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Yamabe Problem

Itahisa Barrios Cubas

Directores: María del Mar González Nogueras y Daniel Faraco Hurtado

Lugar: Facultad de Ciencias - UAM

Departamento: Matemáticas

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Abstract

We study the classical Yamabe problem from the point of view of the Riemannian geometry and PDEs. We discuss the existence and uniqueness of positive solutions and its relation with the sharp form of the Sobolev inequality. Next, we introduce the moving plane method and its applications to obtain radial symmetry results for the solutions of some elliptic equations related to the Yamabe problem and also to deduce a Harnack type estimate for the parabolic case $u_t = \Delta u^{\frac{N+2}{N-2}}$, which is associated to the Yamabe flow. Finally, for this last equation, we describe the asymptotic behaviour near the vanishing time for the positive solutions to a Cauchy problem with an initial condition verifying a certain rate of decay at ∞.

Resumen

Estudiamos el problema clásico de Yamabe desde el punto de vista de la geometría Riemaniana y de las ecuaciones en derivadas parciales. Discutimos la existencia y unicidad de soluciones positivas y su relación con la desigualdad óptima de Sobolev. A continuación, introducimos el método del plano móvil y sus aplicaciones para obtener resultados de simetría radial para las soluciones de algunas ecuaciones elípticas relacionadas con el problema de Yamabe y también para deducir una estimación tipo Harnack para el caso parabólico $u_t = \Delta u^{\frac{N+2}{N-2}}$, que está asociado al flujo de Yamabe. Finalmente, para esta última ecuación, describimos el comportamiento asintótico cerca del tiempo de extinción para las soluciones positivas de un problema de Cauchy con una condición inicial verificando una cierta tasa de decaimiento en ∞.

Contents

Introduction

The main purpose of this Master's thesis is the study of the Yamabe problem. This problem was born while Yamabe was trying to solve the Poincaré conjecture:

A compact simply-connected Riemannian manifold (M,g) *of dimension* $N=3$ *is diffeomorphic to* $\mathbb{S}^3$.

With this aim, he considered a metric with constant scalar curvature and the Yamabe problem arose:

Let (M^N,g) *be a compact Riemannian manifold of dimension* $N\geq 3$ *and non-constant scalar curvature. Is there a metric with constant scalar curvature conformal to g?*

In 1960, Yamabe proved in [26] that his problem is equivalent to the so called Yamabe equation:

$$-C_N\Delta u+R_g u=Ru^{\frac{N+2}{N-2}},$$

where $C>0$ and R are constants, R_g is the scalar curvature of M and Δ is the Laplacian operator. Moreover, he proved the uniqueness of its solutions in the cases where the curvature is equal to zero or negative, but when the curvature is positive, it we can not be proven the uniqueness in general, as we will see later, being the sphere case an exception ([16]).

On the same paper [26], Yamabe used the variational approach in order to prove the existence of its solutions. When $2^*=\frac{2N}{N-2}$, we have the critical case for which the inclusion $H^1(M)\subset L^{2^*}(M)$ is not compact and it is not possible to prove that a minimizing sequence of a certain energy functional has a subsequence converging to an extremal function. To overcame this situation, Yamabe took a collection of perturbed problems under a subcritical case ($s<2^*$) in which this difficulty disappears, and then he solved the problem taking $s\to 2^*$. However, in 1968 Trudinger [23] discovered that there was a gap in Yamabe's proof and he made a modification of Yamabe's work in which he proved that for dimension $N\geq 6$ whenever the so called Yamabe invariant λ verifies $\lambda(M)\leq 0$, showing also that there exists a constant $\alpha(M)$ such that the problem can be solved when $\lambda(M)<\alpha(M)$. In 1976, Aubin [2] improves this result by demonstrating that if $N\geq 6$, we have $\alpha(M)=\lambda(\mathbb{S}^N)$, so when $\lambda(M)<\lambda(\mathbb{S}^N)$, the proof works on any compact Riemannian manifold M. Finally, in 1981 Schöen and Yau [18] proved the positive mass theorem of general relativity for dimensions 3 and 4, which was used by Schöen [17] in 1984 to prove the cases for dimensions $N = 3,4,5$, completing the demonstration of Yamabe problem.

As pointed by Lee and Parker ([16]), the solution of the Yamabe problem marked a milestone in the development of the theory of nonlinear partial differential equations. Semilinear equations of this form with critical exponent arise in many contexts and have long been studied by analysts. This was the first time that such an equation has been completely solved.

Next, we will present the moving plane method which consists on comparing values of the solutions of a PDE at two different points, one point is the reflection of the other over an hyperplane, and this hyperplane is moved until it arrives to a critical position, where it stops. This method was introduced by Alexandrov [1] in 1958 and Serrin [21] in 1971 and it has important applications in the theory of PDEs particularly in the proof of qualitative properties of solutions, as monotonicity, radial symmetry or Harnack type estimates ([6], [8], [11], [27]). Thanks to the use of the moving plane technique done by [8], we will determine the precise form of radially symmetric solutions to the Yamabe problem.

On the other hand, in 1988 Hamilton [12] introduced the Yamabe flow, a tool to generate metrics of constant scalar curvature in a given conformal class, with the aim to solve the Yamabe problem. In 1994, Ye [27] used the referred moving plane technique to get a Harnack inequality for the Yamabe flow. Later, in 2001 Del Pino and Sáez [9] take advantage of this Harnack estimation and applied it to study the asymptotic behaviour of the Yamable flow in $\mathbb{R}^N$ in the critical case, where they were able to do a transformation into a fast diffusion problem posed on the sphere via the stereographic projection. The organization is as follows:

In this Master's thesis we are going to cover the study of all these topics from the point of view of Riemannian geometry and its applications in PDEs.

In chapter 1, we will collect different notions of differential geometry, laplacian operator on manifolds, Sobolev spaces and from the theory of elliptic and non-linear parabolic equations, that are needed for the subsequent analysis.

In chapter 2, first we formally state the Yamabe problem showing that under certain transformations, this is equivalent to the Yamabe equation. Secondly, we initiate the variational approach used to prove the existence of its solutions. Next, we focus on the model case of the sphere which reveals as a benchmark in the study of the Yamabe equation, stating the known Obata's theorem. We will see its relation with the sharp form of the Sobolev inequality on $\mathbb{R}^N$, and using this result we will obtain the inequality $\lambda(M) \le \lambda(\mathbb{S}^N)$ on any compact Riemannian manifold M, where λ is the Yamabe invariant. We finish this chapter solving the Yamabe problem for any compact Riemannian manifold provided that $\lambda(M) < \lambda(\mathbb{S}^N)$, and discussing the uniqueness of solutions.

Chapter 3 is dedicated to the moving-plane method and its applications to obtain some results of radial symmetry of solutions of elliptic equations related to the Yamabe problem, and to obtain a Harnack type estimate for a parabolic equation associated to the Yamabe flow.

In the final chapter, we will study the asymptotic behaviour of the solutions to a Cauchy problem for the equation $u_t = \Delta u^{\frac{N+2}{N-2}}$. When the initial condition has a certain decay rate at ∞, it can be proven that the positive solutions to this problem has a finite vanishing time, and we will describe the asymptotic behaviour of these solutions near its vanishing time.

Chapter 1

Preliminaries

In this chapter we are going to see some results but we are not going into the proofs.

1.1 Notation

Einstein summation convention

In the study of smooth manifolds it is common to use the known Einstein summation convention which consists in omitting the summation sign, that is to say that instead of write $E(x) = \sum_{i=1}^{N} x^i E_i$, we write

$$E(x) = x^i E_i.$$

We are going to use this convention during the whole work.

Notation

We use the following notation:

- $\mathbb{S}^N = \{x \in \mathbb{R}^{N+1} \mid \|x\| = 1\}$, the unit sphere contained in $\mathbb{R}^{N+1}$.
- $B_R(x)$, the ball of radius R centered in x.
- $\langle .,. \rangle_g$, the inner product with respect to the g metric.
- ∇f, the gradient of a scalar function.
- $\langle \nabla f, \nabla h \rangle_g = \nabla^i f \nabla_i h$.
- $|\nabla f|_g^2 = \nabla^i f \nabla_i f$.
- $\Delta = \Delta_g$, the Laplace-Beltrami operator with respect of the g metric.

- $(.)_+$, the positive part, $(x)_+ = \max\{0, x\}$.
- g_0, the Euclidean metric in $\mathbb{R}^N$.
- g_c, the standard metric in $\mathbb{S}^N$.
- g_1, the cylindrical metric in $\mathbb{R} \times \mathbb{S}^{\mathbb{N}-\mathbb{1}}$.
- $2^* = \frac{2N}{N-2}$.
- $C_N = \frac{4(N-1)}{N-2}$.

1.2 Some notions of Differential Geometry

The main references here are [4], [14] and [15].

1.2.1 Differentiable manifolds

Definition 1.2.1. *A topological space M is a Hausdorff space if for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.*

Definition 1.2.2. *M^N is a manifold of dimension N if it is a topological Hausdorff space such that each point $p \in M^N$ has a neighbourhood Ω homeomorphic to $\mathbb{R}^N$ or equivalently to an open set of $\mathbb{R}^N$.*

We are going to work here with connected manifolds of finite dimension.

Proposition 1.2.1. A manifold is locally compact and locally path connected.

Proof. See [4], Proposition 1.2, p. 20. □

Definition 1.2.3. *Let M^N be a manifold of dimension N.*

- *A local chart on M^N is a pair (Ω, φ), where $\Omega \subset M^N$ is open and $\varphi : \Omega \longrightarrow U$ is an homeomorphism, with $U \subset \mathbb{R}^N$ an open set.*
- *A collection of $(\Omega_i, \varphi_i)_{i \in I}$ of local charts such that $\bigcup_{i \in I} \Omega_i = M^N$ is called an atlas.*
- *An atlas of class C^k is an atlas for which all changes of charts $\varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms of class C^k if $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$. If all of them are C^∞, we say that the atlas is smooth.*
- *Two atlases of class C^k or C^∞ are said to be equivalent if their union is an atlas of class C^k or C^∞, respectively.*

The coordinates of some point $P \in \Omega$ related to the local chart (Ω, φ) are the coordinates of the point $\varphi(P) \in \mathbb{R}^N$.

Definition 1.2.4. *A differentiable manifold of class C^k (respectively, C^∞ or smooth) is a manifold together with an equivalence class of C^k (or C^∞) atlases.*

Proposition 1.2.2. Let $\mathbb{S}^N \subset \mathbb{R}^{N+1}$, be the unit sphere. Then $\mathbb{S}^N$ is a compact smooth manifold.

Proof. See [4], Example 1.8, p.23. □

Definition 1.2.5. *Let M be a smooth manifold, let Y be a smooth vector field on M and let φ be the local group of local diffeomorphisms related to Y (also known as flow of Y). For any smooth vector field X on M, we define the Lie derivative of X with respect to Y, denoted $\mathscr{L}_Y X$, as*

$$(\mathscr{L}_Y X)_p = \frac{d}{dt}\,|_{t=0}\, d\left(\varphi_{-t}\right)_{\varphi_t(p)}\left(X_{\varphi_t(p)}\right) = \lim_{t\to 0} \frac{d\left(\varphi_{-t}\right)_{\varphi_t(p)}\left(X_{\varphi_t(p)}\right) - X_p}{t}, \quad p \in M,$$

provided the derivative exists.

1.2.2 Riemannian manifolds

Definition 1.2.6.
- *A smooth Riemannian manifold of dimension N is a pair (M^N, g) where M^N is a smooth manifold and g is a Riemannian metric.*
- *A Riemannian metric is a twice-covariant tensor field g, i.e. a section of $T^*(M) \otimes T^*(M)$, such that at each point $P \in M$, g_P is a positive bilinear symmetric form, i.e. it satisfies:*
  - ∗ $g_P(X,Y) = g_P(Y,X), \ \forall X,Y \in T^*(M) \otimes T^*(M)$.
  - ∗ $g_P(X,X) > 0, \ if\ X \neq 0$.

Riemannian metrics can be written in any smooth local coordinates $\{x_i\}$ as

$$g = g_{ij} dx^i dx^j,$$

where g_{ij} is a positive-definite symmetric matrix of smooth functions.

Definition 1.2.7. *The Riemannian or Levi-Civita connection is the unique connection with vanishing torsion tensor for which the covariant derivative of the metric tensor is zero, i.e.* $\nabla g = 0$.

From now we are going to consider (M^N, g) a connected smooth Riemannian manifold endowed with the Riemannian connection.

For a Riemannian connection, the Christoffel symbols Γ^k_{ij} in a local coordinate system are given by

$$\Gamma^l_{ij} = \frac{1}{2}\left[\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\right] g^{kl},$$

where g^{kl} are the components of the inverse matrix of the matrix $(g_{ij})_{ij}$, i.e. it verifies $g_{ij}g^{kj} = \delta^k_i$, with δ^k_i the Kronecker symbol.

Example 1.2.1. Let us define the euclidean metric on $\mathbb{R}^N$ as

$$g_0 = \delta_{ij} dx^i dx^j = \left(dx^1\right)^2 + \ldots + \left(dx^N\right)^2 = \sum_{i=1}^{N} \left(dx^i\right)^2 = |dx|^2,$$

where δ_{ij} is the Kronecker delta. On $\mathbb{R}^2$, we have $g_0 = dx^2 + dy^2$ and on $\mathbb{R}^3$, $g_0 = dx^2 + dy^2 + dz^2$.

Remark 1.2.1. One example is the Euclidean metric in Polar coordinates in dimension 2. Let (x, y) be the euclidean coordinates and (r, θ) be the polar coordinates such that

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Then the Euclidean metric in polar coordinates can be rewritten as

$$g_0 = |dx|^2 = dr^2 + r^2 d\theta.$$

Another example is the Euclidean metric in cylindrical coordinates. Let (x, y, z) be the Euclidean coordinates and (r, θ, z) be the cylindrical coordinates defined as

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z. \end{cases}$$

Then the euclidean metric in cylindrical coordinates is

$$g_0 = dr^2 + r^2 d\theta^2 + dz^2.$$

Example 1.2.2. **(Standard metric in $\mathbb{S}^N$)** $\mathbb{S}^N \subset \mathbb{R}^{N+1}$ is an embedded submanifold of dimension N. We denote by g_c the standard metric defined on $\mathbb{S}^N$ by the Euclidean metric on $\mathbb{R}^{N+1}$.

Example 1.2.3. **(Cylindrical metric in $\mathbb{R} \times \mathbb{S}^{N-1}$)** We consider the product metric on $\mathbb{R} \times \mathbb{S}^{N-1}$, $g_1 = dt^2 + d\theta^2$, with $t \in \mathbb{R}$ and $\theta \in \mathbb{S}^{N-1}$. Here dt^2 is the Euclidean metric in $\mathbb{R}$ and $d\theta^2$ is the standard metric in $\mathbb{S}^{N-1}$. Sometimes it is interesting to use the cylindrical metric instead of the Euclidean metric in order to facilitate the calculations We are going to see that the cylindrical metric and the Euclidean one are conformal.

If we use the Emden-Fowler coordinates $r = e^{-t}$, then the euclidean metric in $\mathbb{R}^N$ in polar coordinates becomes:

$$g_0 = dr^2 + r^2 d\theta^2 = r^2\left(dt^2 + d\theta^2\right) = r^2 g_1.$$

Definition 1.2.8. *Let (M^N, g) be an oriented $N-$dimensional smooth Riemannian manifold and $\left(x^1, ..., x^N\right)$ a smooth coordinates system, then the Riemannian volume form has the local coordinates expression:*

$$dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge ... \wedge dx^N,$$

where g_{ij} are the components of g in these coordinates. We define the volume of M by

$$Vol(M) = \int_M dV_g.$$

Definition 1.2.9. *$\mathfrak{X}(M)$ is the space of smooth vector fields on M.*

Definition 1.2.10. *Let $\left(M^N, g\right)$ be a $N-$dimensional Riemannian manifold, the (1,3)-curvature tensor R is defined by*

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \quad \forall X, Y, Z \in \mathfrak{X}(M),$$

with D denoting the Riemannian connection.

Definition 1.2.11. *Let (M^N, g) be a $N-$dimensional smooth Riemannian manifold, its Riemannian curvature tensor is defined as* $\mathrm{Riem}(X,Y,Z,T) = g[R(X,Y)T, Z]$*, a 4-covariant tensor whose components are* $\mathrm{Riem}_{ijkl} = g_{lm}\, \mathrm{R}^m_{ijk}$*, with R^m_{ijk} the components of the curvature tensor R, which verify:*

- $\mathrm{Riem}_{ijkl} = -\mathrm{Riem}_{ijlk}$.
- $\mathrm{Riem}_{ijkl} = \mathrm{Riem}_{klij}$.

Definition 1.2.12. *Let (M^N, g) be a $N-$dimensional smooth Riemannian manifold, the Ricci tensor is a covariant 2-tensor field defined by contraction of the curvature or Riemannian curvature tensors. Its components are*

$$\mathrm{Ric}_{ij} = R^k_{kij} = g^{kl} R_{likj}.$$

Remark 1.2.2. Using properties of metrics and of the Riemannian curvature tensor, we obtain that the Ricci tensor is symmetric:

$$\mathrm{Ric}_{ij} = g^{kl}\,\mathrm{Riem}_{likj} = g^{kl}\,\mathrm{Riem}_{kjli} = g^{lk}\,\mathrm{Riem}_{kjli} = \mathrm{Ric}_{ji}\,.$$

Definition 1.2.13. *Let (M^N, g) be a smooth Riemannian manifold with $N \geq 2$, $p \in M$, $X, Y \in T_pM$ linearly independent vectors. The sectional curvature of the plane spanned by X and Y is defined by*

$$\sigma(X,Y) = \frac{\mathrm{Riem}(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Example 1.2.4. $(\mathbb{S}^N, g_c)$, with $N \geq 2$, has a constant sectional curvature 1 (see [15], Theorem 8.34, p. 254).

Definition 1.2.14. *The scalar curvature is defined as the contraction of the Ricci tensor:*

$$R_g = g^{ij}\,\mathrm{Ric}_{ij}\,.$$

Remark 1.2.3. The scalar curvature at $p \in M$ is the sum of all sectional curvatures of the planes spanned by ordered pairs of vectors from an orthonormal basis of T_pM (see [15]), Proposition 8.32, p. 253). Then for $(\mathbb{R}^{\mathbb{N}}, g_0)$, $R_{g_0} = 0$, and for $(\mathbb{S}^N, g_c)$, $R_{g_c} = N(N-1)$, being constant both scalar curvatures.

Definition 1.2.15. *Let (M^N, g) be a smooth Riemannian manifold, the trace-free Ricci tensor is defined as*

$$E_g = \mathrm{Ric} - \frac{R_g}{N} g.$$

Remark 1.2.4. The trace-free Ricci tensor verifies:

- $\mathrm{tr}(E_g) = g^{ij} E_{g_{ij}} = 0$.
- $\mathrm{div}(E_g) = \frac{N-2}{2N} dR_g$, with $\mathrm{div}(E_g) = \mathrm{tr}(\nabla E_g)$.

(see [15], Proposition 7.18, p.209).

Definition 1.2.16. *A Riemannian metric g is said an Einstein metric if there exists $\lambda \in \mathbb{R}$ such that* $\mathrm{Ric} = \lambda g$.

Proposition 1.2.3. (Schur's Lemma) Let (M^N, g) be a smooth connected Riemannian manifold with $N \geq 3$. If $\mathrm{Ric} = fg$, for $f \in C^\infty(M)$, then f is constant and g is an Einstein metric.

Proof. See [15], Proposition 7.19, p. 210. □

Proposition 1.2.4. Let (M^N, g) be a connected smooth Riemannian manifold, with $N \geq 3$. Then g is Einstein if and only if $E_g = 0$.

Proof. See [15], Corollary 7.20, p. 210. □

1.2.3 Conformal geometry

Conformal geometry is the study of transformations preserving angles on surfaces (see Figure 1.1). Some examples of these transformations are translations, orthogonal maps or spherical isometries, and they are important in many areas such as physics (heat diffusion, electric–magnetic fields), general relativity, cartography,...

Figure 1.1: Conformal transformation

Definition 1.2.17. *A normal coordinate system at $P \in M^N$ is a local coordinate system for which the components of the metric tensor at P satisfy $g_{ij}(P) = \delta_i^j$, and, $\partial_k g_{ij}(P) = 0$, or, equivalently, $\Gamma_{ij}^k(P) = 0$, $\forall i, j, k$.*

Definition 1.2.18. *Two metrics g and $\tilde{g}$ on a N−dimensional smooth Riemannian manifold M are said conformal if there is a positive function $f \in C^\infty(M)$, such that $g = f\tilde{g}$.*

Definition 1.2.19. *A diffeomorphism φ between two smooth Riemannian manifolds (M, g) and $(\tilde{M}, \tilde{g})$ is called a conformal diffeomorphism (or conformal transformation) if it pulls $\tilde{g}$ back to a metric conformal to g, i.e., if $\varphi^*\tilde{g} = fg$, for some $f \in C^\infty(M)$, $f > 0$. Two smooth Riemannian manifolds are said conformally equivalent if there is a conformal diffeomorphism between them.*

Theorem 1.2.1. *Let M^N be a Riemannian manifold and $P \in M^N$. For each $N \geq 2$, there is a conformal metric g on M^N such that*

$$\det g_{ij} = 1 + O(|r|^N),$$

where $r = |x|$ in normal coordinates at P.

Proof. See [16], Theorem 5.1, p. 58. □

Definition 1.2.20. *Let (M^N, g) be a smooth Riemannian manifold, with $N \geq 3$, the Weyl tensor is a 4 covariant tensor defined in a local chart by the components:*

$$\begin{aligned} W_{ijkl} = \mathrm{Riem}_{ijkl} - \frac{1}{N-2}\left(\mathrm{Ric}_{ik}\, g_{jl} - \mathrm{Ric}_{il}\, g_{jk} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{jk}\, g_{il}\right) \\ + \frac{R_g}{(N-1)(N-2)}\left(g_{jl}g_{ik} - g_{jk}g_{il}\right). \end{aligned}$$

Remark 1.2.5. The Weyl tensor is conformally invariant.

Definition 1.2.21. *A smooth Riemannian manifold (M^N, g) is locally conformally flat if every $P \in M$ has a neighborhood that is conformally equivalent to $(\mathbb{R}^N, g_0)$.*

Proposition 1.2.5. Let (M^N, g) be a smooth Riemannian manifold of dimension $N \geq 3$ and locally conformally flat, then the Weyl tensor vanishes identically: $W \equiv 0$.

Proof. See [15], Corollary 7.3, p. 218. □

Example 1.2.5. (S^N, g_c) is locally conformally flat (see [15], Corollary 3.6, p. 61). In fact, $g_c = \left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}} |dx|^2$.

Let consider R_g the scalar curvature of (M^N, g), which we assume that is not constant, and, $R_{g'}$, the scalar curvature of (M^N, g'), that by hypothesis is constant with g' conformal to g. If $g' = e^f g$, we can compute the curvature tensor $R_{g'}$ of g' in terms of that of g. In the next theorem we calculate their difference by the transformation laws of the Ricci tensor ([20], p. 184) given by:

$$\mathrm{Ric}^{g'}_{kj} - \mathrm{Ric}^{g}_{kj} = -\frac{(N-2)}{2}\nabla_k\nabla_j f - \frac{1}{2}\Delta f g_{jk} + \frac{N-2}{4}\nabla_k f\nabla_j f - \frac{N-2}{4}\nabla^i f\nabla_i f g_{jk}, \tag{1.1}$$

Theorem 1.2.2. *Under the above hypothesis, we have that:*

$$R_{g'}e^f - R_g = -(N-1)\Delta f - \frac{(N-1)(N-2)}{4}\nabla_i f\nabla^i f. \tag{1.2}$$

Proof. If we contract (1.1) by g^{kj}, we get

$$\mathrm{Ric}^{g'}_{kj}g^{kj} - \mathrm{Ric}^{g}_{kj}g^{kj} = -\frac{(N-2)}{2}\nabla_k\nabla_j f g^{kj} - \frac{1}{2}\Delta f g_{jk}g^{kj} + \frac{N-2}{4}\nabla_k f\nabla_j f g^{kj} - \frac{N-2}{4}\nabla^i f\nabla_i f g_{jk}g^{kj}.$$

Ans using that $g_{jk}g^{kj} = N$ because it is the trace of the identity matrix of order N,

$$\mathrm{Ric}^{g'}_{kj}g^{kj} - \mathrm{Ric}^{g}_{kj}g^{kj} = -\frac{(N-2)}{2}\nabla_k\nabla_j f g^{kj} - \frac{N}{2}\Delta f + \frac{N-2}{4}\nabla_k f\nabla_j f g^{kj} - \frac{N(N-2)}{4}\nabla^i f\nabla_i f.$$

Knowing that $\nabla_j f g^{kj} = \nabla^k f$:

$$\mathrm{Ric}^{g'}_{kj}g^{kj} - \mathrm{Ric}^{g}_{kj}g^{kj} = -\frac{(N-2)}{2}\nabla_k\nabla^k f - \frac{N}{2}\Delta f + \frac{N-2}{4}\nabla_k f\nabla^k f - \frac{N(N-2)}{4}\nabla^i f\nabla_i f.$$

Thus, because $g^{kj} = e^f g'^{kj}$, and using the fact that $\nabla_k\nabla^k f = \Delta f$, we get

$$R_{g'}e^f - R_g = -\frac{(N-2)}{2}\Delta f - \frac{N}{2}\Delta f + \frac{N-2}{4}\nabla^i f\nabla_i f - \frac{N(N-2)}{4}\nabla^i f\nabla_i f.$$

We arrive finally to

$$R_{g'}e^f - R_g = -(N-1)\Delta f - \frac{(N-2)(N-1)}{4}\nabla^i f\nabla_i f.$$

□

1.3 The Laplacian operator on manifolds

Definition 1.3.1. *Let (M^N, g) be a smooth Riemannian manifold. The gradient of a scalar function f is the vector field, denoted ∇f, defined as*

$$\langle\nabla f(x), v_x\rangle_g = d_x f(v_x), \quad \forall v_x \in T_x M.$$

So, in local coordinates, we denote $\partial^i f = g^{ij}\partial_j f$*, and we have* $\nabla f = g^{ij}\partial_j f \partial_i f$*, then the i-th component of* ∇f *is :*

$$\nabla^i f(x) = \partial^i f = g^{ij}\partial_j f.$$

Definition 1.3.2. *Let* (M^N, g) *be a smooth Riemannian manifold. The divergence of a vector field* X *on* M*, denoted* $\operatorname{div} X$*, is defined as the scalar function verifying*

$$(\operatorname{div}(X))\, dV_g = \mathscr{L}_X dV_g,$$

where $\mathscr{L}_X$ *is the Lie derivative along the vector field* X*, and* dV_g *is the Riemannian volume form. So in local coordinates:*

$$\operatorname{div}(X) = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}X^i\right).$$

Definition 1.3.3. *Let* $N > 2$*,* (M^N, g) *be a smooth Riemannian manifold , we define the standard Laplace-Beltrami operator of a scalar smooth function* f *as*

$$\Delta_g f = \operatorname{div}(\nabla f).$$

Combining the above definitions, in local coordinates:

$$\Delta_g f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right).$$

Definition 1.3.4. *Let* (M^N, g) *be a smooth Riemannian manifold. The conformal laplacian operator is defined as*

$$L_g = -C_N \Delta_g + R_g.$$

Proposition 1.3.1. Given $g, \tilde{g}$ two conformal metrics on a smooth Riemannian manifold M^N, with $\tilde{g} = u^{\frac{4}{N-2}} g$, for u a smooth positive function, then the conformal laplacian operator satisfies the following property:

$$L_{\tilde{g}}(\varphi) = u^{-\frac{(N+2)}{(N-2)}} L_g(u\varphi), \quad \forall \varphi \in C^\infty(M^N).$$

If $\varphi = 1$, we have the classical scalar curvature equation:

$$L_g(u) = R_{\tilde{g}} u^{\frac{N+2}{N-2}}.$$

Proof. See [15], Problem 7.11, p. 223. □

1.4 Sobolev spaces

Definition 1.4.1. *Let $p \in \mathbb{R}$, $1 \le p \le \infty$. The Lebesgue space $L^p(M)$ is the set of locally integrable functions u on M for which the following norm is finite: $\|u\|_{L^p(M)} = \left(\int_M |u|^p \, dV_g\right)^{\frac{1}{p}}$.*

Definition 1.4.2. *Let $p \in \mathbb{R}$, $1 \le p \le \infty$, and k a non-negative integer.*

- *The Sobolev space $W^{k,p}(M)$ is defined as*

$$W^{k,p}(M) = \left\{u \in L^p(M) \,|\, D^\alpha u \in L^p(M),\ \forall \alpha \in \mathbb{N}^N : \ |\alpha| \le k\right\},$$

  where $D^\alpha u$ is the α weak derivative of u, $\forall \alpha \in \mathbb{N}^N$. This is a Banach space equipped with the norm:

$$\|u\|_{W^{k,p}(M)} = \left(\sum_{i=0}^{k} \|\nabla^i u\|^p_{L^p(M)}\right)^{\frac{1}{p}},$$

  with $\nabla^0(u) = u$. This space is reflexive for $1 < p < \infty$ and it is separable for $1 \le p < \infty$.

- *We set $H^k(M) = W^{k,2}(M)$, which is a separable Hilbert space. For example, $H^1(M)$ is equipped with the scalar product:*

$$\langle u, v\rangle_{H^k(M)} = \langle u, v\rangle_{L^2(M)} + \langle \nabla u, \nabla v\rangle_{L^2(M)} = \int_M (u \cdot v + \nabla u \cdot \nabla v)\, dV_g,$$

  and with the associated norm:

$$\|u\|_{H^k(M)} = \left(\|u\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)}\right)^{\frac{1}{2}}.$$

Remark 1.4.1. We obviously have $W^{k,p}(M) \subset L^p(M)$. Also, for $\alpha = 0$, $W^{0,p}(M) = L^p(M)$.

Definition 1.4.3. *Given $1 \le p \le \infty$, k a non-negative integer and N the dimension of a smooth Riemannian manifold (M^N, g). We define the critical exponent p^* as the real number which satisfies*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{k}{N}.$$

In particular, for $p = 2$, $k = 1$ and assuming $N \ge 3$, we define

$$2^* = \frac{2N}{N-2}.$$

Theorem 1.4.1. ***(Sobolev embedding for** $\mathbb{R}^N$ **)** Let $M = \mathbb{R}^n$ and let $\frac{1}{p} - \frac{k}{N} > 0$. Then*

$$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N),$$

with a continuous injection. In particular for $p = 2$, $k = 1$ and $N \geq 3$, we have the following Sobolev inequality:

$$\|u\|^2_{L^{2^*}(\mathbb{R}^N)} \leq \sigma_N \|\nabla u\|^2_{L^2(\mathbb{R}^N)}, \quad \forall u \in H^1(\mathbb{R}^N).$$

We will call the smallest such constant σ_N, the $N-$dimensional Sobolev constant.

Proof. See [16], Theorem 2.1, p. 44. □

The following theorem shows us that the Sobolev inequality holds with the same constant on any compact manifold M.

Theorem 1.4.2. ***(Sharp Sobolev inequality)*** *Let (M^N, g) be a compact Riemannian manifold, $N \geq 3$ and let σ_N be the best Sobolev constant defined in the theorem before. Then $\forall \varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\|u\|^2_{L^{2^*}(M)} \leq (1+\varepsilon)\sigma_N \int_M |\nabla u|^2 dV_g + C_\varepsilon \int_M u^2 dV_g, \quad \forall u \in C^\infty(M).$$

Proof. See [16], Theorem 2.3, p.45. □

Theorem 1.4.3. ***(Sobolev embedding for compact manifolds)*** *Let M^N be a $N-$dimensional compact Riemannian manifold, then:*

1. *If $\frac{1}{p^*} \geq \frac{1}{p} - \frac{k}{N}$, then the embedding $W^{k,p}(M^N) \hookrightarrow L^{p^*}(M^N)$ is continuous.*

2. *If $0 < \alpha < 1$, and, $\frac{k-\alpha}{N} \geq \frac{1}{p}$, then the embedding $W^{k,p}(M^N) \hookrightarrow C^\alpha(M^N)$ is continuous.*

Proof. See [3], Theorem 2.20, p. 44. □

Theorem 1.4.4. ***(Rellich-Kondrachov)*** *Let M^N be a $N-$dimensional compact Riemannian manifold (possibly with C^1 boundary), and, $1 \geq \frac{1}{q} > \frac{1}{p} - \frac{k}{N}$, then the embedding, $W^{k,p}(M^N) \hookrightarrow L^q(M^N)$, is compact.*

Proof. See [3], Theorem 2.34, p. 55. □

1.5 Fast diffusion equation

Following [24] and [25], the non-linear heat equation:

$$u_t(x,t) = \Delta u^m(x,t), \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}, m \in \mathbb{R}, \tag{1.3}$$

is called the porous medium equation for $m > 1$ and fast diffusion equation for $m < 1$. The classical heat equation is the case $m = 1$. We can write it in the divergence form as $u_t = \operatorname{div}(D(u)\nabla u)$, where $D(u)$ is the diffusion coefficient, $D(u) = mu^{m-1}$, if $u \geq 0$, and $D(u) = m|u|^{m-1}$, for signed solutions. However, it is a parabolic equation only when $u \neq 0$, while when $u = 0$, we say that the porous medium equation is a slow diffusion equation, because: $D(u) = m|u|^{m-1} \underset{u\to 0}{\longrightarrow} 0$. Instead, in the case of fast diffusion equations we have this name because: $D(u) = m|u|^{m-1} \underset{u\to 0}{\longrightarrow} \infty$.

Note that when $m < 0$, the fast diffusion equation can be written in the following modified form:

$$u_t = \Delta\left(\frac{u^m}{m}\right) = \operatorname{div}\left(u^{m-1}\nabla u\right),$$

to keep the parabolic character of the equation. When $m = 0$, this modified form allows us to write the equation as

$$u_t = \operatorname{div}\left(u^{-1}\nabla u\right) = \Delta \log(u),$$

and it is called logarithmic diffusion.

1.6 Uniformly elliptic equations

1.6.1 Uniformly elliptic equations

Using the Einstein summation convention, let consider the solutions $u \in C^2$ of the elliptic problems of the form

$$Lu = -a^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i} + c(x)u = 0, \quad x \in \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is an open smooth bounded set, $a^{ij}(x)$, $b^i(x)$, and $c(x)$ are $C\left(\overline{\Omega}\right)$ functions called structural functions. Similarly, the equation can be posed on Riemannian manifolds.

Definition 1.6.1. *We say that the operator L is uniformly elliptic if there exists a constant $\theta > 0$, such that for every vector $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and a.e. $x \in \Omega$, the following inequality holds:*

$$\theta\,|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j, \tag{1.5}$$

where a^{ij} are the structural functions in (1.4).

Now we are going to see the following theorems on an open set in $\mathbb{R}^N$ and on a manifold.

Theorem 1.6.1. ***(Local elliptic regularity)*** *Let $p \geq 1$, $\Omega \subset \mathbb{R}^N$ an open set and $u \in L^1_{loc}(\Omega)$ be a weak solution to $\Delta u = f$.*

1. *If $f \in W^{k,p}(\Omega)$, then $u \in W^{k+2,p}(K)$ for any compact set $K \Subset \Omega$, and there exists a constant $C > 0$ such that*

$$\|u\|_{W^{k+2,p}(K)} \leq C \left(\|\Delta u\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)} \right).$$

2. *If $f \in C^{k,\alpha}(\Omega)$, then $u \in C^{k+2,\alpha}(K)$ for any compact set $K \Subset \Omega$, and there exists a constant $C > 0$ such that*

$$\|u\|_{C^{k+2,\alpha}(K)} \leq C \left(\|\Delta u\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^{\alpha}(\Omega)} \right).$$

Proof. See [16], Theorem 2.4, p. 46. □

Theorem 1.6.2. ***(Global elliptic regularity)*** *Let $p \geq 1$, M be a compact Riemannian manifold and $u \in L^1_{loc}(M)$ be a weak solution to $\Delta u = f$.*

1. *If $f \in W^{k,p}(M)$, then $u \in W^{k+2,p}(M)$, and there exists a constant $C > 0$ such that*

$$\|u\|_{W^{k+2,p}(M)} \leq C \left(\|\Delta u\|_{W^{k,p}(M)} + \|u\|_{L^p(M)} \right).$$

2. *If $f \in C^{k,\alpha}(M)$, then $u \in C^{k+2,\alpha}(M)$, and there exists a constant $C > 0$ such that*

$$\|u\|_{C^{k+2,\alpha}(M)} \leq C \left(\|\Delta u\|_{C^{k,\alpha}(M)} + \|u\|_{C^{\alpha}(M)} \right).$$

Proof. See [16], Theorem 2.5, p. 46. □

Proposition 1.6.1. (Weak removable singularities) Let $U \in M$ an open set, $P \in U$ and u be a weak solution to $-\Delta u = h(u)$ on $U \setminus \{P\}$, with $h \in L^{\frac{N}{2}}(U)$ and $u \in L^p(U)$ for some $p > \frac{2^*}{2} = \frac{N}{N-2}$. Then u satisfies $-\Delta u = h(u)$ weakly on all of U.

Proof. See [16], Proposition 2.7, p. 47. □

1.6.2 Maximum principle

Theorem 1.6.3. ***(Weak maximum principle)*** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $u \in C^2(\Omega) \cap C(\overline{\Omega})$, L in (1.4) an uniformly elliptic operator with $c \equiv 0$ in Ω. We have:*

- *If $Lu(x) \le 0$, for $x \in \Omega$, then $\max_{x \in \overline{\Omega}} u = \max_{x \in \partial\Omega} u$.*
- *If $Lu(x) \ge 0$, for $x \in \Omega$, then $\min_{x \in \overline{\Omega}} u = \min_{x \in \partial\Omega} u$.*

Proof. See [10], Theorem 1, p. 344-345. □

We are going to present the well known Hopf Lemma in whose proof we use a suitable comparison function which is the same tool that we will use in subsequent chapters.

Lemma 1.6.1. ***(Hopf)*** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, L in (1.4) an uniformly elliptic operator with $c \equiv 0$ in Ω. If $Lu(x) \le 0$, $\forall x \in \Omega$, and there exists a point $P \in \partial\Omega$ such that*

- *There exists an open ball $B_R(x) \subset \Omega$, $R > 0$, with $p \in \partial B_R(x)$.*
- *$u(x) < u(P)$, $\forall x \in \Omega$.*

Then $\frac{\partial}{\partial \nu} u(P) = u_\nu(P) > 0$, where $\frac{\partial}{\partial \nu}$ is the outward normal to $B_R(x)$ at p. The same conclusion holds if $c \ge 0$ in Ω provided that $u(P) \ge 0$.

Proof. We can assume without loss of generality that $B_R(x) = B_R(0)$. We give the proof for $c \equiv 0$, and we refer to [10], p.348-349, for the case $c \ge 0$. Let us define the function $v(x) = e^{-\lambda|x|^2} - e^{-\lambda R^2}$, then we have by (1.5):

$$\begin{aligned} Lv(x) &= -a^{ij}(x) v_{x_i x_j} + b^i(x) v_{x_i} = e^{-\lambda|x|^2} a^{ij}(x)\left(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}\right) - e^{\lambda|x|^2} b^i(x) 2\lambda x_i \\ &\le e^{-\lambda|x|^2}\left(-4\lambda^2\theta|x|^2 + 2\lambda\,\mathrm{tr}(A) + 2\lambda|b||x|\right), \end{aligned}$$

where $A = (a_{ij}(x))_{i,j=1,\dots,N}$, and, $b = (b^1, \dots, b^N)$. Now, consider the open annulus $W = B_R(0) - \overline{B}_{\frac{R}{2}}(0)$, then

$$Lv(x) \le e^{-\lambda|x|^2}\left(-\lambda^2\theta R^2 + 2\lambda\,\mathrm{tr}(A) + 2\lambda|b|R\right) \le 0, \quad \forall x \in W, \tag{1.6}$$

where $\lambda > 0$ is fixed large enough.

By hypothesis, we have $u(x) < u(p)$, $\forall x \in \Omega$, then there exists $\varepsilon > 0$ sufficiently small such that

$$u(x) + \varepsilon v(x) \le u(p), \quad \forall x \in \partial B_{\frac{R}{2}}(0), \tag{1.7}$$

and since $v(x) \equiv 0$, for $x \in \partial B_R(0)$:

$$u(x) + \varepsilon v(x) \le u(p), \quad \forall x \in \partial B_R(0). \tag{1.8}$$

Now from (1.6) and that $Lu \le 0$ in Ω:

$$L(u(x) + \varepsilon v(x) - u(p)) \le 0, \quad \forall x \in W.$$

Moreover, from (1.7) and (1.8):

$$u(x) + \varepsilon v(x) - u(p) \le 0, \quad x \in \partial W.$$

Then applying the weak maximum principle of Theorem 1.6.3, we get $u(x)+\varepsilon v(x)-u(p) \le 0$ for $x \in W$. As we have $u(p)+\varepsilon v(p)-u(p) = 0$, then

$$u_\nu(p) + \varepsilon v_\nu(p) \ge 0.$$

From where we obtain

$$u_\nu(p) \ge -\varepsilon v_\nu(p) = -\frac{\varepsilon}{R}\nabla v(p)\cdot p = \left(-\frac{\varepsilon}{R}\right)\left(-2\lambda R e^{-\lambda R^2}\right) = 2\varepsilon\lambda e^{-\lambda R^2} > 0.$$

□

Theorem 1.6.4. ***(Strong maximum principle)*** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, L in (1.4) an uniformly elliptic operator with $c = 0$ in Ω, and, $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then:*

- *If $Lu(x) \le 0$, $\forall x \in \Omega$, and u attains its maximum over $\overline{\Omega}$ at an interior point, then $u(x)$ is constant for $x \in \Omega$.*
- *If $Lu(x) \ge 0$, for $x \in \Omega$, and u attains its minimum over $\overline{\Omega}$ at an interior point, then $u(x)$ is constant for $x \in \Omega$.*

The same conclusion holds for $c \ge 0$ in Ω if u attains respectively a non-negative maximum or a non-positive minimum.

Proof. See [10], Theorem 3, p.349-351. □

1.7 Non-linear parabolic equations

Following what we have done on the previous section of non-linear elliptic equations, let us consider $u \in C^2$ a solution of the quasi-linear parabolic equation

$$u_t(x,t) = \left(a^i(x,t,u,\nabla u)\right)_{x_i} + b(x,t,u,\nabla u), \tag{1.9}$$

where $a^i(x,t,u,p_1,\dots,p_N)$, and, $b(x,t,u,p_1,\dots,p_N)$ are bounded and smooth functions, called structural functions.

Definition 1.7.1. *We say that the equation (1.9) is uniformly parabolic if there exist constants $0 < C_1 < C_2 < \infty$, such that for every vector $\xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^N$, the following inequality holds:*

$$C_1|\xi|^2 \le \left(a^i(x,t,u,u_{x_i})\right)_{p_j}\xi_i\xi_j \le C_2|\xi|^2, \tag{1.10}$$

where a_i are the structural functions in (1.9).

Let see the following theorem of regularity of uniformly parabolic equations:

Theorem 1.7.1. *Let $u_0(x)$ be bounded and continuous and (1.9) satisfying the uniformly parabolic condition. Then the Cauchy problem associated to (1.9) with initial condition $u(x,0) = u_0(x)$, can be solved and the solution verifies:*

- *$u(x,t) \in C^\infty\left(\mathbb{R}^N \times (0,\infty)\right)$.*
- *u is unique.*
- *u is continuous down to $t = 0$, that is $u \in C^\infty\left(\mathbb{R}^N \times [0,\infty)\right)$, and, $u(x,0) = u_0(x)$.*

Proof. See [25], (i) in p.31. □

1.7.1 A comparison principle

One important property of parabolic equations is the maximum principle, especially in the form of comparison principle. We state here a comparison principle for the following general non-linear Dirichlet problem for the so-called filtration equation ([25]):

$$\begin{cases} u_t(x,t) = \Delta\phi(u(x,t)) + f(x,t), & in\ \Omega \times (0,T), \\ u(x,0) = u_0(x), & in\ \Omega, \\ \phi(u(x,t)) = h(x,t), & in\ \partial\Omega \times [0,T), \end{cases} \tag{1.11}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with regular boundary $\partial\Omega \in C^{2+\alpha}$ $(0 \leq \alpha < 1)$, u_0 is a measurable function in Ω, $h(x,t)$ is a measurable function on $\partial\Omega \times [0,T)$, $f(x,t)$ is a measurable function in $\Omega \times (0,T)$, and, $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous and strictly increasing in u with $\phi(0_+) = 0$, $\phi(+\infty) = +\infty$, and such that $\phi(u)$ is smooth with $\phi'(u) > 0$, for $u > 0$. The filtration equation includes the particular case: $u_t(x,t) = \Delta u^m(x,t)$, which the porous medium equation explained in Section 1.5. In Chapter 5 of [25], it is proved the existence of weak solutions to this problem and that these solutions also satisfy the following comparison theorem.

Let denote ψ the primitive of ϕ with respect to u:

$$\psi(s) = \int_0^s \phi(\sigma)\, d\sigma,$$

and define $L_\psi(\Omega) \subset L^1(\Omega)$ the space where u_0 is a measurable function such that $\psi(u_0) \in L^1(\Omega)$.

Theorem 1.7.2. ***(Comparison Principle)*** *Let $H \in L^2\big((0,T);H^1(\Omega)\big)$ with, $h = T_{\partial\Omega\times(0,\infty)}(H)$, its trace, $u_0 \in L_\psi(\Omega)$, and, $f \in L^2(\Omega \times (0,T))$ such that there exists $u \in L^\infty\big((0,\infty);L_\psi(\Omega)\big)$, a weak solution for the problem (1.11). Then the comparison principle applies to these solutions, i.e., taking $u, \overline{u}$ two weak solutions that satisfy*

$$\begin{cases} u_0(x) \leq \overline{u}_0(x), & a.e.\ in\ \Omega, \\ f(x,t) \leq \overline{f}(x,t), & a.e.\ in\ \Omega \times (0,T), \\ h(x,t) \leq \overline{h}(x,t), & a.e.\ in\ \partial\Omega \times (0,\infty). \end{cases}$$

Then $u(x,t) \leq \overline{u}(x,t)$, a.e. in $\Omega \times (0,T)$.

Proof. See Theorem 5.14, p. 105 in [25]. □

1.7.2 Asymptotic limit

To address the asymptotic limit of the solutions of non-linear parabolic equations we have followed [22]. Let (M,g) be a compact smooth Riemannian manifold, we start by defining the following energy functional:

Definition 1.7.2. *Let $u(x,t)$ be a C^∞ function on M with $(x,t)\in M\times[0,T)$, $T>0$. We define the energy functional*

$$\mathscr{E}(u)=\int_M E(x,u,\nabla u)\,dV_g,$$

where $E=E(x,z,p)$, with $(x,z,p)\in M\times\mathbb{R}\times T_xM$, is a function which satisfies

- *If p_x depends smoothly on x, then E depends smoothly on $(x,z,p_x)\in M\times\mathbb{R}\times T_xM$.*
- *Let x fixed, then E is uniformly convex in the p variable for $p\in T_xM$ and $|z|+|p|$ sufficiently small, that is to say, there is a constant $C>0$ (independent of x and p), such that*

  $$\frac{d^2}{ds^2}\Big|_{s=0}E(x,0,sp)\geq C|p|^2.$$

- *E has an analytic dependence on $(z,p)\in\mathbb{R}\times T_xM$ uniformly in x for sufficiently small $|z|,|p|$, that is to say, there exists $\beta>0$ such that for $|z|,|u|,|p|,|q|<\beta$,*

  $$E(x,z+\lambda_1 u,p+\lambda_2 q)=\sum_{|\alpha|\geq 0}E_\alpha(x,z,u,p,q)\lambda^\alpha,\quad z,u\in\mathbb{R},\,p,q\in T_xM,$$

  where $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2$ such that $|\lambda|<1$, and, $\sup_{|\lambda|<1}\left|\sum_{|\alpha|=j}E_\alpha(x,z,u,p,q)\lambda^\alpha\right|\leq 1$, for $j\geq 1$.

Let us denote the Euler-Lagrange operator for $\mathscr{E}(u)$ as $W(u)$, i.e.

$$W(u)=-D\mathscr{E}(u),$$

with D the Frechet derivative, which it is uniquely characterized by

$$\langle -W(u),\varphi\rangle_{L^2(M)}=\frac{d}{ds}\Big|_s\mathscr{E}(u+s\varphi),\ \text{for } u,\varphi\in C^2(M). \tag{1.12}$$

We also require $W(0)=0$. By the hypothesis on $E(x,z,p)$, $W(u)$ is a second order quasi-linear operator as the one appearing in the right side of (1.9), being this a uniformly parabolic equation for $\|u\|_{C^1(M)}$ sufficiently small.

Let consider the following equation:

$$u_t(x,t)=W(u(x,t))+f(x,t),\quad (x,t)\in M\times(0,\infty). \tag{1.13}$$

where $f(x,t)$ is a smooth function with exponential decay with respect to t, that is for a given $\varepsilon>0$ there exists a constant $C>0$, such that

$$\|f(x,t)\|_{H^l}+\|f_t(x,t)\|_{H^l}+\|f_{tt}(x,t)\|_{H^l}\leq Ce^{-\varepsilon t},\forall t>0,$$

where l is an integer.

The following theorem states that for initial data sufficiently small, there exists a solution of (1.13) which either reaches a point where $\mathscr{E}(u)$ is negative, or else it is defined for all time and converges asymptotically to a solution of the equation $W(u(x,t))=0$.

Remark 1.7.1. Let l be an integer sufficiently large such that $C^2(M)\subset H^{l-1}(M)$, and $\forall\varphi\in H^{l-1}(M)$, we have $|\varphi|_{C^2(M)}\leq C\|\varphi\|_{H^{l-1}}$, for certain $C>0$.

Theorem 1.7.3. *Given $\varepsilon>0$, there are constants $\delta=\delta(E,\varepsilon)>0$ and $\alpha=\alpha(E)\in\left(0,\frac{1}{2}\right)$ such that for any given $u_0(x)\in C^\infty(M)$ with $\|u_0\|_{H^{l+2}}<\delta$, and, $f(x,t)\in C^\infty(M\times[0,\infty)$ with exponential decay with respect to M, there exist $T_*>0$ and a solution $\overline{u}\in C^\infty(M\times[0,\infty))$ of the equation (1.13) on $[0,T_*)$ satisfying:*

- *$\overline{u}(x,0)=u_0(x)$ on M.*
- $\sup\limits_{[0,T_*)}\|\overline{u}(x,t)\|_{H^p}<\delta^\alpha$.
- *One of the following assertions:*
  - *If $T_*<\infty$,*
$$\lim_{t\to T_*}\mathscr{E}(\overline{u}(x,t))\leq\mathscr{E}(0)-\delta.$$
  - *If $T_*=\infty$,*
$$\lim_{t\to\infty}\left(|\overline{u}_t(x,t)|_{C^1(\Omega)}+|\overline{u}(x,t)-u(x,t)|_{C^2(\Omega)}\right)=0,$$

  where $u(x,t)\in C^\infty(M)$ is a solution of $W(u(x,t))=0$.

Proof. See [22], Theorem 2, p. 535. □

To continue, consider the following Cauchy problem:

$$\begin{cases}u_t(x,t)=W(u(x,t)), & (x,t)\in M\times(0,\infty),\\ u(x,0)=u_0(x),\end{cases}\tag{1.14}$$

where the initial data fulfills the condition $\|u_0\|_{H^{l+2}}<\infty$. From the theorem above, we obtain the two following corollaries which are relevant to prove the uniqueness of the asymptotic limit for the solution of quasilinear parabolic equations. The first one guaranties the existence and uniqueness of the solution of the parabolic problem and the second one the existence of its asymptotic limit.

Corollary 1.7.1. *Let $\varphi(x) \in C^\infty(M)$ such that for a given $\varepsilon > 0$, when $\|\varphi(x) - \tilde{\varphi}(x)\|_{H^l} < \varepsilon$, we have*

$$\mathscr{E}(\varphi(x)) \leq \mathscr{E}(\tilde{\varphi}(x)).$$

Then there exist constants $\delta = \delta(\varepsilon, M, E, \varphi)$, and, $\alpha = \alpha(E, \varphi) \in (0,1)$, such that if $\|u_0(x) - \varphi(x)\|_{H^{l+2}} < \delta$, then there is $u(x,t) \in C^\infty(M \times [0,\infty))$, a solution of the problem (1.14) such that $\lim_{t\to\infty} u(x,t) = \overline{u}(x)$, where:

- $\mathscr{E}(\overline{u}(x)) = \mathscr{E}\left(\varphi(x)\right)$.
- $W(\overline{u}(x)) = 0$.
- $\|\overline{u}(x) - \varphi(x)\|_{H^l} < \min\left\{\delta^\alpha, \frac{\varepsilon}{2}\right\}$.

Proof. See [22], Corollary 1, p.536. □

Corollary 1.7.2. *Let u be a smooth solution of the equation (1.13), $\forall t \in (0,\infty)$, and $f(x,t) \in C^\infty(M \times (0,\infty))$ a function with exponential decay with respect to t. If there exists a sequence $t_k \to \infty$, and $\overline{u}(x)$, with $W(\overline{u}(x)) = 0$, such that*

$$\lim_{k\to\infty} \|u(x,t_k) - \overline{u}(x)\|_{H^{l+2}} = 0.$$

Then we have $\lim_{t\to\infty} \|u(x,t) - \overline{u}(x)\|_{H^{l+2}} = 0$.

Proof. See [22], Corollary 2, p. 536. □

Let see the following estimate for $\|W(u)\|_{L^2(M)}$:

Theorem 1.7.4. *Let $\beta > 0$ be as in Definition 1.7.2, and $\mu \in (0,1)$ arbitrary. There are constants $\theta = \theta\left(E,\beta\right) \in \left(0, \frac{1}{2}\right)$, $\gamma = \gamma(E,\beta) \geq 2$, and, $\sigma = \sigma(E,\beta) \in (0,\beta)$, such that if $u \in C^{2,\mu}(M)$ is an arbitrary function with $\|u\|_{C^{2,\mu}(M)} < \sigma$, then*

$$\|W(u)\|_{L^2(M)} \geq \left(\inf_{\zeta\in\mathscr{Z}} \|u - \zeta\|_{L^2(M)}\right)^\gamma, \quad \textit{and,} \quad \|W(u(x,t))\|_{L^2(M)} \geq |\mathscr{E}(u(x,t)) - \mathscr{E}(0)|^{1-\theta},$$

where $\mathscr{Z} = \left\{\zeta \in C^2(M) : |\zeta|_{C^2(M)} < \beta,\ W(\zeta) = 0\right\}$.

Proof. See [22], Theorem 3, p. 537. □

Chapter 2

The Yamabe problem

2.1 An introduction to the Yamabe problem

As we exposed in the introduction, Yamabe originally formulated his problem in the following terms:

Let (M^N, g) be a compact Riemannian manifold of dimension $N \geq 3$ and non-constant scalar curvature. Is there a metric with constant scalar curvature conformal to g?

Throughout this chapter, we are going to study in depth the resolution of this problem. To simplify the equation (1.2), Yamabe proposed a conformal deformation in the form $g' = u^{\frac{4}{N-2}} g$, with $u \in C^\infty(M)$, $u > 0$. In the next theorem we use this transformation to obtain this simpler equation, in the sense that changing f by u, instead of having the gradients, we have the laplacian.

Theorem 2.1.1. *Under the previous hypothesis,* (1.2) *is equivalent to*

$$R_{g'} u^{\frac{N+2}{N-2}} = -C_N \Delta u + R_g u. \tag{2.1}$$

Proof. From (1.2) we have

$$R_{g'} = e^{-f} \left[-(N-1)\Delta f - \frac{(N-2)(N-1)}{4} \nabla^i f \nabla_i f + R_g \right].$$

We do the change of function $e^f = u^{p-2} = u^{\frac{4}{N-2}}$ for $u \in C^\infty(M)$, $u > 0$. So we get $f = \frac{4}{N-2} \log(u) = -\log(u^{2-p})$, and then

$$R_{g'} = u^{2-p} \left[(N-1)\Delta\left(\log\left(u^{2-p}\right)\right) - \frac{(N-2)(N-1)}{4} \nabla^i \left(\log\left(u^{2-p}\right)\right) \nabla_i \left(\log\left(u^{2-p}\right)\right) + R_g \right].$$

So,

$$R_{g'} = u^{1-p}\left[(N-1)(2-p)\Delta(\log(u))u - \frac{(N-2)(N-1)}{4}(2-p)^2\nabla^i(\log(u))\nabla_i(\log(u))u + R_g u\right].$$

As we know, $\nabla\log(u) = \frac{\nabla u}{u}$, and, $\Delta\log(u) = \nabla_i\nabla^i u = \frac{u\cdot\Delta u - |\nabla u|^2}{u^2}$. Then

$$R_{g'} = u^{1-p}\left[(N-1)(2-p)\left(\frac{\Delta u\cdot u - |\nabla u|^2}{u^2}\right)u - \frac{(N-2)(N-1)}{4}(2-p)^2\left|\frac{\nabla u}{u}\right|^2 u + R_g u\right].$$

Then

$$R_{g'} = u^{1-p}\left[-4\frac{(N-1)}{(N-2)}\Delta u + R_g u\right].$$

As $p-1 = \frac{N+2}{N-2}$, we finally get (2.1). □

Then we have obtained an elliptic semi-linear equation known as the Yamabe equation:

$$-C_N\Delta u + R_g u = R_{g'}u^{\frac{N+2}{N-2}}, \tag{2.2}$$

where $C_N = C(N) = 4\frac{(N-1)}{(N-2)} > 0$ is a constant.

Definition 2.1.1. *Let consider R_g the scalar curvature of (M^N, g) a N-dimensional smooth Riemannian manifold.*

- *If $R_g > 0$, we say that g is scalar positive. For example, the sphere case.*
- *If $R_g < 0$, we say that g is scalar negative. The metric of an hyperbolic space is an example of this.*
- *If $R_g = 0$, we say that g is scalar flat, as what happens in the plane.*

2.2 The variational approach

As we have seen, the Yamabe problem is equivalent to obtaining a smooth and strictly positive solution to (2.2) with $R_{g'}$ constant. To achieve this goal, we follow the variational method as stated in [4] or [16] proposed by Yamabe.

Definition 2.2.1. *Let us define the Yamabe functional as*

$$I[u] = \frac{\int_M \left(C_N|\nabla u|^2 + R_g u^2\right)dV_g}{\left[\int_M |u|^{2^*}dV_g\right]^{\frac{2}{2^*}}}, \tag{2.3}$$

where $u \in H^1(M)$, with $2^* = \frac{2N}{N-2}$.

Lemma 2.2.1. *The functional I is invariant by rescaling.*

Proof. Let λ be a scalar, we have

$$I[\lambda u] = \frac{\int_M \left(C_n |\nabla(\lambda u)|^2 + R_g (\lambda u)^2\right) dV_g}{\left[\int_M |\lambda u|^{2^*} dV_g\right]^{\frac{2}{2^*}}} = \frac{\lambda^2 \int_M \left(C_n |\nabla u|^2 + R_g u^2\right) dV_g}{\lambda^2 \left[\int_M |u|^{2^*} dV_g\right]^{\frac{2}{2^*}}} = I[u].$$

□

Remark 2.2.1. If we take $\lambda = \frac{1}{\int_M |u|^{2^*} dV_g}$, then by the last lemma we have $I[\lambda u] = I[u]$, so we can assume without loss of generality that $\int_M |u|^{2^*} dV_g = 1$.

Lemma 2.2.2. *Let u be solution of* (2.2)*, then $I[|u|] = I[u]$.*

Proof. Since $u \in H^1(M)$, then $|u| \in H^1(M)$, and, $|\nabla u| = |\nabla |u||$ almost everywhere. From this constraint we obtain that:

$$I[|u|] = \frac{\int_M \left(C_n |\nabla |u||^2 + R_g |u|^2\right) dV_g}{\left[\int_M |u|^{2^*} dV_g\right]^{\frac{2}{2^*}}} = \frac{\int_M \left(C_n |\nabla u|^2 + R_g u^2\right) dV_g}{\left[\int_M |u|^{2^*} dV_g\right]^{\frac{2}{2^*}}} = I[u].$$

□

Remark 2.2.2. Lemma 2.2.2 allows us to assume $u \in H^1(M)$, $u \geq 0$ without loss of generality.

Remark 2.2.3. We can define the functional I for conformal metrics, as follows:

$$I[u] = \frac{\int_M R_{g'} \, dV_{g'}}{\left[\int_M dV_{g'}\right]^{\frac{2}{2^*}}} = I[g'], \text{ with } g' \text{ conformal to } g. \tag{2.4}$$

Let $g' = u^{\frac{4}{N-2}} g$, with $u \in C^\infty(M)$, $u > 0$, by the Definition 1.2.8, we have

$$dV_{g'} = \sqrt{|g'|} dx^1 \wedge \cdots \wedge dx^N = \sqrt{u^{\frac{4N}{N-2}} |g|} dx^1 \wedge \cdots \wedge dx^N = u^{2^*} \sqrt{|g|} dV_g = u^{2^*} dV_g. \tag{2.5}$$

So we have by the Green's identity and (2.2),

$$I[u] = \frac{\int_M \left(C_N |\nabla u|^2 + R_g u^2\right) dV_g}{\left[\int_M u^{2^*} dV_g\right]^{\frac{2}{2^*}}} = \frac{\int_M R_{g'} u^{\frac{N+2}{N-2}} u \, dV_g}{\left[\int_M u^{2^*} dV_g\right]^{\frac{2}{2^*}}} = \frac{\int_M R_{g'} u^{2^*} dV_g}{\left[\int_M u^{2^*} dV_g\right]^{\frac{2}{2^*}}}.$$

Using (2.5), we get (2.4).

Do note that $I[u]$ is well defined because Theorem 1.4.3 ensures that $H^1(M) \subset L^{2^*}(M)$, that is to say:

Lemma 2.2.3. *The functional I is bounded below in $\mathscr{A}$.*

Proof. On one hand, we have $C_N\int_M |\nabla u|^2\,dV_g \geq 0$, then for $u\in\mathscr{A}$:

$$I[u]=\int_M \left(C_N|\nabla u|^2+R_g u^2\right)dV_g \geq \int_M R_g u^2\,dV_g \geq \inf_{x\in M}\{R_g,0\}\int_M u^2\,dV_g. \tag{2.6}$$

On the other hand, applying Hölder inequality for $p=\frac{2^*}{2}$, $q=\frac{N}{2}$, we have

$$\int_M u^2\,dV_g \leq \|u^2\|_{L^{\frac{2^*}{2}}(M}\|1\|_{L^{\frac{N}{2}}(M)}=\|u\|^2_{L^{2^*}(M)}Vol(M)^{\frac{2}{N}}=Vol(M)^{\frac{2}{N}}<\infty. \tag{2.7}$$

From (2.6) and (2.7) we deduce the result. □

Remark 2.2.4. Lemma 2.2.3 allows us to define $\lambda(M)=\inf\limits_{u\in H^1(M)}\{I[u]\}$. Since $C^\infty(M)$ is dense in $H^1(M)$, Lemma 2.2.3 guarantees that

$$\lambda(M)=\inf_{u\in C^\infty(M),u>0}\{I[u]\}=\inf_{\text{g' conformal to g}}\{I[g']\}.$$

Thus, the constant $\lambda(M)$ is a conformal invariant.

Definition 2.2.2. *Let (M,g) be a Riemannian manifold, we define the Yamabe invariant as the following constant:*

$$\lambda(M)=\inf_{u\in C^\infty(M),u>0}\{I[u]\}=\inf_{\textit{g' conformal to g}}\{I[g']\}.$$

We consider the solutions of (2.2) as critical points of the functional I, so now we are going to calculate the first variation of this functional.

Lemma 2.2.4. *The Euler-Lagrange equation for the functional I is* (2.2).

Proof. Let $u_0\in H^1(M)$, $u_0\geq 0$ fixed, $\varepsilon\in\mathbb{R}$ and $v\in C_c^\infty(M)$. Let us denote $\|u\|_{2^*}=\left(\int_M |u|^{2^*}dV_g\right)^{\frac{1}{2^*}}$, then we have

$$\begin{aligned}I[u_0+\varepsilon v]&=\frac{\int_M\left(C_N|\nabla(u_0+\varepsilon v)|^2+R_g(u_0+\varepsilon v)^2\right)dV_g}{\|u_0+\varepsilon v\|^2_{2^*}}\\&=\frac{\int_M\left(C_N|\nabla u_0|^2+C_N\varepsilon^2|\nabla v|^2+2C_N\varepsilon\nabla u_0\cdot\nabla v+R_g(u_0^2+2u_0\varepsilon v+\varepsilon^2v^2)\right)dV_g}{\|u_0+\varepsilon v\|^2_{2^*}}.\end{aligned}$$

For the next step, we have: $|u_0+\varepsilon v|^{2^*}=[(u_0+\varepsilon v)(u_0+\varepsilon v)]^{\frac{2^*}{2}}=\left(u_0^2+2u_0\varepsilon v+\varepsilon^2v^2\right)^{\frac{2^*}{2}}$, then

$$\frac{d}{d\varepsilon}|u_0+\varepsilon v|^{2^*}=\frac{2^*}{2}\left(2u_0v+2\varepsilon v^2\right)\left(u_0^2+2u_0\varepsilon v+\varepsilon^2v^2\right)^{\frac{2^*}{2}-1}=\frac{2^*}{2}\left(2u_0v+2\varepsilon v^2\right)|u_0+\varepsilon v|^{2^*-2}.$$

So we can calculate the derivative of the functional:

$$\frac{d}{d\varepsilon}I[u_0+\varepsilon v] = \frac{\int_M \left(2C_N\varepsilon|\nabla v|^2 + 2C_N\nabla u_0\cdot\nabla v + R_g(2u_0v+2\varepsilon v^2)\right)dV_g\,\|u_0+\varepsilon v\|_{2^*}^2}{\|u_0+\varepsilon v\|_{2^*}^4}$$

$$-\frac{\int_M \left(C_N|\nabla(u_0+\varepsilon v)|^2 + R_g(u_0+\varepsilon v)^2\right)dV_g}{\|u_0+\varepsilon v\|_{2^*}^4}$$

$$\cdot\frac{2}{2^*}\left(\int_M \frac{2^*}{2}\left(2u_0v+2\varepsilon v^2\right)|u_0+\varepsilon v|^{2^*-2}\,dV_g\right)\left[\int_M |u_0+\varepsilon v|^{2^*}\,dV_g\right]^{\frac{2}{2^*}-1}.$$

Letting $\varepsilon=0$, we get

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I[u_0+\varepsilon v] = \frac{2}{\|u_0\|_{2^*}^2}\left[\int_M \left(C_N\nabla u_0\cdot\nabla v + R_g u_0 v\right)dV_g - \|u_0\|_{2^*}^{-2^*}\int_M \left(C_N|\nabla u_0|^2 + R_g u_0^2\right)u_0^{2^*-1}v\,dV_g\right].$$

By an integration by parts using the Green's identities, we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I[u_0+\varepsilon v] = \frac{2}{\|u_0\|_{2^*}^2}\left[\int_M \left(-C_N\Delta u_0\cdot v + R_g u_0 v\right)dV_g - \|u_0\|_{2^*}^{-2^*}\int_M \left(-C_N\Delta u_0\cdot u_0 + R_g u_0^2\right)u_0^{2^*-1}v\,dV_g\right].$$

If we denote $E[u_0]=\int_M \left(-C_N\Delta u_0\cdot u_0 + R_g u_0^2\right)dV_g$, then we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I[u_0+\varepsilon v] = \frac{2}{\|u_0\|_{2^*}^2}\int_M \left[-C_N\Delta u_0 + R_g u_0 - \|u_0\|_{2^*}^{-2^*}E[u_0]u_0^{2^*-1}\right]v\,dV_g.$$

Thus, critical points of this functional must satisfy $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I[u_0+\varepsilon v]=0$ and

$$-C_N\Delta u_0 + R_g u_0 - \|u_0\|_{2^*}^{-2^*}E[u_0]u_0^{2^*-1} = 0.$$

Finally,

$$-C_N\Delta u_0 + R_g u_0 = \frac{E[u_0]}{\|u_0\|_{2^*}^{2^*}}u_0^{\frac{N-2}{N+2}}.$$

Let μ be the Lagrange multiplier defined as

$$\mu = \frac{E[u_0]}{\|u_0\|_{2^*}^{2^*}} \in \mathbb{R}.$$

Then we have that u_0 is a critical point of the functional I if it verifies equation (2.2) with μ as above, that is:

$$-C_N\Delta u + R_g u = \mu u^{\frac{N-2}{N+2}}.$$

□

With this goal, we define:

$$\mathscr{A} = \left\{u\in H^1(M)\mid u\geq 0,\ \|u\|_{L^{2^*}(M)}=1\right\}.$$

2.3 The sphere case

As we will see, the Yamabe invariant for the sphere ($\lambda(\mathbb{S}^N)$) will play a critical role in solving the Yamabe problem. In this section, we go deeper in the study of this model case.

2.3.1 The stereographic projection

Definition 2.3.1. *The stereographic projection σ, a conformal diffeomorphism, is defined by*

$$\begin{array}{rcl} \sigma : \mathbb{S}^N \setminus \{q_0\} & \longrightarrow & \mathbb{R}^N \\ (z,\xi) = (z_1,\dots,z_N,\xi) & \longmapsto & (x_1,\dots,x_N), \end{array}$$

where $q_0 = (0,..,0,1)$ is the north pole on $\mathbb{S}^N \subset \mathbb{R}^{N+1}$, and,

$$x_j = \frac{z_j}{1-\xi}, \quad \textit{for } j \in 1,\dots,N, \textit{ and, } (z,\xi) \in \mathbb{S}^N \setminus \{q_0\}.$$

Remark 2.3.1. From the definition of the stereographic projection we also have:

$$\begin{aligned} &\sigma^{-1} : \mathbb{R}^N \longrightarrow \mathbb{S}^N \setminus \{q_0\} \\ &(z_1,\dots,z_N) \longmapsto \frac{\left(2z_1,\dots,2z_N,|z|^2-1\right)}{1+|z|^2} = \left(\frac{2z}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right). \end{aligned}$$

And denoting $\rho = \sigma^{-1}$, we can compute the pullback metric of g_c, the standard metric on $\mathbb{S}^N$ ([15], p. 61):

$$\rho^* g_c = \frac{4\left(dz_1^2 + \dots + dz_N^2\right)}{\left(|z|^2+1\right)^2} = \frac{4}{\left(|z|^2+1\right)^2} g_0,$$

with g_0 the Euclidean metric on $\mathbb{R}^N$.

Since we are denoting conformal change by $g = u^{\frac{4}{N-2}} g_0$, we call

$$u_1(z) = \left(|z|^2+1\right)^{\frac{2-N}{2}}. \tag{2.8}$$

So we have $\rho^* g_c = \left[4^{\frac{N-2}{4}} u_1(z)\right]^{\frac{4}{N-2}} g_0$.

Now we are going to see that we can use the stereographic projection to write conformal diffeomorphisms of the sphere generated by rotations and maps of the form $\sigma^{-1}\tau_v\sigma$ and $\sigma^{-1}\delta_\mu\sigma$, where τ_v is the translation by $v \in \mathbb{R}^N$ given by

$$\begin{array}{rcl} \tau_v : \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \\ x & \longmapsto & x - v, \end{array}$$

and δ_μ is the dilation by $\mu > 0$:

$$\begin{array}{rcl} \delta_\mu : \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \\ x & \longmapsto & \mu^{-1}x. \end{array}$$

By doing a change of variables $z = \mu^{-1}z'$, with $\mu > 0$, then under the dilation, the spherical metric becomes

$$\delta_\mu^* \rho^* g_c = 4\left(\frac{1}{|\mu^{-1}z'|^2 + 1}\right)^{\frac{N-2}{2}\frac{4}{N-2}} \frac{1}{\mu^2} g_0. \tag{2.9}$$

Finally, if we call

$$u_\mu(z) = \left(\frac{\mu}{|z|^2 + \mu^2}\right)^{\frac{N-2}{2}}, \tag{2.10}$$

then we have

$$\delta_\mu^* \rho^* g_c = 4u_\mu(z)^{\frac{4}{N-2}} g_0. \tag{2.11}$$

Remark 2.3.2. The family of radial functions (2.10) are solutions of the Yamabe problem in $\mathbb{R}^N$ and they are called "bubbles". They receive this name, because of their shape, as we can see in Figure 2.1 and play an important role in what follows.

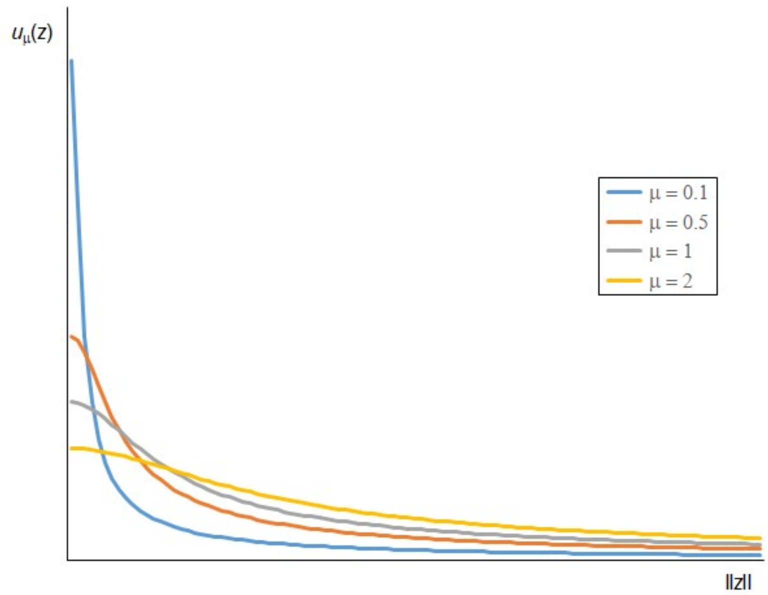

Figure 2.1: Bubbles

2.3.2 Yamabe problem on the sphere

Theorem 2.3.1. ***(Obata)*** *If g is a metric on $\mathbb{S}^N$ conformal to the standard metric g_c and it has constant scalar curvature, then g is a conformal diffeomorphism of the canonical metric given by g_c.*

Proof. We follow [16].

- By hypothesis, $g_c = v^{-2}g$, with $v \in C^\infty(\mathbb{S}^N)$, $v > 0$. Then the transformation laws of the Ricci tensor (1.1), with $e^f = v^{-2}$, becomes

$$\mathrm{Ric}^{g}_{jk} - \mathrm{Ric}^{g_c}_{jk} = v^{-1}\left[(N-2)\nabla_k\nabla_j v - (N-1)v^{-1}\nabla^i v\nabla_i v g_{jk} + \Delta v g_{jk}\right],$$

  with the right hand side computed with respect to g. Since g_c is Einstein, its trace-less Ricci tensor $E^{g_c} = 0$ (Proposition 1.2.4) and thus,

$$0 = E_{g_{c_{jk}}} = E_{g_{c_{jk}}} + (N-2)v^{-1}\left[\nabla_k\nabla_j v - \frac{1}{N}\Delta v g_{jk}\right].$$

Now, because E_g is traceless, integration by parts gives:

$$\int_{\mathbb{S}^N} v|E|_g^2\, dV_g = \int_{\mathbb{S}^N} vE_{g_{jk}}E_g^{jk}\, dV_g = -(N-2)\int_{\mathbb{S}^N} E_g^{jk}\left(\nabla_k\nabla_j v - \frac{1}{N}\Delta v g_{jk}\right) dV_g$$
$$= -(N-2)\int_{\mathbb{S}^N} E_g^{jk}\nabla_k\nabla_j v\, dV_g = (N-2)\int_{\mathbb{S}^N}\langle \operatorname{div}(E_g),\nabla v\rangle_g\, dV_g = 0,$$

where we have used that $\operatorname{div}(E_g) = \frac{N-2}{2N}dR_g = 0$, because g has constant curvature and then $dR_g = 0$. So, $|E_g|^2 = 0$, if and only if, $E_g = 0$, i.e., $\mathrm{Ric}^g = \frac{R_g}{N}g$ and this means that g is Einstein by Proposition 1.2.4.

- Now we are going to prove that the sectional curvature of g is constant. For that, we clear Riem_{ijkl} from the Weyl tensor:

$$\mathrm{Riem}_{ijkl} = W^g_{ijkl} + \frac{1}{(N-2)}\left(\mathrm{Ric}_{ik}\, g_{jl} - \mathrm{Ric}_{il}\, g_{jk} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{jk}\, g_{il}\right)$$
$$- \frac{R_g}{(N-1)(N-2)}\left(g_{jl}g_{ik} - g_{jk}g_{il}\right).$$

Now, because the Weyl tensor is a conformal invariant and g_c is locally flat, we have that $W^g = 0$ by Proposition 1.2.5, and using that $\mathrm{Ric}^g = \frac{R_g}{N}g$, we obtain:

$$\mathrm{Riem}_{ijkl} = \frac{R_g 2}{(N-2)N}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) - \frac{R_g}{(N-1)(N-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$$
$$= \frac{R_g}{N(N-1)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right).$$

Then we obtain

$$\mathrm{Riem}_{ijkl} = \frac{R_g}{N(N-1)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right).$$

Returning to the Definition 1.2.13 of sectional curvature, from this we get that is constant. Then it is well known that $(\mathbb{S}^N, g)$ is isometric to a standard sphere, $\mathbb{R}^N$ or the $N-$dimensional hyperbolic space, depending on wether the sign of its constant scalar curvature is positive, zero or negative, respectively. But $R_{g_c} = N(N-1) > 0$, and Proposition 2.5.1 allows us to deduce that $R_g > 0$. Therefore, $(\mathbb{S}^N, g)$ is isometric to the standard sphere. Then we conclude that g must be the standard metric on the sphere.

□

We can conclude that for the standard sphere $(\mathbb{S}^N, g_c)$, the minimizers of the Yamabe functional (2.3) are unique in the sense that all of them can be obtained from g_c through conformal diffeomorphisms, as we had described in the previous section.

2.3.3 Sobolev inequality on the sphere

We now turn to see the close relationship between the Yamabe's constant and the Sobolev inequality (see Theorem 1.4.1).

Theorem 2.3.2. ***(Sharp Sobolev inequality on the sphere)*** *Let I be the functional* (2.3) *defined on $(\mathbb{S}^N, g_c)$, and u a solution of* (2.2)*. The $N-$dimensional Sobolev constant is $\zeta_N = \frac{C_N}{\Lambda}$ where*

$$\Lambda = \lambda\left(\mathbb{S}^N\right) = I[u] = N(N-1)Vol(\mathbb{S}^N)^{\frac{2}{N}}.$$

Furthermore, the sharp form of the Sobolev inequality on $\mathbb{R}^N$ is

$$\|u\|^2_{L^{2^*}(\mathbb{R}^N)} \leq \frac{C_N}{\Lambda}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}.$$

Equality is attained only by constant multiples and translates of u_μ defined in (2.10)*.*

Proof. We follow [16]. By means of the stereographic projection, the Yamabe problem on $\mathbb{S}^N$ becomes a problem on $\mathbb{R}^N$. Indeed, if we consider $u \in C^\infty(\mathbb{S}^N)$ and denoting $w = u_1\rho^* u \in C^\infty(\mathbb{R}^N)$ the weighted push-forward function, with u_1 the conformal factor (2.8), then by (2.11) we get, for $\mu = 1$,

$$\rho^* g_c = 4u_1^{\frac{4}{N-2}} g_0.$$

By the definition of u_1,

$$\rho^*(u^{\frac{4}{N-2}} g_c) = 4w^{\frac{4}{N-2}} g_0.$$

By the conformal invariance of I and knowing that $R_{g_0} = 0$, we get

$$I[u] = I[w] = \frac{\int_{\mathbb{R}^N}\left(C_N|\nabla w|^2 + R_{g_0}w^2\right)dV_{g_0}}{\left[\int_{\mathbb{R}^N}|w|^{2^*}dV_{g_0}\right]^{\frac{2}{2^*}}} = \frac{\int_{\mathbb{R}^N} C_N|\nabla w|^2\,dV_{g_0}}{\left[\int_{\mathbb{R}^N}|w|^{2^*}dV_{g_0}\right]^{\frac{2}{2^*}}},$$

so we have

$$\lambda(\mathbb{S}^N) = \inf_{u\in C^\infty(\mathbb{S}^N)}\{I[u]\} = \inf_{w\in C^\infty(\mathbb{R}^N)}\{I[w]\} = \inf_{w\in C^\infty(\mathbb{R}^N)}\left\{\frac{\int_{\mathbb{R}^N} C_N|\nabla w|^2\,dV_{g_0}}{\left[\int_{\mathbb{R}^N}|w|^{2^*}dV_{g_0}\right]^{\frac{2}{2^*}}}\right\}.$$

Before we continue, let $\varepsilon > 0$ and denoting $B_\varepsilon = B(0,\varepsilon) \subset \mathbb{R}^N$, we define the smooth cutoff function φ_ε by:

$$\begin{cases} 0 \leq \varphi_\varepsilon \leq 1 \\ \operatorname{supp}(\varphi_\varepsilon) \subset B_{2\varepsilon} \\ \varphi_\varepsilon \equiv 1, \ on\ B_\varepsilon. \end{cases}$$

Now we can define $w^{\varepsilon} = \varphi_{\varepsilon} w$ and we can obtain an approximation of w such that when $\varepsilon \to 0$, we have

$$\lambda(\mathbb{S}^N) = \inf_{w \in C_c^{\infty}(\mathbb{R}^N)} \left\{ \frac{C_N \|\nabla w\|^2_{L^2(\mathbb{R}^N)}}{\|w\|^2_{L^{2^*}(\mathbb{R}^N)}} \right\}.$$

Then, we can apply the Sobolev inequality of Theorem 1.4.1 and conclude that there exists the $N-$ dimensional Sobolev constant $\zeta_N > 0$ such that

$$\lambda(\mathbb{S}^N) = \inf_{w \in C_c^{\infty}(\mathbb{R}^N)} \left\{ \frac{C_N \|\nabla w\|^2_{L^{2^*}(\mathbb{R}^N)}}{\zeta_N \|w\|^2_{L^{2^*}}(\mathbb{R}^N)} \right\} \geq \frac{C_N}{\zeta_N} > 0. \tag{2.12}$$

And we have that $\lambda(\mathbb{S}^N) > 0$. From (2.12), we can conclude that identifying $\lambda(\mathbb{S}^N)$ and the related extremal functions is equivalent to identifying the N-dimensional Sobolev constant and extremal functions for Sobolev inequality.

In [20] (appendix to chapter V, p. 224-230) it is proved the computation of $\lambda(\mathbb{S}^N) = N(N-1)Vol(\mathbb{S}^N)^{\frac{N}{2}}$ and that the best constant in the Sobolev inequality in $\mathbb{R}^N$ is achieved by constant multiples and translations of the family of radial functions (2.10). □

In chapter 3, we will see the moving plane technique in order to obtain radially symmetric solutions in $\mathbb{R}^N$. The difference between this technique and the Obata's theorem is that the first one is on $\mathbb{R}^N$ and the second one is on $\mathbb{S}^N$ which is compact. For this reason, in the moving plane method we will study the asymptotics at ∞.

2.3.4 Upper bound for the Yamabe invariant

The objective of this section is to show that we can compare the invariant of any compact smooth Riemannian manifold with the invariant of the sphere. We will see at the end of this section the proof of this result:

Proposition 2.3.1. Let M be a compact smooth Riemannian manifold of dimension $n \geq 3$, then $\lambda(M) \leq \lambda(\mathbb{S}^N)$.

Lemma 2.3.1. *Let u_μ be the bubble in (2.10) and $u_\mu(r) = \left(\frac{\mu}{r^2+\mu^2}\right)^{\frac{N-2}{2}}$, $r \in \mathbb{R}$, $\mu > 0$. Let $k > -N$, and $\varepsilon > 0$ fixed, then when $\mu \to 0$ we have that,*

$$J[\mu] = \int_0^{\varepsilon} u_\mu^2(r) r^{k+N-1} \, dr$$

is bounded above and below by:

- *positive multiples of μ^{k+2}, if $N > k+4$.*
- $\mu^{k+1}\log(\frac{1}{\mu})$, *if $N = k+4$,*
- μ^{N-2}, *if $N < k+4$.*

Proof. To begin we do a change of variable $y = \frac{r}{\mu}$:

$$\begin{aligned} J[\mu] =& \mu^{k+2}\int_0^{\frac{\varepsilon}{\mu}} y^{k+N-1}(y^2+1)^{2-N}\,dy \le \mu^{k+2}\left(C + 2^{N-2}\int_1^{\frac{\varepsilon}{\mu}} y^{k+N-1}(y^2)^{2-N}\,dy\right) \\ =& \mu^{k+2}\left(C + \int_1^{\frac{\varepsilon}{\mu}} y^{k+3-N}\,dy\right). \end{aligned}$$

where we have applied that for $y \ge 1$, we have $y^2+1 \le 2y^2$.

- If $N > k+4$ we have that this integral is bounded, as $\mu \to 0$.
- If $N < k+4$ then

  $$\begin{aligned} J[\mu] \le& \mu^{k+2}\left(C + 2^{2-N}\int_1^{\frac{\varepsilon}{\mu}} y^{k+3-N}\,dy\right) = \mu^{k+2}\left(C + \left[\frac{1}{k+4-N}y^{k+4-N}\right]_1^{\frac{\varepsilon}{\mu}}\right) \\ =& \mu^{k+2}\left(C + \frac{1}{k+4-N}\left[\left(\frac{\varepsilon}{\mu}\right)^{k+4-N} - 1\right]\right) = \mu^{N-2}\left(C'\mu^{k+4-N} + \frac{\varepsilon^{k+4-N}}{k+4-N}\right). \end{aligned}$$

  So this is bounded by μ^{N-2}, as $\mu \to 0$.
- If $N = k+4$, then

  $$J[\mu] \le \mu^{k+2}\left(C + 2^{2-N}\int_1^{\frac{\varepsilon}{\mu}} \frac{1}{y}\,dy\right) = \mu^{k+2}\left(C + 2^{2-N}\left[\log(y)\right]_1^{\frac{\varepsilon}{\mu}}\right) = \mu^{k+2}\left(C + 2^{2-N}\log\left(\frac{\varepsilon}{\mu}\right)\right).$$

  So this is bounded by $\mu^{k+2}\log\left(\frac{1}{\mu}\right) = \mu^{N-2}\log\left(\frac{1}{\mu}\right)$.

□

Proof of Proposition 2.3.1

Proof. We proceed as in [16]. By Theorem 2.3.2 the functions u_μ satisfy the equality

$$\Lambda\,\|u_\mu\|^2_{L^{2^*}(\mathbb{R}^N)} = C_N\,\|\nabla u_\mu\|^2_{L^2(\mathbb{R}^N)}, \tag{2.13}$$

where $u_\mu(r) = \left(\frac{\mu}{r^2+\mu^2}\right)^{\frac{N-2}{2}}$, and, $\partial_r u_\mu = \partial_r u_\mu(r)\,|_{r=|x|}$. Let now fix $\varepsilon > 0$ and denoting $B_\varepsilon = B(0,\varepsilon) \subset \mathbb{R}^N$, we again define a smooth radial cutoff function $\varphi_\varepsilon(x)$, $\forall x \in \mathbb{R}^N$, by:

$$\begin{cases} 0 \le \varphi_\varepsilon(x) \le 1 \\ \text{supp}(\varphi_\varepsilon(x)) \subset B_{2\varepsilon} \\ \varphi_\varepsilon(x) \equiv 1, \ x \in B_\varepsilon \\ \varphi_\varepsilon(x) = f(r), \ f \in C^\infty(\mathbb{R}) \text{ non-negative}, \ |x| = r. \end{cases}$$

Now we can define $u = \varphi_\varepsilon u_\mu$, which is a radial function and,

$$\begin{aligned}
C_N \|\nabla u\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} C_N |\nabla(\varphi_\varepsilon u_\mu)|^2\, dx \\
&= \int_{B_{2\varepsilon}} \left(C_N \varphi_\varepsilon^2 |\nabla u_\mu|^2 + 2C_N \varphi_\varepsilon u_\mu \langle \nabla\varphi_\varepsilon, \nabla u_\mu\rangle + u_\mu^2 |\nabla\varphi_\varepsilon|^2 \right) dx \\
&= \int_{B_{2\varepsilon}} C_N \varphi_\varepsilon^2 |\nabla u_\mu|^2\, dx + \int_{B_{2\varepsilon}\setminus B_\varepsilon} 2C_N \varphi_\varepsilon u_\mu \langle \nabla\varphi_\varepsilon, \nabla u_\mu\rangle\, dx + \int_{B_{2\varepsilon}\setminus B_\varepsilon} u_\mu^2 |\nabla\varphi_\varepsilon|^2\, dx
\end{aligned}$$

Because we know that $\varphi_\varepsilon \equiv 1$ on B_ε, and then $\nabla\varphi_\varepsilon \equiv 0$ on B_ε, so, we have

$$C_N \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} C_N |\nabla u_\mu|^2\, dx + C\int_{B_{2\varepsilon}\setminus B_\varepsilon} \left(u_\mu |\nabla u_\mu| + u_\mu^2 \right) dx.$$

From where we obtain the following:

$$C_N \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} C_N |\partial_r u_\mu|^2\, dx + C\int_{B_{2\varepsilon}\setminus B_\varepsilon} \left(u_\mu |\partial_r u_\mu| + u_\mu^2 \right) dx. \tag{2.14}$$

Then:

$$\partial_r u_\mu(r) = \frac{N-2}{2}\left(\frac{-2r\mu}{(r^2+\mu^2)^2}\right)\left(\frac{\mu}{r^2+\mu^2}\right)^{\frac{N-2}{2}-1} = (2-N)r\mu^{-1}\left(\frac{\mu}{r^2+\mu^2}\right)^{\frac{N}{2}}.$$

So we get $|\partial_r u_\mu(r)| \le (N-2) r^{1-N} \mu^{\frac{N-2}{2}}$, and we also know that

$$u_\mu \le r^{2-N} \mu^{\frac{N-2}{2}}. \tag{2.15}$$

Then returning to the inequality (2.14), there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
C_N \|\nabla u\|^2_{L^2(\mathbb{R}^N)} &\le \int_{\mathbb{R}^N} C_N |\partial_r u_\mu|^2\, dx + C\int_{B_{2\varepsilon}\setminus B_\varepsilon} \mu^{N-2}\left((N-2) r^{3-2N} + r^{4-2N} \right) dx \\
&= \int_{\mathbb{R}^N} C_N |\partial_r u_\mu|^2\, dx + C_1 \mu^{N-2} \underset{(2.13)}{=} \Lambda \left(\int_{\mathbb{R}^N} |u_\mu|^{2^*}\, dx \right)^{\frac{2}{2^*}} + C_1 \mu^{N-2} \\
&= \Lambda \left(\int_{\mathbb{R}^N\setminus B_\varepsilon} |u_\mu|^{2^*}\, dx + \int_{B_\varepsilon} |u_\mu|^{2^*}\, dx \right)^{\frac{2}{2^*}} + C_1 \mu^{N-2}.
\end{aligned}$$

By (2.15) and as $\mu \to 0$,

$$\begin{aligned}
C_N \|\nabla u\|^2_{L^2(\mathbb{R}^N)} &\le \Lambda \left(\int_{\mathbb{R}^N\setminus B_\varepsilon} \left| r^{2-N} \mu^{\frac{N-2}{2}} \right|^{\frac{2N}{N-2}} dx + \int_{B_{2\varepsilon}} |u_\mu|^{2^*}\, dx \right)^{\frac{2}{2^*}} + C_1 \mu^{N-2} \\
&= \Lambda \left(\int_{\mathbb{R}^N\setminus B_\varepsilon} \mu^N \left| r^{-2N} \right| dx + \int_{B_{2\varepsilon}} |u|^{2^*}\, dx \right)^{\frac{2}{2^*}} + C_1 \mu^{N-2} \\
&= \Lambda \left(\int_{B_{2\varepsilon}} |u|^{2^*}\, dx \right)^{\frac{2}{2^*}} + C_2 \mu^N + C_1 \mu^{N-2} \le \Lambda \|u\|^2_{L^{2^*}(\mathbb{R}^N)} + C\mu^{N-2}.
\end{aligned}$$

Then we have

$$C_N \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \le \Lambda \|u\|^2_{L^{2^*}(\mathbb{R}^N)} + C\mu^{N-2}. \tag{2.16}$$

Now, on a compact manifold M, let $u = \varphi_\varepsilon u_\mu$ in normal coordinates $\{x_i\}$ in a neighborhood of $P \in M$, extended by zero to a smooth function on M. Then we know that $dV_g = (1+O(r^N))\,dx$ in normal coordinates (Theorem 1.2.1), and the above estimate can be corrected as follows:

$$\int_M \left(C_N|\nabla u|^2 + R_g u^2\right) dV_g \le \int_{B_{2\varepsilon}} \left(C_N|\nabla u|^2 + R_g u^2\right)\left(1 + Cr^N\right) dx$$
$$\le (1+C\varepsilon)\left(\Lambda \|u\|^2_{L^{2^*}(\mathbb{R}^N)} + C'\mu^{N-2} + C''\int_0^{2\varepsilon} u_\mu^2(r) r^{N-1}\,dr\right),$$

with $C, C', C'' > 0$ certain constants.

Finally, Lemma 2.3.1 allows to prove that the last term is $O(\mu)$. Therefore, we can select ε and μ small enough to write

$$I[u] \le (1+C\varepsilon)\left(\Lambda + C\mu\right),$$

which prove that

$$\lambda(M) \le \Lambda.$$

□

2.4 Existence of solutions

Notwithstanding, there remains a problem. Due to the fact that precisely $2^* = \frac{2N}{N-2}$ is the critical exponent for which the inclusion $H^1(M) \subset L^{2^*}(M)$ is not compact (see Theorem 1.4.4) it is not possible to straightly prove that a minimizing sequence of $I[u]$ in (2.3) has a subsequence converging to an actual extremal function. Yamabe [26] realized this and his approach to overcome the issue was to address a collection of perturbed problems in which this difficulty disappears. We follow [16] throughout this section. Based on the previous sections, we are now in a position to prove that the Yamabe problem can be solved on any compact smooth Riemannian manifold M provided that $\lambda(M) < \lambda(\mathbb{S}^N)$.

Definition 2.4.1. *Let define the subcritical Yamabe energy functional as*

(2.17) $$I^s[u] = \frac{\int_M \left(C_N|\nabla u|^2 + R_g u^2\right) dV_g}{\left[\int_M |u|^s\,dV_g\right]^{\frac{2}{s}}},$$

where $u \in H^1(M)$, and, $2 \le s \le 2^*$. Also, we define the subcritical Yamabe invariant as

(2.18) $$\lambda_s = \inf_{u \in C^\infty(M),\, u>0} \left\{I^s[u]\right\}.$$

2.4.1 Subcritical equation

If u is such that $\|u\|_{L^s(M)} = 1$ and it is a critical point of the functional (2.4.1), it satisfies the subcritical equation:

$$-C_N\Delta u + R_g u = \lambda_s u^{s-1}. \tag{2.19}$$

Let us see the next theorem about regularity in the subcritical case:

Theorem 2.4.1. *Let $u \in H^1(M)$ be a non-negative weak solution of (2.19) with $2 \le s \le 2^*$ and $K > 0$ a constant, such that $|\lambda_s| \le K$. If there exists $r > \frac{N(s-2)}{2}$, such that $u \in L^r(M)$, then we have one of the following cases:*

- $u \equiv 0$.
- $u > 0$, *smooth and* $\|u\|_{C^{2,\alpha}(M)} \le C$, *where* $C = C\left(M, g, r, \|u\|_{L^r(M)}\right)$ *and* $0 < \alpha < 1$.

Proof. Let $u \in L^r(M)$, with $r > \frac{N(s-2)}{2}$ from (2.19), then $-C_N\Delta u = \lambda_s u^{s-1} - R_g u \in L^q(M)$, with $q = \frac{r}{(s-1)}$. Applying Theorem 1.6.2, we get $u \in W^{2,q}(M)$. By the Sobolev embedding Theorem 1.4.3, we have that $W^{2,q}(M)$ is continuously embedded in $L^{r'}(M)$, where

$$\frac{1}{r'} \ge \frac{1}{q} - \frac{2}{N} = \frac{Ns - N - 2r}{Nr}.$$

Thus, taking $r' = \frac{Nr}{Ns-N-2r}$, then by hypothesis $r' > r$, and $u \in L^{r'}(M)$. Iterating this argument allows to conclude that $u \in W^{2,q}(M)$, $\forall q \ge 1$.

Now, for $0 < \alpha < 1$, there exists $q \ge 1$ such that $\frac{2-\alpha}{N} \ge \frac{1}{q}$, so by the Hölder continuity case of Theorem 1.4.3, we get that $W^{2,q}(M)$ is continuously embedded on $C^\alpha(M)$. Thus $u \in C^\alpha(M)$, and then $u^{s-1} \in C^\alpha(M)$. Applying Theorem 1.6.2, then $u \in C^{2,\alpha}(M)$. By the maximum principle (Theorem 1.6.4), we have two cases:

- If $u = 0$ at some point, then $u \equiv 0$.
- If $u \neq 0$, we have $-C_N\Delta u + R_g u - \lambda_s u^{s-1} > 0$, then

  $$-C_N\Delta u + R_g u - \lambda_s u^{s-1} = \left(-C_N\Delta + R_g - \lambda_s u^{s-2}\right)u > (-C_N\Delta + K)u > 0.$$

  where K is a constant such that $K \ge \sup\left\{R_g - \lambda_s u^{s-2}\right\}$. So we conclude $u > 0$.

  Finally, let $u \neq 0$, and, $u \in C^{2,\alpha}(M)$, from (2.19), then $u^{s-1} \in C^{2,\alpha}(M)$. Applying Theorem 1.6.2, we obtain $u \in C^{4,\alpha}(M)$. Repeating this process successively, we obtain $u \in C^\infty(M)$.

□

We will see in the next proposition the existence of a minimizer sequence of the subcritical Yamabe functional and that the equation (2.19) always has a positive smooth and minimizing solution for $s < 2^*$.

Proposition 2.4.1. For $2 \leq s < 2^*$, there exists u_s a smooth positive solution to (2.19), such that $I^s[u_s] = \lambda_s$, and, $\|u_s\|_{L^s(M)} = 1$.

Proof. Similarly to Section 2.2, it can be proved that there exists $\{u_i\}_{i\in\mathbb{N}} \subset C^\infty(M)$ be a minimizing subsequence of the functional I^s, for $2 \leq s < 2^*$ such that $\|u_i\|_{L^s(M)} = 1$. If we take $|u_i|$ instead of u_i, then we have $I^s[|u_i|] = I^s[u_i]$, so we may suppose that $u_i \geq 0$.

As $\{u_i\}_{i\in\mathbb{N}}$ is a minimizing sequence of I^s, then $I^s[u_i] \longrightarrow \lambda_s(M)$. Knowing that $\|u_i\|_{L^s(M)} = 1$, and by Hölder's inequality:

$$\begin{aligned}\|u_i\|^2_{H^1(M)} = \int_M \left(|\nabla u_i|^2 + u_i^2\right) dV_g = \frac{I^s[u_i]}{C_N} - \int_M \left(1 - \frac{R_g}{C_N}\right) u_i^2\, dV_g \\ \leq \frac{I^s[u_i]}{C_N} + C\|u_i\|^2_{L^{2^*}(M)} < \infty.\end{aligned}$$

Thus, $\{u_i\}_{i\in\mathbb{N}}$ is bounded in $H^1(M)$ and there is a subsequence, that we are going to denote $\{u_i\}_{i\in\mathbb{N}}$ too, converging weakly to a function $u_s \in H^1(M)$. As $2 \leq s < 2^*$, applying Theorem 1.4.4, we have that $H^1(M)$ is compactly embedded in $L^s(M)$, then $u_i \underset{i\to\infty}{\longrightarrow} u_s$ weakly in $H^1(M)$, and, $u_i \underset{i\to\infty}{\longrightarrow} u_s$ strongly in $L^s(M)$, for some $u_s \in H^1(M)$, satisfying $\|u_s\|_{L^s} = 1$. By weak convergence in $H^1(M)$ and by Schwarz's inequality, we have

$$\int_M |\nabla u_s|^2\, dV_g = \lim_{i\to\infty} \int_M \langle \nabla u_i, \nabla u_s \rangle\, dV_g \leq \limsup_{i\to\infty} \left(\int_M |\nabla u_i|^2\, dV_g\right)^{\frac{1}{2}} \left(\int_M |\nabla u_s|^2\, dV_g\right)^{\frac{1}{2}}.$$

From where we get $I^s[u_s] \leq \lim_{i\to\infty} I^s[u_i] = \lambda_s$. However, by Definition 2.18, we deduce that $I^s[u_s] = \lambda_s$, then u_s is a weak solution of (2.19). By Theorem 2.4.1, as $u_s \in H^1(M)$, we have that $u \in C^\infty(M)$, and it is positive. □

2.4.2 Limit when s goes to 2*

After we have seen the subcritical case, it is natural to ask about the critical case, that is to say, what happens when $s \to 2^*$?

As we pointed before, the first issue in the critical case is that with the exponent 2^*, we have a minimizing sequence $\{u_i\}_{i\in\mathbb{N}}$, such that $u_i \underset{i\to\infty}{\longrightarrow} u \in H^1(M)$, but we do not have the compactness of $H^1(M)$ in $L^{2^*}(M)$, so we can not prove that if $\|u_i\|_{L^{2^*}} = 1$, then $\|u\|_{L^{2^*}} = 1$.

Yamabe claimed that the collection of solutions $\{u_s\}_{s\in[2,2^*)}$, verify that $\|u_s\|_{L^{2^*}}$, is uniformly bounded as $s \to 2^*$ in general. However, it is false and this was his gap in the proof. The main result in this section is the following theorem in which we have a solution for the Yamabe problem just supposing $\lambda(M) < \lambda(\mathbb{S}^N)$:

Theorem 2.4.2. *Let $\lambda(M) < \lambda(\mathbb{S}^N)$, and $\{u_s\}$ be the collection of function given by Proposition 2.4.1. As $s \to 2^*$, there exists a subsequence which converges uniformly to a positive function $u \in C^\infty(M)$, such that*

$$I[u] = \lambda(M), \quad \textit{and}, \quad -C_N\Delta u + R_g u = \lambda(M)u^{2^*-1}.$$

As a consequence, the metric $\tilde{g} = u^{\frac{4}{N-2}}g$ has constant scalar curvature $\lambda(M)$.

Before passing to the proof of this theorem, we study the behaviour of λ_s in the following lemma. For this, we fix a metric g such that $Vol(M) = 1$.

Lemma 2.4.1. *Let (M^N, g) be a smooth compact Riemannian manifold such that $\int_M dV_g = 1$. Then the function $s \longmapsto |\lambda_s|$ is non-increasing for $s \in [2, 2^*]$. Moreover, if $\lambda(M) \geq 0$, then λ_s is continuous from the left.*

Proof. Firstly, let see that $s \longmapsto |\lambda_s|$ is non-increasing. For that, note that $\forall s, s' \in [2, 2^*]$ and $u \in C^\infty(M)$, such that $u \neq 0$, then

$$I^{s'}[u] = \frac{\|u\|^2_{L^s(M)}}{\|u\|^2_{L^{s'}(M)}} I^s[u]. \tag{2.20}$$

By hypothesis, as $\int_M dV_g = 1$, then if $s \leq s'$, by Hölder's inequality we have $\|u\|_{L^s(M)} \leq \|u\|_{L^{s'}(M)}$, and by (2.4.1) and (2.20), $|\lambda_{s'}| \leq |\lambda_s|$. So $s \longmapsto |\lambda_s|$ is non-increasing.

Furthermore, if there exists some s such that $\lambda_s < 0$, then there is a $u \in C^\infty(M)$, which verifies $I^s[u] < 0$. By (2.20), we have $I^{s'}[u] < 0$, $\forall s' \in [2, 2^*]$. Then we have $\lambda_s < 0$, $\forall s \in [2, 2^*]$.

To finish, supposing $\lambda(M) \geq 0$, by the previous argument we obtain $\lambda_s \geq 0$, $\forall s \in [2, 2^*]$. Let us choose $s \in [2, 2^*]$, then given $\varepsilon > 0$, there exists $u \in C^\infty(M)$, such that by the definition of the subcritical Yamabe functional (2.17),

$$I^s[u] < \lambda_s + \varepsilon. \tag{2.21}$$

As $s \longmapsto \|u\|_{L^q(M)}$ is continuous, then by (2.18) and (2.20),

$$\lambda_{s'} \leq I^{s'}[u] = I^{s'}[u] - I^s[u] + I^s[u] \leq \frac{I^s[u]}{\|u\|^2_{L^s(M)}} \left(\|u\|^2_{L^s(M)} - \|u\|^2_{L^{s'}(M)} \right) + I^s[u] \leq \lambda_s + 2\varepsilon,$$

for $\forall s' \leq s$ close enough. As we have seen that $|\lambda_s|$ is non-increasing, we conclude that it is continuous from the left. □

We turn now to see the following proposition due to Aubin [2] and Trudinger [23], which shows that the hypothesis $\lambda(M) < \lambda(\mathbb{S}^N)$ is enough to solve the problem.

Proposition 2.4.2. Let (M^N, g) be a smooth compact Riemannian manifold with $\int_M dV_g = 1$ and $\lambda(M) < \lambda(\mathbb{S}^N)$, and $\{u_s\}_{s \in [2, 2^*)}$ the collection of functions given by Proposition 2.4.1. Then there exist constants $s_0 < 2^*$, $r > 2^*$, and, $C > 0$, such that

$$\|u_s\|_{L^r(M)} \leq C, \quad \forall s \geq s_0.$$

Proof. Let $\delta > 0$, if we multiply (2.19) by $u_s^{1+2\delta}$:

$$-C_N \Delta u_s \cdot u_s^{1+2\delta} + R_g u_s^{2+2\delta} = \lambda_s u_s^{s+2\delta}.$$

Integrating over M:

$$-C_N \int_M \Delta u_s \cdot u_s^{1+2\delta} \, dV_g + \int_M R_g u_s^{2+2\delta} \, dV_g = \lambda_s \int_M u_s^{s+2\delta} \, dV_g.$$

Then:

$$C_N \int_M \left\langle \nabla u_s, (1+2\delta) u_s^{2\delta} \nabla u_s \right\rangle dV_g + \int_M R_g u_s^{2+2\delta} \, dV_g = \lambda_s \int_M u_s^{s+2\delta} \, dV_g.$$

And setting $w = u_s^{1+\delta}$:

$$C_N \frac{(1+2\delta)}{(1+\delta)^2} \int_M |\nabla w|^2 \, dV_g + \int_M R_g w^2 \, dV_g = \lambda_s \int_M u_s^{s-2} w^2 \, dV_g,$$

which yields

$$C_N \int_M |\nabla w|^2 \, dV_g = \frac{(1+\delta)^2}{(1+2\delta)} \left(\lambda_s \int_M u_s^{s-2} w^2 \, dV_g - \int_M R_g w^2 \, dV_g \right). \tag{2.22}$$

On the other hand, by the Sharp Sobolev embedding (Theorem 1.4.2), for $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\|w\|^2_{L^{2^*}(M)} \le (1+\varepsilon)\sigma_N \int_M |\nabla w|^2 \, dV_g + C_\varepsilon \int_M w^2 \, dV_g,$$

where σ_N is the N-dimensional Sobolev constant. Then, for $\Lambda = \lambda(\mathbb{S}^N)$, from Theorem 2.3.2 follows:

$$\|w\|^2_{L^{2^*}(M)} \le (1+\varepsilon)\frac{C_N}{\Lambda} \int_M |\nabla w|^2 \, dV_g + C_\varepsilon \int_M w^2 \, dV_g.$$

Applying (2.22) and Hölder's inequality:

$$\begin{aligned}\|w\|^2_{L^{2^*}(M)} &\le (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)} \frac{\lambda_s}{\Lambda} \int_M u_s^{s-2} w^2 \, dV_g + C'_{\varepsilon,\delta} \int_M w^2 \, dV_g \\ &\le (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)} \frac{\lambda_s}{\Lambda} \|u_s\|^{s-2}_{L^{\frac{(s-2)N}{2}}(M)} \|w\|^2_{L^{2^*}(M)} + C'_{\varepsilon,\delta} \|w\|^2_{L^2(M)}.\end{aligned}$$

As $\frac{(s-2)N}{2} < s$, we have $\|u_s\|^{s-2}_{L^{\frac{(s-2)N}{2}}(M)} \le \|u_s\|^{s-2}_{L^s(M)} = 1$, so,

$$\|w\|^2_{L^{2^*}(M)} \le (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)} \frac{\lambda_s}{\Lambda} \|w\|^2_{L^{2^*}(M)} + C_{\varepsilon,\delta} \|w\|^2_{L^2(M)}.$$

- Now, on the one hand, if we suppose $0 \le \lambda(M) < \Lambda$, for some $s_0 < 2^*$, we will have by Lemma 2.4.1, $\frac{\lambda_s}{\Lambda} \le \frac{\lambda_{s_0}}{\Lambda} < 1$, for $s_0 \le s$. On the other hand, we can choose $\delta, \varepsilon > 0$ sufficiently small to obtain $(1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)}\frac{\lambda_s}{\Lambda} < 1$, then there is a constant $C > 0$ such that:

$$\|w\|^2_{L^{2^*}(M)} \le C \|w\|^2_{L^2(M)}. \tag{2.23}$$

- Supposing $\lambda(M) \le 0$, we obtain obviously the same result.

Applying again Hölder's inequality on (2.23), we obtain

$$\|w\|^2_{L^2(M)} = \|u_s\|^{2(1+\delta)}_{L^{2(1+\delta)}(M)} \le C \|u_s\|^{2(1+\delta)}_{L^s(M)} = 1.$$

Then we conclude that from (2.23), $\|w\|_{L^{2^*}(M)} = \|u_s\|^{1+\delta}_{L^{2^*(1+\delta)}}$ is bounded independently of s. □

Proof of Theorem 2.4.2

Proof. By Proposition 2.4.1, we have that $\{u_s\}_{s\in[s_0,2^*)}$ is uniformly bounded in $L^r(M)$ for certain $2 < s_0 < 2^*$, $r > 2^*$. Then applying Theorem 1.6.2, as we have done on the proof of the Theorem 2.4.1, we obtain that $\{u_s\}_{s\in[s_0,2^*)}$ is uniformly bounded in $C^{2,\alpha}(M)$.

Now, by Arzela-Ascoli theorem, we know that there exists a subsequence which converges to $u \in C^2(M)$, such that u verifies

$$I[u] = \lambda, \quad \text{and,} \quad -C_N\Delta u + R_g u = \lambda u^{2^*-1},$$

where we define the limit $\lambda = \lim_{s\to 2^*} \lambda_s$. Furthermore, we have

- If $0 \le \lambda(M)$, by Lemma 2.4.1, λ_s is non-increasing, then $\lambda = \lambda(M)$.
- If $\lambda(M) < 0$, λ_s is increasing, so we have that $\lambda \le \lambda(M)$. However, by (2.18), as it is the infimum of the functional, then $\lambda = \lambda(M)$, too.

Finally, by Theorem 2.4.1, we have that $u \in C^\infty(M)$, and $u > 0$, because

$$\|u\|_{L^{2^*}} \ge \lim_{s\to 2^*} \|u_s\|_{L^s} = 1 > 0.$$

□

2.5 Uniqueness

To continue, we are going to study the uniqueness of the solutions of the semi-linear PDE (2.2).

Proposition 2.5.1. If we have two strictly positive solutions of (2.2), then the constant curvatures of both metrics have the same sign or both solutions are equal to zero.

Proof. Following [3], p. 171, we consider $g' = u^{\frac{4}{N-2}} g$, with $u > 0$, and $\tilde{g} = v^{\frac{4}{N-2}} g$, with $v > 0$, two solutions of the problem, with $R_{g'}$ and $R_{\tilde{g}}$ are their respective constant scalar curvatures.

Then we can compute g' in terms of the metric $\tilde{g}$. For that, we know that as $u, v > 0$ we can set $u = wv$, and we have

$$g' = u^{\frac{4}{N-2}} g = (wv)^{\frac{4}{N-2}} g = w^{\frac{4}{N-2}} \tilde{g}.$$

Then by (2.2):

$$-C_n \Delta u + R_g u = R_{g'} u^{\frac{N+2}{N-2}}.$$

Setting $\widetilde{\Delta} w = -\widetilde{g}^{ij} \widetilde{\nabla}_i \widetilde{\nabla}_j w$, idem as (2.2), we get

$$-C_n \widetilde{\Delta} w + R_{\widetilde{g}} w = R_{g'} w^{\frac{N+2}{N-2}}. \tag{2.24}$$

Now, integrating with respect to $\widetilde{g}$ and knowing that $\int \widetilde{\Delta} w \, dV_{\widetilde{g}} = 0$, we get

$$R_{\widetilde{g}} \int w \, dV_{\widetilde{g}} = R_{g'} \int w^{\frac{N+2}{N-2}} \, dV_{\widetilde{g}}.$$

So we obtain that $R_{\widetilde{g}}$ and $R_{g'}$ have the same sign or both are equal to zero. □

By the above proof, we can conclude:

- If the curvature is equal to zero, i.e. $R_{g'} = R_{\widetilde{g}} = 0$, then by (2.24), $\widetilde{\Delta} w = 0$, so we have $w = constant$ and both solutions are proportional. Then, in this case there is a unique solution up to rescaling.
- If the curvature is negative, i.e. $R_{g'} = R_{\widetilde{g}} < 0$, (2.24) has the unique solution $w \equiv 1$. Indeed, if w has a maximum in $p \in M$, then $\tilde{\Delta} w(p) \geq 0$, and from (2.24):

  $$C_N \tilde{\Delta} w(p) = R_{\tilde{g}} w(p) - R_{g'} w^{\frac{N+2}{N-2}} \geq 0.$$

  Then $w(p)^{\frac{4}{N-2}} \leq \frac{R_{g'}}{R_g} = 1$. So, $w(p) \leq 1$. In the same way, if w has a minimum in $q \in M$, then $\tilde{\Delta} w(q) \leq 0$, and analogously $w(q) \geq 1$. Therefore, $w \equiv 1$, and there is a unique solution.
- If the curvature is positive, i.e. $R_{g'} = R_{\widetilde{g}} > 0$, we can not prove that there is a unique solution.

In the next section, for the cases where g is scalar positive, we are going to see a classical example where we do not have a unique solution. Therefore, if the curvature is positive we can not guarantee uniqueness. In general, although we know that there are few examples where uniqueness holds, as we have seen in a certain sense for the sphere.

2.5.1 Example of non uniqueness

As we have just seen, when the scalar curvature is positive we do not have uniqueness generally. A classical example where there is more than one solution is due to Schöen [19], who found all solutions to the Yamabe problem in the Riemannian manifold $\mathbb{S}^1 \times \mathbb{S}^{N-1}$ with the product metric, being $\mathbb{S}^1 = \mathbb{S}^1(T)$ the 1-sphere of length $T > 0$ and $\mathbb{S}^{N-1}$ the $N-$sphere of radius 1, both considered with their standard metrics.

To address this example, we remember the relation between g_0 and g_1 of the Example 1.2.3 and for convenience we normalize first solutions of the equation (2.2) so that their scalar curvature is $R_{g'} = N(N-1)$, i.e, equal to the scalar curvature of $\mathbb{S}^N$, and we look for solutions of the form $g' = u^{\frac{4}{N-2}} g_1$ on $\mathbb{R} \times \mathbb{S}^{N-1}$ with respect to the product metric $g_1 = dt^2 + d\theta^2$, $(t,\theta) \in \mathbb{R} \times \mathbb{S}^{N-1}$, with scalar curvature $R_{g_1} = (N-1)(N-2)$. In short, (2.2) turns to

$$-\Delta_{g_1} u + \frac{(N-2)^2}{4} u = \frac{N(N-2)}{4} u^{\frac{N+2}{N-2}}. \tag{2.25}$$

When u just depends on t, writing $\Delta_{g_1} = \partial_{tt} + \Delta_\theta$:

$$u_{tt} - \frac{(N-2)^2}{4} u + \frac{N(N-2)}{4} u^{\frac{N+2}{N-2}} = 0, \quad u = u(t),\ t \in \mathbb{R}. \tag{2.26}$$

and we are interested in finding positive solutions of (2.26) defined in $\mathbb{R}$.

Explicit solutions

This ODE has two explicit non-zero solutions:

- One solution is the constant $u_0 = \left(\frac{N-2}{N}\right)^{\frac{N-2}{4}}$, where $u_0^{\frac{4}{N-2}} g_1$ is the multiple of g_1 with scalar curvature $N(N-1)$.

  Proof. This is an immediate calculation from (2.26). □

- The other is a solution of constant sectional curvature. Let us take the spherical metric g_c in $\mathbb{R}^N$ given by $g_c = 4\left(1+|x|^2\right)^{-2} g_0$ (see (2.9)). Writing g_c in the terms of g_1: $g_c = 4\left(\frac{|x|}{1+|x|^2}\right)^2 g = 4\left(|x| + |x|^{-1}\right)^{-2} g$. Then as $|x| = r = e^{-t}$, we have

$$\cosh^{-2}(t) = 4\left(e^t + e^{-t}\right)^{-2} = 4\left(|x| + |x|^{-1}\right)^{-2}.$$

  Thus,

$$g_c = \cosh^{-2}(t) g_1.$$

  From where we have the solution:

$$u_1(t) = \cosh^{-\frac{N-2}{2}}(t).$$

Proof. This is an immediate calculation.

□

System of first order

To continue, we are going to transform (2.26) into a first order system. For that, we take $v = u_t$ and let define

$$X(u,v) = \left(v, \frac{(N-2)^2}{4}u - \frac{N(N-2)}{4}u^{\frac{N+2}{N-2}}\right). \tag{2.27}$$

Then (2.26) become $(u,v)_t = X(u,v)$ a non-linear autonomous system that we are going to study.

Critical points

This system has two critical points: $X(u,v) = 0$ if and only if $v = 0$ and $u = 0$, or, $\frac{(N-2)^2}{4}u + \frac{N(N+2)}{4}u^{\frac{N+2}{N-2}} = 0$. That is, the critical points are $(0,0)$ and $(0,u_0)$.

Linearization of the equation

First, we calculate the Jacobian matrix and then we study each critical point. The Jacobian matrix denoted J is

$$J(u,v) = \begin{pmatrix} \partial_u \partial_t u \mid_{(u,v)} & \partial_v \partial_t u \mid_{(u,v)} \\ \partial_u \partial_t v \mid_{(u,v)} & \partial_v \partial_t u \mid_{(u,v)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{(N-2)^2}{4} - \frac{N(N+2)}{4}u^{\frac{4}{N-2}} & 0 \end{pmatrix}.$$

- For the critical point $(u_0,0)$:

$$J(u_0,0) = \begin{pmatrix} 0 & 1 \\ -(N-2) & 0 \end{pmatrix}.$$

Then, its eigenvalues are given by

$$\lambda^2 + N - 2 = 0.$$

Then we have, $\lambda = \pm i\sqrt{N-2}$. To conclude, as $N \geq 3$ and we have two pure complex eigenvalues, then the critical point $(u_0,0)$ is a (linear) center.

- For the critical point $(0,0)$:

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \frac{(N-2)^2}{4} & 0 \end{pmatrix}.$$

  Then, its eigenvalues are

$$\lambda^2 - \frac{(N-2)^2}{4} = 0.$$

  From where we have, $\lambda = \pm\frac{N-2}{2}$. Finally, as $N \geq 3$, $\lambda_1 > 0$, and, $\lambda_2 < 0$, we conclude that the critical point $(0,0)$ is a saddle point.

As the point $(u_0,0)$ is a center and the system is an ODE of first order we have the next equations for the orbits:

$$u(t) = A\sin\left(t\sqrt{N-2}\right) + B\cos\left(t\sqrt{N-2}\right),$$

where $A,B > 0$ are constants.

We also know that these orbits are constants. Let $u(t+T) = u(t)$ where $T = \frac{2\pi}{\sqrt{N-2}}$ is the fundamental period.

The orbits

Until now, we have transformed the equation of second order (2.26) into the first order system (2.27). The study of linearization yields a critical point which is a (linear) center. Now we study the phase diagram of the non-linear problem. However, in the non-linear case, we will show that there exists a Hamiltonian in order to see in the phase diagram that there are some orbits around this center. In this section we will study how to get the exact equations of these orbits.

To continue, in order to calculate the exact trajectories of the orbits we are going to compute the Hamiltonian. For this, we multiply (2.26) by u_t:

$$u_{tt}\cdot u_t - \frac{(N-2)^2}{4}u\cdot u_t + \frac{N(N-2)}{4}u^{\frac{N+2}{N-2}}\cdot u_t = 0.$$

And knowing that

- $u_{tt}\cdot u_t = \frac{1}{2}\left((u_t)^2\right)_t$.
- $u\cdot u_t = \frac{1}{2}\left(u^2\right)_t$.
- $u^{\frac{N+2}{N-2}}\cdot u_t = \frac{N-2}{2N}\left(u^{\frac{2N}{N-2}}\right)_t$,

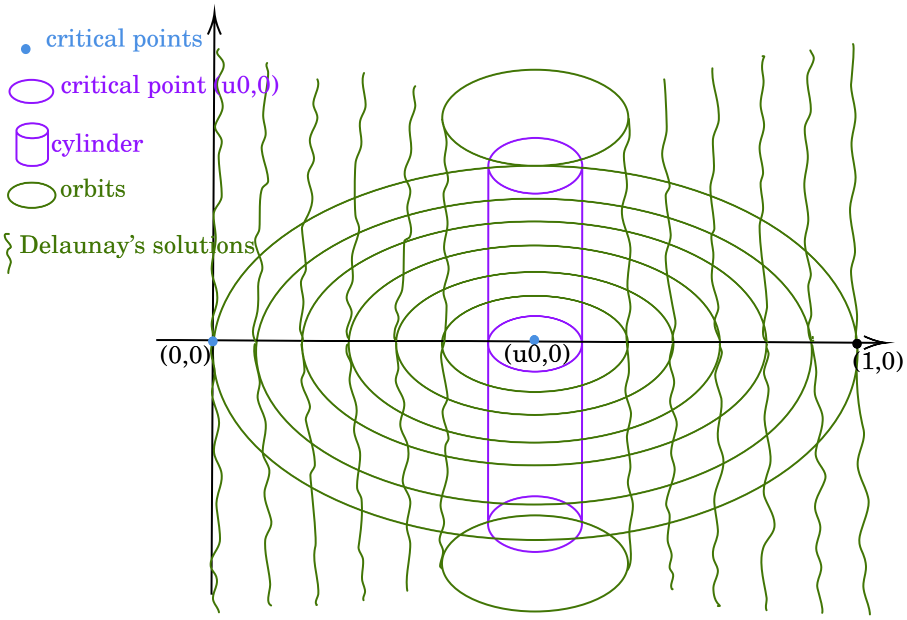

Figure 2.2: Diagram of phases

we have

$$\frac{1}{2}\left((u_t)^2\right)_t - \frac{(N-2)^2}{8}\left(u^2\right)_t + \frac{(N-2)^2}{8}\left(u^{\frac{2N}{N-2}}\right)_t = 0.$$

Integrating with respect to t we obtain

$$H = \frac{1}{2}(u_t)^2 - \frac{(N-2)^2}{8}u^2 + \frac{(N-2)^2}{8}u^{\frac{2N}{N-2}} = \pm C,$$

where $C > 0$ is a constant. Therefore,

$$\frac{1}{2}v^2 - \frac{(N-2)^2}{8}u^2 + \frac{(N-2)^2}{8}u^{\frac{2N}{N-2}} \pm C = 0.$$

Finally we get the exact trajectories of these orbits

$$v = \pm\sqrt{\frac{(N-2)^2}{4}u^2 - \frac{(N-2)^2}{4}u^{\frac{2N}{N-2}} \pm C}.$$

Remark 2.5.1. The intermediate solutions have geometric interpretation, they are small perturbations around the cylinder as we can see in Figure 2.2, and they are known as Delaunay's solutions.

The orbit corresponding to the solution $u_1(t)$ contains the point $(1,0)$, is symmetric under reflection in the u-axis, and when $t \to \infty$ it approaches $(0,0)$. Therefore, this orbit and $(0,0)$ bound a region Ω, which includes the center $(u_0,0)$.

Now taking $\alpha \in [u_0,1]$,we parametrize the orbits in Ω by $\gamma_\alpha(t)$, with $\gamma_\alpha(0) = (\alpha,0)$ where $\alpha \in [u_0,1]$. So we have $\gamma_{u_0}(t) \equiv (u_0,0)$, and $\gamma_1(t) = (u_1(t),(u_1(t))_t)$. For $\alpha \in (u_0,1)$, there exists a first positive time $\frac{1}{2}T(\alpha)$, at which γ_0 intersects the $u-$ axis. Denoting $\gamma_\alpha(t) = (u_\alpha(t), v_\alpha(t))$, thus $\gamma_\alpha(-t) = (u_\alpha(t), -v_\alpha(t))$. From where we get that $\gamma_\alpha(t)$ is periodic with period $T(\alpha)$. Moreover, we have $T(\alpha) \underset{\alpha\to 1}{\longrightarrow} \infty$, and, $T(\alpha) \underset{\alpha\to u_0}{\longrightarrow} \frac{2\pi}{\sqrt{N-2}}$, where, as we have seen before, this is the fundamental period.

To continue, we return to the manifold $\mathbb{S}^1(T)\times\mathbb{S}^{N-1}$, provided with the product metric g. Supposing that $T(\alpha)$ is increasing for $\alpha \in [u_0,1]$, then there exists a $T_0 = \frac{2\pi}{\sqrt{N-2}}$, such that the manifold $\mathbb{S}^1(T)\times\mathbb{S}^{N-1}$, for $T \le T_0$, has a unique solution of (2.26), which is a constant times g_0. So we conclude that for $T \in (T_0, 2T_0]$, (2.26) has two solutions: the constant solution and the solution with fundamental period T. However, these two solutions have different periods so they are inequivalent. Furthermore, the same reasoning leads to assure that for $T \in [2T_0, 3T_0]$, there are three inequivalent solutions, and so on. Then this example stress that we can not guarantee uniqueness of solutions in case of positive scalar curvature.

Note that $\mathbb{R}\times\mathbb{S}^{N-1}$ is conformally equivalent to $\mathbb{R}^N\setminus\{0\}$, and thus the solution of (2.25) in $\mathbb{R}\times\mathbb{S}^{N-1}$ is equivalent to the solution in $\mathbb{R}^N\setminus\{0\}$. Precisely, [11] proved that under suitable conditions, any solution of (2.25) is a radial function. Furthermore, $u(x) = \psi(r)$, with $r = |x|$, $\psi \in C^2(\mathbb{R}^N\setminus\{0\})$ is a solution of (2.25) if and only if:

$$-\psi_{rr} - \psi_r\left(\frac{N-1}{r}\right) + \frac{(N-2)^2}{4}\psi = \frac{N(N-2)}{4}\psi^{\frac{N+2}{N-2}}. \tag{2.28}$$

As pointed by [8], p. 272, equation (2.28) can be simplified by considering $u(t) = \left(r^{\frac{N-2}{2}}\psi(r)\right)|_{r=e^{-t}}$, in which case, such equation becomes (2.26).

Chapter 3

Moving plane

The moving plane technique dates back to applications to geometry by Aleksandrov in 1958 ([1]) and Serrin ([21]) in 1971. After that, this method found important applications in the theory of PDEs, beginning with the seminal papers by Gidas, Ni and Niremberg ([11]) followed by many researchers in the field ([6]), reveling itself as a fundamental tool in the proof of qualitative properties of solutions of certain PDEs, as monotonicity, radial symmetry or even Harnack type estimates. Do note that knowing that if a particular PDE has only a radially symmetric solutions it allows to reduce it to an ODE, simplifying the problem in some sense, as we did before. In this section we introduce this technique, and at the end, we show its relation with the Yamabe problem. Even more, we will see in the last chapter that it will play a main role in the study of parabolic PDEs, as it is the case of Yamabe flow.

3.1 Main idea

The method of moving plane essentially consists on comparing values of the solution to a PDE at two different points, one point is the reflection of the other over a hyperplane $x_1 = \lambda$, and after that, the plane is moved until it reaches a critical position, then the solution will be symmetric with respect to that plane.

From now on, as we are interested in radial symmetry, we select $\gamma = (1,0,\dots,0)$ and we will study symmetry with respect to x_1. After that, reordering variables leads to desired result. In this section we follow [11] and consider the following semi-linear elliptic

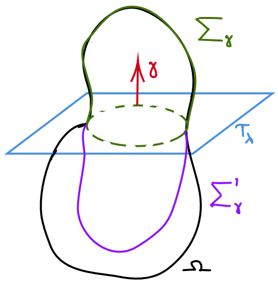

Figure 3.1: Moving plane method

equation:

$$\Delta u(x) + b_1(x) u_{x_1}(x) + f(u) = 0, \tag{3.1}$$

where $b_1 \in C(\overline{\Omega})$, $f \in C^1$, and, x_1 is the first coordinate of $x = (x_1, \ldots, x_N)$. Our objective will be to prove the symmetry of a solution $u \in C^2(\Omega)$ with respect to the hyperplane x_1, with Ω an arbitrary bounded domain with smooth boundary $\partial\Omega$, but first let see which is what we call radial symmetry.

Definition 3.1.1. *A measurable function u defined in $\mathbb{R}^N$ is called radially symmetric if $u(r) = \tilde{u}(r)$, for $r = |x|$.*

Following [11], we present the main idea of the moving plane method. Let $\Omega \subset \mathbb{R}^N$ be bounded with smooth boundary, $\gamma \in \mathbb{R}^N$ be a unit vector, and, $T_\lambda = \{x \in \mathbb{R}^N \mid \lambda = x \cdot \gamma\} \subset \mathbb{R}^N$ be an hyperplane orthogonal to γ, for $\lambda \in \mathbb{R}$. To start the method, let $\overline{\lambda} \in \mathbb{R}$ be large enough such that if $\lambda = \overline{\lambda}$, we have that the hyperplane $T_{\overline{\lambda}}$ is disjoint with $\overline{\Omega}$. Then if the plane moves continuously in the γ direction toward Ω while λ decreases, then T_λ will begin to cut off from Ω an open cap that we define as $\Sigma(\lambda) = \{x \in \Omega \mid \gamma \cdot x > \lambda\}$. On the other side, let denote $\Sigma'(\lambda)$ the reflection of $\Sigma(\lambda)$ with respect to the plane T_λ. It will begin to be inside Ω, since $\partial\Omega$ is smooth, until one of the following situations occurs:

- Let $P \notin T_\lambda$ and $\Sigma'(\lambda)$ is tangent to $\partial\Omega$ at P.
- Let $Q \in T_\lambda$ and Σ'_λ is orthogonal to $\partial\Omega$ at Q.

To continue, we suppose that T_λ reaches one of the above positions in $\lambda = \lambda_1$, then $\Sigma'(\lambda) \subset \Omega$, $\forall \lambda \geq \lambda_1$, and, by definition $\Sigma'(\lambda) \not\subseteq \Omega$, for $\lambda < \lambda_1$. For any unit vector γ, the goal is to prove the reflection symmetry of solutions to (3.1) with respect to the plane $\gamma \cdot x = 0$. In this step it will be crucial some form of the maximum principle.

We are going to use the following notation:

- The maximum of the first coordinate for $x \in \overline{\Omega}$ is $\lambda_0 = \max_{x \in \overline{\Omega}} x_1$.
- The maximal hyperplane

$$T_{\lambda_1} := \{x_1 = \lambda_1\}, \quad \text{with } \lambda_1 < \lambda_0.$$

- The maximal cap

$$\Sigma = \Sigma_\lambda = \Sigma(\lambda_1) = \{x \in \Omega \mid x_1 > \lambda_1\}.$$

- The reflection of this maximal cap on the plane T_{λ_1}: $\Sigma'(\lambda_1)$.

Definition 3.1.2. *We define $u^\lambda(x) = u(x^\lambda)$, with $x^\lambda = (2\lambda - x_1, x_2, \dots, x_N)$, to be the reflection of x with respect to the hyperplane $x_1 = \lambda$, for $\lambda_1 < \lambda < \lambda_0$.*

3.2 Symmetry on an arbitrary bounded domain

Next we see the following symmetry theorem which allows us to obtain the radial symmetry on any bounded domain with smooth boundary.

Theorem 3.2.1. *Let $u \in C^2(\overline{\Omega} \cap \Sigma)$ be a solution of (3.1) and such that*

$$\begin{cases} u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \cap \Sigma. \end{cases}$$

Let $b_1(x) \geq 0$, for $x \in \Sigma \cup \Sigma'$. Then $u_{x_1}(x) < 0$ in Σ. Furthermore, if $u_{x_1} = 0$ at some point in Ω on the plane T_{λ_1}, then u is symmetric on the plane T_{λ_1}, that is to say $\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup (T_{\lambda_1} \cap \Omega)$, and $b_1(x) = 0$.

To prove this theorem, we start the moving plane method in a point of the boundary and while we have

$$u_{x_1}(x) < 0, \quad \text{and} \quad u(x) < u^\lambda(x), \text{ for } x \in \Sigma(\lambda),$$

we can move the plane until we arrive to a critical value μ, which is the higher value that does not allow us to move the plane more. The main idea of the proof is that by contradiction, if $\mu > \lambda_1$, we do not have reached the critical value and therefore we can move the plane a little more.

First, we are going to prove the following lemma which shows that the derivative of u with respect to x_1 has sign, so we can say that it is a kind of Hopf's lemma:

Lemma 3.2.1. *Let $\nu(x)$ be the exterior unit vector of Ω, and, $\overline{x} \in \partial\Omega$ with $\nu_1(\overline{x}) > 0$. For some $\varepsilon > 0$, if $u \in C^2$ is a solution of (3.1) in $\overline{\Omega}_\varepsilon = \Omega \cap \{|x - \overline{x}| < \varepsilon\}$, and it verifies*

$$
\begin{cases} u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in S = \partial\Omega \cap \{|x - \overline{x}| < \varepsilon\}. \end{cases} \tag{3.2}
$$

Then there exists $\delta > 0$ such that $u_{x_1}(x) < 0$, for $x \in \Omega \cap \{|x - \overline{x}| < \delta\}$.

Proof. As $u > 0$ in Ω, and, $u = 0$ in $S = \partial\Omega \cap \{|x - \overline{x}| < \varepsilon\}$, necessarily u increases near the boundary, so $u_v \le 0$, on S. For such ε, as $v_1 > 0$ everywhere, then

$$u_{x_1} \le 0, \quad \text{on } S.$$

We are going to continue the proof by contradiction, so we suppose that the lemma is false, then there would be a sequence of points $\{x_j\}_{j \in \mathbb{N}}$, such that $x^j \to \overline{x}$, and, $u_{x_1}\left(x^j\right) \ge 0$. For j large, the interval in the x_1 direction, going from x^j to $\partial\Omega$, intersects S at a point that we are going to call $\tilde{x}$, where $u_{x_1}(\tilde{x}) \le 0$. Then we have in the direction of x_1, the interval going from x^j to $\overline{x}$ such that

$$u_{x_1}(x^j) \ge 0, \quad \text{and}, \quad u_{x_1}(\tilde{x}) \le 0.$$

By the mean value theorem, there exists a point $\hat{x}^j$ which is in the interval between x^j and $\tilde{x}$, such that $u_{x_1}(\hat{x}^j) = 0$. As $u \in C^2$, we have

$$0 = \lim_{j \to \infty} u_{x_1}(\hat{x}^j) = u_{x_1}\left(\lim_{j \to \infty} \hat{x}^j\right) = u_{x_1}(\overline{x}), \tag{3.3}$$

and we also have

$$u_{x_1 x_1}(\overline{x}) = \lim_{j \to \infty} \frac{u_{x_1}(\hat{x}^j) - u_{x_1}(\overline{x})}{\hat{x}^j - \overline{x}} = 0. \tag{3.4}$$

Suppose $f(0) \ge 0$, then u satisfies

$$\Delta u(x) + b_1(x) u_{x_1}(x) + f(u(x)) - f(0) \le 0, \quad x \in \Omega_\varepsilon. \tag{3.5}$$

By the mean value theorem, for some function $c(x) \ge 0$, we have

$$\Delta u(x) + b_1(x) u_{x_1}(x) + c(x) u(x) \le 0.$$

Taking the function $-u$, we have

$$\Delta u(x) - b_1(x)u_{x_1}(x) - c(x)u(x) \leq 0.$$

Applying the Hopf's Lemma 1.6.1, we find $(-u)_\nu(\overline{x}) > 0$, so $u_\nu(\overline{x}) < 0$. Moreover, as we have $u_\nu(\overline{x}) = u_{x_1}(\overline{x})\nu_1(\overline{x}) < 0$, then

$$u_{x_1}(\overline{x}) < 0,$$

which is a contradiction with (3.3). So we have to suppose $f(0) < 0$. Knowing that as $\overline{x} \in S$ by (3.2), $f(u(\overline{x})) = f(0)$, and, by (3.3), $u_{x_1}(\overline{x}) = 0$, we obtain that u is a solution of (3.5) at $\overline{x}$:

$$\Delta u(\overline{x}) = -f(0).$$

But then if we had $u_{x_ix_j}(\overline{x}) = -f(0)\nu_i(\overline{x})\nu_j(\overline{x})$, then it contradicts (3.4), since $u_{x_1x_1}(\overline{x}) = -f(0)\nu_1(\overline{x})^2 > 0$.

So we need to prove that $u_{x_ix_j}(\overline{x}) = -f(0)\nu_i(\overline{x})\nu_j(\overline{x})$. For that, we follow the same process as before but instead of taking the direction of x_1, we do it in the direction of x_i, then $u_{x_i}(\overline{x}) = 0$, for $1 \leq i \leq N$. Using this and the definition of the derivative with respect to x_j, we have, taking $\overline{x} = 0$ to simplify without loss of generality:

$$u_{x_ix_j}(\overline{x}) = \lim_{h\to 0} \frac{u_{x_i}(0,\dots,h,\dots,0) - u_{x_i}(0,\dots,0)}{h} = \lim_{h\to 0} \frac{u_{x_i}(0,\dots,h,\dots,0)}{h}.$$

By (3.2), $u = 0$ in $\partial\Omega$. As $\overline{x} \in \partial\Omega$, $\nabla u(\overline{x})$ has the direction of ν, then, if $e_j = (0,\dots 1,\dots 0)$:

$$\begin{aligned} u_{x_ix_j}(\overline{x}) =& \lim_{h\to 0} \frac{u_{x_i}(0,\dots,h,\dots,0)}{h} = \lim_{h\to 0} \frac{u_{x_i}(he_j)}{h} = \lim_{h\to 0} \frac{|\nabla u(he_j)| u_{x_i}(he_j)}{|\nabla u(he_j)| h} \\ =& \nu_i \lim_{h\to 0} \frac{|\nabla u(he_j)|}{h} = \nu_i \lim_{h\to 0} \frac{\sqrt{u_{x_1}^2(he_j) + \dots + u_{x_N}^2(he_j)}}{h} \end{aligned}$$

Since we have

$$\begin{aligned} \lim_{h\to 0} \frac{|\nabla u(he_j)|}{h} =& \lim_{h\to 0} \frac{\sqrt{u_{x_1}^2(he_j) + \dots + u_{x_N}^2(he_j)}}{h} \\ =& \lim_{h\to 0} \frac{u_{x_1x_j}(he_j)u_{x_1}(he_j) + \dots + u_{x_Nx_j}(he_j)u_{x_N}(he_j)}{\sqrt{u_{x_1}^2(he_j) + \dots + u_{x_N}^2(he_j)}} \\ =& \lim_{h\to 0} \frac{u_{x_1}(he_j)}{\sqrt{u_{x_1}^2(he_j) + \dots + u_{x_N}^2(he_j)}} u_{x_1x_j}(he_j) + \dots \\ & \dots + \frac{u_{x_N}(he_j)}{\sqrt{u_{x_1}^2(he_j) + \dots + u_{x_N}^2(he_j)}} u_{x_Nx_j}(he_j) \\ =& \sum_{k=1}^N \nu_k(\overline{x}) u_{x_kx_j}(\overline{x}). \end{aligned}$$

Using this we get

$$u_{x_ix_j}(\overline{x}) = \nu_i(\overline{x})\sum_{k=1}^{N} \nu_k(\overline{x})u_{x_kx_j}(\overline{x}).$$

Then $\frac{1}{\nu_i(\overline{x})}u_{x_ix_j}(\overline{x}) = \sum_{k=1}^{N} \nu_k(\overline{x})u_{x_kx_j}(\overline{x})$, and we have $\frac{1}{\nu_1(\overline{x})}u_{x_1x_j}(\overline{x}) = \cdots = \frac{1}{\nu_N(\overline{x})}u_{x_Nx_j}(\overline{x})$. Furthermore, $\forall i,j,k = 1,\dots,N$, it holds

$$u_{x_jx_j} = \frac{\nu_j}{\nu_k}u_{x_jx_k} = \frac{\nu_j}{\nu_k}\frac{\nu_j}{\nu_i}u_{x_ix_k} = \frac{\nu_j^2}{\nu_k\nu_i}u_{x_ix_k}, \text{ all evaluated at } \overline{x}.$$

Thus, fixed i,k,

$$\begin{aligned}\Delta u(\overline{x}) &= \sum_{j=1}^{N} u_{x_jx_j}(\overline{x}) = \sum_{j=1}^{N} \frac{\nu_j^2}{\nu_k\nu_i}u_{x_ix_k}(\overline{x}) = \left(\frac{\nu_1^2}{\nu_k\nu_i} + \cdots + \frac{\nu_N^2}{\nu_k\nu_i}\right)u_{x_ix_k}(\overline{x})\\ &= \frac{1}{\nu_k\nu_i}u_{x_ix_k}(\overline{x}).\end{aligned}$$

From this we have

$$u_{x_ix_k}(\overline{x}) = \nu_k\nu_i\Delta u(\overline{x}) = -f(0)\nu_k\nu_i.$$

Finally, we conclude that there exists $\delta > 0$, such that $u_{x_1} < 0$ in $\Omega \cap \{|x - \overline{x}| < \delta\}$, and this completes the proof. □

The substance of the following lemma is to show that when we are in the case λ_1, we have $u(x) \le u^\lambda(x)$, so it is a Hopf's lemma modified to apply it to our problem:

Lemma 3.2.2. *If $u \in C^2$ is a solution of (3.1). Assume that for some λ with $\lambda_1 \le \lambda < \lambda_0$, we have $b_1(x) \ge 0$ for $x \in \Sigma(\lambda) \cup \Sigma'(\lambda)$, $u_{x_1}(x) \le 0$, and, $u(x) \le u^\lambda(x)$, but $u(x) \not\equiv u^\lambda(x)$, in $\Sigma(\lambda)$. Then*

$$u(x) < u(x^\lambda), \quad \textit{in } \Sigma(\lambda), \tag{3.6}$$

and,

$$u_{x_1}(x) < 0, \quad \textit{on } \Omega \cap T_\lambda. \tag{3.7}$$

Proof. As u is solution of (3.1), then in $\Sigma'(\lambda)$, u^λ verifies

$$-\Delta u^\lambda(x) - b_1(x^\lambda)u^\lambda_{x_1}(x) + f(u^\lambda(x)) = -\Delta u(x^\lambda) + b_1(x^\lambda)u_{x_1}(x^\lambda) + f(u(x^\lambda)) = 0,$$

and by hypothesis, $u^\lambda_{x_1}(x) \geq 0$, for $x \in \Sigma'(\lambda)$. To continue, if we subtract (3.1), we find for $x \in \Sigma'(\lambda)$ that

$$
\begin{aligned}
&-\Delta\left(u^\lambda(x) - u(x)\right) + b_1(x)\left(u^\lambda(x) - u(x)\right)_{x_1} + f(u^\lambda) - f(u) - b_1(x^\lambda)(u^\lambda(x))_{x_1} - b_1(x)(u^\lambda(x))_{x_1}\\
&= -\Delta u^\lambda(x) + \Delta u(x) - b_1(x)(u^\lambda(x))_{x_1} - b_1(x)u_{x_1}(x) + f(u^\lambda) - f(u) - b_1(x^\lambda)(u^\lambda(x))_{x_1}\\
&\quad - b_1(x)(u^\lambda(x))_{x_1} \equiv \left(b_1(x^\lambda) + b_1(x)\right)(u^\lambda(x))_{x_1} \leq 0,
\end{aligned}
$$

where we have used $\left(u^\lambda(x)\right)_{x_1} = u_{x_1}(x^\lambda)(x^\lambda)_{x_1} = -u_{x_1}(x^\lambda)$. Let us define $h(x) = u^\lambda(x) - u(x) \leq 0$, for $x \in \Sigma'(\lambda)$. Then using the mean value theorem, for some function $c(x)$, h verifies

$$-\Delta h(x) + b_1(x)h_{x_1} + c(x)h(x) \leq 0, \quad x \in \Sigma'(\lambda).$$

If $x \in T_\lambda \cap \Omega$, $x = x^\lambda$, then $h(x) = 0$ on $T_\lambda \cap \Omega$. Since $h(x) \leq 0$ for $x \in \Sigma'(\lambda)$, but by hypothesis we have $u^\lambda(x) \not\equiv u(x)$, so $h(x) \not\equiv 0$ in $\Sigma'(\lambda)$, then by the maximum principle (Theorem 1.6.4), we have that

$$h(x) < 0, \quad \text{for } x \in \Sigma'(\lambda).$$

Then we have $u^\lambda(x) < u(x)$, for $x \in \Sigma'(\lambda)$, and thus $u(x) < u^\lambda(x)$, for $x \in \Sigma(\lambda)$.

Since $h(x) = 0$ for $x \in T_\lambda \cap \Omega$, by Hopf's Lemma 1.6.1 we get $h_\nu(x) > 0$, and, $h_{x_1}(x) > 0$, for $x \in T_\lambda \cap \Omega$. Moreover,

$$h_{x_1}(x) = u^\lambda_{x_1}(x) - u_{x_1} = u_{x_1}(x^\lambda) - u_{x_1} = -2u_{x_1}(x), \quad x \in T_\lambda \cap \Omega.$$

Finally, we have

$$u_{x_1}(x) < 0, \quad x \in T_\lambda \cap \Omega.$$

□

Proof of Theorem 3.2.1

Proof. From Lemma 3.2.1, for λ near $\lambda_0, \lambda < \lambda_0$, we obtain

(3.8) $$u_{x_1}(x) < 0, \quad \text{and} \quad u(x) < u^\lambda(x), \text{ for } x \in \Sigma(\lambda)$$

Now, if we decrease λ until a critical value $\mu \geq \lambda_1$. Then (3.6) holds for $\lambda > \mu$, while for $\lambda = \mu$, we have by continuity:

$$u_{x_1}(x) < 0, \quad \text{and} \quad u(x) \leq u^\mu(x), \text{ for } x \in \Sigma(\mu)$$

We will show that $\mu = \lambda_1$. By contradiction, we suppose $\mu > \lambda_1$. For a point $\overline{x} \in \partial\Sigma(\mu)\backslash T_\mu$, then $\overline{x}^\mu \in \Omega$. By (3.8), we have $0 = u(\overline{x}) < u^\mu(\overline{x})$, then we deduce that $u(x) \not\equiv u^\mu(x)$ in $\Sigma(\mu)$. Thus, by Lemma 3.2.2, we get $u(x) < u^\mu(x)$ in $\Sigma(\mu)$, and, $u_{x_1} < 0$ on $\Omega \cap T_\mu$, from where we have (3.8) for $\lambda = \mu$.

As we have $u_{x_1} < 0$ on $\Omega \cap T_\mu$, by continuity, for some $\varepsilon > 0$, we have:

$$u_{x_1} < 0, \text{ in } \Omega \cap \left\{x_1 > \mu - \varepsilon\right\}. \tag{3.9}$$

From the definition of μ, we can conclude that there exists an increasing sequence $\{\lambda^j\}_{j\in\mathbb{N}}$, such that $\lambda_1 < \lambda^j \underset{j\to\infty}{\longrightarrow} \mu$, and there exists a point $x_j \in \Sigma(\lambda^j)$ such that $u(x_j) \geq u^{\lambda^j}(x)$. Then there is a subsequence, which we still call x_j, such that $x_j \underset{j\to\infty}{\longrightarrow} x \in \overline{\Sigma(\mu)}$, and $x_{j_0}^{\lambda_{j_0}} \underset{j_0\to\infty}{\longrightarrow} x^\mu$, with $u(x) \geq u(x^\mu)$. As we have (3.6) for $\lambda = \mu$, then $x \in \partial\Sigma(\mu)$. Furthermore, if $x \notin T_\mu$, then $x^\mu \in \Omega$ and

$$0 = u(x) < u^\mu(x),$$

which is impossible. Therefore, we obtain $x \in T_\mu$ and $x^\mu = x$.

On the other hand, knowing that for j large enough, the segment joining x_j to $x_j^{\lambda_j}$ is in Ω, by the mean value theorem, there exists a point y_j in this segment, such that

$$u_{x_1}\left(y_j\right) \geq 0, \qquad \text{with } y_j \underset{j\to\infty}{\longrightarrow} x = \lim_{j\to\infty} x_j.$$

But this contradicts (3.9), then $\mu = \lambda_1$ and (3.8) holds for $\lambda > \lambda_1$.

As $u \in C^2$, by continuity,

$$u_{x_1}(x) \leq 0, \quad \text{and} \quad u(x) \leq u^{\lambda_1}(x), \quad \text{in } \Sigma = \Sigma(\lambda_1).$$

To finish the proof, let suppose $u_{x_1} = 0$ at some point in $\Omega \cap T_{\lambda_1}$, by Lemma 3.2.2, we have that $u(x) \equiv u^{\lambda_1}(x)$ in $\Sigma(\lambda_1)$. Since $u(x) = 0$, if $x \in \partial\Omega$, and, $x_1 \geq \lambda_1$, then $u^{\lambda_1}(x) = u(x^{\lambda_1}) = 0$ at the reflected point $x = x^{\lambda_1}$, then

$$\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup \left(T_{\lambda_1} \cap \Omega\right).$$

Suppose further that $b_1 > 0$ at some point $x \in \Omega$, and, $x \notin T_{\lambda_1}$, then from (3.1) and the symmetry of the solution in the plane T_{λ_1} that we have just proved, we obtain

$$b_1(x) u_{x_1}(x) = b_1(x^{\lambda_1}) u_{x_1}^{\lambda_1}(x).$$

If $x \in \Sigma(\lambda_1)$, then $u_{x_1}(x) < 0$, and left-hand side is negative, while the right-hand side is non negative, which is impossible. Similarly for $x \in \Sigma'(\lambda_1)$, thus

$$b_1 = 0.$$

□

3.3 Symmetry in $\mathbb{R}^N$

From now on, in (3.1) we take the particular case $f(u) = u^m$, where $m = \frac{N+2}{N-2}$, $N > 2$, and $b_1 \equiv 0$. Therefore, we consider here the non-linear elliptic equation:

$$(3.10) \qquad -\Delta u(x) = u^m(x), \quad \text{in } \mathbb{R}^N,$$

Essentially, we use the method developed by [11] and [27] to prove the main result of this section, which is the following theorem justifies that a C^2 solution of (3.9) is radially symmetric with respect to a plane:

Theorem 3.3.1. *Let $w \in C^2(\mathbb{R}^N)$ be a positive solution of (3.10) on $\mathbb{R}^N$ fulfilling expansions (3.13) and (3.14) as $x \to \infty$, and define $\overline{x} = (\overline{x}_1, \dots, \overline{x}_N)$ to be the center of w, with $\overline{x}_i = -\frac{1}{(N-2)}\frac{a_i}{a_0}$, where a_0 and a_i, for $i = 1, \dots, N$. Then w is symmetric with respect to the plane $x_1 = -\frac{1}{(N-2)}\frac{a_1}{a_0}$, and, $w_{x_1} < 0$, for $x_1 > -\frac{a_1}{(N-2)a_0}$.*

The main idea of the proof of this theorem is to start the technique of moving plane by comparing w with its reflection, i.e.

$$w(x) > w^\lambda(x), \quad \text{for } x_1 < \lambda,\ \forall \lambda \geq \lambda_1,$$

and we will prove that it is true in an open set, which is, we can move the plane a little more.

With the invaluable help of the moving plane method just explained, [8] goes further to generalize Theorem 3.3.1 to equations under the form of (3.1) with $b_1 = 0$ and removing the implicit growth assumption $w = O(|x|^{2-N})$.

To adapt the method of moving plane to $\mathbb{R}^N$, we need to know what happens to the equation (3.1) far from the origin and then some expansion of u in ∞ is required. For that, we exchange 0 by infinity using the Kelvin transform:

$$w(x) = \frac{1}{|x|^{N-2}} u\left(\frac{x}{|x|^2}\right), \text{ for } x \neq 0. \tag{3.11}$$

Doing the change of variable $y = \frac{x}{|x|^2}$,

$$u(y) = \frac{1}{|y|^{N-2}} w\left(\frac{y}{|y|^2}\right), \text{ for } y \neq 0. \tag{3.12}$$

The equation (3.1) is invariant by the Kelvin transform, so we can study it in the same way for u and for w. This allows us to deduce the following Taylor expansions:

Proposition 3.3.1. If $w \in C^2(\mathbb{R}^N)$ is a positive solution of the equation (3.10), verifying $w(x) = O(|x|^{2-N})$, when $|x| \to \infty$, then the following expansions when $|x| \to \infty$ hold:

$$w(x) = \frac{1}{|x|^{N-2}} \left[a_0 + a_i \frac{x_i}{|x|^2} + a_{ij} \frac{x_i x_j}{|x|^4} + O(|x|^{-3})\right], \tag{3.13}$$

$$w_{x_i}(x) = \frac{d}{dx_i} w(x) = -(N-2)\frac{x_i}{|x|^N}\left(a_0 + a_j \frac{x_j}{|x|^2}\right) + \frac{a_i}{|x|^N} - \frac{2x_i}{|x|^{N+2}} a_j x_j + O\left(|x|^{-(N+1)}\right), \tag{3.14}$$

for certain unique a_0, a_i, $a_{ij} \in \mathbb{R}$, $i, j = 1, \dots, N$.

Proof. Consider the Kelvin transform of w given by (3.11). Since $w \in C^2(\mathbb{R}^N)$, $u \in C^2(\mathbb{R} \setminus \{0\})$ and has a removable singularity at zero. Indeed, $u(y)$ verifies also (3.10) for $y \neq 0$, and is bounded near $y = 0$, because $u(y) = |y|^{2-N} w\left(\frac{y}{|y|^2}\right) \leq C|y|^{2-N}\left(\frac{1}{|y|}\right)^{2-N} = C$, for certain $C > 0$ when $y \to 0$ by hypothesis. Then it can be proved that $u(y)$ is a weak solution of (3.10) in $\mathbb{R}^N$ (Proposition 1.6.1), and by regularity theory for this equation guarantees that $u \in C^2$ near $y = 0$ (Theorem 1.6.1).

Then we can write a Taylor expansion for $u(y)$ at $y = 0$ as $u(y) = a_0 + a_i y_i + a_{ij} y_i y_j + O(|x|^3)$, by using Einstein's notation and being $a_0 = u(0) > 0$, $a_i = u_{y_i}(0)$ and $a_{ij} = \frac{1}{2} u_{y_i y_j}(0)$. Putting this expansion in (3.11) proves (3.13). Analogously one can prove (3.14). □

Now, to simplify the Taylor expansions (3.13) and (3.14) we shift the origin by doing a change of variables replacing x by $x - x_0$, with $x_0 = (x_{0\,1}, \dots, x_{0\,N})$, being $x_{0\,j} = -\frac{a_j}{(N-2)a_0}$, $j = 1, \dots, N$, and taking into account that for $q > 0$: $\frac{1}{|x-x_0|^q} = \frac{1}{|x|^q}\left(1 + \frac{q}{|x|^2} x_j x_{0\,j} + \dots\right)$, then we have that the expansions (3.13) and (3.14) turn into:

$$\begin{aligned} w(x) =& \frac{1}{|x|^2}\left(1 + \frac{x_j x_{0\,j}}{|x|^2} + \dots\right)\left(a_0 + \frac{a_i(x_i - x_{0\,i})}{|x - x_0|^2} + \frac{a_{i\,k}(x_k - x_{0\,k})}{|x - x_0|^4} + O\left(\frac{1}{|x|^3}\right)\right) \\ =& \frac{1}{|x|^{N-2}}\left(a_0 + \frac{a_{ij} x_i x_j}{|x|^4} + O\left(\frac{1}{|x|^3}\right)\right), \end{aligned} \tag{3.15}$$

$$
(3.16)\qquad \begin{aligned} w_{x_i}(x) = &-\frac{(N-2)(x_i - x_{0\,i})}{|x|^N}\left(1+\frac{N}{|x|^2}x_j x_{0\,j}+\dots\right)\left(a_0+\frac{a_k(x_k-x_{0\,k})}{|x-x_0|^2}\right)\\ &-\frac{2(x_i-x_{0\,i})}{|x-x_0|^{N+2}}a_j(x_j-x_{0\,j})+O(|x|^{-N-1})\\ =&-\frac{(N-2)}{|x|^N}a_0x_i+O(|x|^{-N-1}). \end{aligned}
$$

With respect to these new coordinates, we will show that any positive solution $w(x)$ of (3.10) and satisfying expansions (3.13) and (3.14) as $x\to\infty$, is rotationally symmetric about the origin, i.e. $w(x) = \tilde{w}(r)$, with $r = |x|$, and that $\tilde{w}'_r < 0$ for $r > 0$. Do note that as (3.10) is rotationally invariant, it is enough to prove Theorem 3.3.1 to reach this conclusion.

With this goal, first observe that from (3.16), there exist constants $C_0, R_N > 0$ such that:

$$
(3.17)\qquad w_{x_1}(x) < 0, \quad \text{for } x_1 \geq \frac{C_0}{|x|} \text{ and } |x| \geq R_N.
$$

To continue, following [11], we are going to prove some results before obtaining the radial symmetry.

Lemma 3.3.1. *Let $w \in C^2(\mathbb{R}^N)$ a positive solution of (3.10) in $\mathbb{R}^N$ fulfilling expansions (3.13) and (3.14) for $x\to\infty$. For any $\lambda > 0$, there exists $R = R(\lambda)$ depending only on* $\min\{1,\lambda\}$ *and also on w, such that, for $x = (x_1, x')$, $y = (y_1, y') \in \mathbb{R}^N$ satisfying:*

- $x_1 < y_1$.
- $x_1 + y_1 \geq 2\lambda$.
- $|x| \geq R$.

Then $w(x) > w(y)$.

Proof. We prove this result by contradiction. So fixing $\lambda > 0$, we consider two points $x = (x_1, x')$, $y = (y_1, y') \in \mathbb{R}^N$, satisfying $x_1 < y_1$, $x_1 + y_1 \geq 2\lambda$, $|x| \geq R$ for some $R > 0$, and such that:

$$
(3.18)\qquad w(x) \leq w(y).
$$

Then we are going to see that there exists $R_1 > 0$ depending only on $\min\{1,\lambda\}$, such that $|x|, |y| < R_1$. First, do note that under a rotation or a translation we can assume that

x and y are both in the x_1 axis with $0 < x_1 < y_1$. Thus, we have $|x| < |y|$ and from (3.16) for large R,

$$\begin{aligned} w(x) - w(y) =& \frac{1}{|x|^{N-2}}\left[a_0 + a_{ij}\frac{x_i x_j}{|x|^4} + O(|x|^{-3})\right] - \frac{1}{|y|^{N-2}}\left[a_0 + a_{ij}\frac{y_i y_j}{|y|^4} + O(|y|^{-3})\right] \\ =& a_0\left(\frac{1}{|x|^{N-2}} - \frac{1}{|y|^{N-2}}\right) + a_{ij}\left(\frac{x_i x_j}{|x|^{N+2}} - \frac{y_i y_j}{|y|^{N+2}}\right) + O(|x|^{-N-1}) + O(|y|^{-N-1}) \le 0. \end{aligned}$$

From where we get that there exist constants $C_1, C > 0$ such that:

$$\begin{aligned} a_0\left(\frac{1}{|x|^{N-2}} - \frac{1}{|y|^{N-2}}\right) <& a_{ij}\left(\frac{y_i y_j}{|y|^{N+2}} - \frac{x_i x_j}{|x|^{N+2}}\right) + O(|x|^{-N-1}) + O(|y|^{-N-1}) \\ \le& a_{ij}\left(\frac{y_i y_j}{|y|^{N+2}} - \frac{x_i x_j}{|x|^{N+2}}\right) + C_1|x|^{-N-1} \le C|x|^{-N}. \end{aligned} \tag{3.19}$$

Now, because $|x| < |y|$, for $p \ge 1$ we have

$$\frac{1}{|x|^p} - \frac{1}{|y|^p} \ge \frac{1}{|x|^{p-1}}\frac{1}{|x|} - \frac{1}{|y|^{p-1}}\frac{1}{|y|} \ge \frac{1}{|x|^{p-1}}\frac{1}{|x|} - \frac{1}{|x|^{p-1}}\frac{1}{|y|} = \frac{1}{|x|^{p-1}}\left(\frac{1}{|x|} - \frac{1}{|y|}\right),$$

then (3.19) implies that

$$\frac{1}{|x|^{N-3}}\left(\frac{1}{|x|} - \frac{1}{|y|}\right) \le \frac{1}{|x|^{N-2}} - \frac{1}{|y|^{N-2}} \le C|x|^{-N}, \text{ for a certain } C > 0. \tag{3.20}$$

Then:

$$\frac{1}{|x|} - \frac{1}{|y|} \le \frac{1}{|x|} - \frac{|x|^{N-3}}{|y|^{N-2}} \le C|x|^{-3}.$$

From where we have:

$$|y| - |x| \le C\frac{|y|}{|x|^2} = C\left(\frac{|y| - |x|}{|x|^2} + \frac{1}{|x|}\right).$$

If we suppose that $C|x|^{-2} \le \frac{1}{2}$, then:

$$|y| - |x| \le \frac{2C}{|x|}, \quad and, \quad |y| \le 2|x|. \tag{3.21}$$

Indeed, we may assume from now on that $C|x|^{-2} \le \frac{1}{2}$, because if $Cx^{-2} \le \frac{1}{2}$, i.e., if $x^2 < 2C$, then as $w(y) \to 0$ when $y \to \infty$, and $w(x) \le w(y)$, we have that $|y| \le R$ for some R independent of λ and the result will be proved. Therefore, returning to (3.19),

$$\begin{aligned} a_0\left(\frac{1}{|x|^{N-2}} - \frac{1}{|y|^{N-2}}\right) \le& b_{11}\left(\frac{y_1^2}{|y|^{N+2}} - \frac{x_1^2}{|x|^{N+2}}\right) + 2\sum_{j>1} b_{1j}x_j\left(\frac{y_1}{|y|^{N+2}} - \frac{x_1}{|x|^{N+2}}\right) \\ &+ \sum_{j,k>1} b_{kj}x_j x_k\left(\frac{1}{|y|^{N+2}} - \frac{1}{|x|^{N+2}}\right) + C|x|^{-N-1}. \end{aligned}$$

By following the same arguments as before, we obtain:

$$\begin{aligned}\frac{1}{|x|^{N-3}}\left(\frac{1}{|x|}-\frac{1}{|y|}\right) \leq & C\left(\frac{y_1^2-x_1^2}{|x|^{N+2}}\right)+C\left(\frac{y_1-x_1}{|x|^{N+1}}\right)+C|x|^2\left(\frac{1}{|x|^{N+2}}-\frac{1}{|y|^{N+2}}\right)+C|x|^{-N-1} \\ = & C\left(\frac{y_1^2-x_1^2}{|x|^{N+2}}\right)+C\left(\frac{y_1-x_1}{|x|^{N+1}}\right)+C\frac{1}{|x|^{N-1}}\left(\frac{1}{|x|}-\frac{1}{|y|}\right)+C|y|^{-N-1}.\end{aligned}$$

Then we have:

$$\frac{1}{|x|}-\frac{1}{|y|} \leq C\left(\frac{y_1^2-x_1^2}{|x|^5}\right)+C\left(\frac{y_1-x_1}{|x|^4}\right)+C\frac{1}{|x|^2}\left(\frac{1}{|x|}-\frac{1}{|y|}\right)+C|x|^{-4}.$$

Multiplying by $|x|$ and $|y|$, and by (3.21):

$$\begin{aligned}|y|-|x| \leq & C|y|\left(\frac{y_1^2-x_1^2}{|x|^4}\right)+C|y|\left(\frac{y_1-x_1}{|x|^3}\right)+C\frac{1}{|x|^2}(|y|-|x|)+C|x|^{-3}|y| \\ \leq & \frac{C}{|x|^3}\left(y_1^2-x_1^2\right)+\frac{C}{|x|^2}(y_1-x_1)+\frac{C(|y|-|x|)}{|x|^2}+\frac{C}{|x|^2}.\end{aligned}$$

We multiply by $|y|+|x|$, and take into account that $|y|^2-|x|^2=y_1^2-x_1^2$:

$$\begin{aligned}|y|^2-|x|^2=y_1^2-x_1^2 \leq & \frac{C}{|x|^3}\left(y_1^2-x_1^2\right)(|y|+|x|)+\frac{C}{|x|^2}(|y|+|x|)(y_1-x_1)+\frac{C}{|x|^2}\left(y_1^2+x_1^2\right) \\ & +\frac{C}{|x|^2}(|y|+|x|).\end{aligned}$$

Remembering that we assume $\frac{C}{|x|^2} \leq \frac{1}{2}$, and by (3.21):

$$y_1^2-x_1^2 \leq \frac{C(y_1-x_1)}{|x|}+\frac{C}{|x|}.$$

As we have that $|y|^2-|x|^2=y_1^2-x_1^2 \geq 2\lambda(y_1-x_1)$, we deduce:

$$2\lambda(y_1-x_1) \leq y_1^2-x_1^2 \leq \frac{C(y_1-x_1)}{|x|}+\frac{C}{|x|}.$$

Then:

$$\left(2\lambda-\frac{C}{|x|}\right)(y_1-x_1) \leq \frac{C}{|x|}.$$

On one side, if we suppose $2\lambda-\frac{C}{|x|} \geq \lambda$, that is to say:

$$\frac{C}{\lambda} \leq |x|, \tag{3.22}$$

we have $y_1-x_1 \leq \frac{C}{\lambda|x|}$. So, by hypothesis, as $2\lambda-x_1 \leq y_1$, we have $2\lambda-2x_1 \leq \frac{C}{\lambda|x|}$, then $\lambda-\frac{C}{2\lambda|x|} \leq x_1$.

On the other side, there exists a constant $C_0 > 0$ which verifies (3.17), then we have that $\frac{C_0}{|x|} \leq \lambda - \frac{C}{2\lambda|x|}$, if and only if:

$$\frac{C}{2\lambda^2} + \frac{C_0}{\lambda} \leq |x|. \tag{3.23}$$

Therefore, if (3.22) and (3.23) take place, then necessarily (3.17) happens and it implies that w is decreasing on the segment joining x and y, but this contradicts (3.20). □

Remembering the Definition 3.1.2, we define the function $w^\lambda(x) = w(x^\lambda)$, with $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_N)$ the reflection of x to the hyperplane $x = \lambda$. Let us see the following result in which we can compare w with its reflection, that it is to say that the moving plane method begins.

Lemma 3.3.2. *Let $w(x)$ be in the hypotheses of Theorem 3.3.1. There exists $\lambda_1 \geq 1$, such that, $\forall \lambda \geq \lambda_1$:*

$$w(x) > w^\lambda(x), \quad \textit{for } x_1 < \lambda. \tag{3.24}$$

Proof. Let $R_1 = R(1)$ be defined as in Lemma 3.3.1, we define $\overline{R}_1 = \max\{1, R_1\}$.

On one hand, by Definition 3.1.2 and by Lemma 3.3.1. If $\lambda \geq 1$, and $x_1 < \lambda$, then:

$$w(x) > w^\lambda(x), \quad \text{for } |x| \geq \overline{R}_1. \tag{3.25}$$

On the other hand, for $|x| \leq \overline{R}_1$, we have that there exists a constant $c_0 > 0$ which verifies:

$$w(x) \geq c_0.$$

But if we take $1 < \overline{R}_2$ sufficiently large, we can get:

$$w(y) < c_0, \quad \forall y \in \mathbb{R}^N \text{ with } |y| \geq \overline{R}_2.$$

From where we deduce that (3.24) occurs if $\overline{R}_2 \leq \lambda$, and, $|x| \leq \overline{R}_1$. Finally, by (3.25), it is enough to take $\lambda_1 = \overline{R}_2$ to conclude (3.24). □

Lemma 3.3.3. *Let $w(x)$ be in the hypotheses of Theorem 3.3.1. If for some $\lambda > 0$, we have $w^\lambda(x) \leq w(x)$, and $w^\lambda(x) \not\equiv w(x)$, for $x_1 < \lambda$. Then:*

$$w^\lambda(x) < w(x), \textit{ for } x_1 < \lambda, \quad \textit{and}, \quad w_{x_1}(x) < 0, \textit{ for the hyperplane } x_1 = \lambda. \tag{3.26}$$

Proof. Firstly, the function w^λ is also a solution of (3.10) for $x_1 < \lambda$, and we know that for $x_1 < \lambda$, we have:

$$w^\lambda(x) \leq w(x).$$

Secondly, we set the function $f(x) = w^\lambda(x) - w(x)$, and so, for $x_1 < \lambda$:

$$\begin{cases} f(x) \leq 0, \\ f(x) \not\equiv 0. \end{cases}$$

Thus $f(x)$ satisfies that $-\Delta f = \left(w^\lambda\right)^m - w^\lambda \leq 0$, for $x_1 \leq \lambda$, and it achieves its maximum (zero) at every point on the hyperplane $x_1 = \lambda$. We can apply the maximum principle and the Hopf's Lemma 1.6.1 to f to deduce that:

$$\begin{cases} f(x) < 0, & \text{for } x_1 < \lambda, \\ 0 < f_{x_1}(x) = -2w_{x_1}(x), & \text{on } x_1 = \lambda. \end{cases}$$

□

Lemma 3.3.4. *In the hypotheses of Theorem 3.3.1, the set of positive λ for which* (3.24) *holds is open.*

Proof. Let us consider the set $I = \left\{\lambda > 0 \mid w(x) > w^\lambda(x), \text{ for } x = (x_1, \ldots, x_N) \text{ with } x_1 < \lambda\right\}$. Then, for $\overline{\lambda} \in I$, (3.24) holds. Now, we take $\overline{R} = R\left(\frac{\overline{\lambda}}{2}\right)$ of Lemma 3.3.1, then it follows that for $\lambda \geq \frac{\overline{\lambda}}{2}$ and $|x| > \overline{R}$, (3.24) holds. But, if we instead consider $|x| \leq \overline{R}$ and (3.24) does not hold, $\forall \lambda$ in a neighborhood of $\overline{\lambda}$, then there exists a sequence $\{x^j\}_{j=\{1,2,\ldots\}}$, such that $|x^j| \leq R$, for a certain $R > 0$, and a sequence $\lambda^j \underset{j\to\infty}{\longrightarrow} \overline{\lambda}$ which satisfies $\lambda^j \geq \frac{\overline{\lambda}}{2}$, with, $x_1^j < \lambda^j$, and:

$$w(x^j) \leq w^{\overline{\lambda}}(x^j). \tag{3.27}$$

Then there exists a subsequence which we denote again $\{x^j\}_{j=\{1,2,\ldots\}}$, such that $x^j \underset{j\to\infty}{\longrightarrow} x$, where $|x| \leq R$, and:

$$w(x) \leq w^{\overline{\lambda}}(x).$$

From (3.24), we have necessarily that $x_1 = \overline{\lambda}$. But from (3.27), we have:

$$0 \leq w_{x_1}(x),$$

which contradicts Lemma 3.3.3. Then we have that $\forall \overline{\lambda} \in I$, there exists a neighbourhood of $\overline{\lambda}$ included in I, so I is open. □

Proof of Theorem 3.3.1

Proof. Lemma 3.3.2, Lemma 3.3.3 and Lemma 3.3.4 guarantee that (3.24) and (3.26) hold in some open maximal interval $(\lambda_1,\infty)\subseteq(0,\infty)$. Also, we have

$$w_{x_1}(x)<0,\quad \text{for } x_1>\lambda_1, \tag{3.28}$$

and by continuity, we have:

$$w(x)\geq w^{\lambda_1}(x),\quad \text{for } x_1<\lambda_1. \tag{3.29}$$

If we now suppose that $\lambda_1>0$, then from Lemma 3.3.3, we deduce that we have one of the following cases:

1. $w(x)\equiv w^{\lambda_1}(x)$, for $x_1<\lambda_1$.

2. $w(x)>w^{\lambda_1}(x)$, for $x_1<\lambda_1$.

However, 1 contradicts Lemma 3.3.1 and 2 is impossible while I is open as we have seen in Lemma 3.3.4. Hence, $\lambda_1=0$, and from (3.29), $w(x)=w^0(x)$, for $x_1<0$, proving this way the symmetry of w with respect to the plane $x_1=0$ or $x_1=-\frac{a_1}{(N-2)a_0}$ in the original coordinates. The remainder of the theorem can be deduced from (3.28). □

Theorem 3.3.1 allows us to prove that for all unitary vectors γ in $\mathbb{R}^N$, every positive solution $w(x)$ of (3.10) fulfilling the hypothesis of this theorem is symmetric to the plane $\gamma(x-\bar{x})=0$, where $\bar{x}\in\mathbb{R}^N$ is the center defined in Theorem 3.3.1, and that $\gamma\cdot\operatorname{grad} w<0$, for $\gamma(x-\bar{x})>0$. Therefore, we can finally conclude that $w(x)$ must be radially symmetric about some point, i.e. $w(x)=\tilde{w}(r)$, with $\tilde{w}_r(r)>0$, for $r>0$, where r is the radial coordinate from that point.

Next, we include this result and refer to [8] for the details of the proof.

Theorem 3.3.2. *Let $N\geq 3$, and $w\in C^2(\mathbb{R}^N\setminus\{0\})$ a solution of:*

$$-\Delta u=f(u),\quad x\in\mathbb{R}^N, \tag{3.30}$$

with an isolated singularity at the origin, and f a locally Lipschitz function verifying:

1. *$f(u)$ is non-decreasing and $f(0)=0$.*

2. $u^{-\frac{(N+2)}{(N-2)}} f(u)$ *is non-increasing .*

3. $f(u) \geq Cu^p$*, for some* $p \geq \frac{N}{N-2}$*, when* $u \to \infty$.

Then,

1. *If the origin is a non-removable singularity, then* u *is radially symmetric about the origin, i.e.* $u = \tilde{u}(r)$*, with* $r = |x|$*, and* $\tilde{u}_r < 0$*, for* $r > 0$.

2. *If the origin is a removable singularity, then* u *is radially symmetric about some point* $\overline{x} \in \mathbb{R}^N$*, i.e.* $u = \tilde{u}(r)$*, with* $r = |x - \overline{x}|$*, and* $\tilde{u}_r < 0$*, for* $r > 0$.

Proof. See [8], Theorem 8.1, p. 294. □

3.4 Relation with the Yamabe problem

As we had illustrated on Chapter 2, equation (3.10) is a Yamabe equation in $\mathbb{R}^N$, where u is defining a conformally flat metric $\overline{g} = u^{\frac{4}{N-2}} g_0$, with $\overline{g}$ a metric with a constant positive scalar curvature. The previous section shows that all solutions must be radially symmetric. The next corollary due to [8] determines the precise form of these radially symmetric solutions in this particular case, but first let us see the following identity that we will use later:

Proposition 3.4.1. Let $u \in C^2(\mathbb{R}^N)$ be a non-negative solution of the problem, we define the Pohozaev identity as follows:

$$R^N (u'(R))^2 + R^N \frac{u^{\alpha+1}(R)}{\alpha+1} + \frac{N-2}{N} R^{N-1} u(R) u'(R) = \left(\frac{N}{\alpha+1} - \frac{N-2}{N} \right) \int_0^R r^{N-1} u^{\alpha+1}(r)\, dr.$$

Corollary 3.4.1. *Let* $u \in C^2(\mathbb{R}^N)$*, be a non-negative solution of the problem:*

$$-\Delta u(x) = u^\alpha(x), \quad x \in \mathbb{R}^N, \text{ with } \frac{N}{N-2} \leq \alpha < \frac{N+2}{N-2}, N \geq 3. \tag{3.31}$$

Then we have:

1. *If* $\frac{N}{N-2} \leq \alpha < \frac{N+2}{N-2}$*, then* $u \equiv 0$.

2. *If* $\alpha = \frac{N+2}{N-2}$,

$$u(x) = \left(\frac{\mu \sqrt{N(N-2)}}{\mu^2 + |x - \overline{x}|^2} \right)^{\frac{N-2}{2}}, \quad \mu > 0,$$

for $\overline{x}$ *some point in* $\mathbb{R}^N$.

Proof. 1. By Theorem 3.3.2, we know that u is radially symmetric about the origin. So, if $r=|x|$, denoting $u(x)=u(r)$, by (3.31), u verifies:

$$(r^{N-1}u')' = r^{N-1}u_{r,r}(r) = -r^{N-1}u^{\alpha}(r). \tag{3.32}$$

Integrating from 0 to r:

$$r^{N-1}u_r(r) = -\int_0^r s^{N-1}u^{\alpha}(s)\,ds.$$

Then, as $u(r)$ is decreasing ($u'<0$ by Theorem 3.3.2), we have

$$r^{N-1}u'(r) = -\int_0^r s^{N-1}u^{\alpha}(s)\,ds \le -\frac{r^N}{N}u^{\alpha}(r).$$

Then $\frac{u'(r)}{u^{\alpha}(r)} \le -\frac{r}{N}$. Integrating from 0 to r:

$$\int_0^r \frac{u'(s)}{u^{\alpha}(s)}\,ds \le -\int_0^r \frac{s}{N}\,ds = -\frac{r^2}{2N}.$$

Now by an integration by parts we solve the left side:

$$\int_0^r \frac{u'(s)}{u^{\alpha}(s)}\,ds = \frac{1}{\alpha-1}\left(\frac{1}{u^{1-\alpha}(r)} - \frac{1}{u^{\alpha-1}(0)}\right) \ge -\frac{r^2}{2N}.$$

Then we get:

$$\frac{1}{u^{\alpha-1}(r)} \ge \frac{1}{u^{\alpha-1}(0)} + \frac{(\alpha-1)}{2N}r^2 \ge \frac{\alpha-1}{2N}r^2.$$

This implies that:

$$\begin{aligned} &u(r) \le Cr^{-\frac{2}{\alpha-1}},\\ &|u'(r)| \le Cr^{-\frac{\alpha+1}{\alpha-1}}. \end{aligned} \tag{3.33}$$

To finish the proof we are going to use the Pohozaev identity of Proposition 3.4.1:

$$R^N(u'(R))^2 + R^N\frac{u^{\alpha+1}(R)}{\alpha+1} + \frac{N-2}{N}R^{N-1}u(R)u'(R) = \left(\frac{N}{\alpha+1} - \frac{N-2}{N}\right)\int_0^R r^{N-1}u^{\alpha+1}(r)\,dr$$

By (3.33), we obtain that the left side of the Pohozaev identity tends to 0, when $R\to\infty$. By hypothesis $\frac{N}{N-2} \le \alpha < \frac{N+2}{N-2}$, and therefore $\frac{N}{\alpha+1} - \frac{N-2}{N} > 0$, then we have $\int_0^R r^{N-1}u^{\alpha+1}(r)\,dr = 0$, and this implies that $u \equiv 0$.

2. Again, by Theorem 3.3.2, we know that u is radially symmetric about some point $\overline{x}\in\mathbb{R}^N$. We are going to prove that $u(x) = \left(\frac{\mu\sqrt{N(N-2)}}{\mu^2+|x-\overline{x}|^2}\right)^{\frac{N-2}{2}}$ is solution of the problem (3.31) for $\alpha = \frac{N+2}{N-2}$.

- The derivative of u with respect to x_i:

$$u_{x_i}(x) = -(N-2)\left(\mu\sqrt{N(N-2)}\right)^{\frac{N-2}{2}} \left(\mu^2 + |x-\overline{x}|^2\right)^{-\frac{N}{2}} (x_i - \overline{x}_i).$$

- The second derivative of u with respect to x_i:

$$u_{x_i,x_i}(x) = -(N-2)\left(\mu\sqrt{N(N-2)}\right)^{\frac{N-2}{2}} \left(\mu^2 + |x-\overline{x}|^2\right)^{-\frac{(N+2)}{2}} \left[-N(x_i - \overline{x}_i)^2 + \mu^2 + |x-\overline{x}|^2\right].$$

- The Laplacian of u:

$$-\Delta u(x) = -\sum_{i=1}^{N} u_{x_i,x_i}(x) = (N-2)\left(\mu\sqrt{N(N-2)}\mu\right)^{\frac{N-2}{2}} \left(\mu^2 + |x-\overline{x}|^2\right)^{-\frac{(N+2)}{2}} \cdot \left[-N|x-\overline{x}|^2 + N\mu^2 + N|x-\overline{x}|^2\right] = \left(\frac{\sqrt{N(N-2)}\mu}{\mu^2 + |x-\overline{x}|^2}\right)^{\frac{N+2}{2}}.$$

- For $\alpha = \frac{N+2}{N-2}$:

$$u^{\alpha}(x) = \left(\frac{\sqrt{N(N-2)}\mu}{\mu^2 + |x-\overline{x}|^2}\right)^{\frac{N+2}{2}}.$$

From where we get $-\Delta u = u^{\alpha}$, so we have that u is a solution to the Yamabe problem in $\mathbb{R}^N$. The result required is finally deduced from the uniqueness of solutions to the ODE (3.32).

□

3.5 A Harnack estimate for a parabolic equation related to the Yamabe problem

Now, to illustrate the versatility of the moving plane technique, we put the focus on how this usefull tool allows to obtain a Harnack type estimate for a parabolic equation closely related to the Yamabe problem which we are going to study in some aspects in greater depth in the final chapter of this Master's thesis.

From a geometrical point of view, let us consider first the well known Yamabe flow ([24], [27]). Originally conceived by Hamilton as a device to solve the Yamabe problem, the Yamabe flow is defined by an evolution equation for a family of conformal metrics on a given Riemannian manifold $(M^N, \overline{g})$ that seeks to decrease the total scalar curvature.

In particular, the objective is to find a family of conformal metrics $g(x,t)$ solution of the evolution problem

$$\begin{cases} g_t = (s_g - R_g)g, \\ g(0,x) = \overline{g}, \qquad \text{for } x \in M^N, \end{cases}$$

when M^N is compact, being R_g the scalar curvature and $s_g = \frac{1}{Vol(M)} \int_M R_g \, dV_g$. If the Yamabe flow exists for all $t > 0$ and converges smoothly as $t \to \infty$, then the limit metric has constant scalar curvature ([27]).

For $M^N = \mathbb{R}^N$, with the Euclidean metric g_0, the Yamabe flow can be defined by the evolution equation

$$\begin{cases} g_t = -R_g g, \\ g(0,x) = g_0, \quad \text{for } x \in \mathbb{R}^N. \end{cases}$$

If we write, as usual, $g = v^{\frac{4}{N-2}} g_0$, with $v > 0$, then this equation turns to $(v^N(x,t))_t = C_N \Delta v(x,t)$, where $C_N = \frac{4(N-1)}{N-2}$. Putting $v^N = u$ and rescaling the time variable, we obtain $u_t(x,t) = \Delta u^m(x,t)$, for $m = \frac{N-2}{N+2}$, which is the fast diffusion equation that we are going to study now on following the paper of Del Pino and Sáez [9].

Therefore, for $N \geq 3$, let consider the Cauchy problem:

$$\begin{cases} u_t(x,t) = \Delta u^m(x,t), \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x), \end{cases} \tag{3.34}$$

with $m = \frac{N-2}{N+2}$, $0 < m < 1$, and $u_0(x)$ non-negative, continuous, not identically zero and satisfying the fast decay condition:

$$\|u_0\|_* = \sup_{x \in \mathbb{R}^N} (1 + |x|^{N+2}) u_0(x) < +\infty. \tag{3.35}$$

As we will see in the next chapter, the decay rate condition (3.35) implies that there exists a finite time $T > 0$ such that the solution of (3.34) vanishes when $t > T$, $\forall x \in \mathbb{R}^N$. Do note that some decay of the initial condition is needed to assure the existence of a vanishing time, because for example, $u \equiv 1$ is a solution of (3.34) with $u_0 = 1$ and it does not vanish anywhere.

The objective of this section is the study of the behaviour of any positive solution u to (3.34) when t is near its vanishing time. For this, we are going to define the following transformation:

$$w(x,s) = (T-t)^{-\frac{m}{1-m}} u^m(x,t)\,|_{t=T(1-e^{-s})}, \quad x \in \mathbb{R}^N,\ s>0. \tag{3.36}$$

Then, for $p = \frac{1}{m} = \frac{N+2}{N-2}$, w satisfies the problem

$$\begin{cases} (w^p)_s = \Delta w + \frac{1}{1-m} w^p, & (x,s) \in \mathbb{R}^N \times (0,\infty), \\ w(x,0) = T^{-\frac{m}{1-m}} u_0^m(x). \end{cases} \tag{3.37}$$

Furthermore, the steady state of this equation are the positive solutions $\overline{w}$ of the elliptic equation analogous to the case studied in the previous section:

$$\Delta \overline{w} + \frac{1}{1-m} \overline{w}^p = 0, \quad x \in \mathbb{R}^N.$$

So, Theorem 3.3.2 guarantees that $\overline{w}$ is radially symmetric around some point of $\mathbb{R}^N$, and as in Corollary 3.4.1, necessarily:

$$\overline{w}(x) = \left(\frac{k_N \mu}{\mu^2 + |x-\overline{x}|^2} \right)^{\frac{N-2}{2}}, \tag{3.38}$$

for $k_N = \left[4N\left(\frac{N-2}{N+2}\right)\right]^{\frac{1}{2}}$ and some $\mu > 0$, $\overline{x} \in \mathbb{R}^N$.

In order to pass from $\mathbb{R}^N$ to $\mathbb{S}^N$ to obtain advantages from the compactness of the sphere, we are going to define another transformation via the stereographic projection (see Remark 2.3.1). We define the function v from w as:

$$w(x,s) = \left(\frac{2}{1+|x|^2} \right)^{\frac{N-2}{2}} v(F(x),s), \quad (x,s) \in \mathbb{R}^N \times (0,\infty), \tag{3.39}$$

or,

$$v(y,s) = \left(\frac{1}{1-y_{N+1}} \right)^{\frac{N-2}{2}} w(F^{-1}(y),s), \quad (y,s) \in \mathbb{S}^N \setminus \{q_0\} \times (0,\infty), \tag{3.40}$$

where $F := \sigma^{-1} : \mathbb{R}^N \longrightarrow \mathbb{S}^N \setminus \{q_0\}$, with $q_0 = (0,\dots,0,1)$ the north pole of $\mathbb{S}^N$, is the stereographic projection and, v satisfies the equation:

$$(v^p)_s = \Delta_{\mathbb{S}^N} v - C(N)v + \frac{1}{1-m} v^p, \quad (y,s) \in \mathbb{S}^N \times (0,\infty), \tag{3.41}$$

for some constant $C(N) > 0$.

In the same way as in the elliptic case, some Taylor expansions are required, so we establish first:

Proposition 3.5.1. Let $s_0 > 0$ be such that $v(y,s)$, defined by (3.39), is positive and smooth in $\mathbb{S}^N \times (0,s_0)$, with $w(x,s)$ a positive solution of (3.37). The following expansions hold when $|x| \to \infty$:

$$w(x,s) = \frac{2^{\frac{N-2}{2}}}{|x|^{N-2}} \left[a_0(s) + a_i(s)\frac{x_i}{|x|^2} + \frac{1}{2}\left(a_{ij}(s) - (N-2)a_0(s)\delta_{ij}\right)\frac{x_i x_j}{|x|^4} + O(|x|^{-3}) \right], \tag{3.42}$$

(3.43)

$$w_{x_i}(x,s) = \frac{d}{dx_i}w(x,s) = -(N-2)2^{\frac{N-2}{2}}\frac{x_i}{|x|^N}\left(a_0(s) + a_j(s)\frac{x_j}{|x|^N}\right) + \frac{a_i(s)}{|x|^N} - \frac{2x_i}{|x|^{N+2}}a_j(s)x_j + O\left(|x|^{-(N+1)}\right),$$

for certain $a_0(s)$, $a_i(s)$, $a_{ij}(s) \in \mathbb{R}$, with $a_0(s) > 0$, $\forall s \in (0,s_0)$, $i,j = 1,\dots,N$.

Proof. Using again the Kelvin transform (3.11)-(3.12) to exchange ∞ by 0, we define:

$$w(x,s) = \frac{1}{|x|^{N-2}} h\left(\frac{x}{|x|^2}, s\right), \quad x \neq 0,\ s \in (0,s_0).$$

Thus, calling $y = \frac{x}{|x|^2}$, and by (3.39):

$$h(y,s) = \frac{1}{|y|^{N-2}} w\left(\frac{y}{|y|^2}, s\right) = 2^{\frac{N-2}{2}}\left(1+|y|^2\right)^{\frac{2-N}{2}} v\left(F(\frac{y}{|y|^2}), s\right), \quad y \neq 0,\ s \in (0,s_0).$$

Now, we can write

$$H(x) = F\left(\frac{x}{|x|^2}\right) : \mathbb{R}^N \longrightarrow \mathbb{S}^N \setminus \{p_0\}$$

$$(x_1,\dots,x_N) \longmapsto \frac{\left(2x_1,\dots,2x_N, 1-|x|^2\right)}{1+|x|^2} = \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right),$$

i.e. $H(x)$ is the stereographic projection from the south pole $p_0 = (0,\dots,0,-1)$, and then $h(y,s) = 2^{\frac{N-2}{2}}\left(1+|y|^2\right)^{\frac{2-N}{2}} v(H(y),s)$, and therefore $h(y,s)$ is smooth in $\mathbb{R}^N \times (0,s_0)$ and we only have to calculate its Taylor expansion at $y = 0$.

1. Calling $a_0(s) = v(H(0),s) = v(q_0,s) > 0$, $\forall s \in (0,s_0)$, then:

$$h(0,s) = 2^{\frac{N-2}{2}} a_0(s), \quad s \in (0,s_0).$$

2. $\forall i = 1,\dots,N$, knowing that $\frac{\partial}{\partial y_i}(|y|^2) = 2|y|\frac{y_i}{|y|} = 2y_i$, and by the chain rule, we have:

$$\begin{aligned}\frac{\partial}{\partial y_i}h(y,s) =& 2^{\frac{N-2}{2}}(2-N)(1+|y|^2)^{-\frac{N}{2}} y_i v(F(y),s)\\ &+ 2^{\frac{N-2}{2}}(1+|y|^2)^{\frac{2-N}{2}}\sum_{k=1}^{N+1} v_{F_k}(F(y),s)\cdot F_{k\,y_i}(y).\end{aligned}$$

Then for $y = 0$, and denoting $a_i(s) = \frac{\partial}{\partial y_i}v(H(y),s)\,|_{y=0}$:

$$\frac{\partial}{\partial y_i}h(0,s) = 2^{\frac{N-2}{2}}\sum_{k=1}^{N+1} v_{F_k}(F(0),s)\cdot F_{k\,y_i}(0) = 2^{\frac{N-2}{2}}\frac{\partial}{\partial y_i}v(H(y),s)\,|_{y=0} = 2^{\frac{N-2}{2}} a_i(s).$$

3. $\forall i,j = 1,\dots,N$,

   (a) If $i \neq j$:

$$\begin{aligned}\frac{\partial^2}{\partial y_j \partial y_i} h(y,s) =& 2^{\frac{N-2}{2}}(2-N)\left(-\frac{N}{2}\right)(1+|y|^2)^{-\frac{N}{2}-1} 2y_i y_i v(F(y),s) \\ &+ 2^{\frac{N-2}{2}} \frac{(2-N)}{2}(1+|y|^2)^{-\frac{N}{2}} 2y_i \frac{\partial}{\partial y_i}(v(H(y),s)) \\ &+ 2^{\frac{N-2}{2}} \frac{(2-N)}{2}(1+|y|^2)^{-\frac{N}{2}} 2y_i \frac{\partial}{\partial y_i}(v(H(y),s)) \\ &+ 2^{\frac{N-2}{2}}(1+|y|^2)^{\frac{2-N}{2}} \frac{\partial^2}{\partial y_j \partial y_i}(v(H(y),s)).\end{aligned}$$

   Then for $y=0$, and denoting $a_{ij}(s) = \frac{\partial^2}{\partial y_j \partial y_i} v(H(y),s)\mid_{y=0}$:

$$\frac{\partial^2}{\partial y_j \partial y_i} h(0,s) = 2^{\frac{N-2}{2}} \frac{\partial^2}{\partial y_j \partial y_i} v(H(y),s)\mid_{y=0} = 2^{\frac{N-2}{2}} a_{ij}(s).$$

   (b) If $i = j$:

$$\begin{aligned}\frac{\partial^2}{\partial y_i \partial y_i} h(y,s) =& 2^{\frac{N-2}{2}}(2-N)\left(-\frac{N}{2}\right)(1+|y|^2)^{-\frac{N}{2}-1} 2y_i y_i v(F(y),s) \\ &+ 2^{\frac{N-2}{2}}(2-N)(1+|y|^2)^{-\frac{N}{2}} v(F(y),s) \\ &+ 2^{\frac{N-2}{2}} \frac{(2-N)}{2}(1+|y|^2)^{-\frac{N}{2}} 2y_i \frac{\partial}{\partial y_i}(v(H(y),s)) \\ &+ 2^{\frac{N-2}{2}} \frac{(2-N)}{2}(1+|y|^2)^{-\frac{N}{2}} 2y_i \frac{\partial}{\partial y_i}(v(H(y),s)) \\ &+ 2^{\frac{N-2}{2}}(1+|y|^2)^{\frac{2-N}{2}} \frac{\partial^2}{\partial y_i \partial y_i}(v(H(y),s)).\end{aligned}$$

   Then for $y=0$ and denoting $a_{ii}(s) = \frac{\partial^2}{\partial y_i \partial y_i} v(H(y),s)\mid_{y=0}$:

$$\frac{\partial^2}{\partial^2 y_i} h(0,s) = 2^{\frac{N-2}{2}}\left[(2-N)a_0(s) + a_{ii}(s)\right] = 2^{\frac{N-2}{2}}\left[a_{ii}(s) - (N-2)a_0(s)\right].$$

So, finally the Taylor expansion of $w(y,s)$ in $y=0$:

$$\begin{aligned}w(y,s) = 2^{\frac{N-2}{2}}|y|^{N-2}&\Bigg[a_0(s) + \sum_{i=1}^{N} a_i(s) y_i + \frac{1}{2}\sum_{i\neq j; i,j=1}^{N} a_{ij}(s) y_i y_j \\ &+ \frac{1}{2}\sum_{i=j, i=1}^{N}(a_{ii}(s) - (N-2)a_0(s))\, y_i^2 + O(|y|^3)\Bigg].\end{aligned}$$

Undoing the change $y = \frac{x}{|x|^2}$:

$$\begin{aligned}w(x,s) = \frac{2^{\frac{N-2}{2}}}{|x|^{N-2}}&\Bigg[a_0(s) + \sum_{i=1}^{N} a_i(s)\frac{x_i}{|x|^2} \\ &+ \frac{1}{2}\left(\sum_{i\neq j; i,j=1}^{N} a_{ij}(s)\frac{x_i}{|x|^2}\frac{x_j}{|x|^2} + \sum_{i=j, i=1}^{N}(a_{ii}(s) - (N-2)a_0(s))\frac{x_i}{|x|^2}\frac{x_i}{|x|^2}\right) + O(|x|^{-3})\Bigg].\end{aligned}$$

By Einstein notation we arrive to (3.42). Analogously, we obtain the Taylor expansion in ∞ for its partial derivatives.

□

Theorem 3.5.1. *Let $s_0 > 0$ be such that $v(y,s)$, defined in (3.39), is positive and smooth in $\mathbb{S}^N \times (0,s_0)$, with $w(x,s)$ a positive solution of (3.37). Let us define $\overline{x}(s) = (\overline{x}_1(s),\dots,\overline{x}_N(s))$, with $\overline{x}_i(s) = \frac{1}{(N-2)}\frac{a_i(s)}{a_0(s)}$, to be the center of $w(x,s)$, for $s \in (0,s_0)$. Then given $0 < s_* < s_0$, there exists $C > 0$ constant such that:*

$$|\overline{x}(s)| \leq C, \quad \forall s \in (s_*, s_0).$$

Proceeding as in the elliptic case, we obtain under the hypotheses of Proposition 3.5.1 that for every $0 < s < s_0$, there exist constants $C_0, R_N > 0$ such that:

$$w_{x_1}(x,s) < 0, \quad \text{for } x_1 \geq \frac{C_0}{|x|} \text{ and } |x| \geq R_N. \tag{3.44}$$

And to prove Theorem 3.5.1, next we establish several intermediate results being analogous their proofs to their respective ones in the elliptic case.

Lemma 3.5.1. *Under the hypotheses of Theorem 3.5.1, for any $\lambda > 0$, there exists $R = R(\lambda)$ depending only on $\min\{1,\lambda\}$, such that, for $x = (x_1,x'), y = (y_1,y') \in \mathbb{R}^N$ satisfying:*

- $x_1 < y_1$.
- $x_1 + y_1 \geq 2\lambda$.
- $|x| \geq R$.

Then $w(x,s) > w(y,s)$, for $0 < s < s_0$.

Proof. See Lemma 3.3.1. □

Definition 3.5.1. *Under the hypotheses of Theorem 3.5.1, we define $w^\lambda(x,s) = w(x^\lambda,s)$, $\forall s \in (0,s_0)$, and, $x^\lambda = (2\lambda - x_1, x_2, \dots, x_N)$, the reflection around the hyperplane $x_1 = \lambda$, for $\lambda > 0$.*

Lemma 3.5.2. *Under the hypotheses of Theorem 3.5.1, there exists $\lambda_1 \geq 1$, such that, $\forall \lambda \geq \lambda_1$ and $0 < s < s_0$:*

$$w(x,s) \geq w^\lambda(x,s), \quad \text{for } x_1 < \lambda. \tag{3.45}$$

Proof. See Lemma 3.3.2. □

Lemma 3.5.3. *Under the hypotheses of Theorem 3.5.1, if for some $\lambda > 0$, we have $w^\lambda(x,s) \leq w(x,s)$, and, $w^\lambda(x,s) \not\equiv w(x,s)$, for, $x_1 < \lambda$, and $0 < s < s_0$. Then:*

$$w^\lambda(x,s) < w(x,s), \text{ for } x_1 < \lambda, \quad \text{and}, \quad w_{x_1}(x_1,s) < 0, \text{ for the hyperplane } x_1 = \lambda.$$

Proof. See Lemma 3.3.3. □

Proof of Theorem 3.5.1

Proof. Given any $0 < s_* < s_0$, let us take $\bar{s} \in [s_*, s_0)$, we can assume under a rotation/reflection if necessary that $\bar{x}_1(\bar{s}) = \max_i |\bar{x}_i(\bar{s})|$, and let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_N)$ be the reflection in the hyperplane $x_1 = \lambda$, for $\lambda > 0$. From Taylor expansions of w and its derivatives in infinity for $s = s_*$ and by Lemma 3.5.2, we obtain that there exists $\lambda_1 > 0$ such that $\forall \lambda \geq \lambda_1$:

$$w(x,s_*) > w^\lambda(x,s_*), \quad x_1 < \lambda.$$

So we have:

- $w(x,s_*) > w^\lambda(x,s_*)$, for $x_1 < \lambda$.
- w^λ is solution of (3.37), for $x_1 \leq \lambda$.
- $w(x,s) = w^\lambda(x,s)$, for $x_1 = \lambda$.

Then applying the Comparison Principle (Theorem 1.7.2), we get:

$$w(x,s) > w^\lambda(x,s), \quad \forall s \in [s_*, \bar{s}],\ x_1 < \lambda, \text{ for } \lambda_1 \leq \lambda. \tag{3.46}$$

Now, let consider the set $I = \left\{ \lambda > \lambda_1 \mid \lambda > \max_{s \in [s_*, \bar{s}]} \bar{x}_1(s), \text{ and (3.46) is verified} \right\}$, which satisfies:

- I is open: the proof is analogous to the elliptic case (Lemma 3.3.4).
- I is relatively close on (λ_1, ∞): let λ be on the closure of I relative to (λ_1, ∞), from (3.46), we have that $w^\lambda \leq w$, for $\lambda \geq \lambda_1$, but we also have $\lambda \geq \max_{s \in [s_*, \bar{s}]} \bar{x}_1(s)$ and we want to prove that $\lambda > \max_{s \in [s_*, \bar{s}]} \bar{x}_1(s)$, so let search it by contradiction.
  Let $\lambda = \max_{s \in [s_*, \bar{s}]} \bar{x}_1(s) = \bar{x}_1(\tilde{s_0})$, with $\tilde{s_0} \in [s_*, \bar{s}]$. By (3.39), we can define v and v^λ from w and w^λ, respectively, and both verify the equation (3.41). Then by the stereographic projection, the region $\{x \in \mathbb{R}^N \mid x_1 < \lambda\}$ becomes $\Omega_\lambda \in \mathbb{S}^N$, such that $q_0 \in \partial\Omega_\lambda$. Then

we have:

$$v^\lambda \leq v, \quad \text{in } \Omega_\lambda,$$
$$v^\lambda = v, \quad \text{in } \partial\Omega_\lambda,\ \forall s \in (s_0, \tilde{s_0}].$$

Then by the parabolic version of Hopf's lemma, we get that

$$\frac{\partial}{\partial \nu} v(y,s)\,|_{(y,s)=(q_0,\tilde{s_0})} < \frac{\partial}{\partial \nu} v^\lambda(y,s)\,|_{(y,s)=(q_0,\tilde{s_0})},$$

where $\frac{\partial}{\partial \nu}$ is the derivative along the outer normal to $\partial\Omega_\lambda$. However, as $\lambda = \overline{x}_1(\tilde{s_0})$, we also have $\frac{\partial}{\partial \nu} v(y,s)\,|_{(y,s)=(q_0,\tilde{s_0})} = \frac{\partial}{\partial \nu} v^\lambda(y,s)\,|_{(y,s)=(q_0,\tilde{s_0})}$, from where we obtain a contradiction.

We conclude that $I = (\lambda_1, \infty)$ and then, $\overline{x}_1(\overline{s}) < \lambda_1$, which yields the result. □

We conclude with the desired Harnack type estimate. We know that the standard Harnack estimate is defined locally on a ball, but here we are going to prove a stronger estimate in the whole sphere $\mathbb{S}^N$.

Proposition 3.5.2. Let $0 < s_* < s_0$, and $v(y,s)$ positive and smooth in $\mathbb{S}^N \times (0, s_0)$. Then there exists a constant $C > 0$ such that:

$$\min_{y \in \mathbb{S}^N} v(y,s) \geq C \max_{y \in \mathbb{S}^N} v(y,s), \quad \forall s \in (s_*, s_0). \tag{3.47}$$

Proof. • STEP 1. Given $0 < s_* < s_0$, let us prove that there exists a constant $C > 0$ such that:

$$\sup_{y \in \mathbb{S}^N} \frac{\left|\nabla_{\mathbb{S}^N} v(y,s)\right|}{v(y,s)} \leq C, \quad \forall s \in (s_*, s_0). \tag{3.48}$$

With this aim, let $q_0 \in \mathbb{S}^N$ which we assume is the north pole of $\mathbb{S}^N$ without loss of generality. By Theorem 3.5.1, we have that there exists a constant $C > 0$ such that $|\overline{z}(s)| \leq C$, $\forall s \in (s_*, s_0)$, where $\overline{z}_i(s) = (N-2)^{-1} \frac{a_i(s)}{a_0(s)}$, $i = 1, \ldots, N$, is the center of $w(x,s)$. From that and expansions (3.42) and (3.43), we get that there exists a constant $C > 0$ such that:

$$\frac{|\nabla w(x,s)|}{w(x,s)} \leq C, \quad s \in (s_*, s_0), \text{ for } |x| \text{ sufficiently large.}$$

By (3.40), we obtain that there is a constant $C > 0$ such that:

$$\frac{|\nabla_{\mathbb{S}^N} v(y,s)|}{v(y,s)} \leq C, \quad s \in (s_*, s_0), \text{ for } y \text{ in a neighbourhood of } q_0.$$

As $q_0 \in \mathbb{S}^N$ is arbitrary, by the compactness of the sphere, we have:

$$\sup_{y \in \mathbb{S}^N} \frac{|\nabla_{\mathbb{S}^N} v(y,s)|}{v(y,s)} \leq C, \quad s \in (s_*, s_0).$$

- STEP 2. Let $y_1, y_2 \in \mathbb{S}^N$ and γ be the geodesic joining y_1 and y_2. Then integrating for each s (3.48) along γ:

$$\int_\gamma \sup_{y \in \mathbb{S}^N} \frac{|\nabla_{\mathbb{S}^N} v(y,s)|}{v(y,s)} \, dy < \int_\gamma C \, dy.$$

If and only if:

$$\log(v(y_2,s)) - \log(v(y_1,s)) < C L(\gamma),$$

where $L(\gamma)$ is the length of γ.

To finish, if we take y_2 and y_1 to be the maximum and minimum of v on $\mathbb{S}^N$, respectively, then we obtain (3.47).

□

Chapter 4

Extinction profile for solutions of $u_t = \Delta u^{\frac{N+2}{N-2}}$

As we saw in the last section of the previous chapter, the Yamabe flow in $\mathbb{R}^N$ with standard metric gives rise in a natural way to the fast diffusion Cauchy problem (3.34), i.e.:

$$\begin{cases} u_t(x,t) = \Delta u^m(x,t), & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

with $N \geq 3$, $m = \frac{N-2}{N+2}$, $0 < m < 1$, and $u_0(x)$ non-negative, continuous, not identically zero and satisfying the fast decay condition:

$$\|u_0\|_* = \sup_{x\in\mathbb{R}^N} (1+|x|^{N+2}) u_0(x) < +\infty. \tag{4.2}$$

As pointed by [9] or [24], it can be proved that solutions of (4.1) are smooth and positive for $m > \frac{N-2}{N}$, while this is not the case for $m \leq \frac{N-2}{N}$. Indeed, for $m < \frac{N-2}{N}$, it can be easily proved that the functions

$$u(x,t) = \left[C|x|^{-2}(T-t)_+\right]^{\frac{1}{1-m}}$$

are non-smooth distributional solutions of (4.1), for a certain $C > 0$. Do note that these functions vanish after time $T > 0$. Moreover, by the comparison principle, these solutions can be used as upper barriers to establish that if $m < \frac{N-2}{N}$, $0 \leq u_0 \in L^\infty(\mathbb{R}^N)$, and

$$u_0(x) = O(|x|^{-\frac{2}{1-m}}), \quad \text{as } |x| \to \infty,$$

then there is a time $T > 0$ such that

$$u(x,t) \equiv 0, \quad \text{for } (x,t) \in \mathbb{R}^N \times (T,\infty). \tag{4.3}$$

We call the vanishing time of the solution u to the infimum of such numbers T. Let remark that assumption (4.2) implies the decay rate (4.3), and therefore, for any positive solution $u(x,t)$ of the problem (4.1), there is a vanishing time $T > 0$.

The rest of the chapter is dedicated to analyzing the asymptotic behaviour of the positive solutions to (4.1) near their vanishing time. Following the pioneering paper of Del Pino and Sáez [9],

Theorem 4.0.1. *Let $u_0(x)$ as in (3.34) which satisfies (3.35). Then there exists $\mu_0 > 0$, and, a point $\overline{x} \in \mathbb{R}^N$ such that:*

$$(T-t)^{-\frac{1}{1-m}} u(x,t) = \left(\frac{k_N \mu_0}{\mu_0^2 + |x-\overline{x}|^2} \right)^{\frac{N+2}{2}} + \vartheta(x,t), \tag{4.4}$$

with $\|\vartheta(\cdot,t)\|_* \underset{t\to T}{\longrightarrow} 0$, and where $T = T(u_0)$ is the vanishing time of the solution u to problem (3.34).

Remark 4.0.1. We talk about vanishing profile of u, because as $\frac{1}{1-m} > 0$, then, $(T-t)^{\frac{1}{1-m}} \underset{t\to T}{\longrightarrow} 0$, and we obtain:

$$u(x,t) = (T-t)^{\frac{1}{1-m}} \left[\left(\frac{k_N \mu_0}{\mu_0^2 + |x-\overline{x}|^2} \right)^{\frac{N+2}{2}} + \vartheta(x,t) \right] \underset{t\to T}{\longrightarrow} 0,$$

from where we get how u vanishes when it is near its vanishing time $T = T(u_0)$. Furthermore, the vanishing profile of u is determined by $(\mu_0, \overline{x}) = (\mu(u_0), \overline{x}(u_0))$, such that $u(x,t) = (T-t)^{\frac{1}{1-m}} \left[\left(\frac{k_N \mu_0}{\mu_0^2 + |x-\overline{x}|^2} \right)^{\frac{N+2}{2}} + \vartheta(x,t) \right]$, where $\mu_0 > 0$, and, $\overline{x} \in \mathbb{R}^N$.

Moreover, if we do not take into account the part of the equation which depends on t, we have the known "bubbles" as in (2.10), that we remember that are the solutions of the Yamabe equation, that is to say that the vanishing profile is a "bubble".

4.1 Short time positivity

The aim of this section is to first prove that every solution u of the problem (4.1) is positive and smooth for every short time.

4.1.1 Tools for the proof

Lemma 4.1.1. *Let u be the solution of (4.1). There exist t_0, η, $R_0 > 0$ and a point $x_0 \in \mathbb{R}^N$ such that:*

$$u(x,t) > \eta, \quad \forall t \in (0,t_0),\ x \in B_{R_0}(x_0). \tag{4.5}$$

Proof. Suppose that $u_0 \equiv 0$ non negative and continuous then by (4.2), there exists $\rho, \delta > 0$ and $x_1 \in \mathbb{R}^N$ such that for $x \in \mathscr{A}_\rho = \left\{x \mid \frac{\rho}{2} < |x - x_1| < \rho\right\}$, we have:

$$\delta < u_0(x). \tag{4.6}$$

Now let suppose that $g(x)$ is a solution of the problem:

$$\left\{ \begin{array}{ll} \Delta g^m(x) + \frac{1}{1-m} g(x) = 0, & \text{in } \mathscr{A}_\rho \; with \; g > 0 \\ g(x) = 0, & \text{in } \partial\mathscr{A}_\rho \end{array} \right\}, \tag{4.7}$$

that we know that it has a positive radially symmetric solution (see [5]).

If we define $\psi(x,t) = (\lambda - t)^{\frac{1}{1-m}} g(x)$, for $\lambda > 0$ and $\forall t \in (0,t_0)$, we get:

$$\begin{aligned} \psi_t(x,t) - \Delta\psi^m(x,t) &= -\frac{1}{1-m}(\lambda - t)^{\frac{1}{1-m}-1} g(x) - (\lambda - t)^{\frac{m}{1-m}} \Delta g^m(x) \\ &= -\frac{1}{1-m}(\lambda - t)^{\frac{m}{1-m}} g(x) + \frac{1}{1+m}(\lambda - t)^{\frac{m}{1-m}} g(x) = 0, \quad \forall x \in \mathscr{A}_\rho,\ \forall t \in (0,\lambda), \end{aligned}$$

where we have used (4.7). Taking $\lambda = \left(\frac{\delta}{\sup\limits_{g \in \mathscr{A}_\rho} g}\right)^{1-m} > 0$, we have:

$$\begin{aligned} \psi(x,0) &= \lambda^{\frac{1}{1-m}} g(x) = \frac{\delta}{\sup\limits_{g \in \mathscr{A}_\rho} g} g(x) \le \delta \underset{(4.6)}{<} u_0(x), \quad \text{in } \mathscr{A}_\rho, \\ \psi(x,t) &\underset{(4.7)}{=} 0 \le u(x,t), \quad \text{in } \partial\mathscr{A}_\rho. \end{aligned} \tag{4.8}$$

Then by the Comparison Principle (Theorem 1.7.2), we get $\psi(x,t) \le u(x,t)$, $\forall x \in \mathscr{A}_\rho$, $\forall t \in (0,\lambda)$.

Finally, choosing $t_0 = \frac{\lambda}{2}$, $0 < R_0 < \frac{\rho}{4}$, $x_0 \in \mathscr{A}_\rho$ with $|x_1 - x_0| = \frac{3}{4}\rho$, and taking $\eta = \inf\limits_{(x,t) \in B_{R_0}(x_0) \times [0,t_0]} \psi(x,t)$, we obtain (4.5). □

Now let see this lemma in which we are going to define a function which bounds u below and which we will use later for comparison purposes.

Lemma 4.1.2. *Let x_0, t_0 and R_0 be as in Lemma 4.1.1 and consider the function $\phi(x)$:*

$$\phi^m(x) = \frac{1}{|x-x_0|^{N-2}} + \frac{1}{|x-x_0|^{\alpha}}, \tag{4.9}$$

where $\alpha = (N-2)p-2 = N$. Then there is a constant $\eta_1 > 0$ such that:

$$u(x,t) \geq \eta_1 t^{\frac{1}{1-m}} \phi(x), \quad \forall t \in (0,t_0),\ |x-x_0| > R_0. \tag{4.10}$$

Proof. Firstly we define $\psi(x,t) = \eta_1 t^{\frac{1}{1-m}} \phi(x)$. Then, using the definition of ϕ on the hypothesis, we obtain:

$$\begin{aligned}
\psi_t(x,t) - \Delta\psi^m(x,t) &= \frac{1}{1-m}\eta_1 t^{\frac{1}{1-m}-1}\phi(x) - \eta_1^m t^{\frac{m}{1-m}} \Delta\phi^m(x) \\
&= \frac{1}{1-m}\eta_1 t^{\frac{m}{1-m}}\phi(x) - \eta_1^m t^{\frac{m}{1-m}} \Delta\left(\frac{1}{|x-x_0|^{N-2}} + \frac{1}{|x-x_0|^{N}}\right) \\
&= \frac{1}{1-m}\eta_1 t^{\frac{m}{1-m}}\phi(x) - \eta_1^m t^{\frac{m}{1-m}} \Delta|x-x_0|^{2-N} - \eta_1^m t^{\frac{m}{1-m}} \Delta|x-x_0|^{-N}.
\end{aligned}$$

Knowing that for $\beta > 0$:

$$\frac{\partial}{\partial x_i}|x-x_0|^{\beta} = \beta|x-x_0|^{\beta-2}(x_i - x_{0\,i}).$$

and,

$$\frac{\partial^2}{\partial^2 x_i}|x-x_0|^{\beta} = \beta(\beta-2)|x-x_0|^{\beta-4}(x_i - x_{0\,i}).$$

Then:

$$\begin{cases} \Delta|x-x_0|^{2-N} = 0, \\ \Delta|x-x_0|^{-N} = 2N|x-x_0|^{-N-2}. \end{cases}$$

If we return to the calculation and by the definition of ϕ in (4.9): Finally we have:

$$\psi_t(x,t) - \Delta\psi^m(x,t) = \eta_1^m t^{\frac{m}{1-m}} \frac{1}{|x-x_0|^{N+2}} \left[\frac{\eta_1^{1-m}}{1-m}\left(1 + \frac{1}{|x-x_0|^2}\right)^{\frac{1}{m}} - 2N\right].$$

So there exists $\eta_1 = \eta_1(t_0, \eta_0, R_0)$ small, such that:

$\psi(x,0) = 0 \leq u_0(x)$, by hypothesis.

$\psi(x,t) - \Delta\psi^m(x,t) \leq 0$, for $|x-x_0| \geq R_0$, and, $\forall t \in (0,t_0)$.

$\psi(x,t) \leq \eta_0$, for $|x-x_0| = R_0$, and, $\forall t \in (0,t_0)$.

Then by the Comparison Principle 1.7.2, we have $\psi(x,t) \leq u(x,t)$, $\forall x \in \mathbb{R}^N$ with $|x-x_0| \geq R_0$, and $\forall t \in (0,t_0)$, so finally we get the inequality (4.10). □

4.1.2 Short time positivity

Now remembering from Section 3.5, how we pass from the problem (4.1) in $\mathbb{R}^N$ to a problem in $\mathbb{S}^N$ by means of the transformations (3.36), which define the function $w(x,s)$ for $(x,s)\in\mathbb{R}^N\times(0,\infty)$, and (3.39), which define $v(y,s)$ for $(y,s)\in\mathbb{S}^N\times(0,\infty)$. The principal result of this section is the following proposition:

Proposition 4.1.1. There exists $s_*>0$ such that:

$$v(y,s)>0, \quad \forall s\in(0,s_*],\ \forall y\in\mathbb{S}^N.$$

Moreover, $v\in C^\infty\left(\mathbb{S}^N\times(0,s_*)\right)$.

Proof. 1. $\underline{v(y,s)>0,\ \forall s\in(0,s_*],\ \forall y\in\mathbb{S}^N.}$

By Lemma 4.1.1 and Lemma 4.1.2, we have that the solution $u(x,t)>0$, $\forall t\in(0,t_0]$. If we take $s_*=\log\left(\frac{T}{t_0}\right)$, $\forall s\in(0,s_*]$:

- For $|x-x_0|\geq R_0$, by Lemma 4.1.2,

$$u(x,t)\geq\eta_1 t^{\frac{1}{1-m}}\phi^m(x).$$

If and only if, using (4.9):

$$u^m(x,t)\geq\eta_1^m t^{\frac{m}{1-m}}\left(\frac{1}{|x|^{N-2}}+\frac{1}{|x|^\alpha}\right).$$

Taking $t=T(1-e^{-s})$, from (3.36) we have $u^m(x,s)=(T-T(1-e^{-s}))^{\frac{m}{1-m}}w(x,s)=(e^{-s})^{\frac{m}{1-m}}w(x,s)$, then:

$$w(x,s)\geq\eta_1^m\left(e^s\right)^{\frac{m}{1-m}}\left(T(1-e^{-s})\right)^{\frac{m}{1-m}}\left(\frac{1}{|x|^{N-2}}+\frac{1}{|x|^\alpha}\right)=\eta_1^m\left(T(e^s-1)\right)^{\frac{m}{1-m}}\left(\frac{1}{|x|^{N-2}}+\frac{1}{|x|^\alpha}\right).$$

From (3.39) we have $w(x,s)=\left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}}v(F(x),s)$, so:

$$v(F(x),s)\geq\eta_1^m\left(T(e^s-1)\right)^{\frac{m}{1-m}}\left(\frac{1}{|x|^{N-2}}+\frac{1}{|x|^\alpha}\right)\left(\frac{1+|x|^2}{2}\right)^{\frac{N-2}{2}}:=P(s)>0,$$

where $P(s)$ is an increasing function.

- For $|x-x_0|<R_0$, by Lemma 4.1.1, we have:

$$u^m(x,t)>\eta^m.$$

By the definition of w in (3.36):

$$w(x,s) \geq \eta^m (e^s)^{\frac{m}{1-m}}.$$

By the definition of v in (3.39):

$$v(F(x),s) \geq \eta^m (e^s)^{\frac{m}{1-m}} \left(\frac{1+|x|^2}{2}\right)^{\frac{N-2}{2}}.$$

If and only if by the definition of the stereographic projection F, we get:

$$v(F^{-1}(x),s) \geq \eta^m (e^s)^{\frac{m}{1-m}} \geq \tilde{\eta} > 0.$$

Finally, we have:

$$v(y,s) \geq \min\{\tilde{\eta}, P(s)\} > 0, \quad \forall s \in (0,s_*], \ \forall y \in \mathbb{S}^N. \tag{4.11}$$

2. $\underline{v \in C^\infty(\mathbb{S}^N \times (0,s_*])}.$

As (4.2) holds for u_0, we have $\sup_{x\in\mathbb{R}^N}(1+|x|^{N+2})u_0(x) < +\infty$, then exists $M > 0$ sufficiently large such that:

$$\sup_{x\in\mathbb{R}^N} \left(1+|x|^{N+2}\right) u_0(x) < M.$$

If and only if for a certain k_N, we have:

$$u_0(x) \leq M^{\frac{1}{1-m}} \left(\frac{k_N}{1+|x|^2}\right), \quad \forall t \in (0,T).$$

Knowing by (3.38) that the right side of the inequality is also solution of (4.1) for $\mu = 1$, then by the Comparison Principle (Theorem 1.7.2) we get:

$$u(x,t) \leq (M-t)^{\frac{1}{1-m}} \left(\frac{k_N}{1+|x|^2}\right), \quad \forall t \in (0,T).$$

Then:

$$u^m(x,t) \leq (M-t)^{\frac{m}{1-m}} \left(\frac{k_N}{1+|x|^2}\right)^m.$$

From (3.36):

$$w(x,s) \leq (e^s)^{\frac{m}{1-m}} (M - T(1-e^{-s}))^{\frac{m}{1-m}} \left(\frac{k_N}{1+|x|^2}\right)^m = (Me^s - T(e^s-1))^{\frac{m}{1-m}} \left(\frac{k_N}{1+|x|^2}\right)^m.$$

From (3.39):

$$v(F(x),s) \leq (Me^s - T(e^s-1))^{\frac{m}{1-m}} \left(\frac{k_N}{1+|x|^2}\right)^m \left(\frac{1+|x|^2}{2}\right)^{\frac{N-2}{2}} := A(s),$$

where $A(s)$ is a continuous function. So we have obtained:

$$v(y,s) \leq A(s), \quad \forall s > 0, \forall y \in \mathbb{S}^N. \tag{4.12}$$

Finally, we can write (4.1) as:

$$u_t(x,t) = mu^{m-1}\Delta u + m(m+1)u^{m-2}|\nabla u|^2 = \operatorname{div}\left(mu^{m-1}\nabla u\right) \tag{4.13}$$

This equation satisfies (1.9) with $a_{ij}(x,t) = mu^{m-1}u_{x_i}$ and $b_i(x,t) = 0$, with $i,j = 1,\dots,N$. Furthermore, this equation will be uniformly parabolic if it verifies (1.10), that is $C_1|\xi|^2 \leq mu^{m-1}|\xi|^2 \leq C_2|\xi|^2$ on $\mathbb{R}^N \times (0,\infty) \times (0,\infty) \times \mathbb{R}^N$. Now, by (4.11) and (4.12), we have:

$$0 < \min\{\overline{\eta}, P(s)\} \leq v(y,s) < A(s), \quad \forall s \in (0,s_*],\ y \in \mathbb{S}^N,$$

where $x_0 \in \mathbb{R}^N$ is arbitrary. By (3.37):

$$0 < \min\{\overline{\eta}, P(s)\} \leq v(F(x),s) = \left(\frac{2}{1+|x|^2}\right)^{\frac{2}{N-2}} w(x,s) < A(s).$$

From where we obtain:

$$\begin{aligned}
0 < B^*(s) = C\min\{\overline{\eta}, P(s)\} &\leq \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} \min\{\overline{\eta}, P(s)\} \\
&\leq w(x,s) = (T-t)^{-\frac{m}{1-m}}\, u^m(x,t)\,|_{t=T-e^{-s}} < A(s)\left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} \leq A(s)2^{\frac{N-2}{2}} = A^*(s),
\end{aligned}$$

where $x \in B(x_0,R)$, $t \in [0,T^*]$, $T^* \leq T$. We know that $s = -\ln\left(1-\frac{t}{T}\right) > 0$ and it increases for $t \in [0,T]$, then doing this change of variable we get:

$$0 < B^*\left(-\ln\left(1-\frac{t}{T}\right)\right)(T-t)^{\frac{m}{1-m}} \leq u^m(x,t)\,|_{t=T-e^{-s}} \leq A^*\left(-\ln\left(1-\frac{t}{T}\right)\right)(T-t)^{\frac{m}{1-m}},$$

for $x \in B(x_0,R)$, and $t \in [0,T^*]$. This implies that there exist constants $0 < \alpha, \beta < \infty$, such that:

$$\alpha \leq u(x,t) \leq \beta, \quad \forall x \in B(x_0,R),\ R > 0,\ t \in [t_0,t_1],\ 0 < t_0 < t_1 \leq T.$$

As mu^{m-1}, it decreases for $u > 0$, because $m < 1$, we have that $m\beta^{m-1} \leq mu^{m-1} \leq m\alpha^{m-1}$. Then we obtain:

$$m\beta^{m-1}|\xi|^2 \leq mu^{m-1}|\xi|^2 \leq m\alpha^{m-1}|\xi|^2,$$

from where we have that (4.13) is uniformly parabolic in $B(x_0,R) \times [t_0,t_1] \times (0,\infty) \times \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$, $0 < t_0 < t_1 \leq T^* < T$, and, $R < 0$. By Theorem 1.7.1, we have that $u(x,t) \in C^\infty(B(x_0,R) \times [t_0,t_1])$, $\forall x \in B(x_0,R)$, $R > 0$, and, $0 < t_0 < t_1 \leq T^*$. Then $v(y,s) \in C^\infty\left(\mathbb{S}^N \times (0,s^*)\right)$, so v is positive and smooth for $s \in (0,s_*)$. □

4.2 Some estimates

4.2.1 Uniform estimates

Let see the following proposition which bounds v below and above in order to discard the cases when $v(y,s) \underset{s\to\infty}{\longrightarrow} \infty$ uniformly and $v(y,s) \underset{s\to\infty}{\longrightarrow} 0$ uniformly.

Proposition 4.2.1. Let $s_* > 0$, then there exist constants $\alpha, \beta > 0$ such that:

$$\alpha \le v(y,s) \le \beta, \quad \forall y \in \mathbb{S}^N,\ s \in [s_*, \infty).$$

Proof. We divide the proof into three steps:

- STEP 1. We are going to prove that if $v(y,s)$ is positive and smooth $\forall s \in (0, s_0)$, then there exists constant $k_1 > 0$ such that:

$$\max_{y\in\mathbb{S}^N} v(y,s) \ge k_1. \tag{4.14}$$

  We are going to prove it by contradiction, so let suppose that for each $\varepsilon > 0$, there is a $s_\varepsilon > s_*$ such that, $v(y,s_\varepsilon) < \varepsilon$, $\forall y \in \mathbb{S}^N$. Now let define for a given $\varepsilon > 0$, and, a constant $K > 0$:

$$U(s) = K(1+s_\varepsilon - s)_+^{\frac{m}{1-m}}.$$

  Then, if we remember that $p = \frac{1}{m}$:

$$\begin{aligned}
&\left(U^p(s)\right)_s + C(N)U(s) - \frac{1}{1-m}U^p(s) \\
&\quad = -\frac{K^{\frac{1}{m}}}{1-m}(1+s_\varepsilon - s)_+^{\frac{m}{1-m}} + C(N)K(1+s_\varepsilon-s)_+^{\frac{m}{1-m}} - \frac{K^{\frac{1}{m}}}{1-m}(1+s_\varepsilon-s)_+^{\frac{1}{1-m}} \\
&\quad = \frac{K^{\frac{1}{m}}}{1-m}(1+s_\varepsilon-s)_+^{\frac{m}{1-m}}\left(-1+(1-m)C(N)K^{1-\frac{1}{m}} - (1+s_\varepsilon - s)\right) \\
&\quad = \frac{K^{\frac{1}{m}}}{1-m}(1+s_\varepsilon-s)_+^{\frac{m}{1-m}}\left[K^{1-\frac{1}{m}}\left((1-m)C(N) - K^{\frac{1}{m}-1}\right) - (1+s_\varepsilon-s)\right] \\
&\quad = \frac{K^p}{1-m}(1+s_\varepsilon-s)_+^{\frac{m}{1-m}}\left[K^{1-p}\left((1-m)C(N) - K^{p-1}\right) - (1+s_\varepsilon - s)\right].
\end{aligned}$$

  This expression is non-negative if $K < [(1-m)C(N)]^{1-p}$, and, $s > s_\varepsilon$. Then $U(s)$ is a solution of (3.41) for $s > s_\varepsilon$. Moreover,

  * Choosing $\varepsilon > 0$ small, we can get $\varepsilon < U(s_\varepsilon)$.
  * By hypothesis we have that $v(y, s_\varepsilon) < \varepsilon$, $\forall y \in \mathbb{S}^N$.

So we get that $v(y,s) < U(s)$, for $s > s_\varepsilon$. Therefore, when $U(s)$ vanishes for $s > s_\varepsilon + 1$, then v vanishes, but by the definition of v in (3.39) this means that u vanishes before $t = T$, that is a contradiction. So (4.14) holds.

- STEP 2. We are going to prove that $v(y,s)$ is strictly positive and smooth $\forall s \in (0,\infty)$. We begin by defining:

$$s_0 = \sup_{s>0}\left\{ \inf_{y\in\mathbb{S}^N} v(y,\tau) > 0 \mid \forall 0 < \tau < s \right\}. \tag{4.15}$$

Then we have that $v(y,s)$ is positive and smooth $\forall s \in (0,s_0)$. Secondly, by (4.12):

$$v(y,s) \le A(s), \quad \forall s \in (0,\infty).$$

where $A(s)$ is a continuous function. Moreover, as v satisfies a parabolic semi linear equation, then we have that $v(y,s)$ is smooth $\forall s \in (0,s_0)$. Our objective is to prove that $s_0 = \infty$ by contradiction.

To start, we suppose that there exists a constant $k > 0$ such that $v(y,s) \ge k > 0$, $\forall s \in (0,s_0)$. By the Harnack estimate (3.47), we have that there exists a sequence $s_n > 0$ such that:

$$0 \le v(y,s_n), \quad \forall s \in (s_n,s_0).$$

On the other hand, by Proposition 4.1.1:

$$v(y,s) > 0, \quad \forall s \in (0,s_0].$$

So we obtain $v(y,s) = 0$, for s sufficient large.

By (4.12):

$$v(y,s) \le A(s), \quad \forall s \in (0,s_0).$$

By the definition of s_0 (4.15) we have that $v(y,s)$ is positive and applying Theorem 1.7.1 as we did in the proof of Proposition 4.1.1, we have that $v(y,s)$ is smooth for some time interval further than s_0, which contradicts (4.15), the definition of s_0. So we conclude that $s_0 = \infty$.

- STEP 3. There exists a constant $k_2 > 0$ such that:

$$\min_{y\in\mathbb{S}^N} v(y,s) \le k_2, \quad \forall s \in (0,\infty). \tag{4.16}$$

Let $\overline{s}>0$, we define the function:

$$f(s)=\left(e^{s-\overline{s}}+C(N)(1-m)\right)^{\frac{m}{1-m}}, \tag{4.17}$$

First, we are going to prove that f satisfies the equation:

$$(f^p(s))_s=-C(N)f(s)+\frac{1}{1-m}f^p(s). \tag{4.18}$$

We know that:

* $(f^p(s))_s=\frac{1}{1-m}e^{s-\overline{s}}\left[e^{s-\overline{s}}+C(N)(1-m)\right]^{\frac{1}{1-m}-1}$.
* $-C(N)f(s)=-C(N)\left[e^{s-\overline{s}}+C(N)(1-m)\right]^{\frac{1}{1-m}-1}$.
* $\frac{1}{1-m}f^p(s)=\frac{1}{1-m}\left[e^{s-\overline{s}}+C(N)(1-m)\right]^{\frac{1}{1-m}}$.

Then (4.18) becomes:

$$\begin{aligned}
&(f^p(s))_s+C(N)f(s)-\frac{1}{1-m}f^p(s)\\
&\quad=\frac{1}{1-m}e^{s-\overline{s}}\left[e^{s-\overline{s}}+C(N)(1-m)\right]^{\frac{1}{1-m}-1}+C(N)\left[e^{s-\overline{s}}+C(N)(1-m)\right]^{\frac{1}{1-m}-1}\\
&\qquad-\frac{1}{1-m}\left[e^{s-\overline{s}}+C(N)(1-m)\right]^{\frac{1}{1-m}}\\
&\quad=\left(e^{s-\overline{s}}+C(N)(1-m)\right)^{\frac{1}{1-m}}\left[\left(e^{s-\overline{s}}+C(N)(1-m)\right)^{-1}\left(\frac{1}{1-m}e^{s-\overline{s}}+C(N)\right)-\frac{1}{1-m}\right]\\
&\quad=\left(e^{s-\overline{s}}+C(N)(1-m)\right)^{\frac{1}{1-m}}\left[\left(e^{s-\overline{s}}+C(N)(1-m)\right)^{-1}\right.\\
&\qquad\left.\left(\frac{1}{1-m}e^{s-\overline{s}}+C(N)-\frac{1}{1-m}e^{s-\overline{s}}-C(N)\right)\right].
\end{aligned}$$

If and only if:

$$\left(f^p(s)\right)_s+C(N)f(s)-\frac{1}{1-m}f^p(s)=0.$$

Then f is a solution for the equation (3.41).

To continue, we suppose that there exists $\overline{s}$ such that:

$$\min_{y\in\mathbb{S}^N}v(y,\overline{s})>f(\overline{s})=(1+C(N)(1-m))^{\frac{m}{1-m}},$$

so by the Comparison principle 1.7.2, we have:

$$v(y,s)>f(s),\quad \forall y\in\mathbb{S}^N,\forall s\in(\overline{s},\infty).$$

By (3.39) we know that $v(F(x),s)=\left(\frac{2}{1+|x|^2}\right)^{-\frac{(N-2)}{2}}w(x,s)$, so:

$$w(x,s)>\left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}}f(s),\quad \forall x\in\mathbb{R}^N,\forall s\in(\overline{s},\infty).$$

By (3.36), $w(x,s) = (Te^{-s})^{-\frac{m}{1-m}} u^m(x, T(1-e^{-s})$, and by (4.17) we have:

$$u^m\left(x, T(1-e^{-s})\right) > \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} (Te^{-s})^{-\frac{1-m}{m}} \left(e^{s-\bar{s}} + C(N)(1-m)\right)^{\frac{m}{1-m}}.$$

If and only:

$$u^m\left(x, T(1-e^{-s+\bar{s}})\right) > \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} \left[T\left(e^{-\bar{s}} + e^{-s}C(N)(1-m)\right)\right]^{\frac{m}{1-m}}.$$

Taking the everything to the power of $\frac{1}{m}$:

$$u\left(x, T(1-e^{-s})\right) > \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}\frac{1}{m}} \left[T\left(e^{-\bar{s}} + e^{-s}C(N)(1-m)\right)\right]^{\frac{1}{1-m}}.$$

If and only if:

$$u\left(x, T(1-e^{-s})\right) > \left(\frac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} \left[T\left(e^{-\bar{s}} + e^{-s}C(N)(1-m)\right)\right]^{\frac{1}{1-m}}, \quad \forall x \in \mathbb{R}^N, \forall s \in (\bar{s}, \infty).$$

Then when $t \to T$, there exists a constant $k > 0$ such that:

$$u(x,t) > \left(\frac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} k, \quad \forall x \in \mathbb{R}^N, \forall s \in (\bar{s}, \infty).$$

But this contradicts the fact that $u(x,t) = 0$ after time T.

In conclusion, by the Harnack estimate (3.47), for a given $s_* > 0$ and $s_0 = \infty$ (we can choose it by STEP 2), we have that there exists a constant $C > 0$ such that:

$$\min_{y \in \mathbb{S}^N} v(y,s) \geq C \max_{y \in \mathbb{S}^N} v(y,s), \quad \forall s \in (s_*, \infty).$$

By STEP 1, we have that there exists a constant $k_1 > 0$ such that $k_1 \leq \max_{y \in \mathbb{S}^N} v(y,s)$, and by STEP 3, we have that there exists $k_2 > 0$ such that $\min_{y \in \mathbb{S}^N} v(y,s) \leq k_2$. To finish, taking $\alpha = \frac{k_1}{C}$ and $\beta = Ck_2$, we obtain the result. □

Remark 4.2.1. As we have said at the beginning of this section, we have discarded the cases when $v(y,s) \xrightarrow[s\to\infty]{} \infty$ uniformly and $v(y,s) \xrightarrow[s\to\infty]{} 0$ uniformly. On STEP 1, we defined a large function such that it bounds the solution of the equation and we did a comparison to get that the solution v vanishes before its vanish time. On STEP 3, we have used the same tool, that is to say that we defined another function which bounds the solution and which allows us to use the Comparison Principle to obtain that the solution vanishes after its vanish time.

4.2.2 Lyapunov estimate

Definition 4.2.1. *We define the Lyapunov functional for a function $z(\cdot,s)$, depending only on s:*

$$J_z = \frac{1}{2}\int_{\mathbb{S}^N} \left|\nabla_{\mathbb{S}^N} z\right|^2 dy + \frac{C(N)}{2}\int_{\mathbb{S}^N} z^2\, dy - \frac{1}{(1-m)(p+1)}\int_{\mathbb{S}^N} z^{p+1}\, dy.$$

It holds:

Lemma 4.2.1. *Let v be the solution of* (3.41)*, then we have:*

$$J_v(s) \geq 0, \quad \forall s > 0.$$

Proof. We are going to prove this by contradiction, so let suppose that there is one $s_0 \in (0,\infty)$ such that:

$$J_v(s_0) < 0, \quad \forall y \in \mathbb{S}^N.$$

To start, let define:

$$F(s) = \int_{\mathbb{S}^N} v^{p+1}(y,s)\, dy \geq 0, \quad s \in (0,\infty). \tag{4.19}$$

Differentiating over s and by an integration by parts:

$$\begin{aligned}
\frac{1}{p+1}\frac{d}{ds}F(s) &= \int_{\mathbb{S}^N} v^p(y,s)(v(y,s))_s\, dy \\
&= -\int_{\mathbb{S}^N} \left(v^p(y,s)\right)_s v(y,s)\, dy \\
&\underset{(3.41)}{=} -\int_{\mathbb{S}^N} \left(\Delta_{\mathbb{S}^N} v(y,s) - C(N)v(y,s) + \frac{1}{1-m}v^p(y,s)\right) v(y,s)\, dy \\
&= \int_{\mathbb{S}^N} \left|\nabla_{\mathbb{S}^N} v(y,s)\right|^2 v(y,s)\, dy + C(N)\int_{\mathbb{S}^N} v^2(y,s)\, dy - \frac{1}{1-m}\int_{\mathbb{S}^N} v^{p+1}(y,s)\, dy
\end{aligned}$$

If and only if, from Definition 4.2.1:

$$\begin{aligned}
\frac{p}{p+1}\frac{d}{ds}F(s) &= -2J_v(s) + \frac{1}{1-m}\left(1 - \frac{2}{p+1}\right)\int_{\mathbb{S}^N} v^{p+1}(y,s)\, dy \\
&\underset{(4.19)}{=} -2J_v(s) + \frac{p-1}{(1-m)(p+1)}F(s) > \frac{p-1}{(1-m)(p+1)}F(s), \quad \forall s > s_0.
\end{aligned}$$

So we get:

$$\frac{p(1-m)}{(p-1)}\frac{d}{ds}F(s) > F(s), \quad \forall s > s_0.$$

As $\frac{p(1-m)}{(p-1)} = 1$:

$$\frac{d}{ds}F(s) > F(s), \quad \forall s > s_0.$$

If there exists one s such that $F(s)=0$, then $v(y,s)=0$ and $J_v(s)(s)=0$, which is a contradiction with respect to the hypothesis $J_v(s)\le J_v(s_0)<0$. So we have $F(s)\neq 0$, $\forall s\ge s_0$, and $F(s_0)>0$. Integrating between s_0 and s:

$$F(s)>F(s_0)e^{s-s_0}, \quad \forall s>s_0.$$

From where we have that $F(s)\underset{s\to\infty}{\longrightarrow}\infty$. However, by Proposition 4.2.1, we have that $v(y,s)$ is bounded $\forall s>s_0$ which implies that $F(s)$ is also bounded $\forall s>s_0$, thus we have a contradiction. □

The following lemma shows that the Lyapunov functional is decreasing over the trajectories:

Lemma 4.2.2. *Let v be the solution of (3.41), and let J_v be the Lyapunov functional, then we have:*

$$\frac{d}{ds}J_v(s)=-\frac{4p}{(p+1)^2}\int_{\mathbb{S}^N}\left|\left(v^{\frac{p+1}{2}}(y,s)\right)_s\right|^2 dy. \tag{4.20}$$

Proof. On one side, we know that:

$$\frac{d}{ds}\left[\frac{1}{2}\int_{\mathbb{S}^N}\left|\nabla_{\mathbb{S}^N}v(y,s)\right|^2 dy\right]=\frac{d}{ds}\left[\frac{1}{2}\int_{\mathbb{S}^N}\sum_{i=1}^{N}\frac{\partial v^2(y,s)}{\partial x_i}dy\right]=\int_{\mathbb{S}^N}\sum_{i=1}^{N}\frac{\partial v(y,s)}{\partial x_i}\frac{d}{ds}\left(\frac{\partial v}{\partial x_i}\right)dy$$
$$=\int_{\mathbb{S}^N}\nabla_{\mathbb{S}^N}v(y,s)\cdot\nabla_{\mathbb{S}^N}(v(y,s))_s\,dy.$$

As $\mathbb{S}^N\subset\mathbb{R}^{N+1}$ is compact, we can apply the Dominated convergence theorem, so we can differentiate J in the Definition 4.2.1 with respect to s:

$$\frac{d}{ds}J_v(s)=\int_{\mathbb{S}^N}\nabla_{\mathbb{S}^N}v(y,s)\cdot\nabla_{\mathbb{S}^N}(v(y,s))_s\,dy+C(N)\int_{\mathbb{S}^N}\nabla(v(y,s))_s\,dy$$
$$-\frac{1}{(1-m)}\int_{\mathbb{S}^N}v^p(y,s)(v(y,s))_s\,dy.$$

Integrating by parts:

$$\frac{d}{ds}J_v(s)=-\int_{\mathbb{S}^N}\left(\Delta_{\mathbb{S}^N}v(y,s)-C(N)v(y,s)+\frac{1}{(1-m)}v^p(y,s)\right)(v(y,s))_s\,dy$$
$$\underset{(3.41)}{=}-\int_{\mathbb{S}^N}\left(v^p(y,s)\right)_s(v(y,s))_s\,dy=-p\int_{\mathbb{S}^N}v^{p-1}(y,s)\left((v(y,s))_s\right)^2 dy.$$

Then:

$$\frac{d}{ds}J_v(s)=-p\int_{\mathbb{S}^N}v^{p-1}(y,s)\left((v(y,s))_s\right)^2 dy. \tag{4.21}$$

On the other side, we have:

$$-\frac{4p}{(p+1)^2}\int_{\mathbb{S}^N}\left[\left(v^{\frac{p+1}{2}}(y,s)\right)_s\right]^2 dy = -\frac{4p}{(p+1)^2}\int_{\mathbb{S}^N}\left[\frac{p+1}{2}v^{\frac{p+1}{2}-1}(y,s)(v(y,s))_s\right]^2 dy$$
$$= -p\int_{\mathbb{S}^N}\left[v^{\frac{p-1}{2}}(y,s)(v(y,s))_s\right]^2 dy.$$

By (4.21), we obtain (4.20). □

4.3 Main results

4.3.1 The vanishing profile of u

Instead of proving Theorem 4.0.1, we know that by (3.39) this theorem can be written in terms of v as the following theorem that we are going to prove:

Theorem 4.3.1. *Let v as in (3.39). Then there exists a unique solution $\overline{v}$ non-trivial and positive of the equation:*

$$\Delta_{\mathbb{S}^N}\overline{v}(y) - C(N)\overline{v}(y) + \frac{1}{1-m}\overline{v}^p(y) = 0, \quad \forall y \in \mathbb{S}^N, \tag{4.22}$$

and that verifies:

$$\sup_{y\in\mathbb{S}^N}|v(y,t)-\overline{v}(y)| \underset{t\to T}{\longrightarrow} 0. \tag{4.23}$$

Proof.

- STEP 1. We are going to prove the existence of $\overline{v}$ such that it verifies (4.23). By Proposition 4.2.1, we have that there exist constants $\alpha,\beta > 0$ such that $\alpha \leq v(y,s) \leq \beta$, away from $s = 0$, then by the Definition 1.7.1 of quasi-linear equation that are uniformly parabolic, $v(y,s)$ is uniformly parabolic away from $s = 0$. By the Theorem 1.7.1, we have uniform estimates for $C^k(\mathbb{S}^N)$, $\forall k > 0$, then $\forall s \in [s_*,\infty)$ with $s_* > 0$, $y \mapsto v(y,s)$ is compact in $C^k(\mathbb{S}^N)$, $\forall k > 0$, and we can define:

  $$\overline{v}(y) = \lim_{s\to\infty} v(y,s) \quad \text{in } C^2(\mathbb{S}^N).$$

  But this limit is not unique. In STEP 3 we will prove uniqueness.

  It follows from Proposition 4.2.1 that $\overline{v}$ verifies (4.23).

- STEP 2. We are going to show that $\overline{v}$ is a solution of the equation (4.22).

  Firstly, we can apply Lemma 4.2.2, which implies :

  $$J_v(s) = -\frac{4p}{(p+1)^2}\int_{\mathbb{S}^N}\left|v^{\frac{p+1}{2}}(y,s)\right|^2 dy. \tag{4.24}$$

  Then let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{s_n\to\infty} v(y,s_n) = \overline{v}(y)$, in $C^2(\mathbb{S}^N)$, integrating this with respect to s from s_n to $s_n+\tau$, with $\tau > 0$, we get:

  $$\begin{aligned}\int_{s_n}^{s_n+\tau}\int_{\mathbb{S}^N}\left|\left(v^{\frac{p+1}{2}}(y,s)\right)_s\right|^2 dyds &= \int_{\mathbb{S}^N}\left|v^{\frac{p+1}{2}}(y,s_n+\tau) - v^{\frac{p+1}{2}}(y,s_n)\right|^2 dy \\ &\le -\frac{(p+1)^2\tau}{4p}[J_v(s_n+\tau)-J_v(s_n)] = \frac{(p+1)^2\tau}{4p}[J_v(s_n)-J_v(s_n+\tau)],\end{aligned}$$

  where we have used Cauchy-Schwarz's inequality and (4.24) again. By Lemma 4.2.1, $J_v(s)$ is decreasing and bounded from below, then it has a limit when $s\to\infty$. This implies that for each $\tau > 0$, we have $\lim_{s_n\to\infty} v(y,s_n+\tau) = \overline{v}(y)$ in $C^2(\mathbb{S}^N)$, and we also have $\lim_{s_n\to\infty} v(y,s_n+\tau) = \overline{v}(y)$ uniformly, for τ in bounded intervals.
  To continue, if we take $\varphi\in C^\infty(\mathbb{S}^N)$ and by (3.41):

  $$\begin{aligned}&\int_{\mathbb{S}^N}\left(v^p(y,s_n+1)-v^p(y,s_n)\right)\varphi(y)dy \\ &\quad = \int_{s_n}^{s_n+1}\int_{\mathbb{S}^N}\left(\Delta_{\mathbb{S}^N}v(y,s_n+\tau) - C(N)v(y,s_n+\tau) + \frac{1}{1-m}v^p(y,s_n+\tau)\right)\varphi(y)dyd\tau.\end{aligned}$$

  From the above, when $n\to\infty$, we obtain:

  $$0 = \int_{\mathbb{S}^N}\left(\overline{v}^p(y)-\overline{v}^p(y)\right)\varphi(y)dy = \int_{\mathbb{S}^N}\left(\Delta_{\mathbb{S}^N}\overline{v}(y) - C(N)\overline{v}(y) + \frac{1}{1-m}\overline{v}^p(y)\right)\varphi(y)dy.$$

  Then $\overline{v}$ verifies (4.22).

- STEP 3. $\overline{v}$ is unique.

  By Corollary 1.7.1 and Corollary 1.7.2, we have that the limit point $\overline{v}$ of this parabolic equation is unique.

□

4.3.2 Continuity of vanishing time and vanishing profile

Theorem 4.3.2. *$T(u_0)$, μ_0 and $\overline{x}$ are continuous functions of u_0 for the norm $\|\cdot\|_*$ in (4.2).*

Proof. Firstly, let us take $u_0^n(x)$, $\forall x \in \mathbb{R}^N$, a sequence which satisfies (4.2) and $\|u_0^n - u_0\|_* \underset{n\to\infty}{\longrightarrow} 0$. Secondly, let denote $u^n(x,t)$ the solution of (4.1) but with the initial condition $u_n(x,0) = u_0^n(x)$, $\forall x \in \mathbb{R}^N$, with vanishing time T_n, and, $u(x,t)$ the solution of (4.1), with vanishing time T_0.

- <u>STEP 1.</u> We are going to prove that $\lim_{n\to\infty} T_n = T_0$.
  Let define the limit point of T_n:

$$T_* = \lim_{n\to\infty} T_n, \tag{4.25}$$

Secondly, let us define w_n as:

$$w_n(x,s) = (T_n - t)^{-\frac{m}{1-m}} u_n^m(x,t)\,|_{t=T_n(1-e^{-s})}, \tag{4.26}$$

which satisfies (3.37). In the last place, let us define v_n as:

$$v_n(F(x),s) = \left(\frac{2}{1+|x|^2}\right)^{-\frac{(N-2)}{2}} w_n(x,s), \tag{4.27}$$

which satisfies (3.41). Now we can apply the Proposition 4.2.1 to v_n, so there exist constants $\alpha,\beta > 0$ such that can be chosen uniform in n:

$$\alpha \le v_n(F(x),s) \le \beta, \quad \forall x \in \mathbb{R}^N, s \in [s_*,\infty).$$

By (4.27),

$$\alpha \le \left(\frac{2}{1+|x|^2}\right)^{-\frac{(N-2)}{2}} w_n(x,s) \le \beta, \quad \forall x \in \mathbb{R}^N, s \in [s_*,\infty).$$

If and only if

$$\left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} \alpha \le w_n(x,s) \le \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} \beta, \quad \forall x \in \mathbb{R}^N, s \in [s_*,\infty).$$

By (4.26):

$$\left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} \alpha \le (T_n - t)^{-\frac{m}{1-m}} u_n^m(x,t) \le \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}} \beta, \quad \forall x \in \mathbb{R}^N, t \in (t_*, T_n).$$

Taking everything to the power of $\frac{1}{m}$:

$$\left(\frac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} \alpha \le (T_n - t)^{-\frac{1}{1-m}} u_n(x,t) \le \left(\frac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} \beta, \quad \forall x \in \mathbb{R}^N, t \in (t_*, T_n).$$

Considering that

$$u_n(x,t) \underset{n\to\infty}{\longrightarrow} u(x,t), \quad \text{uniformly } \forall t \in (0,T_0), \tag{4.28}$$

we can pass the inequality to the limit, by (4.25), to obtain:

$$\left(\frac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} \alpha \le (T_* - t)^{-\frac{1}{1-m}} u(x,t) \le \left(\frac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} \beta, \quad \forall x \in \mathbb{R}^N, t \in (t_*, T_*).$$

Then by (4.28) we have that $T_0 = T_*$, so we have proved that $\lim_{n\to\infty} T_n = T_0$, from where we get the stability of the vanishing time.

- STEP 2. Defining $\lim_{s\to\infty} v(y,s) = \overline{v}(y)$ uniformly, and, $\lim_{s\to\infty} v_n(y,s) = \overline{v}_n(y)$ uniformly, we are going to see that $\lim_{n\to\infty} \overline{v}_n(y) = \overline{v}(y)$ uniformly, $\forall y \in \mathbb{S}^N$.
  1. By STEP 2 in the proof of Theorem 4.0.1, $\overline{v}$ verifies (4.22), then when $s \to \infty$ in (4.24), we have that $J_{\overline{v}_n} = J_{\overline{v}}$, that is to say that the energy does not change for different steady states of (3.41).
  2. As we have seen in the STEP 1 of the proof of Theorem 4.0.1, $\overline{v}_n$ is compact in $C^k(\mathbb{S}^N)$, $\forall k > 0$.
  3. Knowing that for a large s, v gets close to the set of steady states, then by steps 1 and 2, we have that this energy functional for the steady states have the same energy and it is uniformly bounded away from zero, so we can take $M(v(y,s)) = (v^p(y,s))_s$ as in (1.12). By Theorem 1.7.4, there exists $\theta \in \left(0, \frac{1}{2}\right)$ such that:

$$\|M(v(y,s))\|_{L^2} \ge |J_v(s) - J_{\overline{v}}|^{1-\theta}, \quad y \in \mathbb{S}^N, \text{ for } s \text{ sufficiently large.} \tag{4.29}$$

Then by Lemma 4.2.2:

$$\begin{aligned} -\frac{d}{ds} J_v(s) &= \frac{4p}{(p+1)^2} \int_{\mathbb{S}^N} \left| \left(v^{\frac{p+1}{2}}(y,s) \right)_s \right|^2 dy \\ &= \frac{4p}{(p+1)^2} \int_{\mathbb{S}^N} \left(M(v(y,s)), (v^p)_s \right)_{L^2} \ge C \|M(v(y,s))\|_{L^2} \|(v^p)_s\|_{L_2}. \end{aligned}$$

Then we obtain by multiplying on both sides by $[J_v(s) - J_{\overline{v}}]^\theta$:

$$-\frac{d}{ds} [J_v(s) - J_{\overline{v}}]^\theta \ge C [J_v(s) - J_{\overline{v}}]^{\theta - 1} \|M(v(y,s))\|_{L_2} \|(v^p)_s\|_{L^2}.$$

By (4.29)

$$-\frac{d}{ds}[J_v(s)-J_{\overline{v}}]^{\theta} \geq C\|(v^p)_s\|_{L^2}. \tag{4.30}$$

Let $0<s_*<s_1<s_2$, taking v_n instead of v and integrating (4.30) from s_2 to s_1, we obtain

$$\|v_n(y,s_2)-v_n(y,s_1)\|_{L^2} \leq C\left|J_{v_n}(s_1)-J_{\overline{v}_n}\right|^{\theta}, \quad n\geq 1,\, y\in\mathbb{S}^N, \tag{4.31}$$

and

$$\|v(y,s_2)-v(y,s_1)\|_{L^2} \leq C\,|J_v(s_1)-J_{\overline{v}}|^{\theta}, \quad y\in\mathbb{S}^N. \tag{4.32}$$

Then by the definitions at the beginning of this step, we have:

$$\begin{aligned}
\|\overline{v}_n(y)-\overline{v}(y)\|_{L^2} \leq& \|v(y,s_2)-v_n(y,s_2)\|_{L^2}\\
=& \|v(y,s_2)-v(y,s_1)+v(y,s_1)-v_n(y,s_1)+v_n(y,s_1)-v_n(y,s_2)\|_{L^2}\\
\leq& \|v(y,s_2)-v(y,s_1)\|_{L^2}+\|v(y,s_1)-v_n(y,s_1)\|_{L^2}\\
&+\|v_n(y,s_1)-v_n(y,s_2)\|_{L^2}.
\end{aligned}$$

By (4.31) and (4.32) we obtain:

$$\begin{aligned}
\|\overline{v}_n(y)-\overline{v}(y)\|_{L^2} \leq& C\,|J_v(s_1)-J_{\overline{v}}|^{\theta}+\|v(y,s_1)-v_n(y,s_1)\|_{L^2}+C\left|J_{v_n}(s_1)-J_{\overline{v}_n}\right|^{\theta}\\
=& \|v(y,s_1)-v_n(y,s_1)\|_{L^2}+C\left(|J_v(s_1)-J_{\overline{v}}|^{\theta}+\left|J_{v_n}(s_1)-J_{\overline{v}_n}\right|^{\theta}\right).
\end{aligned}$$

As $\lim_{n\to\infty} v_n(y,s)=v(y,s)$ uniformly on compact subsets of $s\in(0,\infty)$, then

$$\limsup_{n\to\infty}\|\overline{v}(y)-\overline{v}_n(y)\|_{L^2} \leq 2C\left(|J_v(s_1)-J_{\overline{v}}|^{\theta}\right).$$

As s_1 is arbitrary, we can choose $s_1=\infty$ and we have the continuity of the vanishing profile.

□

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