

# Reconstruction of discontinuities from backscattering data in 2D \*

J.A. Barceló, Daniel Faraco, Alberto Ruiz and Ana Vargas

## Abstract

We prove that the non smooth part of a non compactly supported potential  $q$  in the Schrödinger Hamiltonian  $-\Delta + q$ , in dimension  $n = 2$ , is contained in its Born approximation  $q_B$  for backscattering data in a precise sense in terms of continuity: Given  $q$  in  $W^{\alpha,2}$  the difference  $q - q_B$  is in the Hölder class  $\Lambda^\beta$  for any  $\beta < \alpha$ .

## 1 Introduction

We study the inverse scattering problem, dealing with the reconstruction of an complex and not compactly supported electrical potential in the Schrödinger equation by means of measurements of the far field pattern of the scattering solutions. In the case of complex valued potentials, the scattering solutions might be defined only for large enough values of the energy. This fact motivates to study the reconstruction of the singularities of the potential from backscattering data, in particular if these singularities coincide with the singularities of the linear term in the Neumann-Born series of the scattering amplitude (Born approximation). A central issue, arising when the backscattering procedure is compared with alternatives ways of interpreting the scattering data, is to determine precisely which singularities are recovered. The problem have received a lot of attention in the last years ([13], [19], [16], [17], [12], [4] [5] and [21].) In particular in the works [4] and [5] it was proved that in odd

---

\*The first three authors were partially supported by the grant MTM2011-28198 (Spain). The second author was partially supported by the ERC 307179 The fourth author is partially supported by the grant MTM2010-16518 (Spain)

$n$  dimension the quadratic term in the Born expansion has a gain up to  $\min\{s - (n - 3)/2, 1\}$  of derivative with respect to the potential  $q$ , for compactly supported potentials  $q \in W^{s,2}$ . This in contrast with the previous works were the typical gain is of half derivative in the framework of Hilbertian Sobolev spaces.

Inspired by the approach in [19], we prove that, in the two dimensional case, this reconstruction of singularities can be achieved with an accuracy corresponding with one derivative in the sense of integrability : the error in the approximation is in the Holder class  $\Lambda^\alpha$ , for  $0 < \alpha < s_0$ , provided that the potential  $q$  is in the Sobolev space  $W^{s_0,2}(\mathbb{R}^2)$ . Notice that this would be a corollary of  $q - q_B \in W^{s_0+1,2}$ , but the existing methods or modifications of them seem unable to yield this result. The best recovery up to now is just  $q - q_B \in W^{s_0+1/2,2}$  (see [19], [13] and [16]). We also have been able to deal with not compactly supported potential. This is of interest in the scattering theory of quantum hamiltonian but absolutely essential to its application to recovery of live loads in the Navier operator arising in elasticity. This is because in that theory one deals with Riesz transforms (Leray projections) of the potential which do not preserve supports [3].

Let us now describe our results more rigorously and discuss the technical innovations in the proof. We consider the generalized eigenfunctions of the Schrödinger operator  $H = -\Delta + q$ , with the electrical potential  $q$ . These functions are the solutions of

$$\begin{cases} (H - k^2)u = 0 \\ u = u_i + u_s \\ \lim_{|x| \rightarrow \infty} \left( \frac{\partial u_s}{\partial r} - ik u_s \right) (x) = o(|x|^{-(n-1)/2}), \end{cases} \quad (1.1)$$

$u(k, \theta, x)$ , which are the responses of the electrical potential to an incoming free plane wave  $u_i(x) = e^{ik\theta \cdot x}$ , with wave number  $k$ , incoming angle  $\theta \in \mathbb{S}^{n-1}$  and energy  $k^2$ . The last identity in (1.1) is known as outgoing Sommerfeld radiation condition.

In the case of non compactly supported potentials, some a priori decay conditions together with integrability, allow to construct the scattering solutions by using a priori estimates of the free outgoing resolvent. In this case the radiation condition has to be understood in the sense that the solution is obtained by taking the outgoing limiting absorption principle.

The data in scattering theory are given by the far field patterns or scattering amplitudes, defined by means of the following integral,

$$u_\infty(k, \theta, \frac{x}{|x|}) = \int_{\mathbf{R}^n} e^{-ik \frac{x}{|x|} \cdot y} q(y) u(k, \theta, y) dy \quad (1.2)$$

If the support of  $q$  is compact, the scattering amplitude is the spherical part of the main term in the asymptotic expansion of the generalized eigenfunction, see [7],

$$u(k, \theta, x) = e^{ik\theta \cdot x} + \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u_\infty(k, \theta, \frac{x}{|x|}) + o(|x|^{-(n-1)/2}).$$

The function  $u_\infty : \mathbf{R} \times \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \rightarrow \mathbf{C}$  represents the measurements or data in inverse scattering.  $\theta$  is the incident direction, and  $\omega = x/|x|$  is the receiver direction. The inverse scattering problem deals with the recovery of  $q$  from some measurements of the far field pattern.

The solution  $u_s$  satisfies the equation

$$(\Delta + k^2)u_s = qu_s + qu_i.$$

By applying the outgoing resolvent of Helmholtz equation,

$$R_+(k^2) := \mathcal{F}^{-1}(-|\xi|^2 + k^2 + i0)^{-1} \mathcal{F},$$

where  $\mathcal{F}$  denotes the Fourier transform, we obtain the so called Lippman-Schwinger integral equation

$$u_s = R_+(k^2)(q(\cdot)u_i(k, \theta, \cdot)) + R_+(k^2)(q(\cdot)u_s(k, \theta, \cdot)). \quad (1.3)$$

If we insert this equation in (1.2), we obtain the Born series expansion for  $u_\infty$

$$\begin{aligned} u_\infty(k, \theta, \omega) &= \int_{\mathbf{R}^n} e^{-ik(\omega-\theta) \cdot y} q(y) dy + \sum_{j=1}^m \int e^{-ik\omega \cdot y} (qR_+(k^2))^j (q(\cdot)u_i(k, \theta, \cdot))(y) dy \\ &+ \int_{\mathbf{R}^n} e^{-ik\omega \cdot y} (qR_+(k^2))^m (q(\cdot)u_s(k, \theta, \cdot))(y) dy. \end{aligned}$$

This can be written as

$$u_\infty(k, \theta, \omega) = \hat{q}(\xi) + \sum_{j=1}^m \widehat{Q_{j+1}(q)}(k, \theta, \omega) + \widehat{Q_{m+1}^R(q)}(k, \theta, \omega), \quad (1.4)$$

where  $\xi = k(\omega - \theta)$  and the  $j$ -th term is given by

$$\widehat{Q_j(q)}(k, \theta, \omega) = \int_{\mathbb{R}^n} e^{-ik\omega \cdot y} (qR_+(k^2))^{j-1} (q(\cdot)e^{ik\theta \cdot (\cdot)})(y) dy. \quad (1.5)$$

The inverse problem for the whole scattering amplitude is formally overdetermined. It makes then sense to consider the recovery of  $q$  from partial knowledge of  $u_\infty$ . The most celebrated partial measurements are

- (a) Fixed angle data. One assumes  $u_\infty(k, \theta_0, \omega)$  known for a fixed  $\theta_0 \in \mathbb{S}^{n-1}$ . This inverse problem is formally well determined.
- (b) Fixed energy data. One assumes known the full data for a fixed energy  $k_0^2 > 0$ . This problem is related to the inverse boundary problem with data being the so called Dirichlet to Neuman map.
- (c) Backscattering data. These data are  $u_\infty(k, \theta, -\theta)$  for any  $\theta \in \mathbb{S}^{n-1}$  and any  $k > 0$ . The inverse problem is formally well determined.

In case (a) and (c) the Born approximation of the potential  $q$  can be defined by using the inverse Fourier transform after an appropriate change of variable.

For instance in (a) with  $\xi = k(\omega - \theta_0)$ , the function  $q_B^{\theta_0}$  is known as the **Born approximation** for fixed angle data.

$$\widehat{q_B^{\theta_0}}(\xi) = u_\infty(k, \theta_0, \omega). \quad (1.6)$$

In (c) the corresponding Born approximation is given by

$$\widehat{q_B}(\xi) = u_\infty(k, \theta, -\theta)$$

where  $\xi = -2k\theta$ .

In practical applications, the actual potential is substituted by its approximation and the procedures to reconstruct the approximation  $\widehat{q_B^\theta} = u_\infty(k, \theta, \omega)$  from scattering data are known as Diffraction Tomography. Nevertheless the basic question on how much information on  $q$  is contained in  $q_B^\theta$  is not completely answered. In this work we study the information of singularities of the actual potential  $q$  that is contained in the Born approximation for backscattering

data, following the investigations of [13], [19], [16], [17], [12], [4] [5] and [21].

By taking  $\omega = -\theta$  in (1.4), we obtain the Born series for backscattering data

$$u_\infty(k, \theta, -\theta) - \hat{q}(\xi) = \sum_{j=1}^m \widehat{Q_{j+1}(q)}(\xi) + \widehat{Q_m^R(q)}(\xi), \quad (1.7)$$

where  $\xi = -2k\theta$  and the  $j$ -th term in the Born series is given by

$$\widehat{Q_j(q)}(\xi) = \int_{\mathbb{R}^n} e^{ik\theta \cdot y} (qR_+(k^2))^{j-1} (q(\cdot) e^{ik\theta \cdot (\cdot)})(y) dy. \quad (1.8)$$

In the case of backscattering data,  $\xi$  is given by the polar coordinates  $\xi = -2k\theta$ . This fact simplifies the Diffraction Tomography and it makes the backscattering inverse problem to be very natural.

Since the scattering amplitude is related to the polar coordinates in the frequency domain, it is expected the information on singularities of the potential to be contained in the scattering amplitude for large values of the frequency  $k$ . This fact allows us to consider complex potential in this problem, since in this case existence and uniqueness of the generalized eigenfunctions might hold only for large values of  $k$  ( $k > k_0$ , depending on the assumption on  $q$ ), see [2] and the references therein.

Following [2], we insert in (1.4) a cutoff function  $\chi$  supported out of  $B(0, 2k_0)$  and identically 1 on the exterior of  $B(0, 4k_0)$ . Then we write for  $k > 0$

$$\begin{aligned} & (u_\infty(k, \theta, -\theta) - \hat{q}(-2k\theta))\chi(-2k\theta) \\ &= \sum_{j=2}^m \widehat{Q_j(q)}(k, \theta, -\theta)\chi(-2k\theta) + \widehat{Q_m^R(q)}(k, \theta, -\theta)\chi(-2k\theta). \end{aligned} \quad (1.9)$$

We define the high frequency (HF) Born approximation for backscattering data

$$q_B(x) = 4 \int_0^\infty \int_{S^{n-1}} e^{i2kx \cdot \theta} u_\infty(k, \theta, -\theta)\chi(-2k\theta) k^{n-1} dk d\sigma(\theta) \quad (1.10)$$

With this expression and (1.9) we may write, modulo a  $C^\infty$  function

$$q(x) = q_B(x) - \sum_{j=2}^m q_j(x) + q_m^R(x) \quad (1.11)$$

where

$$q_j(x) = 4 \int_0^\infty \int_{S^{n-1}} e^{ikx \cdot \theta} Q_j(q)(k, \theta, -\theta) \chi(k) k^{n-1} dk d\sigma(\theta) \quad (1.12)$$

A similar expression gives  $q_m^R(x)$ . In the frequency domain:

$$\hat{q}_j(\eta) = \widehat{Q_j(q)}(2|\eta|, \eta/|\eta|, -\eta/|\eta|) \chi(\eta) \quad (1.13)$$

This expression allows us to reduce estimates of norms in the frequency domains to  $|\eta| > 4$ , by taking  $k_0$  large enough (we are just taking care of singularities of  $q$  and the estimates are modulo  $C^\infty$  functions).

As we said before, the research in this topic has been intense. For real potential, in the 2D case, reconstruction of singularities was studied in [19], [13], [21] and [16]. The basic procedure in these works is to prove that the terms in the Born series are more regular than the potential itself. This fact was achieved in the Sobolev scale in [13] and improved in [19] and [16]. The best result up to now in the Hilbertian Sobolev spaces in 2D is basically the fact that  $q - q_B$  has a gain of half a derivative with respect to the potential  $q$ . The same result was attained in 3D in [19] and [17].

In [5] it was proved that in odd dimensions the quadratic term in the Born expansion has a gain, in some cases up to  $1/2$  of derivative with respect to the potential  $q$ , assuming some decay of  $q$  in the Sobolev scale.

In this paper we study the quadratic term in 2D and we obtain an improvement in the integrability exponent, much better than the obtained in previous works by using Sobolev embedding. The result gives the reconstruction of classical singularities in the sense that  $q_B$  contains the rough singular part of  $q$ , being the error  $q - q_B$  a Hölder continuous function (see the Corollary 1).

**Theorem 1.** *Let  $q \in W^{s_0, 2}(\mathbb{R}^2)$  with  $s_0 > \alpha > 0$  and such that one of the two following conditions holds:*

- $|\cdot|q(\cdot) \in L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$  for some  $p < 2$ ,
- *there exists  $p > 1$  such that  $0 < \frac{2}{p} - 1 < s_0 - \alpha$  and  $|\cdot|q(\cdot) \in L^p(\mathbb{R}^2)$ .*

*Then,  $q_2 \in \Lambda^\alpha(\mathbb{R}^2)$ .*

This theorem allows potentials which do not need to be compactly supported. The conditions on the potentials are related to those in [4] and [5]. The interest in the non compactly supported case is motivated by several problems in Maxwell and elasticity equations in which this kind of potentials are involved.

As a consequence of previous results in [16], where the cubic term in the expansion is proved to be in  $W^{\alpha,2}$  for  $\alpha < s_0 + 1$  and estimates for the general term in the series, we obtain the reconstruction of singularities in the sense of classical continuity moduli:

**Corollary 1.** *Assume that  $q$  is compactly supported and satisfies the hypothesis in Theorem 1. Then we have that  $q - q_B \in \Lambda^\alpha$  for any  $0 < \alpha < s_0$ .*

In [5], they obtain in 3D some improvements of  $1/2$  derivative of gain in the Sobolev scale, mentioned above, when  $q \in W^{s_0,2}$  with  $s_0 > 1/2$ . The gain is close to one derivative when  $q \in W^{1,2}$  and is compactly supported. As discussed at the beginning our result can be understood as a gain of one derivative in 2D.

Let us finish the introduction with some words about the technical novelties in the proof of theorem 1. The standard way to tackle estimates for  $q_2$  is to write  $q_2 = Q + P$ , where  $Q$  is an integration on the Fourier side on a so-called Ewald sphere  $\Gamma(\eta)$ , see Proposition 2.1, and  $P$  is a principal value integral whose singularity lies precisely at the Ewald sphere. Theorem 1 was inspired by a result in [19] where such an estimate is obtained for  $Q$  for the case of compactly supported potential. However, the analysis in [19] for the term  $P$  based on the Hilbert transform does not yield the estimates in terms of Hölder norms. In this paper we argue in a different way by a number of dyadic decompositions and Fubini theorem on manifolds to be able to express  $P$  in terms of Ewald type terms but retaining the cancellation around the singularity. This takes a considerable effort but in the end all the terms match and we obtain Hölder estimates for the principal value term as well.

**Notation.** We use the letter  $C$  to denote any constant that can be explicitly computed in terms of known quantities. The exact value denoted by  $C$  may therefore change from line to line in a given computation.

For  $X, Y \geq 0$ , we write  $X \sim Y$  if there exists a constant  $c > 0$

such that  $c^{-1}Y \leq X \leq cY$ .

Let  $\alpha > 0$  and  $s \geq 0$ , we will use the following notation for the  $L^1$  weighted, hilbertian Sobolev and Hölder spaces:

$$L^1(\langle \cdot \rangle^\alpha) = \left\{ f : \|f\|_{L^1(\langle \cdot \rangle^\alpha)} = \int_{\mathbb{R}^2} (1 + |x|^2)^{\alpha/2} |f(x)| dx < \infty \right\},$$

$$W^{s,2} = \left\{ f : \|f\|_{W^{s,2}} = \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty \right\}.$$

$$\Lambda^\alpha = \left\{ f : \|f\|_{\Lambda^\alpha} \begin{cases} = \|f\|_\infty + \sup_{|t|>0} \frac{\|f(x+t) - f(x)\|_\infty}{|t|^\alpha} < \infty, & 0 < \alpha < 1, \\ = \|f\|_\infty + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{\Lambda^\alpha} < \infty, & \alpha \geq 1 \end{cases} \right\}.$$

We have that

$$\|f\|_{\Lambda^\alpha} \leq C \|\hat{f}\|_{L^1(\langle \cdot \rangle^\alpha)}.$$

## 2 Proof of Theorem 1

Let us state the following expression for  $\hat{q}_2$ , which holds in any dimension:

**Proposition 2.1.** *For any  $\eta \in \mathbb{R}^n$ ,  $\eta \neq 0$ , we have:*

$$\hat{q}_2(\eta) = p.v. \int_{\mathbb{R}^n} \frac{\hat{q}(\eta - \xi) \hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta) + \frac{i\pi}{|\eta|} \int_{\Gamma(\eta)} \hat{q}(\eta - \xi) \hat{q}(\xi) d\sigma_\eta(\xi) \chi(\eta), \quad (2.14)$$

where

$$\Gamma(\eta) = \left\{ \xi \in \mathbb{R}^n : \left| \xi - \frac{\eta}{2} \right| = \frac{|\eta|}{2} \right\},$$

and  $d\sigma_\eta$  denotes the measure of the sphere  $\Gamma(\eta)$ .

This proposition is a consequence of the formula

$$R_+(k^2)f(x) = v.p. \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{\hat{f}(\xi)}{-|\xi|^2 + k^2} d\xi + \frac{i\pi}{2k} \widehat{d\sigma_k} * f(x),$$

which can be found in [10]. Here  $d\sigma_k$  is the measure induced by  $\mathbb{R}^2$  on the sphere of radius  $k$ .

The scheme of the proof is inspired by the Hölder estimates of the spherical integral term obtained in [19] for the non compactly supported case. In the first subsection we improved this estimates to treat non-compactly supported potentials. Then we handle the solid integral by decomposing it into small annuli and a central term.



## 2.1 Estimate for the spherical term

Hölder estimates of the spherical integral term were obtained in [19] in the compactly supported case. In the non compactly supported case we have

**Proposition 2.2.** *We consider the spherical term*

$$\widehat{Q}(q)(\eta) = \frac{i\pi}{|\eta|} \int_{\Gamma(\eta)} \hat{q}(\eta - \xi) \hat{q}(\xi) d\sigma_\eta(\xi) \chi(\eta). \quad (2.15)$$

Let  $0 < \alpha < s_0$  and  $p \in (1, 2)$  then

$$\|Q(q)\|_{\Lambda^\alpha} \leq C(\|q\|_{L^p}^2 + \|q\|_{W^{s_0, 2}}^2). \quad (2.16)$$

*Proof:* By the symmetry in  $\xi$  and  $\eta - \xi$  in (2.15), we may reduce to estimate

$$\widehat{Q_+}(q)(\eta) = \frac{i\pi}{|\eta|} \int_{\Gamma_+(\eta)} \hat{q}(\eta - \xi) \hat{q}(\xi) d\sigma_\eta(\xi) \chi(\eta),$$

where  $\Gamma_+(\eta) = \Gamma(\eta) \cap \{\xi \in \mathbb{R}^2 : |\xi| \geq |\eta - \xi|\}$ .

$$\begin{aligned} \|Q_+(q)\|_{\Lambda^\alpha} &\leq C \|\widehat{Q}(q)\|_{L^1(\langle \eta \rangle^\alpha)} \leq \\ &\int_{\mathbb{R}^2} \left| \frac{1}{|\eta|} \int_{\Gamma_+(\eta)} \hat{q}(\eta - \xi) \hat{q}(\xi) d\sigma_\eta(\xi) \right| \langle \eta \rangle^\alpha d\eta. \end{aligned}$$

We change the order of integration, observing that

$$d\sigma_\eta(\xi) d\eta = \frac{|\eta|}{|\xi|} d\lambda_\xi(\eta) d\xi, \quad (2.17)$$

where  $d\lambda_\xi$  denotes the measure of the segment

$$\Lambda(\xi) = \{\eta \in \mathbb{R}^2 : (\eta - \xi) \perp \xi, |\xi| \geq |\eta - \xi|\}.$$

Hence,

$$\|\widehat{Q}(q)\|_{L^1(\langle \eta \rangle^\alpha)} \leq \int_{\mathbb{R}^2} |\hat{q}(\xi)| \int_{\Lambda(\xi)} |\hat{q}(\eta - \xi)| \langle \eta \rangle^\alpha |\xi|^{-1} d\lambda_\xi(\eta) d\xi.$$

We write the first integral in polar coordinates,

$$= \int_0^\infty \int_{S^1} |\hat{q}(r\theta)| \int_{\Lambda(r\theta)} |\hat{q}(\eta - r\theta)| \langle \eta \rangle^\alpha d\lambda_{r\theta}(\eta) d\sigma(\theta) dr.$$

For fixed  $\theta$ , choose an orthonormal reference  $\left\{\theta = \frac{\xi}{|\xi|}, \theta^\perp\right\}$ . Then, if  $\eta \in \Lambda(r\theta)$ , then,  $\eta = r\theta + s\theta^\perp$ ,  $s\theta^\perp = \eta - r\theta$ , so that, in these coordinates the condition  $|\eta - \xi| \leq |\xi|$  reads as  $s \leq r$ . Furthermore, the measure  $d\lambda_{r\theta}(\eta) = ds$  and thus the last integral is bounded by

$$\leq C \int_0^\infty \int_{S^1} |\hat{q}(r\theta)| \int_0^r |\hat{q}(s\theta^\perp)| (1 + s^2 + r^2)^{\alpha/2} ds d\sigma(\theta) dr.$$

By Cauchy–Schwarz inequality and Minkowski’s integral inequality,

$$\|Q(q)\|_{\Lambda^\alpha} \leq C \int_0^\infty (1 + r^2)^{\frac{\alpha}{2}} \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} F(r) dr, \quad (2.18)$$

where after the change of variables  $\theta^\perp \rightarrow \theta$ ,

$$F(r) = \int_0^r \left( \int_{S^1} |\hat{q}(s\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} ds. \quad (2.19)$$

Next we obtain a pointwise bound for  $F(r)$ . Let us start with the estimate

$$\begin{aligned} F(r) &\leq \left( \int_0^1 + \int_1^r \right) \left( \int_{S^1} |\hat{q}(s\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} ds \chi_{[1, \infty]}(r) \\ &\quad + \int_0^r \left( \int_{S^1} |\hat{q}(s\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} ds \chi_{[0, 1]}(r). \end{aligned}$$

To bound the second integral we use Cauchy-Schwartz inequality

$$\begin{aligned} &\int_1^r (1 + s^2)^{\frac{s_0}{2}} s^{\frac{1}{2}} \left( \int_{S^1} |\hat{q}(s\theta)|^2 d\sigma(\theta) \right)^{1/2} (1 + s^2)^{-\frac{s_0}{2}} s^{-\frac{1}{2}} ds \chi_{[1, \infty]}(r) \\ &\leq C \left( \int_1^r \int_{S^1} |\hat{q}(s\theta)|^2 (1 + s^2)^{s_0} ds d\sigma(\theta) \right)^{1/2} \chi_{[1, \infty]}(r) \\ &= C \|q\|_{W^{s_0, 2}} \chi_{[1, \infty]}(r), \end{aligned} \quad (2.20)$$

for any  $s_0 > 0$ .

For the first and third integrals, let  $p' > 2$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$ . We use Hölder’s and Hausdorff-Young’s inequalities to obtain that for every  $0 < r < 1$ ,

$$\int_0^r \left( \int_{S^1} |\hat{q}(s\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} ds \leq C \int_0^1 s^{\frac{1}{p}-1} s^{\frac{1}{p'}} \left( \int_{S^1} |\hat{q}(s\theta)|^{p'} d\sigma(\theta) \right)^{\frac{1}{p'}} ds$$

$$\leq C \left( \int_0^1 \int_{S^1} s |\hat{q}(s\theta)|^{p'} d\sigma(\theta) ds \right)^{1/p'} \leq C \|\hat{q}\|_{L^{p'}} \leq C \|q\|_{L^p}. \quad (2.21)$$

Hence

$$F(r) \leq C \left( \|q\|_{L^p} \chi_{[0,1]} + (\|q\|_{L^p} + \|q\|_{W^{s_0,2}}) \chi_{[1,\infty]} \right). \quad (2.22)$$

In order to obtain the estimate (2.16), we observe that for  $\alpha < s_0$  we have

$$\begin{aligned} & \int_1^\infty \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{1/2} (1+r^2)^{\alpha/2} dr \\ &= \int_1^\infty (1+r^2)^{\frac{s_0}{2}} r^{\frac{1}{2}} \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{1/2} (1+r^2)^{\frac{\alpha-s_0}{2}} r^{-\frac{1}{2}} dr \leq C \|q\|_{W^{s_0,2}}. \end{aligned}$$

Inserting this expression, (2.22) and (2.21) in (2.18) and using (2.21) to estimate

$$\int_0^1 (1+r^2)^{\frac{\alpha}{2}} \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} dr,$$

we obtain

$$\begin{aligned} & \|Q(q)\|_{\Lambda^\alpha} \leq \\ & C \int_1^\infty \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} (1+r^2)^{\frac{\alpha}{2}} dr (\|q\|_{L^p} + \|q\|_{W^{s_0,2}}) \\ & + \int_0^1 \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} dr \|q\|_{L^p} \leq C (\|q\|_{L^p}^2 + \|q\|_{W^{s_0,2}}^2). \end{aligned}$$

## 2.2 Estimate of the principal value

Now we concentrate in the principal value integral. As before by the symmetry in  $\xi$  and  $\eta - \xi$ :

$$p.v. \int_{\mathbf{R}^2} \frac{\hat{q}(\eta - \xi) \hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta) = 2\widehat{P(q)}(\eta),$$

where

$$\widehat{P(q)}(\eta) = p.v. \int_{\mathbf{R}^n \cap \{\xi \in \mathbf{R}^2: |\xi| \geq |\eta - \xi|\}} \frac{\hat{q}(\eta - \xi) \hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta). \quad (2.23)$$

We split down  $\widehat{P(q)}(\eta)$  to avoid the singularity, which corresponds to  $\xi \in \Gamma(\eta)$ .

$$\widehat{P(q)}(\eta) = \widehat{P_0}(\eta) + \sum_{j=1}^{N(\eta)} \widehat{P_j}(\eta) + \widehat{P_\infty}(\eta). \quad (2.24)$$

Here we define

$$\widehat{P_0}(\eta) = \int_{\Gamma_0} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta) \quad (2.25)$$

and  $\Gamma_0 = \Gamma_0(\eta) = \{\xi \in \mathbb{R}^2 : 2^{-2}|\eta| < \|\xi - \eta/2\| - |\eta|/2\|, |\xi| \geq |\eta - \xi|\}$ ;

$$\widehat{P_j}(\eta) = \int_{\Gamma_j} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta), \quad (2.26)$$

$\Gamma_j = \Gamma_j(\eta) = \{\xi \in \mathbb{R}^2 : 2^{-j-2}|\eta| < \|\xi - \eta/2\| - |\eta|/2\| < 2^{-j-1}|\eta|, |\xi| \geq |\eta - \xi|\}$  and  $N(\eta) = \log_2 |\eta|$  and

$$\widehat{P_\infty}(\eta) = p.v. \int_{\Gamma_\infty} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{\xi \cdot (\eta - \xi)} d\xi \chi(\eta), \quad (2.27)$$

for  $\Gamma_\infty = \Gamma_\infty(\eta) = \{\xi \in \mathbb{R}^2 : \|\xi - \eta/2\| - |\eta|/2\| < 1/4, |\xi| \geq |\eta - \xi|\}$ .

We have

$$\|P(q)\|_{\Lambda^\alpha} \leq \|\widehat{P_0}\|_{L^1(\langle \cdot \rangle^\alpha)} + \left\| \sum_{j=1}^{N(\cdot)} \widehat{P_j} \right\|_{L^1(\langle \cdot \rangle^\alpha)} + \|\widehat{P_\infty}\|_{L^1(\langle \cdot \rangle^\alpha)}. \quad (2.28)$$

### 2.2.1 Estimate of $\widehat{P_0}(\eta)$

We start with the term far from the singularity which is the easiest.

**Proposition 2.3.** *Let  $P_0(q)$  be defined in (2.25). Then, for  $0 < \alpha < s_0$ , there exists a constant  $C$  such that*

$$\|P_0\|_{\Lambda^\alpha} \leq C \|q\|_{W^{s_0,2}}^2,$$

for all  $q \in W^{s_0,2}$ .

*Proof.* Notice that if  $\xi \in \Gamma_0$  we have that  $|\xi \cdot (\eta - \xi)| \geq |\eta|^2$ , then

$$\|P_0\|_{\Lambda^\alpha} \leq C \|\widehat{P_0}\|_{L^1(\langle \cdot \rangle^\alpha)}$$

$$\leq C \int_{|\eta|>2k_0} \frac{(1+|\eta|^2)^{\alpha/2}}{|\eta|^2} \int_{|\xi|\geq|\eta|/2} |\hat{q}(\eta-\xi)\hat{q}(\xi)| d\xi d\eta \quad (2.29)$$

Here  $k_0$  is the constant coming from the cut-off function. Next we use that we are integrating on the domain  $\{|\xi| \geq |\eta|/2\}, |\xi-\eta| \geq |\eta|/4$ .

$$\begin{aligned} &\leq C \int_{|\eta|>2k_0} \frac{1}{(1+|\eta|^2)^{\frac{s_0-\alpha}{2}} |\eta|^2} \int_{\mathbb{R}^2} |\hat{q}(\eta-\xi)| |\hat{q}(\xi)| (1+|\xi|^2)^{s_0/4} (1+|\eta-\xi|^2)^{s_0/4} d\xi d\eta \\ &\leq C \|q\|_{W^{s_0/2,2}}^2 \int_{|\eta|>2k_0} \frac{1}{(1+|\eta|^2)^{\frac{s_0-\alpha}{2}} |\eta|^2} d\eta \leq C \|q\|_{W^{s_0/2,2}}^2. \end{aligned}$$

□

### 2.2.2 Estimate of $\sum_{j=1}^{N(\eta)} \widehat{P}_j$

We address the sum of terms whose Fourier transform is supported in annuli approaching to the singularity.

The key will be the following lemma where in fact we estimate the integrals on "Ewald spheres" scaled by  $\delta$ . We state the lemma, show how to use it to bound the terms  $\sum_{j=1}^{N(\eta)} \widehat{P}_j$ , and finally prove the lemma.

**Lemma 2.1.** *Let us take*

$$\widehat{P^\delta(q)}(\eta) = \chi_{(\delta^{-1}, \infty)}(|\eta|) \int_{\Gamma^\delta} \frac{|\hat{q}(\xi)| |\hat{q}(\eta-\xi)|}{|\eta|^2} d\xi,$$

where

$$\Gamma^\delta = \Gamma^\delta(\eta) = \left\{ \xi \in \mathbb{R}^2 : \left| \left| \xi - \frac{\eta}{2} \right| - \left| \frac{\eta}{2} \right| \right| < \delta |\eta|, |\xi| \geq |\xi - \eta| \right\},$$

and  $\delta > 0$ . Consider  $0 < \alpha < s_0$  and  $p \in (1, 2)$ . Then

$$\|\widehat{P^\delta(q)}\|_{L^1(\langle \cdot \rangle^\alpha)} \leq C \delta^{1+s_0-\alpha} (\|q\|_{L^p}^2 + \|q\|_{W^{s_0,2}}^2).$$

**Proposition 2.4.** *Let  $P_j(q)$  be defined in (2.26). Then there exists a universal constant  $C$  such that*

$$\left\| \sum_{j=1}^{N(\cdot)} \widehat{P}_j(q) \right\|_{L^1(\langle \cdot \rangle^\alpha)} \leq C (\|q\|_{L^p}^2 + \|q\|_{W^{s_0,2}}^2).$$

*Proof.* Let us estimate  $\sum_{j=1}^{N(\eta)} \widehat{P}_j$ . Notice that  $\Gamma_j(\eta) \subset \Gamma^{2^{-j}}(\eta)$  and if  $\xi \in \Gamma_j(\eta)$  we have  $|\xi \cdot (\eta - \xi)| \sim 2^{-j}|\eta|^2$ . Since  $2^j \leq |\eta|$ ,  $j = 1, 2, \dots, N(\eta)$ , and we can suppose that  $k_0 \geq 4$ , then

$$\left| \sum_{j=1}^{N(\eta)} \widehat{P}_j(\eta) \right| \leq \sum_{j=1}^{N(\eta)} |\widehat{P}_j(\eta)| \leq \sum_{j=1}^{\infty} 2^j \widehat{P^{2^{-j}}}(q)(\eta),$$

and

$$\begin{aligned} \left\| \sum_{j=1}^{N(\cdot)} \widehat{P}_j(\cdot) \right\|_{L^1(\langle \cdot \rangle^\alpha)} &\leq C \sum_{j=1}^{\infty} 2^j \left\| \widehat{P^{2^{-j}}}(q) \right\|_{L^1(\langle \cdot \rangle^\alpha)} \quad (2.30) \\ &\leq C \sum_{j=1}^{\infty} 2^{-j(s_0 - \alpha)} (\|q\|_{L^p}^2 + \|q\|_{W^{s_0, 2}}^2) \leq C (\|q\|_{L^p}^2 + \|q\|_{W^{s_0, 2}}^2). \end{aligned}$$

□

*Proof of Lemma 2.1:* We aim to use the ideas developed in subsection 2.1 to deal with the spherical term. Thus we cover  $\Gamma^\delta(\eta)$  by the family  $\Gamma(s\eta)$  where  $s \in (1 - 2\delta, 1 + 2\delta)$  and

$$\Gamma(s\eta) = \left\{ \xi : |\xi - s\eta/2| = |s\eta/2| \text{ and } |\xi| \geq \frac{3}{4}|\xi - \eta| \right\}.$$

Then

$$\left\| \widehat{P^\delta}(q) \right\|_{L^1(\langle \cdot \rangle^\alpha)} = \int_{|\eta| > \delta^{-1}} (1 + |\eta|^2)^{\frac{\alpha}{2}} \int_{\Gamma^\delta} \frac{|\hat{q}(\eta - \xi)| |\hat{q}(\xi)|}{|\eta|^2} d\xi d\eta$$

Now we change variables in the expression of  $\widehat{P^\delta}(q)$ . Notice that  $d\xi = J(\eta, \xi) d\sigma_{s, \eta}(\xi) ds$ , with  $|J(\eta, \xi)| \leq |\eta|$ , where  $d\sigma_{s, \eta}$  is the measure on  $\Gamma(s\eta)$ . Thus  $\left\| \widehat{P^\delta}(q) \right\|_{L^1(\langle \cdot \rangle^\alpha)}$  is bounded by,

$$\begin{aligned} &\int_{|\eta| > \delta^{-1}} (1 + |\eta|^2)^{\alpha/2} \int_{1-2\delta}^{1+2\delta} \int_{\Gamma(s\eta)} \frac{|\hat{q}(\eta - \xi)| |\hat{q}(\xi)|}{|\eta|} d\sigma_{s, \eta}(\xi) ds d\eta \\ &\leq C\delta \sup_{s \in (1-2\delta, 1+2\delta)} \underbrace{\int_{|\eta| > \delta^{-1}} (1 + |\eta|^2)^{\alpha/2} \int_{\Gamma(s\eta)} \frac{|\hat{q}(\eta - \xi)| |\hat{q}(\xi)|}{|\eta|} d\sigma_{s, \eta}(\xi) d\eta}_{I_s}. \end{aligned}$$

We perform the change  $s\eta = \eta'$  to relate  $I_s$  with  $Q$  in (2.15). For aesthetic reasons after the change of variable, we relabel  $\eta' = \eta$

$$I_s = s^{-1} \int_{|\eta| > s\delta^{-1}} (1 + |\eta|^2)^{\alpha/2} |\eta|^{-1} \int_{\Gamma(\eta)} |\hat{q}(\frac{\eta}{s} - \xi)| |\hat{q}(\xi)| d\sigma_\eta(\xi) d\eta. \quad (2.31)$$

Next, we repeat the strategy. First we change the order of integration as in (2.17), noticing that if  $\xi \in \Gamma(s\eta)$  then  $|\xi| \sim |\eta|$ . Thus  $\|\widehat{P^\delta(q)}\|_{L^1(\langle \cdot \rangle^\alpha)}$  is bounded by

$$\leq C\delta \sup_{s \in (1-2\delta, 1+2\delta)} \int_{|\xi| > C\delta^{-1}} (1 + |\xi|^2)^{\alpha/2} |\xi|^{-1} |\hat{q}(\xi)| \int_{\Lambda(\xi)} |\hat{q}(\frac{\eta}{s} - \xi)| d\lambda_\xi(\eta) d\xi.$$

Again we deal with the singularity  $|\xi|^{-1}$  by taking polar coordinates arriving to

$$\leq C\delta \int_{C\delta^{-1}}^\infty (1 + r^2)^{\alpha/2} \int_{S^1} |\hat{q}(r\theta)| \int_{\Lambda(r\theta)} |\hat{q}(\frac{\eta}{s} - r\theta)| d\lambda_{r\theta}(\eta) d\sigma(\theta) dr.$$

We will use  $\Lambda(\xi)$  and the angular variable of  $\xi$  as coordinates to construct the  $\mathbb{R}^2$ - variable  $\tau = \frac{\eta}{s} - \xi$ . Choose an orthonormal reference  $\{\theta = \frac{\xi}{|\xi|}, \theta^\perp\}$ , then  $\eta = r\theta + t\theta^\perp$  and hence  $\frac{\eta}{s} - r\theta = (r/s - r)\theta + t/s\theta^\perp$ . Then, Cauchy-Schwarz and Minkowski integral inequalities yield that,

$$\begin{aligned} & \|\widehat{P^\delta(q)}\|_{L^1(\langle \cdot \rangle^\alpha)} \quad (2.32) \\ & \leq C\delta \sup_{s \in (1-2\delta, 1+2\delta)} \int_{C\delta^{-1}}^\infty (1 + r^2)^{\alpha/2} \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{1/2} F_s(r) dr. \end{aligned}$$

where

$$F_s(r) = \int_0^{2r} \left( \int_{S^1} |\hat{q}((\frac{r}{s} - r)\theta + \frac{t}{s}\theta^\perp)|^2 d\sigma(\theta) \right)^{1/2} dt \quad (2.33)$$

Now we use a sort of twisted polar coordinates to emulate the arguments in the spherical term. We are entitled to do this as long as the change of variables  $(t, \theta) \rightarrow \tau$  given in complex notation  $\theta = e^{i\alpha}$  and  $\tau = \tau_1 + i\tau_2$  by

$$\tau = \left(\frac{r}{s} - r\right)e^{i\alpha} + i\frac{t}{s}e^{i\alpha},$$

does not degenerate. In fact, it has Jacobian  $d\tau = \frac{t}{s^2}d\alpha dt \equiv \frac{t}{s^2}d\sigma(\theta)dt$ ,

Now we need to pointwise bound  $F_s(r)$ . Since  $r > 1$ , let us split the r.h.s of (2.33)

$$F_s\left(\frac{1}{2}\right) + G_s(r) \quad (2.34)$$

$$= \left(\int_0^1 + \int_1^{2r}\right) \left(\int_{S^1} |\hat{q}\left(\left(\frac{r}{s} - r\right)\theta + \frac{t}{s}\theta^\perp\right)|^2 d\sigma(\theta)\right)^{1/2} dt.$$

To study  $G_s(r)$  we use the Cauchy-Schwartz inequality

$$\begin{aligned} G_s(r) &\leq \left(\int_1^{2r} \int_{S^1} \left(1 + \left(\frac{r}{s} - r\right)^2 + \frac{t^2}{s^2}\right)^{-s_0} \frac{s^2}{t} d\sigma(\theta)dt\right)^{1/2} \\ &\left(\int_1^{2r} \int_{S^1} \left(1 + \left(\frac{r}{s} - r\right)^2 + \frac{t^2}{s^2}\right)^{s_0} \frac{t}{s^2} \left|\hat{q}\left(\left(\frac{r}{s} - r\right)\theta + \frac{t}{s}\theta^\perp\right)\right|^2 d\sigma(\theta)dt\right)^{1/2} \\ &\leq C \left(\int_{\mathbb{R}^2} (1 + |\tau|^2)^{s_0} |\hat{q}(\tau)|^2 d\tau\right)^{1/2} \left(\int_1^{2r} (1 + t^2)^{-s_0} t^{-1} dt\right)^{1/2} \\ &\leq C \|q\|_{W^{s_0,2}}, \end{aligned} \quad (2.35)$$

for all  $s_0 > 0$ .

To bound  $F_s(\frac{1}{2})$  we will proceed as in (2.21). Let  $p' > 2$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$ . We will use the change of variable  $(t, \theta) \rightarrow \tau$ .

$$\begin{aligned} F_s\left(\frac{1}{2}\right) &\leq \int_0^1 t^{\frac{1}{p}-1} t^{\frac{1}{p'}} \left(\int_{S^1} |\hat{q}\left(\left(\frac{r}{s} - r\right)\theta + \frac{t}{s}\theta^\perp\right)|^{p'} d\sigma(\theta)\right)^{1/p'} dt \\ &\leq C \left(\int_0^1 t^{1-p} dt\right)^{1/p} \left(\int_0^1 t \int_{S^1} |\hat{q}\left(\left(\frac{r}{s} - r\right)\theta + \frac{t}{s}\theta^\perp\right)|^{p'} d\sigma(\theta)dt\right)^{1/p'} \\ &\leq C \|\hat{q}\|_{L^{p'}} \leq C \|q\|_{L^p}. \end{aligned} \quad (2.36)$$

Hence,

$$F_s(r) \leq C(\|q\|_{L^p} + \|q\|_{W^{s_0,2}}). \quad (2.37)$$

Inserting (2.37) into (2.32) we arrive to

$$\|\widehat{P^\delta(q)}\|_{L^1(\langle \eta \rangle^\alpha)} \quad (2.38)$$



$$\leq C\delta \int_{C\delta^{-1}}^{\infty} (1+r^2)^{\alpha/2} \left( \int_{S^1} |\hat{q}(r\theta)|^2 d\sigma(\theta) \right)^{1/2} (\|q\|_{L^p} + \|q\|_{W^{s_0,2}}) dr.$$

Finally, using Cauchy-Schwarz's inequality, this can be bounded if  $\alpha < s_0$  by

$$C\delta(\|q\|_{L^p} + \|q\|_{W^{s_0,2}})\|q\|_{W^{s_0,2}} \left( \int_{C\delta^{-1}}^{\infty} r^{-1}(1+r^2)^{\alpha-s_0} dr \right)^{1/2},$$

which gives the desired estimate

$$\|\widehat{P^\delta(q)}\|_{L^1(\langle \cdot \rangle^\alpha)} \leq C\delta^{1+s_0-\alpha} (\|q\|_{L^p}^2 + \|q\|_{W^{\beta,2}}^2).$$

□

### 2.2.3 Estimate of $\widehat{P_\infty}$

In this section we treat  $\widehat{P_\infty}$ , where the singularity of the kernel lies. The strategy of the proof is first to use the cancellation to cancel the singularity on the Ewald sphere. Then the new term can be treated with an extension of our previous ideas to deal with spherical terms, though the technicalities here are more complicated (This is done in Lemma 2.2). Let us first state the corresponding proposition.

**Proposition 2.5.** *Let  $P_\infty(q)$  be defined in (2.27). There exists universal constants  $C, K_0$  such that if  $k_0 > K_0$  it holds that,*

$$\|\widehat{P_\infty(q)}\|_{L^1(\langle \cdot \rangle^\alpha)} \leq C(\|\cdot\| \cdot \|q\|_{L^p}^2 + \|q\|_{W^{s_0,2}}^2),$$

for  $p < 2$ ,  $0 < 1 - \frac{2}{p'} < s_0 - \alpha$ , and

$$\|\widehat{P_\infty(q)}\|_{L^1(\langle \cdot \rangle^\alpha)} \leq C(\|\cdot\| \cdot \|q\|_{L^p}^2 + \|\cdot\| \cdot \|q\|_{L^2}^2 + \|q\|_{W^{s_0,2}}^2),$$

for any  $p < 2$ .

After the cancellation argument the proof of (2.5) will rely on the following bilinear estimate for "Ewald" type operators which we postpone.

**Lemma 2.2.** *Let  $g \in W^{s_0,\alpha}$  with  $0 < \alpha < s_0$  and  $\hat{f} \in L^{p'} \cap L^2$  if  $p' > 2$  or  $\hat{f} \in L^{p'}$  if  $0 < 1 - \frac{2}{p'} < s_0 - \alpha$ . Let us define the bilinear operator  $J$  by*

$$\widehat{J(f,g)}(\eta) = \int_{\Gamma_0^+} \frac{|\hat{f}(\eta - \Phi_\eta(\xi))| |\hat{g}(\xi)|}{|\eta|} d\xi \chi(\eta),$$

where

$$\Gamma_0^+ = \{\xi \in \mathbb{R}^2 : 0 < |\xi - \frac{\eta}{2}| - |\frac{\eta}{2}| < \frac{1}{4}, |\xi| \geq |\xi - \eta|\},$$

and

$$\Phi_\eta(\xi) = \eta - \xi + |\eta| \frac{\xi - \eta/2}{|\xi - \eta/2|}.$$

We have the a priori estimates

$$\|\widehat{J(f, g)}\|_{L^1(\langle \cdot \rangle^\alpha)} \leq C \|g\|_{W^{s_0, 2}} (\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}), \quad (2.39)$$

for any  $p' > 2$ , and

$$\|\widehat{J(f, g)}\|_{L^1(\langle \cdot \rangle^\alpha)} \leq C \|g\|_{W^{s_0, 2}} \|\hat{f}\|_{L^{p'}}, \quad (2.40)$$

for  $p'$  such that  $0 < 1 - \frac{2}{p'} < s_0 - \alpha$ ,

*Proof of Proposition 2.5.* Let us start with the cancellation argument. We define

$$\Gamma_\epsilon^+ = \Gamma_\epsilon^+(\eta) = \{\xi \in \mathbb{R}^2 : \epsilon < |\xi - \frac{\eta}{2}| - |\frac{\eta}{2}| < \frac{1}{4}, |\xi| \geq |\xi - \eta|\},$$

$$\Gamma_\epsilon^- = \Gamma_\epsilon^-(\eta) = \{\xi \in \mathbb{R}^2 : \epsilon < |\frac{\eta}{2}| - |\xi - \frac{\eta}{2}| < \frac{1}{4}, |\xi| \geq |\xi - \eta|\},$$

then

$$\widehat{P}_\infty(\eta) = \lim_{\epsilon \rightarrow 0} \left( \int_{\Gamma_\epsilon^+} + \int_{\Gamma_\epsilon^-} \right) \frac{\hat{q}(\eta - \xi) \hat{q}(\xi)}{|\xi - \frac{\eta}{2}|^2 - |\frac{\eta}{2}|^2} d\xi \chi(\eta).$$

The change of variable taking  $\xi \in \Gamma_\epsilon^-$  to the symmetric  $\xi' \in \Gamma_\epsilon^+$ , given by

$$\xi' = \Phi_\eta(\xi) = \eta - \xi + |\eta| \frac{\xi - \eta/2}{|\xi - \eta/2|}, \quad (2.41)$$

satisfies the following properties:

$$|\Phi_\eta(\xi) - \eta/2| - |\eta/2| = -(|\xi - \eta/2| - |\eta/2|), \quad (2.42)$$

its Jacobian determinant is given by the expression

$$|D\Phi_\eta(\xi)| = 1 + 2 \frac{(|\eta/2| - |\xi - \eta/2|)}{|\xi - \eta/2|}, \quad (2.43)$$

and

$$|\Phi_\eta(\xi) - \xi| = 2(|\eta/2| - |\xi - \eta/2|). \quad (2.44)$$

Thus, changing variables in  $\Gamma_\epsilon^+$  and taking advantage of (2.42) we can write  $\widehat{P}_\infty(\eta)$  as

$$\widehat{P}_\infty(\eta) = \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^-} \left( -\frac{\hat{q}(\eta - \Phi_\eta(\xi))\hat{q}(\Phi_\eta(\xi))|D\Phi_\eta(\xi)|}{|\Phi_\eta(\xi) - \frac{\eta}{2}| + |\frac{\eta}{2}|} + \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{|\xi - \frac{\eta}{2}| + |\frac{\eta}{2}|} \right) \cdot \frac{1}{|\frac{\eta}{2}| - |\xi - \frac{\eta}{2}|} d\xi \chi(\eta).$$

If we insert in this expression the explicit value for the Jacobian of  $\Phi_\eta$  from (2.43) and rearrange the terms,  $\widehat{P}_\infty(\eta)$  is equal to the following sum of three terms:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^-} \frac{(\hat{q}(\eta - \Phi_\eta(\xi))\hat{q}(\Phi_\eta(\xi)) - \hat{q}(\eta - \xi)\hat{q}(\xi))|D\Phi_\eta(\xi)|}{|\frac{\eta}{2}|^2 - |\Phi_\eta(\xi) - \frac{\eta}{2}|^2} d\xi \chi(\eta) \\ & - 2 \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^-} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{(|\Phi_\eta(\xi) - \frac{\eta}{2}| + |\frac{\eta}{2}|)|\xi - \frac{\eta}{2}|} d\xi \chi(\eta) \\ & + 2 \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^-} \frac{\hat{q}(\eta - \xi)\hat{q}(\xi)}{(|\Phi_\eta(\xi) - \frac{\eta}{2}| + |\frac{\eta}{2}|)(|\xi - \frac{\eta}{2}| + |\frac{\eta}{2}|)} d\xi \chi(\eta). \end{aligned} \quad (2.45)$$

If  $\xi \in \Gamma_\epsilon^-$  we have

$$\begin{aligned} & \left( |\Phi_\eta(\xi) - \frac{\eta}{2}| + \left| \frac{\eta}{2} \right| \right) \left| \xi - \frac{\eta}{2} \right| \sim |\eta|^2, \\ & \left( |\Phi_\eta(\xi) - \frac{\eta}{2}| + \left| \frac{\eta}{2} \right| \right) \left( \left| \xi - \frac{\eta}{2} \right| + \left| \frac{\eta}{2} \right| \right) \sim |\eta|^2, \end{aligned}$$

Thus, the second and third terms behave exactly as  $P_0$ . Hence, the same arguments used in (2.29) yield the bounds,

$$\begin{aligned} & \left\| \int_{\Gamma_\epsilon^-} \frac{\hat{q}(\cdot - \xi)\hat{q}(\xi)}{(|\Phi_\cdot(\xi) - \frac{\cdot}{2}| + |\frac{\cdot}{2}|)(|\xi - \frac{\cdot}{2}| + |\frac{\cdot}{2}|)} d\xi \chi(\cdot) \right\|_{L^1(\langle \cdot, \cdot \rangle^\alpha)} \quad (2.46) \\ & + \left\| \int_{\Gamma_\epsilon^-} \frac{\hat{q}(\cdot - \xi)\hat{q}(\xi)}{(|\Phi_\cdot(\xi) - \frac{\cdot}{2}| + |\frac{\cdot}{2}|)(|\xi - \frac{\cdot}{2}| + |\frac{\cdot}{2}|)} d\xi \chi(\cdot) \right\|_{L^1(\langle \cdot, \cdot \rangle^\alpha)} \leq C \|q\|_{W^{s_0, 2}}^2, \end{aligned}$$

with  $C$  independent of  $\epsilon$ .

By inserting the explicit expression for  $|D\Phi_\eta(\xi)|$  obtained in (2.43) the first integral in

$$\int_{\Gamma_\epsilon^-} \frac{(\hat{q}(\eta - \Phi_\eta(\xi))\hat{q}(\Phi_\eta(\xi)) - \hat{q}(\eta - \xi)\hat{q}(\xi))|D\Phi_\eta(\xi)|}{|\Phi_\eta(\xi) - \frac{\eta}{2}|^2 - |\frac{\eta}{2}|^2} d\xi \chi(\eta)$$

is splitted down in the sum of two terms analogous to

$$\widehat{J}_1(\eta) = \int_{\Gamma_\epsilon^-} \frac{(\hat{q}(\eta - \Phi_\eta(\xi)) - \hat{q}(\eta - \xi)) \hat{q}(\Phi_\eta(\xi))}{|\Phi_\eta(\xi) - \frac{\eta}{2}|^2 - |\frac{\eta}{2}|^2} d\xi \chi(\eta),$$

and a third one of the type

$$2 \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^-} \frac{\hat{q}(\eta - \xi) \hat{q}(\xi)}{(|\Phi_\eta(\xi) - \frac{\eta}{2}| + |\frac{\eta}{2}|) |\xi - \frac{\eta}{2}|} d\xi \chi(\eta). \quad (2.47)$$

Since the singularity in (2.47) is run out, it is a harmless term as discussed in (2.46)

To deal with  $J_1$  we take advantage of the cancellation. Namely, recall Calderón pointwise inequality for Sobolev functions,

$$|f(x) - f(y)| \leq C(M(\nabla f)(x) + M(\nabla f)(y))|x - y|, \quad (2.48)$$

where  $M$  is the Hardy-Littlewood maximal operator. If we combine this inequality and (2.44), we may estimate  $|\widehat{J}_1(\eta)|$  by

$$2 \int_{\Gamma_0^-} \frac{(M(\nabla \hat{q})(\eta - \Phi_\eta(\xi)) + M(\nabla \hat{q})(\eta - \xi)) |\hat{q}(\Phi_\eta(\xi))|}{|\Phi_\eta(\xi) - \frac{\eta}{2}| + |\frac{\eta}{2}|} d\xi \chi(\eta), \quad (2.49)$$

where

$$\Gamma_0^- = \{\xi \in \mathbb{R}^2 : 0 < |\frac{\eta}{2}| - |\xi - \frac{\eta}{2}| < \frac{1}{4}, |\xi| \geq |\xi - \eta|\}.$$

Now the singularity is milder and  $J_1$  can be treated similarly as we did with other Ewald terms with the help of Lemma 2.2.

In fact, we may decompose (2.49) in two integrals in the obvious way. The first one is

$$\int_{\Gamma_0^-} \frac{(M(\nabla \hat{q})(\eta - \Phi_\eta(\xi))) |\hat{q}(\Phi_\eta(\xi))|}{|\Phi_\eta(\xi) - \frac{\eta}{2}| + |\frac{\eta}{2}|} d\xi \chi(\eta).$$

This one can be reduced by the change of variable  $\xi' = \Phi_\eta(\xi)$  to estimate

$$\int_{\Gamma_0^+} \frac{(M(\nabla \hat{q})(\eta - \xi)) |\hat{q}(\xi)|}{|\eta|} d\xi \chi(\eta). \quad (2.50)$$

Notice that here we do not retain the explicit expression for the Jacobian (2.43) as we did before in order to kill the singularity but rather we simply notice that  $1 \leq |D\Phi_\eta(\xi)| \leq 2$ .

The second one, which is the most difficult,

$$\int_{\Gamma_0^-} \frac{(M(\nabla \hat{q})(\eta - \xi)) |\hat{q}(\Phi_\eta(\xi))|}{|\Phi_\eta(\xi) - \frac{\eta}{2}| + |\frac{\eta}{2}|} d\xi \chi(\eta),$$

can be reduced by the same change of variables to estimate

$$\int_{\Gamma_0^+} \frac{(M(\nabla \hat{q})(\eta - \Phi_\eta(\xi))) |\hat{q}(\xi)|}{|\eta|} d\xi \chi(\eta). \quad (2.51)$$

To finish we apply Lemma 2.2 to  $\hat{f} = M(\nabla \hat{q}), g = \hat{q}$  and by (2.40), we obtain

$$\begin{aligned} & \left\| \int_{\Gamma_0^+} \frac{(M(\nabla \hat{q})(\cdot - \Phi_\cdot(\xi))) |\hat{q}(\xi)|}{|\cdot|} d\xi \chi(\cdot) \right\|_{L^1(\langle \cdot \rangle^\alpha)} \\ & \leq C(\|q\|_{W^{s_0,2}}^2 + \|M(\nabla \hat{q})\|_{L^{p'}}^2) \end{aligned} \quad (2.52)$$

$$\leq C(\|q\|_{W^{s_0,2}}^2 + \|\nabla \hat{q}\|_{L^{p'}}^2) \leq C(\|q\|_{W^{s_0,2}}^2 + \|\cdot\| q\|_{L^p}^2).$$

□

*Proof of Lemma 2.2:*

As in the proof of Lemma 2.1, we want to use our understanding of spherical terms and thus we foliate  $\Gamma^\delta$  in the corresponding spherical regions. Namely, for  $|\eta| > 2k_0$  we can cover  $\Gamma_0^+ = \Gamma_0^+(\eta)$  by the family  $\Gamma(s\eta)$  where  $s \in (1, 1 + 1/4|\eta|) \sim (1, 1 + 1/|\eta|)$  and

$$\Gamma(s\eta) = \{\xi \in \mathbb{R}^2 : |\xi - s\eta/2| = |s\eta/2| \text{ and } |\xi| \geq \frac{3}{4}|\xi - \eta|\}.$$

Since,

$$\|\widehat{J(f, g)}\|_{L^1(\langle \cdot \rangle^\alpha)} = \int_{\mathbb{R}^2} (1 + |\eta|^2)^{\alpha/2} \int_{\Gamma_0^+} \frac{|\hat{f}(\eta - \Phi_\eta(\xi))| |\hat{g}(\xi)|}{|\eta|} d\xi \chi(\eta) d\eta$$

An easy comparison between the volume elements yields,

$$d\xi \leq \frac{|\eta|}{2} d\sigma_{s,\eta}(\xi) ds \quad (2.53)$$

Thus we obtain that,

$$\|\widehat{J(f, g)}\|_{L^1(\langle \cdot \rangle^\alpha)} \leq \int_{|\eta| \geq 2k_0} (1 + |\eta|^2)^{\alpha/2} \int_1^{1+1/|\eta|} \int_{\Gamma(s\eta)} |\hat{f}(\eta - \Phi_\eta(\xi))| |\hat{g}(\xi)| d\sigma_{s,\eta}(\xi) ds d\eta$$

$$\leq C \sum_{j=\log k_0}^{\infty} \int_{|\eta| \sim 2^j} 2^{j\alpha} \int_1^{1+2^{-j}} \int_{\Gamma(s\eta)} |\hat{f}(\eta - \Phi_\eta(\xi))| |\hat{g}(\xi)| d\sigma_{s,\eta}(\xi) ds d\eta.$$

Now, we control each of the terms in the sum. First, we get rid of the integral in  $s$  by taking supremum norm and reduce to estimate the expression

$$\sup_{s \in (1, 1+2^{-j})} 2^{j(\alpha-1)} \int_{|\eta| \sim 2^j} \int_{\Gamma(s\eta)} |\hat{f}(\eta - \Phi_\eta(\xi))| |\hat{g}(\xi)| d\sigma_{s,\eta}(\xi) d\eta. \quad (2.54)$$

The change  $s\eta = \eta'$ , allows us to write the integral in (2.54) as

$$s^{-2} \int_{|\eta| \sim 2^j/s} 2^{j(\alpha-1)} \int_{\Gamma(\eta)} |\hat{f}(\frac{\eta}{s} - \Phi_{\frac{\eta}{s}}(\xi))| |\hat{g}(\xi)| d\sigma_\eta(\xi) d\eta. \quad (2.55)$$

If  $\xi \in \Gamma(\eta)$  then  $|\xi| \sim |\eta|$ . Hence by changing the order of integration and using (2.17) we obtain that (2.55) is bounded by

$$\leq C \int_{|\xi| \sim 2^j} 2^{j(\alpha-1)} |\hat{g}(\xi)| \int_{\Lambda(\xi)} |\hat{f}(\frac{\eta}{s} - \Phi_{\frac{\eta}{s}}(\xi))| d\lambda_\xi(\eta) d\xi. \quad (2.56)$$

We use polar coordinates to bound the above integral by

$$\leq C \int_{r \sim 2^j} 2^{j\alpha} \int_{S^1} |\hat{g}(r\theta)| \int_{\Lambda(r\theta)} |\hat{f}(\frac{\eta}{s} - \Phi_{\frac{\eta}{s}}(r\theta))| d\lambda_{r\theta}(\eta) d\sigma(\theta) dr. \quad (2.57)$$

We will use  $\Lambda(\xi)$  and the angular variable of  $\xi$  as coordinates to construct the  $\mathbb{R}^2$ - variable  $\tau = \frac{\eta}{s} - \Phi_{\frac{\eta}{s}}(\xi)$ . We express the variable  $\eta \in \Lambda(r\theta)$  as  $\eta = r\theta + t\theta^\perp$ ,  $|t| \leq 2r$ .

We write

$$\tau(s, r, t, \theta) = \frac{\eta}{s} - \Phi_{\frac{\eta}{s}}(r\theta) = r\theta - \frac{|\eta|}{s} |r\theta - \frac{\eta}{2s}|^{-1} (r\theta - \frac{\eta}{2s}).$$

We parametrize  $\Lambda(r\theta)$  with the variable  $t$ ,  $|t| \leq 2r$ , and use Minkowsky integral inequality. We have that (2.57) is bounded by

$$\leq C \int_{r \sim 2^j} 2^{j\alpha} \left( \int_{S^1} |\hat{g}(r\theta)|^2 d\sigma(\theta) \right)^{1/2} F_s^\infty(r) dr. \quad (2.58)$$

where

$$F_s^\infty(r) = \int_0^{2r} \left( \int_{S^1} |\hat{f}(\tau(s, r, t, \theta))|^2 d\sigma(\theta) \right)^{1/2} dt \quad (2.59)$$

Now we are left to check that our new "twisted" polar coordinates do not degenerate. The change of variable  $(t, \theta) \rightarrow \tau$  given in complex notation  $\theta = e^{i\alpha}$  and  $\tau = \tau_1 + i\tau_2$  by

$$\tau = \left( 1 - \frac{2s-1}{s} \left( \frac{r^2+t^2}{r^2(2s-1)^2+t^2} \right)^{1/2} \right) r e^{i\alpha} + i \left( \frac{r^2+t^2}{r^2(2s-1)^2+t^2} \right)^{1/2} \frac{t}{s} e^{i\alpha},$$

has Jacobian

$$\begin{aligned} J(t, \alpha, r, s) &= \Im \left( \frac{\partial \tau}{\partial \alpha} \frac{\partial \bar{\tau}}{\partial t} \right) \\ &= \frac{1}{2} \left( \left( \frac{2s-1}{s} \right)^2 r^2 - \frac{2s-1}{s} B^{-1/2} r^2 + \frac{t^2}{s^2} \right) B'(t) + \frac{t}{s^2} B(t), \end{aligned}$$

where  $B(t) = B(t, r, s) = \frac{r^2+t^2}{r^2(2s-1)^2+t^2}$  and  $B'(t) = 4(s-1) \frac{r^2 t s}{(r^2(2s-1)^2+t^2)^2}$ . Now,  $B(t)$  satisfies

$$\frac{1}{(2s-1)^2} \leq B(t) \leq 1, |B'(t)| \leq \frac{8(s-1)t}{r^2}. \quad (2.60)$$

Thus when we incorporate these bounds in the expression of the Jacobian we see that

$$J(t, \alpha, r, s) = \frac{B(t)}{s^2} t + (s-1) t a(t, s, r), \quad (2.61)$$

with

$$|a(t, s, r)| \leq C,$$

$C$  independent of  $t$ ,  $r$  and  $s$ .

Since  $s \in \left( 1, 1 + \frac{1}{2k_0} \right)$ , and  $k_0$  is arbitrarily large we can choose it large enough to obtain from (2.61) uniform bounds

$$\frac{t}{2} \leq J(t, \alpha, r, s) \leq \frac{3t}{2} \quad (2.62)$$

Let us split the r.h.s of (2.59)

$$\left( \int_0^1 + \int_1^{2r} \right) \left( \int_{S^1} |\hat{f}(\tau(s, r, t, \theta))|^2 d\sigma(\theta) \right)^{1/2} dt = F_s^\infty \left( \frac{1}{2} \right) + G_s^\infty(r). \quad (2.63)$$

Next let  $1 < l \leq 2$  and  $l'$  its Hölder conjugate,  $\frac{1}{l} + \frac{1}{l'} = 1$ . We start by incorporating the Jacobian in the expression of  $G_s^\infty$ .

$$G_s^\infty(r) \leq C \int_1^{2r} |J(t, \alpha, r, s)|^{\frac{1}{l}-1} |J(t, \alpha, r, s)|^{\frac{1}{l}} \cdot \left( \int_{S^1} |\hat{f}(\tau(s, r, t, \theta))|^{l'} d\sigma(\theta) \right)^{\frac{1}{l'}} dt$$

Now we are entitled to use Hölder's inequality in combination with (2.62) and a change of variables to obtain

$$\begin{aligned} G_s^\infty(r) &\leq C \left( \int_1^{2r} J(t, \alpha, r, s)^{1-l} dt \right)^{\frac{1}{l}} \cdot \left( \int_1^{2r} |J(t, \alpha, r, s)| \int_{S^1} |\hat{f}(\tau(s, r, t, \theta))|^{l'} d\sigma(\theta) dt \right)^{\frac{1}{l'}} \\ &\leq C \left( \int_1^{2r} t^{1-l} dt \right)^{\frac{1}{l}} \|\hat{f}\|_{L^{l'}} \end{aligned}$$

For  $l = 2$ , we have

$$G_s^\infty(r) \leq C \log r \|f\|_{L^2}, \quad (2.64)$$

and for  $l' = p' > 2$

$$G_s^\infty(r) \leq Cr^{1-\frac{2}{p'}} \|f\|_{L^{p'}}. \quad (2.65)$$

To bound  $F_s^\infty(\frac{1}{2})$ , we will use again (2.62). Let  $p > 2$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ . Then again Hölder's inequality, (2.62) and a change of variables lead to

$$\begin{aligned} F_s^\infty\left(\frac{1}{2}\right) &\leq C \int_0^1 \left( \int_{S^1} |\hat{f}(\tau(s, r, t, \theta))|^{p'} d\sigma(\theta) \right)^{1/p'} dt \\ &\leq C \left( \int_0^1 \int_{S^1} J(t, \alpha, r, s) |\hat{f}(\tau(s, r, t, \theta))|^{p'} d\sigma(\theta) dt \right)^{1/p'} \\ &\quad \cdot \left( \int_0^1 J(t, \alpha, r, s)^{-\frac{p'}{p}} dt \right)^{\frac{1}{p}} \leq C \|\hat{f}\|_{L^{p'}}. \end{aligned}$$

Hence

$$F_s^\infty(r) \leq C(\|\hat{f}\|_{L^{p'}} + \log r \|f\|_{L^2}) \quad (2.66)$$

or

$$F_s^\infty(r) \leq Cr^{1-\frac{2}{p'}} \|f\|_{L^{p'}}. \quad (2.67)$$



These pointwise bounds will yield the required estimates. We start using the bound for  $F_s^\infty(r)$  given by (2.66). From (2.58) and using Cauchy-Schwartz

$$\begin{aligned}
& \|\widehat{J(f, g)}\|_{L^1(\langle \cdot \rangle^\alpha)} \\
& \leq C(\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}) \sum_{j=\log k_0}^{\infty} j2^{j\alpha} \int_{r \sim 2^j} \left( \int_{S^1} |\hat{g}(r\theta)|^2 d\sigma(\theta) \right)^{1/2} dr \\
& \leq C(\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}) \sum_{j=\log k_0}^{\infty} j2^{j(\alpha-s_0)} \\
& \quad \cdot \left( \int_{r \sim 2^j} r(1+r^2)^{s_0} \int_{S^1} |\hat{g}(r\theta)|^2 d\sigma(\theta) dr \right)^{1/2} \\
& \leq C(\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}) \|g\|_{W^{s_0, 2}} \sum_{j=\log k_0}^{\infty} j2^{j(\alpha-s_0)} \\
& \leq C(\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}) \|g\|_{W^{s_0, 2}},
\end{aligned}$$

and we get (2.39).

In a similar way, if we consider the bound (2.67)

$$\begin{aligned}
& \leq C(\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}) \|g\|_{W^{s_0, 2}} \sum_{j=\log k_0}^{\infty} 2^{j(\alpha-s_0+1-\frac{2}{p'})} \\
& \leq C(\|\hat{f}\|_{L^{p'}} + \|f\|_{L^2}) \|g\|_{W^{s_0, 2}},
\end{aligned}$$

and we obtain (2.40). □

### 2.3 Proof of the Theorem 1

The proof of Theorem 1 follows from choosing  $k_0$  large enough and then apply Propositions 2.1, 2.2, 2.3, 2.4 and 2.5 together with the decomposition (2.24), and by observing that if  $1 < p < 2$  we have

$$\|q\|_{L^p} \leq C(\|q\|_{W^{s_0, 2}} + \|\cdot\| \cdot \|q\|_{L^p}).$$

## 2.4 Proof of Corollary 1

If  $q$  is compactly supported the cubic term  $q_3$  is in  $W^{s_0+1,2}$ , see [16], hence by Sobolev embedding it belongs to the Hölder space  $\Lambda^\alpha$  with  $\alpha < s_0$ .

The rest of the term in the series are in better Sobolev spaces as can be seen in [19] Proposition 4.3. Checking the values  $s_j$  in there for  $j = 4$  and Sobolev embedding theorem we have that for  $s_0 < 1/2$ ,  $q_4 \in W^{s,2}$  for  $s < 3/2 + 3s_0/4$ , hence in  $\Lambda^\alpha$  for  $\alpha < 1/2 + 3s_0/4 > s_0$ . If  $1/2 \leq s_0 \leq 1$ ,  $q_4 \in W^{s,2}$  for  $s < 3/2 + 3s_0/4$ , hence in  $\Lambda^\alpha$  for  $\alpha < 3/4 + s_0/4 \geq s_0$ . The case  $s_0 > 1$  follows from Leibniz type formula for the  $j$ -adic term in the series (see Theorem 6 in [17]).

## References

- [1] S. Agmon, *Spectral properties of Schrodinger operators and scattering theory*, Ann. Sc. Norm. Sup. Pisa, serie 4, vol 2, 121-218, (1975).
- [2] J.A. Barceló, D. Faraco, A. Ruiz and A. Vargas, *Reconstruction of singularities from full scattered data by new estimates of bilinear Fourier multipliers*. Math. Ann. 3 (2010), 505-544.
- [3] J.A. Barceló, M. Folch-Gabayet, S. Pérez-Esteve, A. Ruiz, M. Vilela *A Born approximation for live loads in Navier elasticity*. SIAM J. MATH. ANAL. c 2012 Society for Industrial and Applied Mathematics Vol. 44, No. 4, pp. 28242846
- [4] I. Beltita, and A. Melin, *Local smoothing for the backscattering transform*. Comm. Partial Differential Equations 34 (2009), no. 1-3, 233–25
- [5] I. Beltita and A. Melin, *Analysis of the quadratic term in the backscattering transformation*. Math. Scand. 105 (2009), no.
- [6] D. Colton and R. Kress, *Inverse acoustic and electromagnetic Scattering theory*, Springer-Verlag (1992).
- [7] G. Eskin and J. Ralston, *The inverse backscattering problem in 3 dimension*. Comm. Math. Phys. 124 (1989), 169-215.

- [8] G. Eskin and J. Ralston, *Inverse backscattering in two dimension*. Comm. Math. Phys. 138 (1991), 451-486.
- [9] G. Eskin and J. Ralston, *Inverse backscattering*, *J. d'Analyse Math.* 58 (1992), 177-190.
- [10] I. M. Gel'fand and G. E. Shilov, *Generalized Functions I*. Academic Press Inc. New York (1964).
- [11] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*. Ann. of Math. 121 (1985), 463-494.
- [12] R. Lagergren, *Backscattering in three dimensions*. Ph. D.Thesis. Lund University. 2001.
- [13] P. Ola, L. Päivärinta and V. Serov, *Recovering singularities from backscattering in two dimension*. Comm. Partial Differential Equations 26 (2001), no. 3-4, 697715.
- [14] L. Päivärinta and E. Somersalo, *Inversion of discontinuities for the Schrödinger equation in three dimensions*. SIAM J. Math. Anal. 22 (1991), 480-499.
- [15] R. T. Prosser, *Formal solutions of inverse scattering problems*. J. Math. Phys. 23 (1982), 2127-2130.
- [16] J.M. Reyes, *Inverse backscattering for the Schrödinger equation in 2D*. Inverse Problems 23 (2007), no. 2, 625-643.
- [17] J.M. Reyes, A. Ruiz, *Reconstruction of the singularities of a potential from backscattering data in 2D and 3D*. Inverse Problem and Imaging. Volume 16, No2, 2012,321-355
- [18] A. Ruiz, *Recovery of the singularities of a potential from fixed angle scattering data*. Comm. P.D.E. 26 (9, 10) 1721-1738 (2001).
- [19] A. Ruiz and A.Vargas, *Partial recovery of a potential from backscattering data*. Comm. Partial Differential Equations 30 (2005), no. 1-3, 67-96.
- [20] A. Ruiz and L. Vega, *On local regularity of Schrödinger equations*. Interna. Math. Res. Notices 1, 1993, 13-27.

- [21] V. Serov, *Inverse fixed angle scattering and backscattering problems in two dimensions*. Inverse Problems 24 (2008), no. 6, 065002, 14 pp.
- [22] P. Stefanov, *Generic uniqueness for two inverse problems in potential scattering*. Comm. PDE 17 (1992),55-68.
- [23] P. Stefanov and G. Uhlmann, *Inverse backscattering for the acoustic equation*. SIAM J. Math. Anal. 28 (1997), 1191-1204.
- [24] Z. Sun and G. Uhlmann, *Generic uniqueness for an inverse boundary value problem*. Duke Math. J. 62 (1991), no. 1, 131–155.
- [25] Z. Sun and G. Uhlmann, *Recovery of singularities for formally determined inverse problems*. Comm. Math. Phys. 153(1993), 431-445.
- [26] J.Sylvester , G. Uhlmann, *A global Uniqueness theorem for an inverse boundary value problem*. Ann. of Math. 125.1987,153-169).
- [27] G. Uhlmann, *Time-dependent approach to the inverse backscattering problem*. Special issue to celebrate Pierre Sabatier’s 65th birthday (Montpellier, 2000). Inverse Problems 17 (2001), no. 4, 703–716.