UNBOUNDED POTENTIAL RECOVERY IN THE PLANE

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Dedicated to the memory of Tuulikki

Abstract. We reconstruct compactly supported potentials with only half a derivative in L^2 from the scattering amplitude at a fixed energy. For this we draw a connection between the recently introduced method of Bukhgeim, which uniquely determined the potential from the Dirichlet-to-Neumann map, and a question of Carleson regarding the convergence to initial data of solutions to time-dependent Schrödinger equations. We also provide examples of compactly supported potentials, with s derivatives in L^2 for any $s < 1/2$, which cannot be recovered by these means. Thus the recovery method has a different threshold in terms of regularity than the corresponding uniqueness result.

1. Introduction

We consider the Schrödinger equation $\Delta u = Vu$ on a bounded domain Ω in the plane. For each solution u, we are given the value of both u and $\nabla u \cdot n$ on the boundary $\partial\Omega$, where n is the exterior unit normal on $\partial\Omega$. The goal is then to recover the potential V from this information.

We suppose throughout that $V \in L^2$ is supported on Ω and that 0 is not a Dirichlet eigenvalue for the Hamiltonian $-\Delta + V$. Then for each $f \in H^{1/2}(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ to the Dirichlet problem

$$\begin{cases} \Delta u = Vu \\ u\big|_{\partial\Omega} = f, \end{cases} \tag{1}$$

and the Dirichlet-to-Neumann (DN) map Λ_V can be formally defined by

$$\Lambda_V \,:\, f \mapsto \nabla u \cdot n|_{\partial\Omega}.$$

Then a restatement of our goal is to recover V from knowledge of Λ_V.

We come to this problem via a question of Calderón regarding impedance tomography [14], where f is the electric potential and $\nabla u \cdot n$ is the boundary current, however the DN map $\Lambda_{V-\kappa^2}$ and the scattering amplitude at energy κ^2 are uniquely determined by each other, and indeed the DN map can be recovered from the scattering amplitude (see the appendix for explicit formulae). Thus we are also addressing the question of whether it is

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possible to recover a potential from the scattering data at a fixed positive energy.

In higher dimensions, Sylvester and Uhlmann proved that smooth potentials are uniquely determined by the DN map [57] (see [44, 45, 16] for nonsmooth potentials and [11, 47, 28] for the conductivity problem). The uniqueness result was extended to a reconstruction procedure by Nachman [39, 40]. The planar case is quite different mathematically as it is not overdetermined. Here the first uniqueness and reconstruction algorithm was proved by Nachman [41] via $\overline{\partial}$-methods for potentials of conductivity type (see also [12] for uniqueness with less regularity). Sun and Uhlmann [53, 55] proved uniqueness for potentials satisfying nearness conditions to each other. Isakov and Nachman [31] then reconstructed the real valued L^p-potentials, $p > 1$, in the case that their eigenvalues are strictly positive. The $\overline{\partial}$-method in combination with the theory of quasiconformal maps gave the uniqueness result for the conductivity equation with measurable coefficients [3]. The problem for the general Schrödinger equation was solved only in 2008 by Bukhgeim [13] for C^1-potentials. Bukhgeim's result has since been improved and extended to treat related inverse problems (see for example [9, 25, 26, 27, 46, 29, 30]).

The aim of this article is to emphasise a surprising connection between the pioneering work of Bukhgeim [13] and Carleson's question [15] regarding the convergence to initial data of solutions to time-dependent Schrödinger equations. Elaborating on this new point of view we obtain a reconstruction theorem for general planar potentials with only half a derivative in L^2, which is sharp with respect to the regularity. The precise statements are given in the forthcoming Corollary 1.3 and Theorem 1.4.

To describe the results in more detail, we recall that the starting point in [13] was to consider solutions to $\Delta u = Vu$ of the form $u = e^{i\psi}\big(1 + w\big)$, where from now on

$$\psi(z) \equiv \psi_{k,x}(z) = \tfrac{k}{8}(z - x)^2, \qquad z \in \mathbb{C}, \quad x \in \Omega.$$

Solutions of this type have a long history (see for example [23, 57, 34, 18]), and in this form they were considered first by Bukhgeim. We will recover the potential by measuring a countable number of times on the boundary, so we take $k \in \mathbb{N}$. We will require the homogeneous Sobolev spaces with norm given by $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_{L^2}$, where $(-\Delta)^{s/2}$ is defined via the Fourier transform as usual. In Section 3.2, we prove that if the potential V is contained in $\dot{H}^s$ with $0 < s < 1$, and k is sufficiently large, then we can take $w \equiv w_{k,x} \in \dot{H}^s$ with a bound for the norms which is decreasing to zero in k. We write $u_{k,x} = e^{i\psi}\big(1 + w\big)$ for these $w \in \dot{H}^s$.

The definition of the DN map yields the basic integral formula in inverse problems; Alessandrini's identity. Indeed, if $u, v \in H^1(\Omega)$ satisfy $\Delta u = Vu$ and $\Delta v = 0$, then the formula states that

$$\Big\langle (\Lambda_V - \Lambda_0)[u], v \Big\rangle := \int_{\partial\Omega} (\Lambda_V - \Lambda_0)[u]\, v = \int_{\Omega} V u\, v.$$

Taking $u = u_{k,x}$, which is also in $H^1(\Omega)$, and $v = e^{i\overline{\psi}}$ this yields

$$\Big\langle (\Lambda_V - \Lambda_0)[u_{k,x}], e^{i\overline{\psi}} \Big\rangle = \int_\Omega e^{i(\psi+\overline{\psi})} V(1+w)\,, \tag{2}$$

and so the integral over Ω can be obtained from information on the boundary.

The bulk of the article is concerned with recovering the potential from the integral on the right-hand side of (2). However, in order to calculate the value of the integral, without knowing the value of the potential V inside Ω, we need to calculate the value of the left-hand side of (2). That is to say, we must determine the values of $u_{k,x}$ on the boundary from the DN map. In the case of linear phase, this was achieved by Nachman [41] for L^p-potentials V, with $p > 1$, and Lipschitz boundary. For C^1-potentials, with C^2-boundary, the result was extended by Novikov and Santacesaria to quadratic phases [46]. Here we show that for quadratic phases almost no regularity is needed. We consider potentials in the inhomogeneous L^2-Sobolev space H^s, defined as before with $(-\Delta)^{s/2}$ replaced by $(\mathrm{I} - \Delta)^{s/2}$. Our starting point is similar to [41] but we give a shorter argument, avoiding single layer potentials.

Theorem 1.1. *Let $V \in H^s$ with $s > 0$ and suppose that Ω is Lipschitz. Then we can identify compact operators $\Gamma_{k,x} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$, depending only on k, x and $\Lambda_V - \Lambda_0$, such that*

$$u_{k,x}|_{\partial\Omega} = (\mathrm{I} - \Gamma_{k,x})^{-1}\big[e^{i\psi}|_{\partial\Omega}\big].$$

For C^1-potentials, Bukhgeim [13] proved that the right-hand side of (2), multiplied by $(4\pi)^{-1}k$, converges to $V(x)$ for all $x \in \Omega$, when k tends to infinity. In Section 4, we obtain this convergence for potentials in H^s with $s > 1$. For discontinuous potentials we are no longer able to recover at each point. Instead we bound the fractal dimension of the sets where the recovery fails. As Sobolev spaces are only defined modulo sets of zero Lebesgue measure, we consider first the potential spaces $L^{s,2} = (-\Delta)^{-s/2}L^2(\mathbb{R}^2)$, and bound the Hausdorff dimension of the points where the recovery fails.

Theorem 1.2. *Let $V \in L^{s,2}$ with $1/2 \leqslant s < 1$. Then*

$$\dim_H \Big\{ x \in \Omega \,:\, \tfrac{k}{4\pi}\Big\langle (\Lambda_V - \Lambda_0)[u_{k,x}], e^{i\overline{\psi}} \Big\rangle \not\to V(x) \quad \text{as} \quad k \to \infty \Big\} \leqslant 2 - s.$$

As the members of H^s coincide almost everywhere with members of $L^{s,2}$, we see that rough and unbounded potentials can be recovered almost everywhere from information on the boundary. Note that these results are stable in the sense that $k \in \mathbb{N}$ can be replaced by any sequence $\{n_k\}_{k\in\mathbb{N}}$ such that n_k tends to infinity as k tends to infinity.

Corollary 1.3. *Let $V \in H^{1/2}$. Then*

$$\lim_{k\to\infty} \tfrac{k}{4\pi}\Big\langle (\Lambda_V - \Lambda_0)[u_{k,x}], e^{i\overline{\psi}} \Big\rangle = V(x), \quad \text{a.e. } x \in \Omega.$$

In Section 5, we will prove that this is sharp in the sense of the following theorem. Note that even though there is divergence on a set of full Hausdorff dimension when $s < 1/2$, the dimension of the divergence set is bounded above by $3/2$ when $s \geqslant 1/2$.

Theorem 1.4. *Let $s < 1/2$. Then there exists a potential $V \in H^s$, supported in Ω, for which*

$$\Big|\Big\{x \in \Omega \,:\, \tfrac{k}{4\pi}\Big\langle(\Lambda_V - \Lambda_0)[u_{k,x}], e^{i\overline{\psi}}\Big\rangle \not\to V(x) \quad as \quad k \to \infty\Big\}\Big| \neq 0.$$

Blåsten [9] proved that potentials in H^s with $s > 0$ are uniquely determined by the DN map (see also [30] for uniqueness with L^p-potentials, $p > 2$). It is a curious phenomenon that, within the Bukhgeim approach, uniqueness and reconstruction have different smoothness barriers.

The DN map $\Lambda_{V-\kappa^2}$ can be recovered from the scattering amplitude at a fixed energy $\kappa^2 > 0$ (see the appendix), from which we are able to recover the potential $V - \kappa^2\chi_\Omega$ rather than V. We are free to choose the domain Ω. Taking Ω to be a square, we obtain the following recovery formula. Here $U_{k,x}$ are Bukhgeim solutions which solve $\Delta u = (V - \kappa^2)u$ in Ω.

Theorem 1.5. *Let $V \in H^{1/2}$ be supported in a square Ω. Then*

$$\lim_{k\to\infty} \tfrac{k}{4\pi}\Big\langle(\Lambda_{V-\kappa^2} - \Lambda_0)[U_{k,x}], e^{i\overline{\psi}}\Big\rangle + \kappa^2 = V(x), \quad \text{a.e. } x \in \Omega.$$

Interpreting the problem acoustically, it is unsurprising that we are unable to recover potentials in H^s with $s < 1/2$. Taking

$$V(x) = \kappa^2(1 - c^{-2}(x)),$$

where $c(x)$ denotes the speed of sound at x, the scattered solutions u also satisfy $c^2\Delta u + \kappa^2 u = 0$. Now there are potentials in H^s, with $s < 1/2$, which are singular on closed curves (see for example [59]). Thus the speed of sound is zero on the curve and so a continuous solution u would be zero. That is to say, the continuous incident waves cannot pass through the curve and we should not expect to be able to detect modifications of the interior of the potential which is cloaked in some sense (see [24] for more sophisticated types of cloaking). From this point of view, the uniqueness results [9, 30] reflect the tunneling phenomenon in quantum mechanics.

2. The Bukhgeim solutions

Writing $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, we consider the complex analytic interpretation of the Schrödinger equation $4\partial_z\partial_{\overline{z}}u = Vu$. When looking for solutions of the form $u = e^{i\psi}\big(1 + w\big)$, the equation is equivalent to the system

$$2\partial_{\overline{z}}w = e^{-i(\psi+\overline{\psi})}v, \qquad 2\partial_z v = e^{i(\psi+\overline{\psi})}V(1 + w),$$

which is solved in Ω whenever

$$w = \tfrac{1}{4}\partial_{\overline{z}}^{-1}\Big[e^{-i(\psi+\overline{\psi})}\chi_Q\, \partial_z^{-1}\big[e^{i(\psi+\overline{\psi})}\, V(1 + w)\big]\Big].$$

Here, we take Q to be a fixed, auxiliary, axis-parallel square which properly contains Ω. Thus, defining the operator $\mathrm{S}_V^k \equiv \mathrm{S}_V^{k,x}$ by

$$\mathrm{S}_V^k[F] = \tfrac{1}{4}\partial_{\overline{z}}^{-1}\Big[e^{-i(\psi+\overline{\psi})}\chi_Q\,\partial_z^{-1}\big[e^{i(\psi+\overline{\psi})}\chi_Q\,VF\big]\Big],$$

we see that as soon as $\|\mathrm{S}_V^k\|_{\dot{H}^s\to\dot{H}^s} < 1$, we can treat $(\mathrm{I}-\mathrm{S}_V^k)^{-1}$ by Neumann series to deduce that it is a bounded operator on $\dot{H}^s$. This yields a solution $u_{k,x} \equiv e^{i\psi}\big(1+w\big)$ where

$$w \equiv w_{k,x} = (\mathrm{I}-\mathrm{S}_V^k)^{-1}\mathrm{S}_V^k[1] \in \dot{H}^s. \tag{3}$$

In what remains of this section, we prove that S_V^k is contractive for sufficiently large k. This property will be crucial in the proof of Theorem 1.1 as well as in Section 4. We write $\mathrm{S}_V^k[f] = \frac{1}{4}\mathrm{S}_1^k[Vf]$, where

$$\mathrm{S}_1^k = \partial_{\overline{z}}^{-1}\circ \mathrm{M}^{-k}\circ\partial_z^{-1}\circ\mathrm{M}^k$$

and the multiplier operators $\mathrm{M}^{\pm k}$ are defined by $\mathrm{M}^{\pm k}[F] = e^{\pm i(\psi+\overline{\psi})}\chi_Q\,F$. The key ingredient in the proof of the following estimate, is the classical lemma of van der Corput [20].

Lemma 2.1. *Let* $0 \leqslant s_1, s_2 < 1$. *Then*

$$\|\mathrm{M}^{\pm k}[F](\cdot,x)\|_{\dot{H}^{-s_2}} \leqslant Ck^{-\min\{s_1,s_2\}}\|F(\cdot,x)\|_{\dot{H}^{s_1}}, \quad x\in\Omega,\ k\geqslant 1.$$

Proof. By the Hölder and Hardy–Littlewood–Sobolev inequalities, we have

$$\|\mathrm{M}^{\pm k}[F]\|_2 \leqslant C\|F\|_{\dot{H}^{s_1}}, \tag{4}$$

and

$$\|\mathrm{M}^{\pm k}[F]\|_{\dot{H}^{-s_2}} \leqslant C\|F\|_2, \tag{5}$$

with $0 \leqslant s_1, s_2 < 1$. So by complex interpolation, it will suffice to prove that

$$\|\mathrm{M}^{\pm k}[F]\|_{\dot{H}^{-s}} \leqslant Ck^{-s}\|F\|_{\dot{H}^s}. \tag{6}$$

Indeed, if $s_2 < s_1$ we interpolate with (4), taking $s = s_1$, and if $s_1 < s_2$ we interpolate with (5), taking $s = s_2$. Now by real interpolation with the trivial L^2 bound, (6) would follow from

$$\|\mathrm{M}^{\pm k}F\|_{\dot{B}^{-1}_{2,\infty}} \leqslant Ck^{-1}\,\|F\|_{\dot{B}^1_{2,1}} \tag{7}$$

(see Theorem 6.4.5 in [7]), where the Besov norms are defined as usual by

$$\|f\|_{\dot{B}^{-1}_{2,\infty}} = \sup_{j\in\mathbb{Z}} 2^{-j}\|\mathrm{P}_jf\|_{L^2} \qquad \text{and} \qquad \|f\|_{\dot{B}^1_{2,1}} = \sum_{j\in\mathbb{Z}} 2^j\|\mathrm{P}_jf\|_{L^2}.$$

Here, $\widehat{\mathrm{P}_jf} = \vartheta(2^{-j}|\cdot|)\widehat{f}$ with ϑ satisfying $\operatorname{supp}\vartheta\subset(1/2,2)$ and

$$\sum_{j\in\mathbb{Z}}\vartheta(2^{-j}\cdot) = 1.$$

As $\|F\|_{\dot{B}^{-1}_{2,\infty}} \leqslant C\|\widehat{F}\|_\infty$ and $\|\widehat{F}\|_1 \leqslant C\|F\|_{\dot{B}^1_{2,1}}$, the estimate (7) would in turn follow from

$$\|\widehat{\mathrm{M}^{\pm k}F}\|_\infty \leqslant Ck^{-1}\,\|\widehat{F}\|_1. \tag{8}$$

Now, by the Fourier inversion formula and Fubini's theorem,

$$\begin{aligned}|\widehat{\mathrm{M}^{\pm k}F}(\xi)| &= \frac{1}{(2\pi)^2}\Big|\int_Q e^{\pm i(\psi(z)+\overline{\psi(z)})}\int \widehat{F}(\omega)\,e^{iz\cdot\omega}d\omega\,e^{-iz\cdot\xi}dz\Big|\\ &\leqslant \int\Big|\int_Q e^{\pm ik\frac{(z_1-x_1)^2-(z_2-x_2)^2}{4}}\,e^{iz\cdot(\omega-\xi)}dz\Big||\widehat{F}(\omega)|\,d\omega\end{aligned}$$

so that (8) follows by two applications of van der Corput's lemma [20] (factorising the integral into the product of two integrals). □

In the following lemma, we optimise the decay in k, which will be important in Section 4.

Lemma 2.2. *Let* $0<s<1$. *Then*

$$\|\mathrm{S}_1^k[F](\cdot,x)\|_{\dot{H}^s} \leqslant Ck^{-1}\|F(\cdot,x)\|_{\dot{H}^s}, \quad x\in\Omega,\ k\geqslant 1.$$

Proof. By two applications of Lemma 2.1,

$$\begin{aligned}\|\mathrm{S}_1^k\|_{\dot{H}^s\to\dot{H}^s} \leqslant \|\mathrm{M}^{-k}\circ\partial_z^{-1}\circ\mathrm{M}^k\|_{\dot{H}^s\to\dot{H}^{s-1}} &\leqslant Ck^{s-1}\|\partial_z^{-1}\circ\mathrm{M}^k\|_{\dot{H}^s\to\dot{H}^{1-s}}\\ &\leqslant Ck^{s-1}\|\mathrm{M}^k\|_{\dot{H}^s\to\dot{H}^{-s}}\\ &\leqslant Ck^{s-1-s} = Ck^{-1},\end{aligned}$$

and we are done. □

In the following lemma, we use Lemma 2.1 only once, and gain some integrability using the Hardy–Littlewood–Sobolev theorem. By taking k sufficiently large, we obtain our contraction and thus our Bukhgeim solution $u=u_{k,x}$ as described above.

Lemma 2.3. *Let* $0<s<1$. *Then*

$$\|\mathrm{S}_V^k[F](\cdot,x)\|_{\dot{H}^s} \leqslant Ck^{-\min\{2s,1-s\}}\|V\|_{\dot{H}^s}\|F(\cdot,x)\|_{\dot{H}^s}, \quad x\in\Omega,\ k\geqslant 1.$$

Proof. By the Cauchy–Schwarz and Hardy–Littlewood–Sobolev inequalities,

$$\|VF\|_q \leqslant \|V\|_{2q}\|F\|_{2q} \leqslant C\|V\|_{\dot{H}^s}\|F\|_{\dot{H}^s},$$

where $q=\frac{1}{1-s}$. Thus, as $\mathrm{S}_V^k[F]=\mathrm{S}_1^k[VF]$, it will suffice to prove that

$$\|\mathrm{S}_1^k\|_{L^q\to\dot{H}^s} \leqslant Ck^{-\min\{2s,1-s\}}.$$

When $0<s<1/3$, by Lemma 2.1, we have

$$\begin{aligned}\|\mathrm{S}_1^k\|_{L^q\to\dot{H}^s} \leqslant \|\mathrm{M}^{-k}\circ\partial_z^{-1}\circ\mathrm{M}^k\|_{L^q\to\dot{H}^{s-1}} &\leqslant Ck^{-2s}\|\partial_z^{-1}\circ\mathrm{M}^k\|_{L^q\to\dot{H}^{2s}}\\ &\leqslant Ck^{-2s}\|\mathrm{M}^k\|_{L^q\to\dot{H}^{2s-1}}\\ &\leqslant Ck^{-2s}\|\mathrm{M}^k\|_{L^q\to L^q}.\end{aligned}$$

When $s \geqslant 1/3$, we also use Hölder's inequality at the end;

$$\begin{aligned}\|\mathrm{S}_1^k\|_{L^q\to \dot H^s} \leqslant \|\mathrm{M}^{-k}\circ \partial_z^{-1}\circ \mathrm{M}^k\|_{L^q\to \dot H^{s-1}} &\leqslant Ck^{1-s}\|\partial_z^{-1}\circ \mathrm{M}^k\|_{L^q\to \dot H^{1-s}}\\ &\leqslant Ck^{1-s}\|\mathrm{M}^k\|_{L^q\to \dot H^{-s}}\\ &\leqslant Ck^{1-s}\|\mathrm{M}^k\|_{L^q\to L^{q^*}},\end{aligned}$$

where $q^* = \frac{2}{s+1}$, and so we are done. □

Remark 2.4. Note that van der Corput's estimate is independent of the size of Q and so, when $s \geqslant 1/3$, the potential need not be compactly supported for the results of this section to hold (when $s < 1/3$ we used the compact support in a less obviously removable way).

3. Proof of Theorem 1.1

In this section we show that the boundary values of our Bukhgeim solution $u_{k,x}$ can be determined from knowledge of Λ_V. The argument is inspired by [41, Theorem 5] but we replace the Faddeev green function G_k by its analogue in terms of the operator S_V^k and avoid the use of single layer potentials.

Indeed, considering the kernel representation of S_1^k, we can write $\mathrm{S}_V^k[F]$ in the form

$$\mathrm{S}_V^k[F](z) = \int_\Omega g_\psi(z,\eta)V(\eta)F(\eta)\,d\eta.$$

where g_ψ, the kernel of S_1^k, is given by

$$g_\psi(z,\eta) = \chi_Q(\eta)\frac{e^{i\left(\psi(\eta)+\overline{\psi(\eta)}\right)}}{4\pi^2}\int_Q \frac{1}{\overline{(\omega-\eta)}(z-\omega)}e^{-i\left(\psi(\omega)+\overline{\psi(\omega)}\right)}\,d\omega.$$

In order to work directly with exponentially growing solutions we conjugate g_ψ with the exponential factors, so that

$$\int_\Omega G_\psi(z,\eta)V(\eta)F(\eta)\,d\eta = e^{i\psi(z)}\mathrm{S}_V^k[e^{-i\psi}F](z), \tag{9}$$

where $G_\psi(z,\eta) = e^{i\psi(z)}g_\psi(z,\eta)e^{-i\psi(\eta)}$. Notice also that when $z\in Q\backslash\Omega$ and $\eta\in\Omega$, we have that

$$\Delta_\eta G_\psi(z,\eta) = 0.$$

Thus, if we take (9) with $F = P_V(f)$, where $P_V(f)$ solves $\Delta u = Vu$ with $u|_{\partial\Omega} = f$, using Alessandrini's identity we obtain that, for each $z\in Q\setminus\Omega$,

$$\Big\langle(\Lambda_V-\Lambda_0)[f], G_\psi(z,\cdot)|_{\partial\Omega}\Big\rangle = e^{i\psi(z)}\mathrm{S}_V^k[e^{-i\psi}P_V(f)](z). \tag{10}$$

In particular the right-hand side belongs to $H^1(Q\setminus\Omega)$ and hence we can define the operator $\Gamma_\psi : H^{1/2}\to H^{1/2}$ by

$$\Gamma_\psi[f] = T_r\circ\Big\langle(\Lambda_V-\Lambda_0)[f], G_\psi|_{\partial\Omega}\Big\rangle,$$

where $T_r : H^1(Q \setminus \Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator. Now, by the definitions of $u_{k,x}$ and w, we also deduce from (9) and (3) that

$$\int_\Omega G_\psi(\cdot,\eta)V(\eta)\, u_{k,x}(\eta)\, d\eta = e^{i\psi}\mathrm{S}_V^k[1+w] = e^{i\psi}w = u_{k,x} - e^{i\psi}. \tag{11}$$

Combining (9), (10), and (11) we obtain the integral identity

$$(\mathrm{I} - \Gamma_\psi)[u_{k,x}|_{\partial\Omega}] = e^{i\psi}|_{\partial\Omega}.$$

Thus, we can determine $u_{k,x}$ on the boundary if we can invert $(\mathrm{I} - \Gamma_\psi)$. By the Fredholm alternative it will suffice to show that Γ_ψ is compact and that $(\mathrm{I} - \Gamma_\psi)$ has a trivial kernel on $H^{1/2}(\partial\Omega)$.

Theorem 3.1. *Let $V \in \dot{H}^s$ with $0 < s < 1$. Then*
(i) *Γ_ψ is compact*
(ii) *$\Gamma_\psi[f] = f \Rightarrow f = 0$.*

Proof of (i). We have that

$$\Gamma_\psi[f] = T_r\big[e^{i\psi}\mathrm{S}_V^k[e^{-i\psi}P_V(f)]\big].$$

As the set of compact operators is a left and right ideal, we consider the boundedness properties of each component of the composition. Firstly, $P_V : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ is bounded. Secondly, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ compactly for all $2 < p < \infty$. Now taking p sufficiently large and $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$, by the boundedness of the Cauchy transform followed by the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}\|\mathrm{S}_V^k[e^{-i\psi}G]\|_{H^1(Q\setminus\Omega)} &\leqslant C\|VG\|_{L^2(\Omega)} \leqslant C\|V\|_{L^q(\Omega)}\|G\|_{L^p(\Omega)}\\ &\leqslant C\|V\|_{\dot{H}^s}\|G\|_{L^p(\Omega)}.\end{aligned}$$

Finally, $T_r : H^1(Q \setminus \Omega) \to H^{1/2}(\partial\Omega)$ is bounded. Since the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact, it follows that Γ_ψ is compact.

Proof of (ii). Letting $\rho = \mathrm{S}_V^k[e^{-i\psi}P_V(f)]$, we have that

$$\partial_{\bar z}[e^{i\psi}\rho] = \tfrac{1}{4}e^{-i\overline{\psi}}\chi_Q\partial_z^{-1}[e^{i\overline{\psi}}VP_V(f)],$$

so that

$$4\partial_z\partial_{\bar z}[e^{i\psi}\rho] = VP_V(f) \quad \text{ on } \Omega.$$

This can be rewritten as $\Delta[e^{i\psi}\rho - P_V(f)] = 0$ on Ω. Now by hypothesis $\Gamma_\psi[f] = f$, so that by (10) we have $e^{i\psi}\rho = f$ on $\partial\Omega$. Combining the two, we see that

$$e^{i\psi}\rho = P_V(f) \quad \text{ on } \Omega.$$

From the definition of ρ we see that $\rho = \mathrm{S}_V^k[\rho]$, and as soon as S_V^k is strictly contractive, that $\rho = 0$. This of course follows from Lemma 2.3 for large enough k. Thus, $f = e^{i\psi}\rho = 0$, so that $\mathrm{I} - \Gamma_\psi$ is injective as desired. □

Remark 3.2. We need not suppose that the potential is compactly supported here as long as we suppose that $\chi_\Omega V \in H^\varepsilon$ and then the Bukhgeim solutions which we identify are associated to this potential instead. For $0 < \varepsilon < 1/2$ and Ω Lipschitz, we have $\chi_\Omega V \in H^\varepsilon$ as long as $V \in H^s$ with $s > 1/2 + \varepsilon$. To see this, note that by the fractional Leibnitz rule (see for example [33]),

$$\|\chi_\Omega V\|_{H^\varepsilon} \leqslant \|\chi_\Omega\|_4 \|V\|_{W^{\varepsilon,4}} + \|\chi_\Omega\|_{W^{\varepsilon,p}} \|V\|_{L^q}$$

with $p < \frac{4}{1+2\varepsilon}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then the remark follows by the Hardy–Littlewood–Sobolev inequality, combined with the fact that $\chi_\Omega \in H^s$ for all $s < 1/2$ (see for example [22]).

4. Potential recovery

In order to recover the potential at $x \in \Omega$, it remains to show that the right-hand side of Alessandrini's identity (2) converges to $V(x)$. That is to say $\mathrm{T}^k_{1+w} V(x)$ converges to $V(x)$ as k tends to infinity, where

$$\mathrm{T}^k_{1+w}[F](x) = \frac{k}{4\pi} \int_{\mathbb{R}^2} e^{i(\psi(z)+\overline{\psi(z)})}\, F(z)\big(1 + w(z)\big)\, dz.$$

First we show that $\mathrm{T}^k_w V$ can be considered to be a remainder term.

Theorem 4.1. *Let $V \in \dot{H}^s$ with $0 < s < 1$. Then*

$$\lim_{k\to\infty} \mathrm{T}^k_w[V](x) = 0, \quad x \in \Omega.$$

Moreover, if $k \geqslant (1 + c\|V\|_{\dot{H}^s})^{\max\{\frac{1}{2s},\frac{1}{1-s}\}}$, then

$$\sup_{x\in\Omega} |\mathrm{T}^k_w[V](x)| \leqslant Ck^{-s}\|V\|^2_{\dot{H}^s}.$$

Proof. By Lemma 2.1,

$$\begin{aligned} |\mathrm{T}^k_w[V](x)| &\leqslant Ck\|\mathrm{M}^k[V]\|_{\dot{H}^{-s}}\|w\|_{\dot{H}^s} \\ &\leqslant Ck^{1-s}\|V\|_{\dot{H}^s}\|(\mathrm{I} - \mathrm{S}^k_V)^{-1}\mathrm{S}^k_V[1]\|_{\dot{H}^s}. \end{aligned}$$

By Lemma 2.3, we can treat $(\mathrm{I} - \mathrm{S}^k_V)^{-1}$ by Neumann series to deduce that it is a bounded operator on $\dot{H}^s$ when $k \geqslant 1$ and $Ck^{-\min\{2s,1-s\}}\|V\|_{\dot{H}^s} \leqslant \frac{1}{2}$. Then

$$\begin{aligned} |\mathrm{T}^k_w[V](x)| &\leqslant Ck^{1-s}\|V\|_{\dot{H}^s}\|\mathrm{S}^k_1[V]\|_{\dot{H}^s} \\ &\leqslant Ck^{-s}\|V\|^2_{\dot{H}^s}, \end{aligned}$$

by an application of Lemma 2.2, which is the desired estimate. □

Noting that $e^{i(\psi(z)+\overline{\psi(z)})} = \exp\big(ik\frac{(z_1-x_1)^2-(z_2-x_2)^2}{4}\big)$, it remains to prove

$$\lim_{k\to\infty} \mathrm{T}^k_1[V](x) = V(x), \tag{12}$$

where T^k_1 is defined by

$$\mathrm{T}^k_1[F](x) = \frac{k}{4\pi} \int \exp\big(ik\tfrac{(z_1-x_1)^2-(z_2-x_2)^2}{4}\big) F(z)\, dz.$$

Now when F is a Schwartz function, this is equal to $e^{i\frac{1}{k}\Box}F(x)$, where

$$e^{i\frac{1}{k}\Box}[F](x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\, e^{-i\frac{1}{k}(\xi_1^2-\xi_2^2)}\,\widehat{F}(\xi)\,d\xi.$$

This follows easily, making use of the distributional formula

$$\frac{k}{4\pi}\int e^{ik\frac{z_1^2-z_2^2}{4}}\phi(z)\,dz = \int e^{-i\frac{1}{k}(\xi_1^2-\xi_2^2)}\widehat{\phi}(\xi)\,d\xi,$$

which holds for Schwartz functions ϕ. We see that when V is a Schwartz function, $\mathrm{T}_1^k V$ solves the time-dependent nonelliptic Schrödinger equation,

$$i\partial_t u + \Box u = 0,$$

where $\Box = \partial_{x_1x_1} - \partial_{x_2x_2}$, with initial data V at time $1/k$. When $V \in H^s$ with $s > 1$, both V and its Fourier transform are integrable, and so both $\mathrm{T}_1^k V$ and $e^{i\frac{1}{k}\Box}V$ are continuous functions which are again equal pointwise. Thus, in the following lemma we obtain the convergence (12) and therefore complete the reconstruction for potentials in H^s with $s > 1$.

Lemma 4.2. *Let $V \in H^s$ with $1 < s < 3$. Then*

$$|e^{i\frac{1}{k}\Box}V(x) - V(x)| \leqslant Ck^{\frac{1-s}{2}}\|V\|_{H^s}, \quad x \in \Omega.$$

Proof. By the Fourier inversion formula and the Cauchy–Schwarz inequality,

$$\begin{aligned}
|e^{it\Box}V(x) - V(x)| &= \frac{1}{(2\pi)^2}\Big|\int \widehat{V}(\xi)\,e^{i\xi\cdot x}\big(e^{-i\frac{1}{k}(\xi_1^2-\xi_2^2)} - 1\big)\,d\xi\Big| \\
&\leqslant \|V\|_{H^s}\Big(\int \frac{|e^{-i\frac{1}{k}(\xi_1^2-\xi_2^2)} - 1|^2}{|\xi|^{2s}}d\xi\Big)^{1/2} \\
&= \|V\|_{H^s}\Big(\int \frac{2 - 2\cos\big(\frac{1}{k}(\xi_1^2-\xi_2^2)\big)}{|\xi|^{2s}}d\xi\Big)^{1/2} \\
&= 2k^{\frac{1-s}{2}}\|V\|_{H^s}\Big(\int \frac{\sin^2\big(\frac{1}{2}(\xi_1^2-\xi_2^2)\big)}{|\xi|^{2s}}d\xi\Big)^{1/2} \\
&\leqslant 2k^{\frac{1-s}{2}}\|V\|_{H^s}\Big(\int_{\mathbb{D}} \frac{1}{|\xi|^{2(s-2)}}d\xi + \int_{\mathbb{R}^2\setminus\mathbb{D}} \frac{1}{|\xi|^{2s}}\Big)^{1/2},
\end{aligned}$$

where we have used the trigonometric identity $2\sin^2\theta = 1 - \cos 2\theta$ and the fact that $\sin\theta \leqslant |\theta|$. $\Box$

Altogether we see that $|\mathrm{T}_{1+w}^k V(x) - V(x)| \leqslant Ck^{\frac{1-s}{2}}$ for all $x \in \Omega$ and $V \in H^s$ with $1 < s < 3$, which improves upon the decay rate of [46] where they recovered C^2 potentials. Note that there can be no decay rates, at least for the main term, for the potentials of H^s with $s \leqslant 1$ as they would then be uniform limits of continuous functions and thus continuous.

For discontinuous potentials we are no longer able to recover at each point. Instead we bound the fractal dimension of the sets where the recovery fails. This point of view has its origins in the work of Beurling who bounded the capacity of the divergence sets of Fourier series [8] (see also [4]). Now

Sobolev spaces are only defined modulo sets of zero Lebesgue measure, and so we consider first the potential spaces

$$L^{s,2} = \{\, I_s * g \,:\, g \in L^2(\mathbb{R}^2) \,\},$$

where I_s is the Riesz potential $|\cdot|^{s-2}$. As $\widehat{I_s}(\xi) = C_s|\xi|^{-s}$, we have that $I_s * g$ is also a member of (an equivalence class of) H^s.

To bound the dimension of the sets where the recovery fails, we will prove maximal estimates with respect to fractal measures. We say that a positive Borel measure μ is α-dimensional if

$$c_\alpha(\mu) := \sup_{x\in\mathbb{R}^2,\, r>0} \frac{\mu\big(B(x,r)\big)}{r^\alpha} < \infty, \qquad 0 \leqslant \alpha \leqslant 2, \tag{13}$$

and denote by $\mathcal{M}^\alpha(\Omega)$ the α-dimensional probability measures which are supported in Ω. For $0 < s < 1$, we will require the elementary inequality

$$\|I_s * g\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|g\|_{L^2(\mathbb{R}^2)}, \quad \alpha > 2 - 2s, \tag{14}$$

which holds whenever $\mu \in \mathcal{M}^\alpha(\Omega)$ and $g \in L^2(\mathbb{R}^2)$. To see this, we note that by Fubini's theorem and the Cauchy–Schwarz inequality,

$$\|I_s * g\|_{L^1(d\mu)} \leqslant \|I_s * \mu\|_{L^2} \|g\|_{L^2},$$

so that (14) follows by proving

$$\|I_s * \mu\|_{L^2}^2 \lesssim c_\alpha(\mu), \quad \alpha > 2 - 2s.$$

Now by Plancherel's theorem,

$$\begin{aligned}\|I_s * \mu\|_{L^2}^2 = (2\pi)^{-2}\|\widehat{I_s}\widehat{\mu}\|_{L^2}^2 \lesssim \int \widehat{\mu}(\xi)\,\overline{\widehat{\mu}(\xi)}\,\widehat{I}_{2s}(\xi)\,d\xi \lesssim \int \mu * I_{2s}(y)\,d\mu(y)\\ = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^{2-2s}},\end{aligned}$$

which is nothing more than the $(2-2s)$-energy. Then, by an appropriate dyadic decomposition,

$$\iint \frac{d\mu(x)d\mu(y)}{|x-y|^{2-2s}} \lesssim \int \sum_{j=0}^\infty c_\alpha(\mu) 2^{-j\alpha} 2^{j(2-2s)} d\mu(y) \lesssim c_\alpha(\mu)$$

whenever $\alpha > 2 - 2s$ and $\mu \in \mathcal{M}^\alpha(\Omega)$.

The Fourier transform of less regular potentials V is not necessarily integrable, and so in that case $e^{i\frac{1}{k}\Box}V$ is not even well-defined. Instead we make do with the pointwise limit

$$\mathrm{T}_1^k[V](x) = \lim_{N\to\infty} G_N * \mathrm{T}_1^k[V](x) = \lim_{N\to\infty} e^{i\frac{1}{k}\Box}[G_N * V](x), \quad x \in \Omega, \tag{15}$$

where $G_N = N^2 G(N\cdot)$ and G is the Gaussian $e^{-|\cdot|^2}$. This formula holds as V is compactly supported and integrable; conditions which the initial data in the time-dependent theory does not normally satisfy. We will also require the following lemma due, in this form, to Sjölin [50].

Lemma 4.3. [50] *Let $x, t \in \mathbb{R}$, $\gamma \in [1/2, 1)$ and $N \geqslant 1$. Then*

$$\left| \int_{\mathbb{R}} \frac{\eta(N^{-1}\xi)\, e^{i(x\xi - t\xi^2)}}{|\xi|^\gamma}\, d\xi \right| \lesssim \frac{1}{|x|^{1-\gamma}},$$

where the constant implied by the symbol $\lesssim$ depends only on γ and the Schwartz function η.

In the following theorem, we employ the Kolmogorov–Seliverstov–Plessner method, as used by Carleson [15] for the one-dimensional Schrödinger equation. Dahlberg and Kenig [21] proved that the result of Carleson is sharp and noted that his argument could be applied to the higher dimensional problem (for which the argument is no longer sharp for the elliptic equation, see [10]). We refine their argument, which extends to the nonelliptic case, by proving estimates which hold uniformly with respect to fractal measures.

Theorem 4.4. *Let $1/2 \leqslant s < 1$. Then*

$$\Big\| \sup_{k \geqslant 1} \sup_{N \geqslant 1} |e^{i\frac{1}{k}\Box}[G_N * I_s * g]| \Big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)} \|g\|_{L^2(\mathbb{R}^2)}, \quad \alpha > 2 - s,$$

whenever $\mu \in \mathcal{M}^\alpha(\Omega)$ and $g \in L^2$.

Proof. By linearising, it will suffice to prove

$$\left| \int_\Omega e^{it(x)\Box}[G_{N(x)} * I_s * g]\, w(x)\, d\mu(x) \right|^2 \lesssim c_\alpha(\mu)\, \|g\|_{L^2}^2, \quad \alpha > 2 - s, \tag{16}$$

uniformly in measurable functions $t : \Omega \to \mathbb{R}$, $N : \Omega \to \mathbb{N}$ and $w : \Omega \to \mathbb{D}$. By Fubini's theorem and the Cauchy–Schwarz inequality, the left-hand side of (16) is bounded by

$$\int |\widehat{g}(\xi)|^2 d\xi \int \left| \int G\Big(\frac{\xi}{N(x)}\Big)\, e^{it(x)(\xi_1^2 - \xi_2^2)} e^{ix\cdot\xi} w(x)\, d\mu(x) \right|^2 \frac{d\xi}{|\xi|^{2s}}.$$

Writing the squared integral as a double integral, and applying Fubini's theorem again, it will suffice to show that

$$\begin{aligned} \iiint G\Big(\frac{\xi}{N(x)}\Big)\, G\Big(\frac{\xi}{N(y)}\Big)\, e^{i(t(x)-t(y))(\xi_1^2 - \xi_2^2)} e^{i(x-y)\cdot\xi} \frac{d\xi}{|\xi|^{2s}} \times \\ w(x)w(y)\, d\mu(x) d\mu(y) \lesssim c_\alpha(\mu) \end{aligned} \tag{17}$$

uniformly in the functions t, N and w. Now, as $|\xi|^{2s} \geqslant |\xi_1|^s |\xi_2|^s$, the left-hand side of (17) is bounded by

$$\begin{aligned} \prod_{j=1}^{2} \Big| \int G\Big(\frac{\xi_j}{N(x)}\Big)\, G\Big(\frac{\xi_j}{N(y)}\Big)\, e^{i(-1)^{j+1}(t(x)-t(y))\xi_j^2} e^{i(x_j - y_j)\xi_j} \frac{d\xi_j}{|\xi_j|^s} \Big| \times \\ w(x)w(y)\, d\mu(x) d\mu(y), \end{aligned}$$

and by Lemma 4.3, we have

$$\left|\int \frac{G\left(\frac{\xi_j}{N(x)}\right) G\left(\frac{\xi_j}{N(y)}\right) e^{i(-1)^{j+1}(t(x)-t(y))\xi_j^2} e^{i(x_j-y_j)\xi_j}}{|\xi_j|^s} d\xi_j\right| \lesssim \frac{1}{|x_j-y_j|^{1-s}}.$$

Substituting in, we see that the left-hand side of (17) is bounded by

$$C\iint \frac{|w(x)w(y)|d\mu(x)d\mu(y)}{|x_1-y_1|^{1-s}|x_2-y_2|^{1-s}} \leqslant C\iint \frac{d\mu(x)d\mu(y)}{|x_1-y_1|^{1-s}|x_2-y_2|^{1-s}} \tag{18}$$

To complete the proof, we are required to bound (18) by $c_\alpha(\mu)$. This will require a dyadic decomposition which lends itself to the singularities along the axis-parallel lines A_y defined by

$$A_y = \{x \in \Omega \,:\, x_1 = y_1 \quad \text{or} \quad x_2 = y_2\}, \quad y \in \Omega.$$

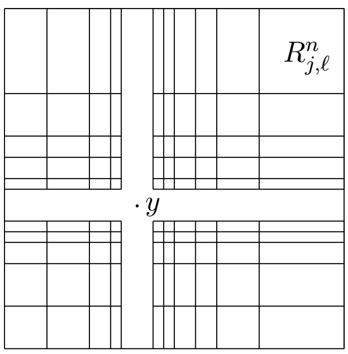
Covering A_y by balls $\{B_j\}_{j\geqslant 1}$ of radius r_j and using the definition (13) of $c_\alpha(\mu)$, we have

$$\mu(A_y) \leqslant \sum_{j\geqslant 1} \mu(B_j) \leqslant c_\alpha(\mu) \sum_{j\geqslant 1} r_j^\alpha.$$

Taking the infimum over all such coverings and using the fact that the α-Hausdorff measure of A_y is zero when $\alpha > 1$, we see that $\mu(A_y) = 0$ for all $\mu \in \mathcal{M}^\alpha(\Omega)$. Thus we can ignore the sets A_y when decomposing the inner integral of (18).

For each $j, \ell \in \mathbb{Z}$ we break up $Q \supset \Omega$ into dyadic rectangles of dimensions $2^{-j} \times 2^{-\ell}$ and consider the unique rectangle $R_{j,\ell}$ which contains y. We call the unique rectangles $R_{j-1,\ell-1}$, $R_{j-1,\ell}$, and $R_{j,\ell-1}$ that contain $R_{j,\ell}$, the mother, the father, and the stepfather respectively. We write $R^n_{j,\ell} \sim R_{j,\ell}$ if their mothers touch, but their fathers and stepfathers do not. As $\mu(A_y) = 0$, we can write

$$\int F(x,y)\, d\mu(x) = \sum_{j,\ell\geqslant 0} \sum_{n: R^n_{j,\ell}\sim R_{j,\ell}} \int_{R^n_{j,\ell}} F(x,y)\, d\mu(x),$$



The rectangles of dimensions $2^{-j} \times 2^{-\ell}$, with $1 \leqslant j, \ell \leqslant 3$, associated with a single point y.

which yields

$$(18) \;\leqslant\; C\int \sum_{j,\ell\geqslant 0}\ \sum_{n:R^n_{j,\ell}\sim R_{j,\ell}} 2^{j(1-s)}2^{\ell(1-s)}\mu(R^n_{j,\ell})\,d\mu(y).$$

Without loss of generality, we can suppose that

$$\sum_{\ell>j\geqslant 0}\ \sum_{n:R^n_{j,\ell}\sim R_{j,\ell}} 2^{j(1-s)}2^{\ell(1-s)}\mu(R^n_{j,\ell}) \leqslant \sum_{j\geqslant\ell\geqslant 0}\ \sum_{n:R^n_{j,\ell}\sim R_{j,\ell}} 2^{j(1-s)}2^{\ell(1-s)}\mu(R^n_{j,\ell}),$$

so that

$$(18) \;\leqslant\; C\int \sum_{j\geqslant\ell\geqslant 0}\ \sum_{n:R^n_{j,\ell}\sim R_{j,\ell}} 2^{j(1-s)}2^{\ell(1-s)}\mu(R^n_{j,\ell})\,d\mu(y).$$

Now by covering each rectangle by discs of radius 2^{-j}, and using the definition (13) of $c_\alpha(\mu)$, we see that

$$\mu(R^n_{j,\ell}) \;\lesssim\; 2^{j-\ell}c_\alpha(\mu)2^{-j\alpha},$$

and for each rectangle $R_{j,\ell}$ there are exactly nine rectangles $R^n_{j,\ell}$ which satisfy $R^n_{j,\ell}\sim R_{j,\ell}$. Thus

$$(18) \;\lesssim\; c_\alpha(\mu)\sum_{j\geqslant\ell\geqslant 0} 2^{j(2-s-\alpha)}2^{-\ell s} \;\lesssim\; c_\alpha(\mu),$$

when $\alpha > 2-s$, and so we are done. $\square$

Proof of Theorem 1.2. By Alessandrini's identity (2) and Frostman's lemma (see for example [38]), it will suffice to prove that

$$\mu\Big\{x\,:\,\limsup_{k\to\infty}|\mathrm{T}^k_{1+w}[V](x)-V(x)|\neq 0\Big\}=0 \tag{19}$$

whenever $\mu\in \mathrm{M}^\alpha(\Omega)$ and $V\in L^{s,2}(\Omega)$ with $\alpha>2-s$. By Theorem 4.1 and (15), this would follow from

$$\mu\Big\{x\,:\,\limsup_{k\to\infty}\limsup_{N\to\infty}|e^{i\frac{1}{k}\Box}[G_N*V](x)-V(x)|\neq 0\Big\}=0.$$

Writing $V=I_s*g$, where $g\in L^2$, we take a Schwartz function h so that $\|g-h\|_{L^2}<\epsilon$. Then

$$\begin{aligned}
&\mu\Big\{\,x\,:\,\limsup_{k\to\infty}\limsup_{N\to\infty}|e^{i\frac{1}{k}\Box}[G_N*V](x)-V(x)|>\lambda\,\Big\}\\
\leqslant\;&\mu\Big\{\,x\,:\,\sup_{k\geqslant 1}\sup_{N\geqslant 1}|e^{i\frac{1}{k}\Box}[G_N*I_s*(g-h)](x)|>\lambda/3\,\Big\}+\\
&\mu\Big\{\,x\,:\,\limsup_{k\to\infty}\limsup_{N\to\infty}|e^{i\frac{1}{k}\Box}[G_N*I_s*h](x)-I_s*h(x)|>\lambda/3\,\Big\}+\\
&\mu\Big\{\,x\,:\,|I_s*(h-g)(x)|>\lambda/3\Big\}.
\end{aligned}$$

As the terms involving h are continuous in all parameters, the second set of the three is empty, so by the elementary inequality (14) and Theorem 4.4, we see that

$$\mu\Big\{x\,:\,\limsup_{k\to\infty}|\mathrm{T}^k_{1+w}V(x)-V(x)|>\lambda\Big\}\lesssim\lambda^{-1}\sqrt{c_\alpha(\mu)}\,\|g-h\|_{L^2}$$
$$\lesssim\lambda^{-1}\sqrt{c_\alpha(\mu)}\,\epsilon,$$

for all $\epsilon>0$, which yields (19), and so we are done. □

Proof of Theorem 1.5. This follows by applying Corollary 1.3 to the potential $q=V-\kappa^2\chi_\Omega$. For $V\in H^{1/2}$, the potentials $q=V-\kappa^2\chi_\Omega$ are contained in $\dot{H}^s$ for $0<s<1/2$ (see for example [22]) and so we find Bukhgeim solutions $U_{k,x}$, associated to q, and recover their value on the boundary as before. However, Corollary 1.3 requires the potential q to be contained in $H^{1/2}$ which is not satisfied for any domain. However, it is clear from the proof of Theorem 4.4 that we can relax this condition further to

$$\big\|\big(i\tfrac{\partial}{\partial_{x_1}}\big)^{1/4}\big(i\tfrac{\partial}{\partial_{x_2}}\big)^{1/4}q\big\|_{L^2(\mathbb{R}^2)}<\infty,$$

which is satisfied when Ω is a axis-parallel square, but not when it is a disc. □

Remark 4.5. As in the previous sections we can consider potentials which are not compactly supported. Here we can recover the potentials on Ω if $V\in H^s$ with $s>3/4$. Indeed, the arguments of this section require that

$$\big\|\big(i\tfrac{\partial}{\partial_{x_1}}\big)^{1/4}\big(i\tfrac{\partial}{\partial_{x_2}}\big)^{1/4}(\chi_\Omega V)\big\|_{L^2(\mathbb{R}^2)}<\infty,$$

for which it is again convenient to take Ω to be an axis-parallel square. Then arguing as in Remark 3.2, by the fractional Leibnitz rule,

$$\big\|\big(i\tfrac{\partial}{\partial_{x_2}}\big)^{1/4}(\chi_\Omega V)(x_1,\cdot)\big\|_{L^2(\mathbb{R})}\leqslant\|\chi_\Omega(x_1,\cdot)\|_4\|\big(i\tfrac{\partial}{\partial_{x_2}}\big)^{1/4}V(x_1,\cdot)\|_4$$

By factorising the integral using Fubini's theorem and applying the argument of Remark 3.2 in the x_2-variable, this holds if

$$\big\|\big(i\tfrac{\partial}{\partial_{x_1}}\big)^{1/4}\big(i\tfrac{\partial}{\partial_{x_2}}\big)^{s_0}V\big\|_{L^2(\mathbb{R}^2)}<\infty,$$

with $s_0>1/2$. Thus if a noncompactly supported potential is in H^s with $s>3/4$, we can recover it on any compact domain.

Finally we note that the uniqueness result of Blåsten [9] can be observed using the connection with the time-dependent Schrödinger equation. Indeed if the scattering data or boundary measurements are the same for two potentials V_1 and V_2, then by Alessandrini's identity (2),

$$\|V_2-V_1\|_{L^2}=\|V_2-\mathrm{T}^k_{1+w}V_2+\mathrm{T}^k_{1+w}V_1-V_1\|_{L^2},$$

so that by the triangle inequality and Lemma 4.1, it suffices to prove

$$\|V-\mathrm{T}^k_1V\|_{L^2}\to 0\quad\text{as}\quad k\to\infty,$$

which is a well-known property of the Schrödinger flow.

5. Proof of Theorem 1.4

First we construct a real potential V, supported in Ω, and contained in H^s with $s<1/2$, for which

$$\Big|\Big\{x\in\Omega\,:\,\lim_{k\to\infty} e^{i\frac{1}{k}\Box}[V](x)\not\to V(x)\Big\}\Big|\neq 0.$$

Throughout this section we work with a different set of coordinates from the previous sections. Indeed, for Schwartz functions F, we now write

$$e^{it\Box}[F](x)=\frac{1}{(2\pi)^2}\int e^{ix\cdot\xi}e^{-i2t\xi_1\xi_2}\,\widehat{F}(\xi)\,d\xi.$$

Let $\phi_{\rm o}$ be a positive Schwartz function, compactly supported in $[-1/4,1/4]$, and consider $\phi=\phi_{\rm o}*\phi_{\rm o}$, which is supported in $[-1/2,1/2]$. Note that $\widehat{\phi}=(\widehat{\phi}_{\rm o})^2\geqslant 0$. We consider the potential V defined by

$$\begin{aligned}V(x)&=\sum_{j\geqslant 2}V_j(x)=\sum_{j\geqslant 2}2^{(1-\beta)j+1}\cos(2^jx_2)\phi(2^jx_1)\phi(x_2)\\&=\sum_{j\geqslant 2}2^{(1-\beta)j}e^{i2^jx_2}\phi(2^jx_1)\phi(x_2)+\sum_{j\geqslant 2}2^{(1-\beta)j}e^{-i2^jx_2}\phi(2^jx_1)\phi(x_2)\\&=\sum_{j\geqslant 2}V_j^+(x)+\sum_{j\geqslant 2}V_j^-(x),\end{aligned}$$

which is supported in $[-\frac18,\frac18]\times[-\frac12,\frac12]$. If $\beta\in(1/2+s,1)$, by changes of variables,

$$\|V\|_{H^s}^2\leqslant C\sum_{j\geqslant 2}2^{(1-2\beta+2s)j}\int|\widehat{\phi}(\xi_1)\widehat{\phi}(\xi_2)|^2\,(1+|\xi|^2)^sd\xi<\infty.$$

Thus V is finite almost everywhere, and we will show that $e^{i\frac{1}{k}\Box}V$ diverges on $[\frac{1}{16},\frac14]\times[-\frac{1}{16},\frac{1}{16}]$.

This potential is an adaptation of an initial datum for the time-dependent nonelliptic Schrödinger equation considered in [48]. The initial datum there was not real, the diverging sequence of time was allowed to depend on the point x, and more crucially, the initial datum was not compactly supported. Thus our arguments will have a different flavour, working on the frequency and spatial side simultaneously.

By changes of variables and the Fourier inversion formula,

$$\begin{aligned}e^{it\Box}[V_j^+](x)&=\frac{2^{(1-\beta)j}e^{i2^jx_2}}{(2\pi)^2}\int\widehat{\phi}(\xi_1)\widehat{\phi}(\xi_2)\,e^{-i2^{j+1}t\xi_1\xi_2}e^{i(2^j\xi_1(x_1-2^{j+1}t)+\xi_2x_2)}d\xi\\&=\frac{2^{(1-\beta)j}e^{i2^jx_2}}{2\pi}\int\phi\big(2^j(x_1-2^{j+1}t-2t\xi_2)\big)\widehat{\phi}(\xi_2)\,e^{i\xi_2x_2}d\xi_2.\end{aligned}$$

Taking $t = 1/k$ with k the nearest natural number to $2^{j+1}/x_1$,

$$e^{i\frac{1}{k}\Box}[V_j^+](x) = \frac{2^{(1-\beta)j}e^{i2^jx_2}}{2\pi}\int \phi\big(\zeta(x_1,j) - \tfrac{2^{j+1}}{k}\xi_2\big)\widehat{\phi}(\xi_2)\, e^{i\xi_2x_2}d\xi_2,$$

where $|\zeta(x_1,j)| \leqslant \frac{1}{4}$ when $x_1 \in [\frac{1}{16}, \frac{1}{4}]$, so that, using the compact support of ϕ, we see that

$$\begin{aligned}|e^{i\frac{1}{k}\Box}[V_j^+](x)| &= \Big|\frac{2^{(1-\beta)j}}{2\pi}\int_{-16}^{16} \phi\big(\zeta(x_1,j) - \tfrac{2^{j+1}}{k}\xi_2\big)\widehat{\phi}(\xi_2)\, e^{i\xi_2x_2}d\xi_2\Big|\\ &\geqslant \frac{2^{(1-\beta)j}}{2\pi}\Big|\int_{-16}^{16} \phi\big(\zeta(x_1,j) - \tfrac{2^{j+1}}{k}\xi_2\big)\widehat{\phi}(\xi_2)\, \cos(\xi_2x_2)\, d\xi_2\Big|.\end{aligned}$$

Now when $x_2 \in [-\frac{1}{16}, \frac{1}{16}]$, we have $|\xi_2x_2| \leqslant 1$, so that $|\cos(\xi_2x_2)| > \cos(1)$. Using the fact that ϕ and $\widehat{\phi}$ are nonnegative, we obtain

$$\begin{aligned}|e^{i\frac{1}{k}\Box}[V_j^+](x)| &\geqslant \frac{2^{(1-\beta)j}\cos(1)}{2\pi}\int_{-16}^{16} \phi\big(\zeta(x_1,j) - \tfrac{2^{j+1}}{k}\xi_2\big)\widehat{\phi}(\xi_2)\, d\xi_2\\ &\geqslant C_12^{(1-\beta)j}.\end{aligned}$$

It remains to bound from above the solution associated to the other pieces of the potential. Again, by the Fourier inversion formula,

$$\begin{aligned}|e^{i\frac{1}{k}\Box}[V_\ell^\pm](x)| &= \frac{2^{(1-\beta)\ell}}{(2\pi)^2}\Big|\int \widehat{\phi}(\xi_1)\widehat{\phi}(\xi_2)\, e^{-i\frac{2}{k}\xi_1\xi_2}e^{i(2^\ell\xi_1(x_1\mp\frac{2^{\ell+1}}{k})+\xi_2x_2)}d\xi\Big|\\ &= \frac{2^{(1-\beta)\ell}}{2\pi}\Big|\int \phi\big(2^\ell(x_1\mp\tfrac{2^{\ell+1}}{k} - \tfrac{2}{k}\xi_2)\big)\widehat{\phi}(\xi_2)\, e^{i\xi_2x_2}d\xi_2\Big|.\end{aligned}$$

Using the fact that $\phi(y) \leqslant C|y|^{-1/2}$, we obtain

$$|e^{i\frac{1}{k}\Box}[V_\ell^\pm](x)| \leqslant C2^{(1/2-\beta)\ell}\int \frac{|\widehat{\phi}(\xi_2)|}{|x_1\mp\frac{2^{\ell+1}}{k} - \frac{2}{k}\xi_2|^{1/2}}\, d\xi_2.$$

Taking $0 < \epsilon < \min\{1/4, 1-\beta\}$, and using the rapid decay of $\widehat{\phi}$, we see that

$$|e^{i\frac{1}{k}\Box}[V_\ell^\pm](x)| \leqslant C2^{(1/2-\beta)\ell}\Big(\int_{|\xi_2|<2^{\epsilon j}} \frac{1}{|x_1\mp\frac{2^{\ell+1}}{k} - \frac{2}{k}\xi_2|^{1/2}}\, d\xi_2 + C2^{-j}\Big).$$

Now one can check that when $\ell \neq j$ or $j = \ell$ and $\mp$ is an addition,

$$|\tfrac{2}{k}\xi_2| \leqslant \tfrac{3}{4}|x_1\mp\tfrac{2^{\ell+1}}{k}|$$

when $|\xi_2| \leqslant 2^{j\epsilon}$. Indeed, when $j > \ell$, the left-hand side is less than $\frac{1}{4}|x_1|$ which is less than the right-hand side. On the other hand, when $j < \ell$ or $j = \ell$ and $\mp$ is an addition, the left-hand side is less than $\frac{1}{2}|x_1|$ which is less than the right-hand side. Thus, the integrand of the final integral is nonsingular so that the integral is bounded by $C|x_1|^{-1/2}2^{\epsilon j} \leqslant C2^{\epsilon j}$.

By summing a geometric series in ℓ, we obtain

$$\Big|\sum_{\ell\neq j} e^{i\frac{1}{k}\Box}V_\ell^\pm(x) + e^{i\frac{1}{k}\Box}V_j^-(x)\Big| \leqslant C_22^{\epsilon j},$$

and we can conclude that on $[\frac{1}{16},\frac{1}{4}]\times[-\frac{1}{16},\frac{1}{16}]$,

$$|e^{i\frac{1}{k}\Box}[V]|\geqslant|e^{i\frac{1}{k}\Box}[V_j^+]|-\Big|\sum_{\ell\neq j}e^{i\frac{1}{k}\Box}[V_\ell^\pm]+e^{i\frac{1}{k}\Box}V_j^-(x)\Big|\geqslant C_1 2^{j(1-\beta)}-C_2 2^{j\epsilon},$$

which diverges as j tends to infinity. Considering forty-five degree rotations of the V_j, which are Schwartz functions, via the pointwise equality, this yields

$$|\mathrm{T}_1^k[V]|\geqslant|\mathrm{T}_1^k[V_j^+]|-\Big|\sum_{\ell\neq j}\mathrm{T}_1^k[V_\ell^\pm]+\mathrm{T}_1^k[V_j^-]\Big|\geqslant C_1 2^{j(1-\beta)}-C_2 2^{j\epsilon}$$

on a forty-five degree rotation of $[\frac{1}{16},\frac{1}{4}]\times[-\frac{1}{16},\frac{1}{16}]$, so that $|\mathrm{T}_1^k[V]|$ diverges as k tends to infinity. Thus, by Theorem 4.1, combined with Alessandrini's identity (2),

$$\Big\{x\ :\ \tfrac{k}{4\pi}\Big\langle(\Lambda_V-\Lambda_0)[u_{k,x}|_{\partial\Omega}],e^{i\overline{\psi}}|_{\partial\Omega}\Big\rangle\not\to V(x)\ \text{ as }\ k\to\infty\Big\}$$

contains a forty-five degree rotation of $[\frac{1}{16},\frac{1}{4}]\times[-\frac{1}{16},\frac{1}{16}]$, which has nonzero Lebesgue measure. □

Note that this result is stable in the sense that $k\in\mathbb{N}$ can be replaced by any sequence $\{n_k\}_{k\in\mathbb{N}}$ satisfying $n_k\in[k,k+1)$.

Remark 5.1. In [51], Sjölin asked for which values of s is it true that

$$\lim_{k\to\infty}e^{i\frac{1}{k}\Delta}f(x)=0,\quad \text{a.e. } x\in\mathbb{R}^d\backslash(\mathrm{supp}f),$$

for all $f\in H^s$. In principle, this question could have stronger positive results and weaker negative results than Carleson's question: for which values of s is it true that

$$\lim_{k\to\infty}e^{i\frac{1}{k}\Delta}f(x)=f(x),\quad \text{a.e. } x\in\mathbb{R}^d,$$

for all $f\in H^s$? Indeed, before Bourgain's recent breakthrough [10], Sjölin proved a stronger positive result for his question than what was known for Carleson's question in three dimensions. Here we solve Sjölin's question completely for the nonelliptic equation in two dimensions. That is to say,

$$\lim_{k\to\infty}e^{i\frac{1}{k}\Box}f(x)=0,\quad \text{a.e. } x\in\mathbb{R}^2\backslash(\mathrm{supp}f),$$

for all $f\in H^s$ if and only if $s\geqslant 1/2$.

Appendix A. The DN map from the scattering amplitude

It is well–known that in the absence of zero Dirichlet eigenvalues there is a unique weak solution to the Dirichlet problem (1) that satisfies

$$\|u\|_{H^1(\Omega)}\leqslant C\|f\|_{H^{1/2}(\partial\Omega)} \tag{20}$$

(see for example [19] - in two dimensions $L^{n/2}(\mathbb{R}^n)$ can be replaced by $L^2(\mathbb{R}^2)$). Here $H^{1/2}(\partial\Omega) := H^1(\Omega)/H^1_0(\Omega)$, where $H^1_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$. The DN map Λ_V is then defined by

$$\Big\langle \Lambda_V[f], \psi \Big\rangle = \int_{\partial\Omega} \Lambda_V[f]\,\psi = \int_\Omega V u \Psi + \nabla u \cdot \nabla \Psi,$$

for all $\Psi \in H^1(\Omega)$ with $\psi = \Psi + H^1_0(\Omega)$. When the solution and boundary are sufficiently regular, this definition coincides with that of the introduction by Green's formula. To see that Λ_V maps from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, the dual of $H^{1/2}(\partial\Omega)$, we note that by Hölder's inequality and the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}\Big|\Big\langle \Lambda_V[f], \psi \Big\rangle\Big| &\leqslant \|u\|_{H^1(\Omega)}\|\Psi\|_{H^1(\Omega)} + \|V\|_2\|u\|_{L^4(\Omega)}\|\Psi\|_{L^4(\Omega)}\\ &\leqslant (1 + C\|V\|_2)\|u\|_{H^1(\Omega)}\|\Psi\|_{H^1(\Omega)}\end{aligned}$$

whenever $\Psi \in H^1(\Omega)$, so that by (20), we obtain

$$\Big|\Big\langle \Lambda_V[f], \psi \Big\rangle\Big| \leqslant C(1 + \|V\|_2)\|f\|_{H^{1/2}(\partial\Omega)}\|\psi\|_{H^{1/2}(\partial\Omega)}.$$

There are a number of different approaches to showing that the scattering amplitude at a fixed energy $\kappa^2 > 0$ uniquely determines the DN map $\Lambda_{V-\kappa^2}$ and *vice versa* (see for example [6, 39, 58, 54, 56]). Here we follow a constructive argument due to Nachman [40, Section 3]. We must additionally assume that κ^2 is not a Dirichlet eigenvalue of $-\Delta + V$. This can be arranged by taking Ω sufficiently large as the eigenvalues decrease strictly as the domain grows [43] (the result of [37] can be extended to L^2-potentials using the unique continuation of [32]). We also additionally suppose that V is real.

Let G_V and G_0 be the outgoing Green's functions that satisfy

$$(-\Delta + V - \kappa^2)G_V(x, y) = \delta(x - y), \quad (-\Delta - \kappa^2)G_0(x, y) = \delta(x - y),$$

and let S_V and S_0 be the corresponding near-field operators defined via single layer potentials;

$$S_V[f](x) = \int_{\partial\Omega} G_V(x, y)f(y)\,dy, \quad S_0[f](x) = \int_{\partial\Omega} G_0(x, y)f(y)\,dy.$$

These are bounded and invertible, mapping $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$ (the two–dimensional proof can be found in [31, Proposition A.1]). Then Nachman's formula [39],

$$\Lambda_{V-\kappa^2} = \Lambda_{-\kappa^2} + S_V^{-1} - S_0^{-1},$$

allows us to recover the DN map on Lipschitz domains.

Thus it remains to recover the single layer potential S_V from the scattering amplitude A_V at energy κ^2. For $\omega \in \mathbb{S}^1$, the outgoing scattering solution $v(\cdot, \omega, \kappa)$ is the unique solution to the Lippmann–Schwinger equation

$$v(y, \omega, \kappa) = e^{i\kappa y\cdot\omega} - \int_{\mathbb{R}^2} G_0(y, z)V(z)v(z, \omega, \kappa)\,dz. \tag{21}$$

For $(\sigma,\omega)\in\mathbb{S}^1\times\mathbb{S}^1$, the scattering amplitude then satisfies

$$A_V(\sigma,\omega,\kappa)=\int_{\mathbb{R}^2}e^{-i\kappa\sigma\cdot z}V(z)v(z,\omega,\kappa)\,dz. \tag{22}$$

When Ω is a disc, Nachman recovers S_V via formulae given by expansions in spherical harmonics as below. Otherwise he uses a density argument (we remark that Sylvester [56] also invokes density in order to recover). Since we have been obliged to work with Ω a square, at this point we deviate and instead follow an argument of Stefanov [52], obtaining an explicit formula for the Green's function G_V in terms of A_V. Alternatively it seems likely that we could pass to the DN map on the square from that on the disc via the argument in [41, Section 6] for the conductivity problem, but we prefer this more direct approach.

Stefanov worked in three dimensions, with bounded potentials, and a number of details change in two dimensions, so we present the argument. We recover G_V outside of a disc which contains the potential, but which is contained in the domain, so that S_V can be obtained by integrating along the sides of our square Ω.

First we require the following asymptotics.

Lemma A.1.

$$G_V(x,y)-G_0(x,y)=\frac{-i}{8\pi\kappa}\frac{e^{i\kappa|x|}}{|x|^{\frac12}}\frac{e^{i\kappa|y|}}{|y|^{\frac12}}A_V\Big(-\frac{x}{|x|},\frac{y}{|y|},\kappa\Big)+o\Big(\frac{1}{|x|^{\frac12}|y|^{\frac12}}\Big).$$

Proof. It is well–known (see for example (3.66) in [42]) that G_V satisfies

$$G_V(x,z)=\frac{e^{i\frac{\pi}{4}}}{(8\pi)^{\frac12}}\frac{e^{i\kappa|x|}}{\kappa^{\frac12}|x|^{\frac12}}v\Big(z,-\frac{x}{|x|},\kappa\Big)+o\Big(\frac{1}{|x|^{\frac12}}\Big), \tag{23}$$

and, in particular,

$$G_0(y,z)=\frac{e^{i\frac{\pi}{4}}}{(8\pi)^{\frac12}}\frac{e^{i\kappa|y|}}{\kappa^{\frac12}|y|^{\frac12}}e^{-i\kappa\frac{y}{|y|}\cdot z}+o\Big(\frac{1}{|y|^{\frac12}}\Big). \tag{24}$$

On the other hand, it is easy to verify that

$$G_V(x,y)-G_0(x,y)=-\int_{\mathbb{R}^2}G_V(x,z)V(z)G_0(y,z)\,dz. \tag{25}$$

Substituting in (23) and (24), see that $G_V(x,y)-G_0(x,y)$ is equal to

$$\frac{-i}{8\pi\kappa}\frac{e^{i\kappa|x|}}{|x|^{\frac12}}\frac{e^{i\kappa|y|}}{|y|^{\frac12}}\int e^{-i\kappa\frac{y}{|y|}\cdot z}V(z)v\Big(z,-\frac{x}{|x|},\kappa\Big)dz+o\Big(\frac{1}{|x|^{\frac12}|y|^{\frac12}}\Big),$$

so that by (22) we obtain the result. □

In the following, J_n and $H_n^{(1)}$ denote the Bessel and Hankel functions of the first kind of nth order, respectively (see for example [36]). We also write x in polar coordinates as $(|x|,\phi_x)$.

Theorem A.2. *Let $V \in H^s$ with $s > 0$ be supported in the disc of radius ρ, centred at the origin, and consider the Fourier series*

$$A_V(\sigma, \omega, \kappa) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_{n,m} e^{in\phi_\sigma} e^{im\phi_\omega}.$$

Then

$$G_V(x, y) - G_0(x, y) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{(-1)^n}{16} i^{n+m} a_{n,m} H_n^{(1)}(\kappa|x|) H_m^{(1)}(\kappa|y|) e^{in\phi_x} e^{im\phi_y},$$

where the series is uniformly, absolutely convergent for $|x| > |y| > R > \frac{3}{2}\rho$.

Proof. We can expand $G_0(x, y) = \frac{i}{4} H_0^{(1)}(\kappa|x - y|)$ as

$$G_0(x, y) = \frac{i}{4}\Big(H_0^{(1)}(\kappa|x|)J_0(\kappa|y|) + 2\sum_{n\geqslant 1} H_n^{(1)}(\kappa|x|)J_n(\kappa|y|) \cos(\phi_x - \phi_y)\Big),$$

(see for example [17, Section 3.4] or [49, Theorem 3.4]). As $H_{-n}^{(1)} = (-1)^n H_n^{(1)}$ and $J_{-n} = (-1)^n J_n$, in order to separate variables it will be convenient to write this as

$$G_0(x, y) = \frac{i}{4} \sum_{n\in\mathbb{Z}} H_n^{(1)}(\kappa|x|)J_n(\kappa|y|)e^{in\phi_x} e^{-in\phi_y}.$$

As before, it is easy to check that

$$G_V(x, y) - G_0(x, y) = -\int_{\mathbb{R}^2} G_0(x, z)V(z)G_V(z, y)\, dz,$$

and so substituting (25) into this we obtain $G_V - G_0 = -I_1 + I_2$, where

$$I_1 = \int G_0(x, z)V(z)G_0(z, y)\, dz$$
$$I_2 = \int G_0(x, z_1)V(z_1) \int G_V(z_1, z_2)V(z_2)G_0(y, z_2)\, dz_2 dz_1.$$

Now in both integrals we introduce the expansion of G_0 (note that $G_0(x, y) = G_0(y, x)$), extracting the terms independent of z, z_1, z_2. In this way we get

$$I_1 = -\frac{1}{16} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \alpha_{n,m} H_n^{(1)}(\kappa|x|)H_m^{(1)}(\kappa|y|)e^{in\phi_x} e^{im\phi_y}, \tag{26}$$

$$I_2 = -\frac{1}{16} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \beta_{n,m} H_n^{(1)}(\kappa|x|)H_m^{(1)}(\kappa|y|)e^{in\phi_x} e^{im\phi_y}, \tag{27}$$

where

$$\alpha_{n,m} = \int_{\mathbb{R}^2} V(z)J_n(\kappa|z|)J_m(\kappa|z|)e^{-i(n+m)\phi_z}\, dz,$$
$$\beta_{n,m} = \int_{\mathbb{R}^4} J_n(\kappa|z_1|)V(z_1)G_V(z_1, z_2)V(z_2)J_m(\kappa|z_2|)e^{-in\phi_{z_1}} e^{-im\phi_{z_2}}\, dz_1 dz_2.$$

It remains to show that the sums (26) and (27) converge uniformly and absolutely for $|x| > |y| > R > \frac{3}{2}\rho$. Once we know that this is the case, we can take limits and use the asymptotics of the Hankel functions for large r;

$$H_n^{(1)}(r) = e^{-i(n\frac{\pi}{2}+\frac{\pi}{4})}\Big(\frac{2}{\pi r}\Big)^{\frac{1}{2}} e^{ir} + o\Big(\frac{1}{r^{\frac{1}{2}}}\Big)$$

(see for example [36, Section 5.16]), and then Lemma A.1 tells us that

$$-\frac{1}{16}(-i)^{n+m+1}\frac{2}{\pi}(\beta_{n,m} - \alpha_{n,m}) = -i\frac{(-1)^n}{8\pi}a_{n,m}.$$

To see that the sums converge note that, by Hölder's inequality, we have

$$\begin{aligned}
|\alpha_{n,m}| &\leqslant C_\rho \|V\|_{L^2} \|J_n(\kappa|\cdot|)\|_{L^\infty(B_\rho)} \|J_m(\kappa|\cdot|)\|_{L^\infty(B_\rho)},\\
|\beta_{n,m}| &\leqslant \|G_V\|_{L^2(B_\rho\times B_\rho)} \|V\|_{L^2}^2 \|J_n(\kappa|\cdot|)\|_{L^\infty(B_\rho)} \|J_m(\kappa|\cdot|)\|_{L^\infty(B_\rho)}.
\end{aligned}$$

At this point we deviate from [52] as there seems to be less local knowledge regarding G_V in two dimensions. Instead we can rewrite (25) as

$$G_V(\cdot, y) = G_0(\cdot, y) - (-\Delta + V - \kappa^2 - i0)^{-1}[VG_0(\cdot, y)],$$

and use that the resolvent is bounded from $L^2((1+|\cdot|^2)^\delta)$ to $L^2((1+|\cdot|^2)^{-\delta})$ with $\delta > 1/2$ (see [1, Theorem 4.2]). Thus, using that V is compactly supported, and taking $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ with large p so that $1 - \frac{2}{q} = s$,

$$\begin{aligned}
\|G_V(\cdot, y)\|_{L^2(B_\rho)} &\leqslant \|G_0(\cdot, y)\|_{L^2(B_\rho)} + C_\rho \|VG_0(\cdot, y)\|_{L^2(B_\rho)}\\
&\leqslant \|G_0(\cdot, y)\|_{L^2(B_\rho)} + C_\rho \|V\|_q \|G_0(\cdot, y)\|_{L^p(B_\rho)}\\
&\leqslant \|G_0(\cdot, y)\|_{L^2(B_\rho)} + C_\rho \|V\|_{H^s} \|G_0(\cdot, y)\|_{L^p(B_\rho)},
\end{aligned}$$

by the Hardy–Littlewood–Sobolev inequality. Integrating again with respect to y, and recalling that the singularity of $H_0^{(1)}$ at the origin is logarithmic, we see that $\|G_V\|_{L^2(B_\rho\times B_\rho)} \leqslant C$. Then, using the Taylor series expansion for the Bessel function,

$$|J_n(r)| = \Big|\sum_{j\geqslant 0} \frac{(-1)^j}{j!(|n|+j)!}\Big(\frac{r}{2}\Big)^{2j+|n|}\Big| \leqslant C_\rho \frac{1}{|n|!}\Big(\frac{\rho}{2}\Big)^{|n|}, \quad 0 \leqslant r \leqslant \rho,$$

we see that

$$\begin{aligned}
|\alpha_{n,m}| &\leqslant C_\rho \|V\|_{L^2} \frac{1}{|n|!}\Big(\frac{\rho}{2}\Big)^{|n|} \frac{1}{|m|!}\Big(\frac{\rho}{2}\Big)^{|m|},\\
|\beta_{n,m}| &\leqslant C_\rho (1 + \|V\|_{H^s}^3) \frac{1}{|n|!}\Big(\frac{\rho}{2}\Big)^{|n|} \frac{1}{|m|!}\Big(\frac{\rho}{2}\Big)^{|m|}.
\end{aligned}$$

Finally, we require the Hankel function estimate,

$$|H_n^{(1)}(r)| \leqslant C_R |n|!\Big(\frac{3}{R}\Big)^{|n|}, \quad R \leqslant r,$$

which is proven in [2, Lemma 2.3]. The sums (26) and (27) are then bounded by a constant multiple of

$$\sum_{n\geqslant 0}\sum_{m\geqslant 0}\Big(\frac{3\rho}{2R}\Big)^n\Big(\frac{3\rho}{2R}\Big)^m$$

which is convergent when $|x|>|y|>R>\frac{3}{2}\rho$, and so we are done. □

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References

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), no. 2, 151–218.

[2] K. Astala, D. Faraco and K.M. Rogers, Recovery of the Dirichlet-to-Neumann map from scattering data in the plane, Kokyuroku Bessatsu, to appear.

[3] K. Astala and L. Päivärinta, Calderón's inverse conductivity problem in the plane, Ann. of Math. **163** (2006), no. 1, 265–299.

[4] J.A. Barceló, J. Bennett, A. Carbery and K.M. Rogers, On the dimension of divergence sets of dispersive equations, Math. Ann. **349** (2011), no. 3, 599–622.

[5] J.A. Barceló, A. Ruiz and L. Vega, Weighted estimates for the Helmholtz equation and some applications, J. Funct. Anal. **150** (1997), no. 2, 356–382.

[6] Ju. M. Berezanskiĭ, The uniqueness theorem in the inverse problem of spectral analysis for the Schrödinger equation, Trudy Moskov. Mat. Obšč. **7** (1958), 1–62.

[7] J. Bergh and J. Löfström, *Interpolation spaces*, Springer, Berlin, 1976.

[8] A. Beurling, Ensembles exceptionnels, Acta Math. **72** (1940), 1–13.

[9] E. Blåsten, Stability and uniqueness for the inverse problem of the Schrödinger equation in 2D with potentials in $W^{\epsilon,p}$, arXiv:1106.0632 (2011).

[10] J. Bourgain, On the Schrödinger maximal function in higher dimension, ArXiv:1201.3342 (2011).

[11] R. Brown and R. Torres, Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in L^p, J. Fourier Analysis and Appl. **6** (2003), 563-574.

[12] R. Brown and G. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, Comm. in P.D.E. **22**, (1997), 1009–1027.

[13] A.L. Bukhgeim, Recovering a potential from Cauchy data in the two-dimensional case, J. Inverse Ill-Posed Probl. **16** (2008), no. 1, 19–33.

[14] A.P. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matematica, Rio de Janeiro, (1980), 65-73.

[15] L. Carleson, Some analytic problems related to statistical mechanics, Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), Lecture Notes in Math. **779**, 5–45, Springer, Berlin, 1980.

[16] S. Chanillo, A problem in electrical prospection and an n-dimensional Borg-Levinson theorem, Proc. Amer. Math. Soc. **108** (1990), no. 3, 761–767.

[17] D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*, Applied Mathematical Sciences, 93, Springer, Berlin, 1992.

[18] D. Dos Santos Ferreira, C.E. Kenig and M. Salo and G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems, Invent. Math. **178** (2009), no. 1, 119–171.

[19] D. Dos Santos Ferreira, C.E. Kenig and M. Salo, Determining an unbounded potential from Cauchy data in admissible geometries, Comm. Partial Differential Equations **38** (2013), no. 1, 50–68.

[20] J.G. van der Corput, 'Zahlentheoretische Abschätzungen', Math. Ann. **84** (1921), 53–79.

[21] B.E.J. Dahlberg and C.E. Kenig, A note on the almost everywhere behavior of solutions to the Schrödinger equation, Harmonic analysis (Minneapolis, Minn., 1981), Lecture Notes in Math. **908**, 205–209, Springer, Berlin, 1982.

[22] D. Faraco and K.M. Rogers, The Sobolev norm of characteristic functions with applications to the Calderón inverse problem, Q. J. Math. **64** (2013), 133–147.

[23] L.D. Faddeev, Increasing solutions of the Schrödinger equation (in Russian), Dokl. Akad. Nauk SSSR **165** (1965), 514 – 517; English translation in Sov. Phys. Dokl. **10** (1966), 1033 1035. 85, 100

[24] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Invisibility and inverse problems, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 1, 55–97.

[25] C. Guillarmou, M. Salo and L. Tzou, Inverse scattering at fixed energy on surfaces with Euclidean ends, Comm. Math. Phys. **303** (2011), no. 3, 761–784.

[26] C. Guillarmou and L. Tzou, Calderón inverse problem with partial data on Riemann surfaces, Duke Math. J. **158** (2011), no. 1, 83–120.

[27] C. Guillarmou and L. Tzou, Identification of a connection from Cauchy data on a Riemann surface with boundary, Geom. Funct. Anal. **21** (2011), 393–418.

[28] B. Haberman and D. Tataru, Uniqueness in Calderón's problem with Lipschitz conductivities, Duke Math. J. 162 (2013), 496–516.

[29] O.Y. Imanuvilov, G. Uhlmann and M. Yamamoto, The Calderón problem with partial data in two dimensions. J. Amer. Math. Soc. **23** (2010), no. 3, 655–691.

[30] O.Y. Imanuvilov and M. Yamamoto, Inverse boundary value problem for Schrödinger equation in two dimensions, arXiv:1208.3775 (2012).

[31] V. Isakov and A. I. Nachman, Global uniqueness for a two-dimensional semilinear elliptic inverse problem, Trans. Amer. Math. Soc. **347** (1995), no. 9, 3375–3390.

[32] D. Jerison and C.E. Kenig, Unique continuation and absence of positive eigenvalues for Schrödinger operators, Ann. of Math. (2) **121** (1985), no. 3, 463–494.

[33] C.E. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle, Comm. Pure Appl. Math., **46**, 1993, 527–620.

[34] C.E. Kenig, J. Sjöstrand and G. Uhlmann, The Calderón problem with partial data, Ann. of Math. (2) **165** (2007), no. 2, 567–591.

[35] G.M. Khenkin and R.G. Novikov, The $\overline{\partial}$-equation in the multidimensional inverse scattering problem. (Russian) Uspekhi Mat. Nauk **42** (1987), 93–152.

[36] N.N. Lebedev, *Special functions and their applications*, Revised English edition. Prentice Hall, Englewood Cliffs, NJ, 1965.

[37] R. Leis, Zur Monotonie der Eigenwerte selbstadjungierter elliptischer Differentialgleichungen, Math. Z. **96** (1967), 26–32.

[38] P. Mattila, Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability, Cambridge University Press, 1995.

[39] A.I. Nachman, Reconstructions from boundary measurements. Ann. of Math. **128** (1988), no. 3, 531–576.

[40] A.I. Nachman, Inverse scattering at fixed energy, in *Mathematical physics, X (Leipzig, 1991)*, 434–441, Springer, Berlin.

[41] A.I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. of Math. **142** (1995), 71–96.

[42] A.I. Nachman, L. Päivärinta and A. Teirilä, On imaging obstacles inside inhomogeneous media, J. Funct. Anal. **252** (2007), no. 2, 490–516.

[43] A.I. Nachman, personal communication.

[44] A.I. Nachman, J. Sylvester and G. Uhlmann, An n-dimensional Borg-Levinson theorem, Comm. Math. Phys. **115** (1988), no. 4, 595–605.
[45] R.G. Novikov, A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. (Russian) Funktsional. Anal. i Prilozhen. **22** (1988), no. 4, 11–22, 96; translation in Funct. Anal. Appl. **22** (1988), no. 4, 263–272 (1989).
[46] R.G. Novikov and M. Santacesaria, Global uniqueness and reconstruction for the multi-channel Gel'fand-Calderón inverse problem in two dimensions. Bull. Sci. Math. **135** (2011), no. 5, 421–434.
[47] L. Päivärinta, A. Panchenko and G. Uhlmann, Complex geometrical optics solutions for Lipschitz conductivities. Rev. Mat. Iberoamericana **19** (2003), no. 1, 57–72.
[48] K.M. Rogers, A. Vargas and L. Vega, Pointwise convergence of solutions to the nonelliptic Schrödinger equation. Indiana Univ. Math. J. **55** (2006), no. 6, 1893–1906.
[49] A. Ruiz, Harmonic analysis and inverse problems, www.uam.es/gruposinv/inversos/publicaciones/inverseproblems.pdf.
[50] P. Sjölin, Maximal estimates for solutions to the nonelliptic Schrödinger equation, Bull. Lond. Math. Soc. **39** (2007), 404–412.
[51] P. Sjölin, Some remarks on the localization of Schrödinger means. Bull. Sci. Math. **136** (2012), 638-647.
[52] P. Stefanov, Stability of the inverse problem in potential scattering at fixed energy, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 4, 867–884 (1991).
[53] Z.Q. Sun and G. Uhlmann, Generic uniqueness for an inverse boundary value problem, Duke Math. J. **62** (1991), no. 1, 131–155.
[54] Z.Q. Sun and G. Uhlmann, Inverse scattering for singular potentials in two dimensions, Trans. Amer. Math. Soc. **338** (1993), no. 1, 363–374.
[55] Z.Q. Sun and G. Uhlmann, Recovery of singularities for formally determined inverse problems, Comm. Math. Phys. **153** (1993), no. 3, 431–445.
[56] J. Sylvester, The Cauchy data and the scattering amplitude, Comm. Partial Differential Equations **19** (1994), no. 9-10, 1735–1741.
[57] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem. Annals of Math. **125**, (1987), 153–169.
[58] G. Uhlmann, Inverse boundary value problems and applications, Astérisque No. 207 (1992), 6, 153–211.
[59] D. Žubrinić, Singular sets of Sobolev functions, C. R. Math. Acad. Sci. Paris **334** (2002), no. 7, 539–544.

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