimum over the set on the left is greater or equal to the one over the set on the right.

- The last equality is true because of (4.28).

Since $\max_{u \in W \backslash \{0\}} R(u) \geqslant \lambda_j$ for every $w \in \mathcal{U}_j^A$, with $\max_{u \in W_j \backslash \{0\}} R(u) = \lambda_j$ with $W_j \in \mathcal{U}_j^A$ (because $u_1, \ldots, u_j$ are linearly independent due to Lemma 4.3.10) we have that

$$\lambda_j = \min_{w \in \mathcal{U}_j^A} \left(\max_{u \in W \backslash \{0\}} R(u) \right),$$

which is the first equality of the statement of the theorem that we wanted to prove.

Proof of the second equality of (4.25)

To prove the second equality of the theorem, we reason in a similar way to what we have done for $W \in \mathcal{U}_j^A$ before.

Let $W \in \mathcal{U}_{j-1}^A$ be a subspace of $[A(N(B))]^\perp$ of dimension $j-1$. Reasoning as before, $W_j \cap (AW)^\perp \neq \{0\}$. So

$$\min_{u\in[A(W)+A(N(B))]^\perp\backslash\{0\}} R(u) \leqslant \min_{u\in W_j\cap(AW)^\perp\backslash\{0\}} R(u) \leqslant \max_{u\in W_j\cap(AW)^\perp\backslash\{0\}} R(u) \leqslant \max_{u\in W_j\backslash\{0\}} R(u) = \lambda_j,$$

having used on the first equality that[4] $W_j \cap (AW)^\perp \subset [AW + A(N(B))]^\perp$.

Since $\min_{u\in[A(W)+A(N(B))]^\perp\backslash\{0\}} R(u) \leqslant \lambda_j$ for every $W \in \mathcal{U}_{j-1}^A$ and $\min_{u\in[A(W_{j-1})+A(N(B))]^\perp\backslash\{0\}} R(u) = \lambda_j$ because of the second equality of (4.28) with $W_{j-1} \in \mathcal{U}_{j-1}^A$, then

$$\lambda_j = \max_{w\in\mathcal{U}_{j-1}^A} \left(\min_{u\in[A(W+N(B))]^\perp\backslash\{0\}} R(u) \right),$$

proving thus the second equality of the theorem. □

The following corollary shows that it is possible to remove the dependence on A in the choice of the subspaces in the min-max principle for the eigenvalues λ_j.

Corollary 4.3.16. *Let A, B, λ_j and R be as in Theorem 4.3.13. Then*

$$\lambda_j = \min_{W\subset\mathcal{U}_j} \left(\max_{u\in W\backslash\{0\}} R(u) \right), \tag{4.30}$$

where $\mathcal{U}_j$ denotes the set of all j-dimensional subspaces W of X such that $W \cap N(B) = \{0\}$.

Proof. To clarify the notation, recall that

- $\mathcal{U}_j$ is the set of all the W vector subspaces of X of dimension j such that $W \cap N(B) = \{0\}$.
- $\mathcal{U}_j^A$ is the set of all the subspaces of dimension j of $[A(N(B))]^\perp$.

Let us see that $\mathcal{U}_j^A \subset \mathcal{U}_j$. Let $W \in \mathcal{U}_j^A$, i.e., a subspace of $[A(N(B))]^\perp \subset X$ of dimension j. To check that $W \in \mathcal{U}_j$, we just have to see that $W \cap N(B) = \{0\}$.

Let $v \in W \cap N(B)$.

- Since $v \in W$, then $v \in [A(N(B))]^\perp$. So $(v, w) = 0$ for every $w \in (A(N(B))$.
- Since $v \in N(B)$, then $Av \in A(N(B))$.

[4]This is true because, in general, given A, B vector subspaces of X, and given $A_1 \subset A^\perp$ a vector subspace, we have that

$$A_1 \cap B^\perp \subseteq [A + B]^\perp,$$

because if $v \in A_1 \cap B^\perp$, given $w = w_A + w_B \in A + B$, we have that

$$(v, w) = (v, w_A) + (v, w_B) = 0$$

where the first term of the sum is 0 because $v \in A_1 \subset A^\perp$ and $w_A \in A$; and the second is 0 because $v \in B^\perp$ and $w_A \in B$. So it is enough to take $A = A(N(B))$, $A_1 = W_j$ y $B = AW$.

So $(v, Av) = 0$. Since $(v, Av) = (Av, v) = [v, v]_A = 0$, and $[\cdot,\cdot]_A$ is a scalar product (because of Lemma 4.3.9), then $v = 0$. So $W \cap N(B) = \{0\}$ and, therefore, $W \in \mathcal{U}_j$.

In brief,

$$\mathcal{U}_j^A \subset \mathcal{U}_j.$$

From Theorem 4.3.13 and the fact that $\mathcal{U}_j^A \subset \mathcal{U}_j$, we have that

$$\lambda_j \underset{\text{Thm. 4.3.13}}{\underset{\uparrow}{=}} \min_{W \in \mathcal{U}_j^A} \left(\max_{u \in W \setminus \{0\}} R(u) \right) \underset{\mathcal{U}_j^A \subset \mathcal{U}_j}{\underset{\uparrow}{\geqslant}} \min_{W \in \mathcal{U}_j} \left(\max_{u \in W \setminus \{0\}} R(u) \right).$$

So to prove (4.30), it is enough to prove the other inequality, that is, it is enough to see that

$$\lambda_j \leqslant \min_{W \subset \mathcal{U}_k} \left(\max_{u \in W \setminus \{0\}} R(u) \right). \tag{4.31}$$

Proof of (4.31)

Let $W \in \mathcal{U}_j$ and let $v_1, \ldots, v_j$ a basis of W.

We want to prove the previous theorem, and in order to do it we need to work with j-dimensional vector subspaces of $[A(N(B))]^\perp$, that is, elements of $\mathcal{U}_j^A$. To do so, we are going to try to obtain a subspace $\widetilde{W} \in \mathcal{U}_j^A$ from the given subspace W.

Because of the direct sum, every v_l $(l = 1, \ldots, j)$ of the basis of W can be decomposed as a sum

$$v_l = v_l^0 + \tilde{v}_l$$

with $\tilde{v}_l \in [A(N(B))]^\perp$ y $v_l^0 \in N(B)$.

The vectors $\{\tilde{v}_l\}_{l=1}^j$ are linearly independent, since $W \in \mathcal{U}_j \implies W \cap N(B) = \{0\}$, so, if $\sum_1^j a_l \tilde{v}_l = 0$, we have that

$$W \ni \sum_1^j a_l v_l \underset{v_l = v_l^0 + \tilde{v}_l}{\underset{\uparrow}{=}} \sum_1^j a_l (v_l^0 + \tilde{v}_l) = \sum_1^l a_l v_l^0 + \sum_1^l a_l \tilde{v}_l \underset{\text{Hypothesis}}{\underset{\uparrow}{=}} \sum_1^j a_l v_l^0 \in N(B) \underset{W \cap N(B) = \{0\}}{\underset{\uparrow}{\Longrightarrow}}$$

$$\sum_1^j a_l v_l = 0 \underset{v_1, \ldots, v_j \text{ lin. indep.}}{\underset{\uparrow}{\Longrightarrow}} a_1 = \ldots = a_j = 0$$

Therefore, the space $\widetilde{W} := span\{\tilde{v}_1, \ldots, \tilde{v}_j\}$ has dimension j. Besides, $\widetilde{W} \subset [A(N(B))]^\perp$ (since $\tilde{v}_1, \ldots, \tilde{v}_j \in [A(N(B))]^\perp$ by definition). So $\widetilde{W} \in \mathcal{U}_j^A$.

Let $\tilde{u} \in \tilde{u} \in \widetilde{W} \setminus \{0\}$. Then, there exist scalars $a_1, \ldots, a_j$ such that

$$\tilde{u} = \sum_1^j a_l \tilde{v}_l = \sum_1^j a_l (v_l - v_l^0) = \sum_1^j a_l v_l - \sum_1^j a_l v_l^0.$$

Denoting $u := \sum_1^j a_l v_l \in W$, $u^0 := \sum_1^j a_l v_l^0 \in N(B)$, we have that every $\tilde{u} \in \widetilde{W}$ can be written as

$$\tilde{u} = u - u^0$$

for some $u \in W$, $u^0 \in N(B)$.

We have that:

- Since $u^0 \in N(B)$, $Bu^0 = 0$.
- Since $u^0 \in N(B)$, then $Au^0 \in A(N(B))$. Since $\tilde{u} \in [A(N(B))]^\perp$, then $(Au^0, \tilde{u}) = 0$.

So

$$R(u) \underset{\substack{\uparrow \\ \text{Def.}}}{=} \frac{(Au,u)}{(Bu,u)} \underset{\substack{\uparrow \\ u=u^0+\tilde{u};\ (Au^0,\tilde{u})=0}}{=} \frac{(A\tilde{u},\tilde{u})+(Au^0,u^0)}{(B\tilde{u},\tilde{u})+(Bu^0,u^0)} \underset{\substack{\uparrow \\ Bu^0=0}}{=} \frac{(A\tilde{u},\tilde{u})}{(B\tilde{u},\tilde{u})} + \frac{(Au^0,u^0)}{(B\tilde{u},\tilde{u})} \underset{\substack{\uparrow \\ \text{Def.}}}{=} R(\tilde{u}) + \frac{(Au^0,u^0)}{(B\tilde{u},\tilde{u})}.$$

Since A is stricly positive and B is non-negative, i.e., $(Au^0, u^0) \geqslant c\|u^0\|^2$, $(B\tilde{u},\tilde{u}) \geqslant 0$ (and $\neq 0$ because $\tilde{u} \neq 0$ and because of Remark 4.3.14) then

$$R(u) = R(\tilde{u}) + \frac{(Au^0,u^0)}{(B\tilde{u},\tilde{u})} \geqslant R(\tilde{u}).$$

So for every $\tilde{u} \in \widetilde{W}\backslash\{0\}$ there exists $u \in W\backslash\{0\}$ such that $R(\tilde{u}) \leqslant R(u)$. Therefore,

$$\max_{\tilde{u}\in\widetilde{W}} R(\tilde{u}) \leqslant \max_{u\in W\backslash\{0\}} R(u). \tag{4.32}$$

Besides, by Theorem 4.3.13 we also have that

$$\max_{\tilde{u}\in\widetilde{W}} R(\tilde{u}) \underset{\substack{\uparrow \\ \widetilde{W}\in\mathcal{U}_j^A}}{\geqslant} \min_{\widetilde{W}\in\mathcal{U}_j^A} \left(\max_{\tilde{u}\in\widetilde{W}} R(u) \right) \underset{\substack{\uparrow \\ \text{Thm. } 4.3.13}}{=} \lambda_j. \tag{4.33}$$

So, combining (4.32) and (4.33), we have

$$\lambda_j \leqslant \max_{u\in W\backslash\{0\}} R(u).$$

This is true for every $W \in \mathcal{U}_j$ (because that one we have chosen is arbitrary), so

$$\lambda_j \leqslant \min_{W\in\mathcal{U}_j} \left(\max_{u\in W\backslash\{0\}} R(u) \right),$$

as we wanted to prove. □

Until now, we have stablished general results about the spectral decomposition of an operator A (bounded, self-adjoint and stricly positive definite) with respect to a self-adjoint nonnegative compact operator. The following theorem is a corollary of all the results we have proved in this sections, and it is exactly what we will use to prove the existence of transmission eigenvalues. The two hypothesis enumerated in it may look a little bit rare at first sight, but they are used to apply Bolzano's Theorem.

Theorem 4.3.17. *Let $\tau \mapsto A_\tau$ be a continuous mapping from $(0,\infty)$ to the set of bounded, self-adjoint, and strictly positive definite operators on the Hilbert space X and let B be a self-adjoint and nonnegative compact linear operator on X. We assume that there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$A_{\tau_0} - \tau_0 B$ is strictly positive on X.*
2. *$A_{\tau_1} - \tau_1 B$ is non-positive on an l-dimensional subspace W_l of X.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \dots, l$ has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue (counting multiplicity) of A_τ with respect to B, that is, $N(A_\tau - \lambda_j(\tau)B) \neq \{0\}$.

Proof. The idea of the proof is to apply Bolzano's Theorem on the closed and bounded interval $[\tau_0, \tau_1]$ to the continuous function $\lambda_j(\tau) - \tau$. So the proof is divided in two steps: first, we prove the continuity using Corollary 4.3.16. It will be important to use it instead of Theorem 4.3.13, because we will use that the subspaces over which we take the minimum do not depend on A_τ. The second part of the proof is to check that $\lambda_j(\tau_0) - \tau_0 > 0$ and that $\lambda_j(\tau_1) - \tau_1 < 0$, so that we can apply Bolzano's Theorem.

First step: $\lambda_j(\tau)$ is a continuous function of τ.

Let us see the continuity in $\tau_0 \in (0, \infty)$. From Lemma 5.1.1 and the fact that $\inf(f) = -\sup(-f)$ we have that $|\inf(f) - \inf(g)| \leqslant \inf(|f - g|)$, so

$$
\begin{aligned}
|\lambda_j(\tau) - \lambda_j(\tau_0)| &= \left| \min_{W \in \mathcal{U}_j} \left(\max_{u \in W \backslash \{0\}} R_\tau(u) \right) - \min_{W \in \mathcal{U}_j} \left(\max_{u \in W \backslash \{0\}} R_{\tau_0}(u) \right) \right| \\
&\leqslant \min_{W \in \mathcal{U}_j} \left(\left| \max_{u \in W \backslash \{0\}} R_\tau(u) - \max_{u \in W \backslash \{0\}} R_{\tau_0}(u) \right| \right) \\
&\leqslant \min_{W \in \mathcal{U}_j} (\max_{u \in W \backslash \{0\}} |R_\tau(u) - R_{\tau_0}(u)|)
\end{aligned}
$$

We have that

$$
\begin{aligned}
|R_\tau(u) - R_{\tau_0}(u)| &= \left| \frac{(A_\tau u, u)}{(Bu, u)} - \frac{(A_{\tau_0} u, u)}{(Bu, u)} \right| \\
&= \left| \frac{((A_\tau - A_{\tau_0})u, u)}{(Bu, u)} \right| \\
&\leqslant \frac{\|(A_\tau - A_{\tau_0})u)\| \cdot \|u\|}{(Bu, u)} \\
&\leqslant \frac{\|A_\tau - A_{\tau_0}\| \cdot \|u\|^2}{(Bu, u)}
\end{aligned}
$$

Given $\varepsilon > 0$, we want to choose $\delta > 0$ independent of u in such a way that the last expression is $< \varepsilon$ for every $u \in W \backslash \{0\}$, and this for every $W \in \mathcal{U}_j$. However, this turns out to be complicated, so we take a different path. We have that

$$
|\lambda_j(\tau) - \lambda_j(\tau_0)| \leqslant \min_{W \in \mathcal{U}_j} (\max_{u \in W \backslash \{0\}} |R_\tau(u) - R_{\tau_0}(u)|) \leqslant \max_{u \in W_0 \backslash \{0\}} |R_\tau(u) - R_{\tau_0}(u)|)
$$

for any subspace $W_0 \in \mathcal{U}_j$. We are interested in chooosing one in which we can bound $\frac{\|A_\tau - A_{\tau_0}\| \cdot \|u\|^2}{(Bu,u)}$ independently of u, so it is convenient to choose a subspace in which we can work explicitly.

So let $(u_k)_{k \geqslant 1}$ be the sequence of vectors given by Theorem 4.3.7 associated to the operators A_{τ_0} and B. Let $W_0 := span\{u_1, \dots, u_j\}$. Then, every $u \in W_0$ can be written as $u = \sum_{k=1}^{j} \alpha_k u_k$. Therefore

$$
(Bu, u) = \sum_{k=1}^{j} |\alpha_k|^2.
$$

We are interested in obtaining a lower bound for this which has the form of a constant multiplied by $\|u\|^2$ so that, when we take the inverse, we can obtain an upper bound for $\frac{\|A_\tau - A_{\tau_0}\| \cdot \|u\|^2}{(Bu,u)}$ and the terms

$\|u\|^2$ in the numerator and denominator simplify, leaving just a constant multiplied by $\|A_\tau - A_{\tau_0}\|$. We have that

$$\|u\|^2 = \left\|\sum_1^j \alpha_k u_k\right\|^2 \leqslant \left(\sum |\alpha_k| \cdot \|u_k\|\right)^2 \leqslant C(j) \cdot \sum |\alpha_k|^2 \cdot \|u_k\|^2 \leqslant C(j) \cdot \sup_{k=1,\dots,j} \|u_k\|^2 \cdot \sum_1^j |\alpha_k|^2$$

where $C(j) > 0$ is a constant that depends on j.

So

$$\frac{1}{(Bu,u)} = \frac{1}{\sum_1^j |\alpha_k|^2} \leqslant \frac{1}{C(j)} \cdot \frac{1}{\sup_{k=1,\dots,j} \|u_k\|^2} \frac{1}{\|u\|^2}.$$

Therefore,

$$\frac{\|A_\tau - A_{\tau_0}\| \cdot \|u\|^2}{(Bu,u)} \leqslant \frac{\|A_\tau - A_{\tau_0}\| \cdot \|u\|^2}{\sup_{k=1,\dots,j} \|u_k\|^2} \frac{1}{\|u\|^2} = C(\tau_0, j) \cdot \|A_\tau - A_{\tau_0}\|$$

for every $u \in W_0 \backslash \{0\}$, with $C(\tau_0, j) > 0$ a constant that depends on τ_0 and j (it depends on τ_0 because the sequence $(u_k)_{k \geqslant 1}$ is associated to the operator A_{τ_0}).

So, given $\varepsilon > 0$, since $\tau \mapsto A_\tau$ is continuous, there exists $\delta > 0$ such that if $|\tau - \tau_0| < \delta$, then $\|A_\tau - A_{\tau_0}\| \leqslant \frac{\varepsilon}{C}$, and therefore $|\lambda_j(\tau) - \lambda_j(\tau_0)| \leqslant C \cdot \frac{\varepsilon}{C} = \varepsilon$.

In brief, $\lambda_j(\tau)$ is a continuous function of τ for every $j \geqslant 1$.

Second step: to apply Bolzano's theorem.

Since $\lambda_j(\tau)$ is continuous, we can apply to it the well-known theorems of analysis of one real variable about continuous functions on compact intervals (more specifically, we are going to use Bolzano's theorem).

Hypothesis 1. tells us that $A_{\tau_0} - \tau_0 B$ is stricly positive on X. Since B is non-negative, we have that[5] $A_{\tau_0} - \tau B$ is positive for every $\tau \leqslant \tau_0$, so $N(A_{\tau_0} - \tau B) = \{0\}$ for every $\tau \leqslant \tau_0$. Therefore, in order to have $\tau = \lambda_j(\tau_0)$ (that is, in order for τ to be an eigenvalue of A_{τ_0} in the sense that $N(A_{\tau_0} - \tau B) \neq \{0\}$) a necessary condition is that

$$\lambda_j(\tau_0) > \tau_0. \tag{4.34}$$

Hypotesis 2. gives us that $A_{\tau_1} - \tau_1 B$ is non-positive in a l-dimensional subspace of X, namely W_l.

That is,

$$((A_{\tau_1} - \tau_1 B)u, u) \leqslant 0 \quad \forall u \in W_l \iff (A_{\tau_1} u, u) \leqslant \tau_1 (Bu, u) \quad \forall u \in W_l. \tag{4.35}$$

Since A_{τ_1} is strictly positive, $(A_{\tau_1} u, u) \geqslant C\|u\|^2$ for every $u \in X$, in particular for every $u \in W_l$.

So $\tau_1(Bu,u) > 0$ for every $u \in W_l$, which implies that

- $\frac{(A_{\tau_1} u, u)}{(Bu,u)} \leqslant \tau_1$ para todo $u \in W_l$.
- $Bu \neq 0$ for every $u \in W_l$, that is,
$$W_l \cap N(B) = \{0\}.$$

[5] In detail, the explanation is the following:

$$C\|u\|^2 \leqslant (A_{\tau_0} u - \tau_0 Bu, u) = (A_{\tau_0} u, u) - \tau_0 (Bu, u) \leqslant (A_{\tau_0} u, u) - \tau (Bu, u)$$

for every $\tau \leqslant \tau_0$, since $A_{\tau_0} - \tau_0 B$ is stricly positive and B is non-negative.

From the second item we deduce that $W_l \in \mathcal{U}_l$ and, in particular, any j-dimensional subspace of W_l (with $1 \leqslant j \leqslant l$) is in $\mathcal{U}_j$.

The min-max principle (equation (4.30)) gave us that

$$\lambda_j(\tau) = \min_{W \in \mathcal{U}_j} \left(\max_{u \in W\setminus\{0\}} \frac{(A_\tau u, u)}{(Bu, u)} \right).$$

So given $W_j \in \mathcal{U}_j$ a j-dimensional subspace of W_l and using (4.35), we have that

$$\lambda_j(\tau_1) = \min_{W \in \mathcal{U}_j} \left(\max_{u \in W\setminus\{0\}} \frac{(A_{\tau_1} u, u)}{(Bu, u)} \right) \leqslant \max_{u \in W_j\setminus\{0\}} \frac{(A_{\tau_1} u, u)}{(Bu, u)} \leqslant \max_{u \in W_j\setminus\{0\}} \tau_1 = \tau_1.$$

In brief, we have seen that

$$\begin{cases} \lambda_j(\tau_0) > \tau_0 \\ \lambda_j(\tau_1) \leqslant \tau_1 \end{cases}$$

for every $1 \leqslant j \leqslant l$. So, if we define the continuous function $f(\tau) := \lambda_j(\tau) - \tau$, we have that $f(\tau_0) > 0$ y $f(\tau_1) \leqslant 0$. So, by Bolzano's theorem, there exists $\tau \in [\tau_0, \tau_1]$ (or $[\tau_1, \tau_0]$ if $\tau_1 < \tau_0$) such that $f(\tau) = 0$, that is, such that $\lambda_j(\tau) = \tau$. □

We are now ready to prove the main theorem of this chapter:

Theorem 4.3.18. *If $n \in L^\infty(D)$ with $1 < n_{min} \leqslant n(x) \leqslant n_{max} < \infty$, the there exists an infinit set of transmission eigenvalues with $+\infty$ as its only accumulation point.*

Proof. We have that

$$0 < \frac{1}{n_{max} - 1} \leqslant \frac{1}{n(x) - 1} \leqslant \frac{1}{n_{min} - 1} < \infty.$$

By Proposition 4.3.3, A_τ is self-adjoint and stricly positive (as well as bounded), and B is self-adjoint, non-negative and compact. Therefore, they satisfy the first necessary conditions to apply Theorem 4.3.17.

By Lemma 4.3.5, $A_\tau - \tau B$ is strictly positive on $X = H_0^2(D)$ for $0 < \tau < \frac{\lambda_1(D)}{n_{max}}$. So hypothesis 1. of Theorem 4.3.17 is satisfied if we take $\tau_0 < \frac{\lambda_1(D)}{n_{max}}$.

So the only thing left to check is hypothesis 2. of Theorem 4.3.17.

In order to do so, let $k_{1,n_{min}}$ be the smallest transmission eigenvalue for the ball B of radius $R = 1$ and let $n(x) := n_{min}$ (constant). Notice that this transmission eigenvalue exists because of Theorem 4.1.1, and there is a minimum eigenvalue because of Theorem 4.2.3.

Then $k_{\varepsilon,n_{min}} := \frac{k_{1,n_{min}}}{\varepsilon}$ is the least transmission eigenvalue corresponding to the ball of radius ε with refractive index n_{min}. This is due to the following scaling argument:

$$\int_{B_1} \frac{1}{n_{min} - 1} (\Delta u + k^2 u)(\Delta \overline{v} + k^2 n_{min} \overline{v})\, dx = 0 \quad \forall v \in H_0^2(D) \iff$$

$$\int_{B_\varepsilon} \frac{1}{n_{min} - 1} \left(\Delta u \left(\frac{y}{\varepsilon}\right) + k^2 u \left(\frac{y}{\varepsilon}\right) \right) \left(\Delta \overline{v} \left(\frac{y}{\varepsilon}\right) + k^2 n_{min} \overline{v} \left(\frac{y}{\varepsilon}\right) \right) \cdot \frac{1}{\varepsilon^3}\, dy = 0 \quad \forall v \in H_0^2(D) \iff$$

$$\int_{B_\varepsilon} \frac{1}{n_{min} - 1} \left(\frac{1}{\varepsilon^2} \Delta u \left(\frac{y}{\varepsilon}\right) + \frac{k^2}{\varepsilon^2} u \left(\frac{y}{\varepsilon}\right) \right) \cdot \left(\frac{1}{\varepsilon^2} \Delta \overline{v} \left(\frac{y}{\varepsilon}\right) + \frac{k^2}{\varepsilon^2} \overline{v} \left(\frac{y}{\varepsilon}\right) \right) dy = 0 \quad \forall v \in H_0^2(D)$$

and since the laplacian of $w(x) = u(\frac{x}{\varepsilon})$ is $\Delta w(x) = \frac{1}{\varepsilon^2}\Delta u(\frac{x}{\varepsilon})$, we have that w is a non-trivial[6] solution of the interior transmission problem on the ball B_ε with $n(x) = n_{min}$, so $\frac{k}{\varepsilon}$ is a transmission eigenvalue for this medium.

Therefore,

$$k \text{ is trans. eig. of the media } B_1, n_{min} \iff \frac{k}{\varepsilon} \text{ is trans. eig. of the media } B_\varepsilon, n_{min}.$$

Therefore $k_{\varepsilon,n_{min}} := \frac{k_{1,n_{min}}}{\varepsilon}$ is the smallest transmission eigenvalue corresponding to the ball of radius ε and index of refraction n_{min}.

Given $\varepsilon > 0$ (sufficiently small), let $m := m(\varepsilon) \geqslant 1$ be the maximum number of disjoint balls $B_\varepsilon^1, \dots, B_\varepsilon^m$ of radius ε that are contained in D. That is,

$$\overline{B_\varepsilon^j} \subset D \quad \forall j = 1, \dots, m \quad \text{ and } \quad \overline{B_\varepsilon^j} \cap \overline{B_\varepsilon^i} = \varnothing \text{ if } j \neq i.$$

Then $k_{\varepsilon,n_{min}} := \frac{k_{1,n_{min}}}{\varepsilon}$ is the smallest transmission eigenvalue for each one of these balls with index of refraction n_{min} and let $u^{B^j_{\varepsilon},n_{min}} \in H_0^2(B_\varepsilon^j)$ $(j = 1, \dots, m)$ the corresponding eigenfunctions.

The extension of $\tilde{u}^j$ from $u^{B^j_{\varepsilon},n_{min}}$ to all D by zero is on $H_0^2(D)$ since $u^{B^j_{\varepsilon},n_{min}} \in H_0^2(B_\varepsilon^j)$ (see Theorem 5.1.21 of the appendix).

Besides, the vectors $\{\tilde{u}^1, \dots, \tilde{u}^m\}$ are orthogonal in $H_0^2(D)$ because they have disjoint supports. In particular, they are linearly independent.

By (4.17), as they are a solution of the interior transmission problem for $n \equiv n_{min}$, they satisfy

$$\begin{aligned} 0 &= \int_D \frac{1}{n_{min} - 1}(\Delta\tilde{u}^j + k^2_{\varepsilon,n_{min}}\tilde{u}^j)(\Delta\overline{\tilde{u}^j} + k^2_{\varepsilon,n_{min}} \cdot n_{min}\overline{\tilde{u}^j})\,dx \\ &= \int_D \frac{1}{n_{min} - 1}|\Delta\tilde{u}^j + k^2_{\varepsilon,n_{min}}\tilde{u}^j|^2\,dx + k^2_{\varepsilon,n_{min}}\int_D |\tilde{u}^j|^2\,dx - k^2_{\varepsilon,n_{min}}\int_D |\nabla\tilde{u}^j|\,dx \end{aligned} \tag{4.36}$$

for each $j = 1, \dots, m$, where the second equality comes from the calculation made before to see that (4.17) is equivalent to $(A_\tau u - \tau Bu, v) = 0$ for all $v \in H_0^2(D)$.

Let $\mathcal{U} := span\{\tilde{u}^1, \dots, \tilde{u}^m\}$ a m-dimensional subspace of $H_0^2(D)$.

Since every $\tilde{u}^j$, $j = 1, \dots, m$ satisfies (4.36) and they have disjoint supports, for $\tau_1 := k^2_{\varepsilon,n_{min}}$ and for every $\tilde{u} \in \mathcal{U}$ it is true that

$$\begin{aligned} (A_{\tau_1}\tilde{u} - \tau_1 B\tilde{u}, \tilde{u})_{H^2(D)} &= \int_D \frac{1}{n - 1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\,dx + \tau_1^2\int_D |\tilde{u}|^2\,dx - \tau_1\int_D |\nabla\tilde{u}|^2\,dx \\ &\leqslant \int_D \frac{1}{n_{min} - 1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\,dx + \tau_1^2\int_D |\tilde{u}|^2\,dx - \tau_1\int_D |\nabla\tilde{u}|^2\,dx = 0. \end{aligned}$$

Hence, hypothesis 2 of Theorem 4.3.17 is also satisfied.

Therefore, we can apply it to conclude that there are $m(\varepsilon)$ transmission eigenvalues (counting multiplicity) in $[\tau_0, k_{\varepsilon,n_{min}}]$. Notice that $m(\varepsilon)$ and $k_{\varepsilon,n_{min}}$ go to $+\infty$ when $\varepsilon \to 0$. Since the multiplicity of each eigenvalue is finite (because of Theorem 4.2.3) taking $\varepsilon \to 0$ we see that there exists an infinte set (countable, because of Theorem 4.2.3) of transmission eigenvalues that accumulates at $+\infty$ (that is, they can be ordered in an increasing sequence that goes to infinity). □

In a similar way (see Theorem 4.12.2 of [7], pages 135-136) it is possible to prove the following theorem:

Theorem 4.3.19. *Assume that $0 < n_{\min} \leqslant n_{\max} < 1$. Then there exist an infinite number of transmission eigenvalues with ∞ as the only accumulation point.*

[6] Non-trivial because $u \neq 0$.

Chapter 5

Appendix

5.1 Results from Analysis

5.1.1 Technical lemma: supremum of the difference of two functions

Lemma 5.1.1. *Let $f, g\colon A \to \mathbb{R}$ be two real-valued functions. Then*

$$\sup_{x\in A}(f(x) - g(x)) \geqslant \sup_{x\in A} f(x) - \sup_{x\in A} g(x).$$

Proof. First proof: we begin by proving that

$$\sup_{x\in A}(f(x) + g(x)) \leqslant \sup_{x\in A} f(x) + \sup_{x\in A} g(x). \tag{5.1}$$

This is a particular case of the fact that the supremum over a set is greater or equal to the supremum over a subset. The left-hand side of (5.1) is

$$\sup_{x,y\in A; x=y} (f(x) + g(y)) \tag{5.2}$$

whilst the right-hand side of (5.1) is

$$\sup_{x,y\in A} (f(x) + g(y)). \tag{5.3}$$

The set of the x, y considered in (5.2) is a subset of the x and y considered in (5.3), so (5.1) follows.

Using (5.1), we have that $\sup((f - g) + g) \leqslant \sup(f - g) + \sup(g)$. That is,

$$\sup(f - g) \geqslant \sup(f) - \sup(g).$$

Second proof: we want to prove

$$\sup(f - g) \geqslant \sup(f) - \sup(g).$$

To express everything in terms of addition, we can write the minus sign inside the supremum by changing it to an infimum, i.e., using that

$$-\sup(g) = +\inf(-g).$$

Let us define $h(x) = -g(x)$. The problem then reduces to prove that

$$\sup(f + h) \geqslant \sup(f) + \inf(h).$$

This is simple, because for every x we have that $h(x) \geqslant \inf(h)$, so

$$\sup(f + h) \geqslant \sup(f + \inf(h)) = \sup(f) + \inf(h).$$ □

5.1.2 Basic results from Measure Theory

Differentiation under the integral sign

Theorem 5.1.2. *Suppose that* $f\colon X\times[a,b]\to\mathbb{C}$, *with* $-\infty<a<b<\infty$ *and that* $f(\cdot,t)\colon X\to\mathbb{C}$ *is integrable for every* $t\in[a,b]$. *Let* $F(t)=\int_X f(x,y)\,d\mu(x)$.

1. *Suppose that there exists* $g\in L^1(\mu)$ *such that* $|f(x,t)|\leqslant g(x)$ *for every* x,t. *If* $\lim_{t\to t_0} f(x,y)=f(x,t_0)$ *for every* x, *then* $\lim_{t\to t_0}F(t)=F(t_0)$; *in particular, if* $f(x,\cdot)$ *is continuous for every* x, *then* F *is continuous.*

2. *Suppose that* $\frac{\partial f}{\partial t}$ *exists and there is* $g\in L^1(\mu)$ *such that* $\left|\frac{\partial f}{\partial t}(x,t)\right|\leqslant g(x)$ *for every* x,t. *Then* F *is differentiable and* $F'(x)=\int_X\frac{\partial f}{\partial t}(x,t)\,d\mu(x)$.

Proof. See Theorem 2.27 of [18], page 56. □

Approximations of the identity

Theorem 5.1.3. *Suppose* $|\phi(x)|\leqslant C(1+|x|)^{-n-\varepsilon}$ *for some* $C,\varepsilon>0$ *and* $\int\phi(x)\,dx=a$. *If* $f\in L^p(\mathbb{R}^n)$ $(1\leqslant p\leqslant\infty)$ *and* $\phi_t(x):=t^{-n}\phi(t^{-1}x)$, *then* $f*\phi_t(x)\to af(x)$ *as* $t\to 0^+$ *for every* x *in the Lebesgue set of* f *(in particular for almost every* x, *and for every* x *at which* f *is continuous).*

Proof. See Theorem 8.15 of [18], pages 243-245. □

5.1.3 Integration by parts

In this section, we assume that D is a bounded, open subset of $\mathbb{R}^n$ and ∂D is C^1.

Theorem 5.1.4 (Gauss-Green Theorem)**.** 1. *Suppose* $u\in C^1(\overline{D})$. *Then*

$$\int_D \partial_{x_i}u\,dx=\int_{\partial D}u\nu^i\,ds\quad(i=1,\dots,n).$$

2. *We have*

$$\int_D \operatorname{div}\boldsymbol{u}\,dx=\int_{\partial D}\boldsymbol{u}\cdot\nu\,ds$$

for each vector field $\boldsymbol{u}\in C^1(\overline{D};\mathbb{R}^n)$.

Proof. See Theorem 1 of Section C.2 of [16], pages 711-712. □

The second assertion is called the Divergence Theorem and follows from the first one applied to each component of $\boldsymbol{u}=(u^1,\dots,u^n)$.

Theorem 5.1.5 (Integration by parts formula)**.** *Let* $u,v\in C^1(\overline{D})$. *Then*

$$\int_D\partial_{x_i}uv\,dx=-\int_D u\partial_{x_i}v\,dx+\int_{\partial D}uv\nu^i\,ds\quad(i=1,\dots,n).$$

Proof. It follows from applying the first item of the previous theorem to uv. □

Green's Formulas

To begin with, we recall Green's formulas, which will be used repeatedly throughout the essay. First, we have the classical result for C^2 functions up to the boundary (see [13], page 19).

Lemma 5.1.6 (Green's Formulas)**.** *Let D be a bounded domain with boudnary of class C^1, and let ν be its unitary exterior normal. Let $v \in C^2(\overline{D})$.*

1. *If $u \in C^1(\overline{D})$, then*
$$\int_D (u\Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial D} u \cdot \frac{\partial v}{\partial \nu}\, ds. \tag{5.4}$$

2. *If we make the stronger assumption that $u \in C^2(\overline{D})$, then*
$$\int_D (u\Delta v - v\Delta u)\, dx = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds. \tag{5.5}$$

Proof. To prove (5.4), use the theorem of integration by parts replacing ∂_{x_i} by v. To obtain (5.5), apply (5.4) to u and v, then apply it again interchanging them, and the substract them. $\square$

This theorem can be generalized to the setting of Sobolev spaces.

Lemma 5.1.7. *Let D be a bounded domain with boundary of class C^1, and let ν be its unitary exterior normal. Let $v \in H^2(D)$.*

1. *If $u \in H^1(D)$, then*
$$\int_D (u\Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial D} u\frac{\partial v}{\partial \nu}\, ds. \tag{5.6}$$

2. *If $u \in H^2(D)$, then*
$$\int_D (u\Delta v - v\Delta u)\, dx = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial v} \right) ds. \tag{5.7}$$

Proof. In this TFM we will only use the second one, so we only prove this one. We will use that $C^2(\overline{D})$ is dense in $H^2(D)$ and apply the classical Lemma 5.1.6.

Since $C^2(\overline{D})$ is dense in $H^2(D)$, given $u, v \in H^2(D)$, we can approximate them by a sequence of $C^2(\overline{D})$ functions. Let $\{u_n\}_n, \{v_n\}_n \subset C^2(\overline{D})$ be two sequences such that $u_n \xrightarrow{H^2(D)} u$ and $v_n \xrightarrow{H^2(D)} v$.

By (5.5), we have
$$\int_D (u_n\Delta v_n - v_n\Delta u_n)\, dx = \int_{\partial D} \left(u_n\frac{\partial v_n}{\partial \nu} - v_n\frac{\partial u_n}{\partial \nu} \right) ds$$
for every $n \in \mathbb{N}$.

We would like to see that the left-hand side tends to $\int_D u\Delta v - v\Delta u)\, dx$ when $n \to \infty$ and that the right-hand side tends to $\int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds$, obtaining thus (5.7).

Let us check this. We have that
$$\begin{aligned}
(u_n\Delta v_n - v_n\Delta u_n) - (u\Delta v - v\Delta u) &= (u_n\Delta v_n - u\Delta v) - (v_n\Delta u_n - v\Delta u) \\
&= [u_n\Delta v - u\Delta v_n + u\Delta v_n - u\Delta v] + [v_n\Delta u_n - v\Delta u_n + v\Delta u_n - v\Delta u] \\
&= (u_n - u)\Delta v + u(\Delta v_n - \Delta v) + \Delta u_n(v_n - v) + v(\Delta u_n - \Delta u).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\int_D [(u_n\Delta v_n - v_n\Delta u_n) - (u\Delta v - v\Delta u)]\,dx\right| \\
&\leqslant \int_D \big[|u_n-u||\Delta v| + |u||\Delta v_n - \Delta v| + |\Delta u_n||v_n - v| + |v||\Delta u_n - \Delta u|\big]\,dx \\
&\leqslant \left(\int_D |u_n-u|^2\right)\left(\int |\Delta v_n|^2\right) + \left(\int_D |\Delta v_n - \Delta v|^2\right)\left(\int |u|^2\right) \\
&+ \left(\int_D |v_n - v|^2\right)\left(\int |\Delta u_n|^2\right) + \left(\int_D |\Delta u_n - \Delta u|^2\right)\left(\int |v|^2\right) \\
&= \|u_n - u\|^2_{L^2(D)}\|\Delta v_n\|^2_{L^2(D)} + \|\Delta v_n - \Delta v\|^2_{L^2(D)}\|u\|^2_{L^2(D)} \\
&+ \|v_n - v\|^2_{L^2(D)}\|\Delta u_n\|^2_{L^2(D)} + \|\Delta u_n - \Delta u\|^2_{L^2(D)}\|v\|^2_{L^2(D)}
\end{aligned}$$

Since $u_n \xrightarrow{H^2(D)} u$ and $v_n \xrightarrow{H^2(D)}$, we have that $u_n \xrightarrow{L^2(D)} u$, $v_n \xrightarrow{L^2(D)} v$, $\Delta u_n \xrightarrow{L^2(D)} \Delta u$ and $\Delta v_n \xrightarrow{L^2(D)} \Delta v$. So the above expression tends to 0 as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \int_D [u_n\Delta v_n - v_n\Delta u_n]\,dx = \int_D [u\Delta v - v\Delta u]\,dx.$$

It remains to see that

$$\lim_{n\to\infty} \int_{\partial D} \left(u_n\frac{\partial v_n}{\partial\nu} - v_n\frac{\partial u_n}{\partial\nu}\right)\,ds = \int_{\partial D}\left(u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right)\,ds. \tag{5.8}$$

We have

$$\begin{aligned}
&\left(u_n\frac{\partial v_n}{\partial\nu} - v_n\frac{\partial u_n}{\partial\nu}\right) - \left(u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right) \\
&= \left(u_n\frac{\partial v_n}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right) - \left(v_n\frac{\partial u_n}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right) \\
&= \left(u_n\frac{\partial v_n}{\partial\nu} - u_n\frac{\partial v}{\partial\nu} + u_n\frac{\partial v}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right) \\
&- \left(v_n\frac{\partial u_n}{\partial\nu} - v_n\frac{\partial u}{\partial\nu} + v_n\frac{\partial u}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right)
\end{aligned}$$

So, reasoning as before, we have

$$\begin{aligned}
&\left|\int_{\partial D}\left[\left(u_n\frac{\partial v_n}{\partial\nu} - v_n\frac{\partial u_n}{\partial\nu}\right) - \left(u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right)\right]\,ds\right| \\
&\leqslant \left(\int_{\partial D}|u_n|^2\right)^{1/2}\left(\int_{\partial D}\left|\frac{\partial v_n}{\partial\nu} - \frac{\partial v}{\partial\nu}\right|^2\right)^{1/2} + \left(\int_{\partial D}\left|\frac{\partial v}{\partial\nu}\right|^2\right)^{1/2}\left(\int_{\partial D}|u_n - u|^2\right)^{1/2} \\
&+ \left(\int_{\partial D}|v_n|^2\right)^{1/2}\left(\int_{\partial D}\left|\frac{\partial u_n}{\partial\nu} - \frac{\partial u}{\partial\nu}\right|^2\right)^{1/2} + \left(\int_{\partial D}\left|\frac{\partial u}{\partial\nu}\right|^2\right)^{1/2}\left(\int_{\partial D}|v_n - v|^2\right)^{1/2}
\end{aligned}$$

Because the trace operator is continuous, the fact that $u_n \xrightarrow{H^2(D)} u$ and $v_n \xrightarrow{H^2(D)} v$ imply that $\|u_n - u\|_{L^2(\partial D)} \to 0$, $\|v_n - v\|_{L^2(\partial D)} \to 0$, $\|\frac{\partial u_n}{\partial\nu} - \frac{\partial u}{\partial\nu}\|_{L^2(\partial D)} \to 0$ and $\|\frac{\partial v_n}{\partial\nu} - \frac{\partial v}{\partial\nu}\|_{L^2(\partial D)} \to 0$. So the above expression tends to 0 when $n \to \infty$, obtaining thus (5.8). □

5.1.4 Convergence of the derivatives

Proposition 5.1.8. *Let (f_n) be a sequence of functions in $H^1(\mathbb{R}^n)$, and let $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. If $f_n \rightharpoonup f$ in $L^2(\mathbb{R}^n)$ and $f_n' \rightharpoonup g$ in $L^2(\mathbb{R}^n)$, then $g = f'$.*

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
\int \eta(g - f') &= \int \eta g + \int \eta' f && \text{Int. by Parts} \\
&= \lim_n \int \eta f_n' + \lim_n \int \eta' f_n && f_n' \rightharpoonup g, f_n \rightharpoonup f \text{ in } L^2 \\
&= \lim_n -\int \eta' f_n + \lim_n \int \eta' f_n && \text{Int. by Parts} \\
&= \lim_n \int \eta'(f_n - f_n) = \lim_n 0 = 0.
\end{aligned}$$

Therefore, by Theorem 5.1.3, taking η so that the product is a convolution with an approximation of the identity we obtain that $g - f' = 0$ almost everywhere. So $g = f'$ in $L^2(\mathbb{R}^n)$. $\square$

5.1.5 Poincaré's Inequality

Theorem 5.1.9. *Assume D is a bounded open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(D)$ for some $1 \leqslant p < n$. Then we have the estimate*

$$\|u\|_{L^q(D)} \leqslant C\|\nabla u\|_{L^p(D)}$$

for each $q \in [1, \frac{pn}{n-p}]$, the constant C depending only on p, q, n and D. Notice that $\|\nabla u\|_{L^p(D)}$ means $\big\||\nabla u|\big\|_{L^p(D)}$ where

$$|\nabla u| = \left(\sum_{j=1}^n |\partial_{x_j} u|^2\right)^{1/2}.$$

In particular, for all $1 \leqslant p \leqslant \infty$,

$$\|u\|_{L^p(D)} \leqslant C\|\nabla u\|_{L^p(D)}. \tag{5.9}$$

In fact, for $p = 2$, the greatest possible value of the constant $\frac{1}{C^2}$ is

$$\inf_{u \in H_0^1(D): \|u\|_{L^2(D)}=1} \frac{\|\nabla u\|_{L^2(D)}^2}{\|u\|_{L^2(D)}^2} = \inf_{u \in H_0^1(D): \|u\|_{L^2(D)}=1} \|\nabla u\|_{L^2(D)}^2 = \lambda_1(D),$$

where $\lambda_1(D)$ denotes the smallest Dirichlet eigenvalue of the negative Laplacian. That is, the smallest constant C for which (5.9) is true for $p = 2$ is $C = \frac{1}{\sqrt{\lambda_1(D)}}$

Proof. For a proof of (5.9), see Theorem 3 of Section 5.6 of [16], page 279. The value of the sharpest constant is a consequence of Theorem 2 of Section 6.5 of [16], page 356, using that the bilinear form associated to the negative Laplacian is $B[u, u] = \|\nabla u\|_{L^2(D)}^2$ for $u \in H_0^1(D)$. $\square$

5.1.6 Equivalent norms on $H_0^2(D)$

Proposition 5.1.10. *Let D be an open bounded set of $\mathbb{R}^n$. Then, the $H^2(D)$-norm of a function $u \in H_0^2(D)$ and the $L^2(D)$-norm of its Laplacian are equivalent.*

Proof. Note that $u \in H_0^2(D)$ implies that $\nabla u \in H_0^1(D)$, so Poincaré inequality (Theorem 5.1.9) gives us :

$$\|u\|^2_{H_0^2(D)} \leqslant C \left\|D^2 u\right\|^2_{L^2(D)}$$

Besides, by definition of the H^2 norm, we trivially have

$$\|D^2 u\|^2_{L^2(D)} \leqslant \|u\|_{H_0^2(D)}.$$

So, if we define the norm

$$\|u\|^2_* := \left\|D^2 u\right\|^2_{L^2(D)}$$

on $H_0^2(D)$, it is equivalent to the to the standard $H_0^2(D)$ norm. We now claim that

$$\|\Delta u\|_{L^2(D)} = \left\|D^2 u\right\|_{L^2(D)} = \|u\|_*$$

for any $u \in H_0^2(D)$. If we check this, we end the proof of the proposition. To see this, first consider $u \in C_c^\infty(D)$. Then integration by parts and commutativity of partial derivatives for smooth functions implies

$$\int_D u_{x_i x_i} u_{x_j x_j} dx = -\int_D u_{x_i} u_{x_j x_j x_i} dx = -\int_D u_{x_i} u_{x_j x_i x_j} dx = \int_D u_{x_i x_j} u_{x_i x_j} dx$$

for every $1 \leqslant i, j \leqslant n$. Since

$$|\Delta u|^2 = \sum_{i=1}^n \partial_{x_i x_i} u + 2 \sum_{1 \leqslant i < j \leqslant n} \partial_{x_i x_j} u$$

and

$$|D^2 u|^2 = \sum_{1 \leqslant i,j \leqslant n} |\partial_{x_i x_j} u|^2,$$

we have that

$$\|\Delta u\|_{L^2(D)} = \left\|D^2 u\right\|_{L^2(D)}$$

for all $u \in C_c^\infty(D)$. Since $C_c^\infty(D)$ is dense in $H_0^2(D)$, passing to limits we find that

$$\|\Delta u\|_{L^2(D)} = \left\|D^2 u\right\|_{L^2(D)} \quad \text{for all } u \in H_0^2(D)$$

□

5.1.7 Rellich-Kondrachov Compactness Theorem

Definition 5.1.11. Let X and Y be two Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, written

$$X \subset\subset Y,$$

provided that

1. $\|u\|_Y \leqslant C\|u\|_X$ $(u \in X)$ for some constant C.
2. Each bounded sequence in X is precompact in Y.

Theorem 5.1.12 (Rellich-Kondrachov Compactness Theorem)**.** *Assume U is a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $1 \leqslant p < n$. Then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \leqslant q < \frac{pn}{n-p}$.

Proof. See Theorem 1 of Section 5.7 of [16], pages 286-289. □

5.1.8 Trace Theorems

For a definition of Sobolev Spaces of arbitrary order, see [24], page 76. We use them in the following result.

Theorem 5.1.13. *Define the trace operator* $\gamma\colon C_c^\infty(\overline{D}) \to C_c^\infty(\partial D)$ *by*

$$\gamma u = u|_{\partial D}\,.$$

If D *is a* $C^{k-1,1}$ *domain, and if* $\frac{1}{2} < s \leqslant k$*, then* γ *has a unique extension to a bounded linear operator*

$$\gamma\colon H^s(D) \to H^{s-\frac{1}{2}}(\partial D),$$

and this extension has a continuous right inverse.

Proof. See Theorem 3.37 of [24], page 102. □

5.1.9 Spectral Theory

Relation between the spectrum and the eigenvalues

We state here the fact that, when the operator is compact, its spectrum is composed only by eigenvalues.

In what follows, given a Hilbert space X over a field $\mathbb{F}$ (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and a linear operator $T\colon X \to X$, let $\rho(T)$ denote its spectrum. We collect here the ones needed in the essay.

The first one says that nonzero eigenvalues of compact operators have finite multiplicity.

Theorem 5.1.14. *Suppose* T *is a compact operator on a Hilbert space and* $\alpha \in \mathbb{F}$ *with* $\alpha \neq 0$*. Then* $N(T - \alpha I)$ *is finite-dimensional.*

Proof. See Theorem 10.82 of [1], page 318. □

Theorem 5.1.15. *Suppose* T *is a compact operator on a Hilbert space and* $\alpha \in \mathbb{F}$ *with* $\alpha \neq 0$*. Then the following are equivalent:*

1. $\alpha \in \rho(T)$.
2. α *is an eigenvalue of* T.
3. $T - \alpha I$ *is not surjective.*

Proof. See Theorem 10.85 of [1], page 319. □

The next theorem states that the eigenvalues of an operator form at most a discrete set that accumulates only at 0.

Theorem 5.1.16. *Suppose* T *is a compact operator on a Hilbert space* X*. Then*

$$\{\alpha \in \rho(T) : |\alpha| \geqslant \delta\}$$

is a finite set for every $\delta > 0$*.*

Proof. See Theorem 10.93 of [1], page 322. □

Spectral decomposition

Theorem 5.1.17. *Let X be a Hilbert Space and let $A\colon X \to X$ be a compact self-adjoint operator, $A \neq 0$. Then, every eigenvalue of A is real. A has at least one eigenvalue different from 0 and at most a countable set of eigenvalues that accumulate only at 0. All eigenspaces $N(\lambda I - A)$ for nonzero eigenvalues λ have finite dimension and eigenspaces to different eigenvalues are orthogonal. Assume the sequence (λ_n) of the nonzero eigenvalues to be ordered such that*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \dots$$

and denote by $P_n\colon X \to N(\lambda_n I - A)$ the orthogonal projection operator onto the eigenspace for the eigenvalue λ_n. Then

$$A = \sum_{n=1}^{\infty} \lambda_n P_n \tag{5.10}$$

in the sense of norm convergence. Let $Q\colon X \to N(A)$ denote the orthogonal projection operator onto the nullspace $N(A)$. Then

$$\varphi = \sum_{n=1}^{\infty} P_n\varphi + Q\varphi \tag{5.11}$$

for all $\varphi \in X$. (When there are only finitely many eigenvalues, the series (5.10) and (5.11) degenerate into finite sums).

Proof. See Theorem 15.12 of [22], pages 305-306. □

Integral operators with weakly singular kernels

Definition 5.1.18. Let $G \subset \mathbb{R}^m$ be a nonempty compact Lebesgue measurable set. We say that a function $K\colon G \times G \backslash \{(x,x) : x \in G\} \to \mathbb{R}$ is a **weakly singular kernel** if K is defined and continuous for all $x, y \in G \subset \mathbb{R}^m$, $x \neq y$, and there exist positive constants M and $\alpha \in (0, m]$ such that

$$|K(x,y)| \leqslant M|x-y|^{\alpha-m}, \quad x, y \in G, \quad x \neq y.$$

Theorem 5.1.19. *Integral operators*

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)\,dy$$

with weakly singular kernel K are compact linear operators on $C(G)$, where $C(G)$ denotes the Banach space of continuous functions with the maximum norm $\|u\| = \max_{x\in G}|u(x)|$ for $u \in C(G)$.

Proof. See Theorem 2.29 of [22], page 29. □

Properties of the adjoint operator

Recall that for every bounded linear operator $A\colon X \to Y$ between Hilbert Spaces X and Y there exists a unique bounded linear operator $A^*\colon Y \to X$ called the *adjoint operator* of A, given by the relation $(A\varphi, \psi) = (\varphi, A^*\psi)$ for every $\varphi \in X$ and $\psi \in Y$. In the TFM, we make use of the following basic connection between the kernels and ranges of A and A^*.

Theorem 5.1.20. *Let $A\colon X \to Y$ be a bounded linear operator between Hilbert Spaces X and Y. Then*

$$A(X)^{\perp} = N(A^*) \text{ y } N(A^*)^{\perp} = \overline{A(X)}.$$

Proof. $g \in A(X)^\perp$ means $(A\varphi, g) = 0$ for every $\varphi \in X$. This is equivalent to $(\varphi, A^*g) = 0$ para todo $\varphi \in X$, which is equivalent to $A^*g = 0$, that is, $g \in N(A^*)$. So

$$A(X)^\perp = N(A^*).$$

Let us call $U := A(X)$. We have $\overline{U} \subset (U^\perp)^\perp$. Denote by $P\colon Y \to \overline{U}$ the orthogonal projection operator. Then, for $\varphi \in (U^\perp)^\perp$ arbitrary, we have the orthogonality relation $P\varphi - \varphi \perp U$. We also have $P\varphi - \varphi \perp U^\perp$, because we now that $\overline{U} \subset (U^\perp)^\perp$. Therefore, it follows that $\varphi = P\varphi \in \overline{U}$, so $\overline{U} = (U^\perp)^\perp$; that is,

$$\overline{A(X)} = N(A^*)^\perp.$$

□

Theorem 5.1.21. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then*

$$H_0^2(\Omega) = \{u|_\Omega : u \in H^2(\mathbb{R}^3), u = 0 \text{ in } \mathbb{R}^3\backslash\Omega\}.$$

Proof. See Lemma 7.4 of [19], page 245. □

5.1.10 More facts from Functional Analysis

Theorem 5.1.22. *Let E and F be two Banach spaces and let T be a continuous linear operator from E into F that is bijective. Then T^{-1} is also continuous from F to E.*

Proof. See Corollary 2.7 of [4], page 35. □

Theorem 5.1.23. *Assume that E is a reflexive Banach space and let (x_n) be a bounded sequence in E. Then there exists a subsequence (x_{n_k}) that converges in the weak topology.*

Proof. See Theorem 3.18 of [4], page 69. □

5.2 Normal derivative and radial derivative

Suppose we have two vectors $\vec{v}$ and $\vec{n}$, forming an angle α. Let a be the length (i.e. the module) of the projection of the vector $\vec{v}$ over the vector $\vec{n}$.

We then have that $\cos\alpha = \frac{a}{\|\vec{v}\|}$. So $a = \cos\alpha \cdot \|\vec{v}\|$.

Therefore $\vec{v} \cdot \vec{n} = \|\vec{v}\| \cdot \|\vec{n}\| \cdot \cos\alpha = a \cdot \|\vec{n}\|$. So

$$a = \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|}.$$

If $\vec{n}$ is unitary, then the previous formula reduces to

$$a = \vec{v} \cdot \vec{n}.$$

That is, the scalar product $\vec{v} \cdot \vec{n}$ gives the lenth of the projection of the vector $\vec{v}$ over the unitary vector $\vec{n}$

Hence, if we define

$$\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu,$$

what we obtain is the module of the projection of the vector ∇u over the unit exterior normal vector ν. That is, it gives how much the function u grows in the direction of ν. That is why $\frac{\partial u}{\partial \nu}$ is called the **normal derivative**.

On the other hand, we have the **radial derivative**

$$\frac{\partial u}{\partial r}(x) := \nabla u(x) \cdot \frac{x}{|x|}$$

which can be understood as the normal derivative when the surface is a sphere $S(0, R)$.

In this way, $\frac{\partial u}{\partial r}$ gives us the module of the projection fo u over $\frac{x}{|x|}$, that is, over the radial unit exterior vector. I.e., it gives how much u grows in the radial direction.

5.3 Spherical Harmonics

Spherical harmonics can appear when we separate variables in polar coordinates, taking $x = r\hat{x}$ with $r = |x| \geqslant 0$ and $\hat{x} \in \mathbb{S}^{n-1}$.

For a detailed treatment of the subject, we refer to Chapter 2 of [17], pages 98-110, or Chapter 5 of [2], pages 73-109. See also Section 2.3 of [21], pages 36-48, for an explicit approach on $\mathbb{S}^2$ using Legendre polynomials.

We present here just the basic results that we need for our purposes.

Definition 5.3.1. Let $\mathcal{P}_k$ be the space of homogeneous polynomials of degree k on $\mathbb{R}^n$ and let

$$\mathcal{H}_k = \{P \in \mathcal{P}_k : \Delta P = 0\}$$

be the space of homogeneous harmonic polynomials of degree k and

$$H_k = \{\, P|_{\mathbb{S}^{n-1}} : P \in \mathcal{H}_k\}$$

be the space of their restriction to the unit sphere.

The elements of H_k are called **spherical harmonics** of degree k.

Remark 5.3.2. The restriction map from $\mathcal{H}_k$ to H_k is an isomorphism, its inverse map being $Y \in H_k \mapsto P \in \mathcal{H}_k$, $P(x) = |x|^k Y(|x|^{-1}x)$.

Proposition 5.3.3. *$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{P}_{k-2}$, where $r^2\mathcal{P}_{k-2} = \{r^2 P : P \in \mathcal{P}_{k-2}\}$.*

Proof. See Proposition 2.49 of [17], page 98, or Proposition 5.5 of [2], page 76. □

Corollary 5.3.4. $\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{H}_{k-2} \oplus r^4 \oplus \mathcal{H}_{k-4} \oplus \ldots$

Corollary 5.3.5. *The restriction to the unit sphere of any element of $\mathcal{P}_k$ is a sum of spherical harmonics of degree at most k.*

Theorem 5.3.6. *$L^2(\mathbb{S}^{n-1}) = \oplus_0^\infty H_k$, the expression of the right being an orthogonal direct sum with respect to the scalar product on $L^2((\mathbb{S}^{n-1})$.*

Proof. See Theorem 2.53 of [17], page 99, or Theorem 5.12 of [2], page 81. □

Explicit bases of $\mathcal{H}_m$ and H_m in $\mathbb{R}^3$

For explicit bases of $\mathcal{H}_m$ and H_m in the n-dimensional case we refer to pages 92-94 of [2].

For the 3-dimensional case, we have an explicit basis, which we can define as

$$Y_n^m(\theta,\varphi) := \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}}\, P_n^{|m|}(\cos(\theta))e^{im\varphi}$$

with $-n \leqslant m \leqslant n$ and $n = 0, 1, \ldots$ and where $P_n^{|m|}$ are the Legendre Polynomials (see Section 2.3 of [13] for a description of them). They form an orthonormal system in $L^2(\mathbb{S}^2)$. We will identify $Y_n^m(x)$ with $Y_n^m(\theta,\varphi)$ for $x = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^T \in \mathbb{S}^2$.

The result that we will use is the following:

Theorem 5.3.7. *The functions* $\left\{Y_n^m : -n \leqslant m \leqslant n, n \in \mathbb{N} \cup \{0\}\right\}$ *are complete in* $L^2\left(\mathbb{S}^2\right)$*; that is, every function* $f \in L^2\left(\mathbb{S}^2\right)$ *can be expanded into a generalized Fourier series in the form*

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(f, Y_n^m\right)_{L^2\left(\mathbb{S}^2\right)} Y_n^m$$

The series can also be written as

$$f(x) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \int_{\mathbb{S}^2} f(y) P_n(y \cdot x) ds(y), \quad x \in \mathbb{S}^2$$

The convergence of these series is in the L^2*-sense. Furthermore, on bounded sets in* $C^1\left(\mathbb{S}^2\right)$ *the series converge even uniformly; that is, for every* $M > 0$ *and* $\varepsilon > 0$ *there exists* $N_0 \in \mathbb{N}$*, depending only on* M *and* ε*, such that*

$$\left\| \sum_{n=0}^{N} \sum_{m=-n}^{n} \left(f, Y_n^m\right)_{L^2\left(\mathbb{S}^2\right)} Y_n^m - f \right\|_{\infty} = \max_{\hat{x} \in \mathbb{S}^2} \left| \sum_{n=0}^{N} \sum_{m=-n}^{n} \left(f, Y_n^m\right)_{L^2\left(\mathbb{S}^2\right)} Y_n^m(\hat{x}) - f(\hat{x}) \right| \leqslant \varepsilon$$

for all $N \geqslant N_0$ *and all* $f \in C^1\left(\mathbb{S}^2\right)$ *with* $\|f\|_{1,\infty} \leqslant M$*, and, analogously for (2.19b).*

Here, the space $C^1\left(\mathbb{S}^2\right)$ *consists of those functions* f *such that (with respect to spherical coordinates* θ *and* φ*) the functions* $f, \partial f/\partial\theta$*, and* $\frac{1}{\sin\theta}\partial f/\partial\varphi$ *are continuous and periodic with respect to* φ *with the norm defined by* $\|f\|_{1,\infty} = \max\left\{\|f\|_\infty, \|\partial f/\partial\theta\|_\infty, \left\|\frac{1}{\sin\theta}\partial f/\partial\varphi\right\|_\infty\right\}$

Proof. See Theorem 2.19 of [21], pages 44-47. □

Eigenfunctions of the Laplace-Beltrami operator

One important property about spherical harmonics is that they are eigenfunctions of the Laplace-Beltrami operator:

Theorem 5.3.8. *If* $Y \in H_k$*, then* $\Delta_{\mathbb{S}^{n-1}} Y = -k(k+n-2)Y$*. In particular, for dimension* 3 *and the normalized spherical harmonics defined above, we have*

$$\Delta_{\mathbb{S}^2} Y_n^m + n(n+1) Y_n^m = 0$$

for all $|m| \leqslant n$ *(*n *now denotes the degree of the spherical harmonic; above, the degree is* k*, and the dimension is* n*).*

Proof. See Lemma 2.62 [17], pages 104-105, or page 41 of [21]. □

5.4 Bessel functions

The **spherical Bessel differential equation** is the following

$$z^2\hat{v}''(z) + 2z\hat{v}'(z) + \left[z^2 - n(n+1)\right]\hat{v}(z) = 0. \tag{5.12}$$

We consider this linear differential equation of second order for arbitrary $z \in \mathbb{C}$. For $z \neq 0$ the differential equation is equivalent to

$$\hat{v}''(z) + \frac{2}{z}\hat{v}'(z) + \left(1 - \frac{n(n+1)}{z^2}\right)\hat{v}(z) = 0 \quad \text{in } \mathbb{C}\backslash\{0\}.$$

The coefficients of this differential equation are holomorphic in $\mathbb{C}\backslash\{0\}$ and have poles of first and second order at 0 . As in the case of real z one can show that in every simply connected domain $\Omega \subseteq \mathbb{C}\backslash\{0\}$ there exist at most two linearly independent solutions of (5.12). See Theorem 2.25 of [21] for a proof.

It can be deduced that a pair of linearly independent solutions is the following:

Definition 5.4.1. For all $z \in \mathbb{C}$ the spherical Bessel functions of first and second kind and order $n \in \mathbb{N}_0$ are defined by

$$j_n(z) = (2z)^n \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \frac{(n+\ell)!}{(2n+2\ell+1)!} z^{2\ell}, \quad z \in \mathbb{C},$$
$$y_n(z) = \frac{2(-1)^{n+1}}{(2z)^{n+1}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \frac{(\ell-n)!}{(2\ell-2n)!} z^{2\ell}, \quad z \in \mathbb{C},$$

where - in the definition of y_n-a quantity $\frac{(-k)!}{(-2k)!}$ for positive integers k is defined by

$$\frac{(-k)!}{(-2k)!} = (-1)^k \frac{(2k)!}{k!}, \quad k \in \mathbb{N}$$

The functions

$$h_n^{(1)} = j_n + i y_n,$$
$$h_n^{(2)} = j_n - i y_n,$$

are called Hankel functions of first and second kind and order $n \in \mathbb{N}_0$.

For many applications the Wronskian of these functions is important.

Theorem 5.4.2. *For all $n \in \mathbb{N}_0$ and $z \in \mathbb{C}\backslash\{0\}$ we have*

$$W(j_n, y_n)(z) := j_n(z)y_n'(z) - j_n'(z)y_n(z) = \frac{1}{z^2}.$$

Proof. See Theorem 2.27 of [21]. □

Remark 5.4.3. From this theorem the linear independence of $\{j_n, y_n\}$ follows immediately and thus also the linear independence of $\left\{j_n, h_n^{(1)}\right\}$. Therefore, they span the solution space of the differential equation (2.27).

In the TFM we need the asymptotic behavior of $h_n^{(1)}$.

Theorem 5.4.4. *For every $n \in \mathbb{N}$ and $z \in \mathbb{C}$ we have*

$$h_n^{(1)}(z) = \frac{\exp\left[i\left(z - \frac{\pi}{2}(n+1)\right)\right]}{z}\left[1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right] \qquad \text{for } |z| \to \infty$$

$$\frac{d}{dz}h_n^{(1)}(z) = \frac{\exp\left[i\left(z - \frac{\pi}{2}n\right)\right]}{z}\left[1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right] \qquad \text{for } |z| \to \infty$$

uniformly with respect to $z/|z|$. The corresponding formulas for j_n and y_n are derived by replacing $\exp[i(\ldots)]$ *by* $\cos(\ldots)$ *and* $\sin(\ldots)$*, respectively.*

Proof. See Theorem 2.30 of [21], pages 59-61. □

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