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TRANSLATION INVARIANT OPERATORS

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[I] GENERAL REMARKS

In the following we shall be concerned mainly with the analysis of certain operators acting on functions defined on $\mathbb{R}^n$. These are "multipliers" or translation invariant operators T i.e. operators which conmute with the family of translations of $\mathbb{R}^n$ and are bounded with respect to some norms $\| \|_p$, $\| \|_q$. Such operators can be represented in two equivalent forms, that is $Tf = K * f$ or $\widehat{Tf} = m.\hat{f}$ for every $f \in \mathcal{S}(\mathbb{R}^n)$, where K, the Kernel, is a tempered distribution and $m = \hat{K}$ is a measurable function. We shall consider mainly the case $p=q$, obviously such an operator T has a bounded extension to $L^p(\mathbb{R}^n)$ and we shall design by $|T|_p$ or $|m|_p$ the norm of the extension. A duality argument shows that if T is bounded on $L^p(\mathbb{R}^n)$ then it must also be bounded on $L^{p'}(\mathbb{R}^n)$, $1/p + 1/p' = 1$ and, therefore, by interpolation, on every $L^s(\mathbb{R}^n)$, s between p and p'. We shall consider also the periodic analogue, that is operators $\tilde{T}$ bounded on some $L^p(T^n)$ and, similarly to the nonperiodic case, we may realize such $\tilde{T}$ as convolution with an appropiate tempered distribution or as a multiplier, we have:

$\tilde{T}$ bounded from $L^p(T^n)$ to $L^q(T^n)$, $1 \le p,q \le \infty \Longrightarrow$ there exists a bounded function $m(\nu)$, $\nu \in Z^n$, such that if $f \sim \Sigma\, a_\nu\, e^{2\pi i \nu . x}$ then $\tilde{T}f \sim \Sigma\, m(\nu) a_\nu\, e^{2\pi i \nu . x}$. The first problem that we may ask is to characterize these operators, more concretely: given the kernel K or the multiplier m, how can we decide the boundedness properties of the associated operator T?.

We list general results which throw some light on this problem.

Theorem 1. $Tf = K * f$ is a multiplier of $L^2(\mathbb{R}^n)$ if and only if $\hat{K} \in L^\infty$, furthermore $|T|_2 = \|K\|_\infty$.

Theorem 2. T is a multiplier of $L^1(\mathbb{R}^n)$ if and only if there exists μ, a finite Borel measure, such that $Tf = \mu * f$ and $|T|_1$ = Total variation of μ.

Theorem 3. Given $m_1 \in L^p(\mathbb{R}^n)$, $m_2 \in L^q(\mathbb{R}^n)$, $1/p + 1/q = 1$, then $m_1 * m_2$ is a multiplier of $L^r(\mathbb{R}^n)$ for every r, $\frac{2p}{3p-2} \leq r \leq \frac{2p}{2-p}$.

Theorem 4. If $m(\, , \,)$ is a Fourier multiplier of $L^p(\mathbb{R}^{n+m})$ then for almost every ξ, $m(\xi, —)$ is a multiplier of $L^p(\mathbb{R}^m)$ and

$$|m(\xi, —)|_p \leq |m|_p$$

Theorem 5.

(a) Let $m(\xi)$ be a bounded continuous function on $\mathbb{R}^n$ s.t. $\widehat{Tf}(\xi) = m(\xi)\,\hat{f}(\xi)$ is bounded on $L^p(\mathbb{R}^n)$, then $\tilde{T}$ defined by $\{m(\nu)\}_{\nu \in Z^n}$ is a bounded operator and $|\tilde{T}|_p \leq |T|_p$.

(b) Suppose that for every $\varepsilon > o$ the operator $\tilde{T}_\varepsilon$ given by $\{m(\varepsilon\nu)\}_{\nu \in Z^n}$ is bounded on $L^p(T^n)$ and $\sup_{\varepsilon > o} |T_\varepsilon|_p < \infty$ then the multiplier $\widehat{Tf} = m.\hat{f}$ defined on $\mathcal{S}(\mathbb{R}^n)$ has a bounded extension to L^p and satisfies $|T|_p \leq \sup_{\varepsilon > o} |\tilde{T}_\varepsilon|_p$.

Proofs ol 1,2 and 5 can be found in [1], [2] and [3], theorem 3 is an exercise in interpolation [4]; here we present the proof of de Leew's theorem 4 given by M. Jodeit [5].

Proof of Theorem 4.

Suppose first that $m(\xi,\eta)$ is continuous, and take

$f_1, g_1 \in \mathcal{S}(\mathbb{R}^n)$,

$f_2, g_2 \in \mathcal{S}(\mathbb{R}^m)$

Then $\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} m(\xi,\eta)\ \hat{f}_1(\xi)\ \hat{f}_2(\eta)\ \hat{g}_1(\xi)\ \hat{g}_2(\eta)\ d\xi\ d\eta\right| \leq$

$$\leq |m|_p\ \|f_1\|_p\ \|f_2\|_p\ \|g_1\|_q\ \|g_2\|_q \qquad 1/p + 1/q = 1$$

Writing the first part of this inequality in the form

$$\left|\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} m(\xi,\eta)\ \hat{f}_2(\eta)\ \hat{g}_2(\eta)d\eta\right)\ \hat{f}_1(\xi)\ \hat{g}_1(\xi)d\xi\right|$$

we may conclude that

$M(\xi) = \int_{\mathbb{R}^m} m(\xi,\eta)\ \hat{f}_2(\eta)\ \hat{g}_2(\eta)d\eta$ is a multiplier of $L^p(\mathbb{R}^n)$,

with norm $\leq |m|_p\ \|f_2\|_p\ \|g_2\|_q$.

Since every multiplier M satisfies

$$\|M\|_\infty = |M|_2 \leq M_p, \qquad 1 \leq p \leq \infty, \quad \text{we get}$$

$$\left|\int_{\mathbb{R}^n} m(\xi,\eta)\ \hat{f}_2(\eta)\ \hat{g}_2(\eta)d\eta\right| \leq |m|_p\ \|f_2\|_p\ \|g_2\|_q$$

that is, $|m(\xi,\text{—})|_p \leq |m|_p$.

To remove the restriction about the continuity of m we proceed as follows: Let $\xi \in \mathbb{R}^n$ so that (ξ,η) is a Lebesgue point of m for almost every $\eta \in \mathbb{R}^m$ and write $m_\varepsilon(\xi,\eta) = m * \phi_\varepsilon(\xi,\eta)$.

Where $\phi_\varepsilon = \varepsilon^{-(n+m)}\ \chi_Q\left(\frac{\xi}{\varepsilon}, \frac{\eta}{\varepsilon}\right)$, and Q is the unit cube in $\mathbb{R}^{n+m}$ centered at the origen.

By the preceeding argument we get

$$\left|\int_{\mathbb{R}^m} m_\varepsilon(\xi,\eta)\ \hat{f}_2(\eta)\ \hat{g}_2(\eta)d\eta\right| \leq |m_\varepsilon|_p\ \|f_2\|_p\ \|g_2\|_q \leq$$

$$\leq |m|_p \quad \|f_2\|_p \quad \|g_2\|_q$$

An application of the dominated convergence theorem yields,

$$\left| \int_{\mathbb{R}^m} m(\xi,\eta) \ \hat{f}_2(\eta) \ \hat{g}_2(\eta) d\eta \right| \leq |m|_p \quad \|f_2\|_p \quad \|g_2\|_q$$

for every, $f_2 \in L^p(\mathbb{R}^m)$, $g_2 \in L^q(\mathbb{R}^m)$. q.e.d.

The important information contained in these theorems is, nevertheless, very unsatisfactory for p different from 1 or 2. And for these two cases is very hard, in general, to decide when a bounded function is the Fourier transform of a finite measure or when a tempered distribution has Fourier transform in the space L^∞. Since the Fourier transform of an L^1-function is continuous, at least we have a necessary condition for an L^1-multiplier $m(\xi)$: m has to be continuous. Nevertheless the Hilbert transform in $\mathbb{R}^1$ given by $m(\xi) = i \ sig(\xi)$ is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$, and therefore, our necessary condition is only valid for L^1. However this example leads us to the next stage of the theory which contains the important contributions of Marcinkiewicz, Besicovitch, Calderón-Zygmund, Paley-Littlewood, Mihlin, Hörmander, etc. and which we shall englobe under the name of Calderón-Zygmund theory.

[II] The Calderón-Zygmund Theory.

This theory has been discussed at length in many places (see [7], [6], [2],... for example). Here we just mention some important features of it, as a motivation and reference for the further developments of section [III].

Theorem 6. Suppose that K is a tempered distribution which coincides with a C^1-function outside the origen verifying $|\hat{K}(\xi)| \leq B$, $|\nabla K(x)| \leq B/|x|^{n+1}$. Then the operator $Tf(x) = \text{p.v.} \int K(x-y)\, f(y)dy$ has a bounded extension to $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and it is of weak type (1,1).

(* The second condition $|\nabla K(x)| \leq B/|x|^{n+1}$ can be replaced by the more general $\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leq B$, $|y| > 0$).

Theorem 7. Suppose that $m \in L^\infty(\mathbb{R}^n)$ satisfies the following estimates

$$|D^\alpha m(\xi)| \leq B|\xi|^{-|\alpha|}$$

for every α such that $|\alpha| \leq \theta(n) = \left[\frac{n}{2}\right] + 1$. Then m is a multiplier for $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Important multipliers which may be considered as examples of these two theorems are given by homogeneous kernels:

$K(x) = \dfrac{\Omega(x)}{|x|^n}$, where Ω is homogeneous of degree zero, has mean value zero on the unit sphere and is smooth outside the origen, like the Riesz's transform defined by $R_j(x) = \dfrac{x_j}{|x|^{n+1}}$ on $\mathbb{R}^n$.

Theorem 8. Let K be a tempered distribution on $\mathbb{R}^n$, with compact support, equal to a locally integrable function away from zero and such that $\hat{K}$ is a function. Assume that

(i) $|\hat{K}(\xi)| \leq (1+|\xi|)^{-\frac{n\theta}{2}}$, for $\xi \in \mathbb{R}^n$

(ii) $\displaystyle\int_{|x|>2|y|^{1-\theta}} |K(x) - K(x-y)|\,dx \leq B$, for all $y \in \mathbb{R}^n$ $(|y| \leq 1)$.

Then the operator $Tf = K * f$ has a bounded extension to $L^p(\mathbb{R}^n)$, $1 < p < \infty$ and it is of weak type $(1,1)$.

An example of a multiplier which falls under the scope of this theorem but is not an application of the preceeding is the following one: $f \to K * f$, where $K(x) = |x|^{-n} e^{i|x|^{-r}}$, in $\mathbb{R}^n$, and $r > 0$.

In the proofs of these results, plays an important role the so-called Hardy-Littlewood maximal function, which is defined on locally integrable functions by the formula:

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$ where the supremum is taken over all the cubes which contain the point x. This positive operator which is bounded on $L^p(\mathbb{R}^n)$, $p > 1$, and satisfies a weak type $(1,1)$ inequality, is used to control the more complicated operators considered in the above theorems, together with an orthogonality argument. Perhaps the next theorem that we list contains the more clear expression of the orthogonality that we have just mentioned.

Consider in $\mathbb{R}^n$ the family of dyadic rectangles Δ (in $\mathbb{R}^1$ Δ is given by the family of intervals $(2^k, 2^{k-1})$, $(-2^{k+1}, -2^k)$, $-\infty < k < +\infty$, the dyadic rectangles of $\mathbb{R}^n$ are then obtained as products of these intervals). Consider for each $\rho \in \Delta$ the operator

$\widehat{S_\rho f}(\xi) = \chi_\rho(\xi).\hat{f}(\xi)$, where χ_ρ stands for the characteristic function of the rectangle ρ, and $Sf(x) = (\sum_{\rho\in\Delta} |S_\rho f(x)|^2)^{1/2}$, we have.

Theorem 9. For each p, $1 < p < \infty$, there exist constants $0 < A_p$, $B_p < \infty$ so that $A_p \|f\|_p \leq \|Sf\|_p \leq B_p \|f\|_p$.

The operators which fall under the scope of these theorems (except perhaps for theorem 8) have, basically, kernels with singularities near 0 or ∞. There are however other operators with more complicated sets of singularities which may be consider as "composition" of operatorsof that class, like the double Hilbert transform:

$$Hf(x,y) = p.v \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{f(x-s,\ y-t)}{s.t}\ ds\ dt$$

The following theorem takes care of this situation.

Theorem 10. Let m be a bounded function in $\mathbb{R}^n$ such that:

(a) for each $0 < k \leq n$

$$(*) \qquad \sup_{x_{k+1},\dots,x_n} \int_\rho \left| \frac{\partial^k m}{\partial x_1\ \partial x_2 \dots \partial x_k} \right| dx_1 \dots dx_k \leq B < \infty$$

as ρ ranges over dyadic rectangles of $\mathbb{R}^k$ (we consider $\mathbb{R}^k$ embedded in $\mathbb{R}^n$ in the following way $(x_1,\dots,x_k,0,\dots 0)$).

(b) The condition analogous to $(*)$ is valid for every permutation of the variables $x_1, x_2, \dots, x_n$.

Then m is a multiplier of $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

One can find proofs and excellent discussions of these theorems in [6], [2], [7]. Here we do not resist the temptation of presenting a proof of theorem 6 under the (irrelevant) hypothesis $K \in L^1 + L^2$, in order to illustrate such an important paradigm as the Calderón-Zygmund decomposition.

<u>Proof of theorem 6</u>. T is bounded on $L^2(\mathbb{R}^n)$ because $\hat{K} \in L^\infty(\mathbb{R}^n)$. We will show that T is of weak type $(1,1)$ and the theorem will follow from the Marcinkiewicz's interpolation theorem.

Given $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$, we can find a closed set F and an open set Ω satisfying:

$F \cap \Omega = \phi$, $\mathbb{R}^n = F \cup \Omega$, $\Omega = \bigcup_j Q_j$, where Q_j's are disjoint cubes such that,

$$\alpha \leq \frac{1}{|Q_j|} \int_{Q_j} |f(y)|dy \leq 2^n \alpha$$

$$|f(x)| \leq \alpha, \quad \text{a.e. in } F.$$

Then we decompose the function f as follows: $f = g+b$.

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in F \\ \frac{1}{|Q_j|} \int_{Q_j} f(y)dy, & \text{if} \quad x \in Q_j \end{cases}$$

$$b(x) = \begin{cases} 0 & \text{if} \quad x \in F \\ f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y)dy, & \text{if} \quad x \in Q_j \end{cases}$$

(i) $g \in L^2(\mathbb{R}^n)$, because $\int g(x)^2\,dx = \int_F g(x)^2 dx + \sum_j \int_{Q_j} g(x)^2 dx \leq$

$$\leq \alpha \int_F |g(x)|dx + \sum_j \left(\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx\right)^2 |Q_j| \leq$$

$$\leq \alpha \int_F |f(x)|dx + \sum_j 2^{2n}\alpha^2 |Q_j| \leq \alpha \left\{\int_F |f(x)\,dx +\right.$$

$$\left. + 2^{2n} \sum_j \int_{Q_j} |f(x)|dx\right\} \leq 2^{2n}\alpha \int_{\mathbb{R}^n} |f(x)|dx$$

Therefore $|\{x : |Tg(x)| > \frac{\alpha}{2}\}| \leq 2^{2n} B^2 \dfrac{\|f\|_1}{\alpha}$

(ii) Let $b_j(x) = b(x)\,\chi_{Q_j}(x)$, then $Tb(x) = \sum_j T\,b_j(x)$;

with $\Omega^* = \bigcup Q_j^*$ (Q^* is the double of Q) and $F^* = \mathbb{R}^n - \Omega^*$, we have,

(1) $F^* \subset F$, $\Omega^* \supset \Omega$ and $|\Omega^*| \leq 2^n |\Omega|$

(2) If $x \notin Q_j^*$, then $|x-y_j| \geq 2|y-y_j|$, for every $y \in Q_j$ where y_j is the center of Q_j.

Then $Tb_j(x) = \int_{Q_j} K(x-y)\,b_j(y)dy = \int_{Q_j} (K(x-y) -$

$- K(x-y_j))\,b_j(y)dy$

and

$$\int_{F^*} |Tb(x)|dx \leq \sum_j \int_{x\notin Q_j^*} \int_{y\in Q_j} |K(x-y) - K(x-y_j)|\,|b(y)|dy\,dx$$

$$\leq \sum_j \int_{Q_j} |b(y)|dy \left\{\int_{x\notin Q_j^*} |K(x-y) - K(x-y_j)|dx\right\} \leq$$

$$\leq B \sum_j \int_{Q_j} |b(y)|dy \leq 2B\,\|f\|_1$$

Therefore $|\{x : |Tb(x)| > \frac{\alpha}{2}\}| \leq |\Omega^*| + 4B \dfrac{\|f\|_1}{\alpha} \leq (2+4B)\dfrac{\|f\|_1}{\alpha}$

Proof of theorem 7.

By choosing an adequate partition of unity $1 = \sum_{-\infty}^{+\infty} \phi(2^k\xi)$, $\forall\xi \neq 0$, where ϕ is a smooth function with support in the annulus $\frac{1}{2} \leq |\xi| \leq 2$ we decompose the multiplier $m = \Sigma m_j$, where $m_j(\xi) = \phi(2^{-j}\xi).m(\xi)$. It is easy to see, using the hypothesis of the theorem, that each piece is a nice multiplier, that is: $K_j = \hat{m}_j \in L^1(\mathbb{R}^n)$ and $\|K_j\|_1 \leq C.B$, for every j, where C is a universal constant. Next we need an "orthogonality" argument in order to put together those estimates and, in this case, we may use the previous theorem and observe that

$$K^N(x) = \sum_{-N}^{+N} K_j(x) \quad \text{satisfies:}$$

$$\int_{|x|\geq 2|y|} |K^N(x-y) - K^N(x)|\,dx \leq CB$$

uniformly on N and $|y| \neq 0$.

(a) We have $\int |K_j(x)|\,dx \leq CB$ uniformly on j:

$$\int |K_j(x)|\,dx = \int_{|x|\leq 2^{-j}} |K_j(x)|\,dx + \sum_{s=o}^{\infty} \int_{2^s 2^{-j}\leq|x|\leq 2^{s+1}2^{-j}} |K_j(x)|\,dx$$

and observe that

$$\int_{|x|\sim 2^{s-j}} |K_j(x)|\,dx \leq 2^{(s-j)n/2} \left| \int_{|x|\sim 2^{s-j}} |K_j(x)|^2 dx \right|^{1/2} \leq$$

$$\leq 2^{(s-j)n/2}\, 2^{-(s-j)|\alpha|} \left| \int_{|x|\sim 2^{s-j}} |x|^{2|\alpha|}\, |K_j(x)|^2 dx \right|^{1/2}$$

$$\leq c\, 2^{(s-j)\left[\frac{n}{2} - |\alpha|\right]} \|D^\alpha m_j\|_2 \leq CB\, 2^{s(\frac{n}{2} - |\alpha|)}, \quad \text{for every,}$$

α, $0 < |\alpha| \leq \theta(n) = \left[\frac{n}{2}\right] + 1$.

(b) Want to estimate $\displaystyle\int_{|x|\geq 2|y|} |K^N(x-y) - K^N(x)|\,dx$. We shall consider for each j the integral $\displaystyle\int_{|x|\geq 2|y|} |K_j(x-y) - K_j(x)|\,dx$ and observe that "roughly speaking" we may consider K_j "supported" on $|x| \leq 2^{-j}$. Therefore if $2^j|y| \geq 1$ one should not expect cancellations in $K_j(x-y) - K_j(x)$ and we must evaluate each term separately:

<u>Claim</u> $$\int_{|x|\geq 2|y|} |K_j(x)|\,dx \leq CB\ (2^j\ |y|)^{\left(\frac{1}{2}n - \theta(n)\right)}$$

The proof is contained in the computations of part (a).

Assume now that $2^j|y| \leq 1$. Then one expect that the integral of the difference $|K_j(x-y) - K_j(x)|$ should be "smaller" than the integral of each term separately. We have

$$|K_j(x-y) - K_j(x)| \leq \sum_{1\leq|\alpha|<\theta[n]} \frac{1}{\alpha!}\ |D^\alpha K_j(x)|\ \ |y|^{|\alpha|} +$$

$$+ \sum_{|\alpha|=\theta[n]} \frac{1}{\alpha!} \int_0^1 |D^\alpha K_j(x-ty)|\,dt\ \ |y|^{\theta[n]}$$

let us consider a typical term $|D^\alpha K_j(x)|\ |y|^{|\alpha|}$, $|\alpha| < \theta[n]$ we have:

(i) $$\int_{2^{1+s}|y|\geq|x|\geq 2^s|y|} |y|^\alpha\ |D^\alpha K_j(x)|\,dx \leq |y|^\alpha\left(2^s\ |y|\right)^{n/2} \left|\int |D^\alpha K_j(x)|^2 dx\right|^{1/2} \leq$$

$$\leq CB\ |y|^{|\alpha|}\ \left(2^s|y|\right)^{n/2}\ 2^{+j|\alpha|}\ 2^{j\frac{n}{2}} \leq C.B\ \ 2^j|y|\left(2^s|y|2^j\right)^{n/2}$$

which is a good estimate in the case $2^s\ |y|\ 2^j \leq 1$ i.e. $|x| \leq 2^{-j}$

(ii) On the other hand,

$$D^{\alpha}K_j(x) = \frac{i^{\alpha}}{(i\ x)^{\beta}} \int_{\mathbb{R}^n} e^{ix.\xi}\ D^{\beta}(\xi^{\alpha}\ m_j(\xi))d\xi, \quad |\beta| \leq \theta[n]$$

which implies

$$|y|^{\alpha} \int_{|x|\sim 2^s|y|} |D^{\alpha}K_j(x)|dx \leq |y|^{\alpha}\left(2^s|y|\right)^{n/2} \left| \int_{|x|\sim 2^s|y|} |D^{\alpha}K_j(x)|^2 dx\right|^{1/2} \leq$$

$$\leq C.B\ |y|^{\alpha} \left(2^s|y|\right)^{n/2} \left(2^s|y|\right)^{-\theta[n]|}\ 2^{j\left(|\alpha|-\theta[n]\right)+\frac{n}{2}}$$

$$\leq C.B \left[|y|\ 2^j\right]^{|\alpha|} . \left[2^s|y|2^j\right]^{-\theta[n]+\frac{n}{2}}$$

which takes care of the cases $2^s\ |y|\ 2^j \geq 1$.

The remainder can be treated in the same way, therefore me have from (a) and (b):

$$\int_{|x|\geq 2|y|} |K_j(x-y) - K_j(x)| \leq C.B.\ \min\ \{2^j|y|,\ \left[2^j|y|\right]^{-\theta[n]+\frac{n}{2}}\}$$

which implies for each N:

$$\int_{|x|\geq 2|y|} |K^N(x-y) - K^N(x)|dx \leq \tilde{C}.B \quad \text{uniformly}$$

q.e.d.

We shall be concerned now with certain developments connected with the so called weighted inequalities. That is, we study the boundedness properties of singular integral and maximal operators with respect to absolutely continuous measures $d\mu = \omega(x)dx$ (dx = Lebesgue's measure in $\mathbb{R}^n$). Roughly speaking, the history of this subject begins with the 1960 paper of Helson and Szegö [8] where they proved that the Hilbert transform is bounded on $L^2(d\mu)$ iff $d\mu = \omega dx$ and $\omega = \exp(u + Hv)$, where $u \in L^\infty$ and $\|v\|_\infty < \pi/2$. Hardy's inequality

$$\left|\int_0^\infty \left[\int_0^x f(y)dy\right]^p x^{-r-1} dx\right|^{1/p} \leq \frac{p}{r} \left|\int_0^\infty (y\ f(y))^p\ y^{-r-1} dy\right|^{1/p}$$

may also be considered as a precedent of the modern theory. Then we have [9] where the following theorem is proved.

Theorem 11. There exists a constant c_r, $0 < c_r < \infty$, such that

$$\int_{\mathbb{R}^n} |f^*(x)|^r\ \omega(x)dx \leq c_r \int_{\mathbb{R}^n} |f(x)|^r\ \omega^*(x)dx$$

for locally integrable functions f, and $r > 1$. Where * denotes the Hardy-Littlewood maximal function. Furthermore,

for every $\alpha > 0$, $$\int_{\{x:f^*(x)>\alpha\}} \omega(x)dx \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)|\ \omega^*(x)dx$$

Proof.

Since $f \longrightarrow f^*$ is bounded from $L^\infty(\omega^*(x)dx)$ to $L^\infty(\omega(x)dx)$ it is enough to prove the weak-type estimate part of the theorem. To simplify the geometry we shall consider only dyadic cubes in $\mathbb{R}^n$. Given $\alpha > 0$, $\{f^*(x) > \alpha\} = \bigcup Q_i$, where the Q_i's are disjoint dyadic cubes such that

$$2^n \alpha \geq \frac{1}{|Q_i|} \int_{Q_i} f \geq \alpha$$

For $x \in Q_i$ we have

$$\omega^*(x) \geq \frac{1}{|Q_i|} \int_{Q_i} \omega \quad \text{and}$$

$$\int_{Q_i} f(x)\omega^*(x)dx \geq \frac{1}{|Q_i|} \int_{Q_i} \omega \int_{Q_i} f \geq \alpha \int_{Q_i} \omega$$

therefore

$$\int f(x)\ \omega^*(x)dx \geq \alpha \sum_i \int_{Q_i} \omega = \alpha \int_{\{f^*(x)>\alpha\}} \omega(x)dx$$

q.e.d.

<u>Definition</u>. $\omega \in A_1$ if there exists a constant c such that $\omega^*(x) \leq c\ \omega(x)$.

The previous theorem can be rephrased as follows: $A_1 \Longrightarrow$ Maximal function is weak type (1,1) with respect to $\omega(x)dx$, and bounded on each $L^p(\omega\ dx)$, $p > 1$.

This observation has a converse: if f^* is weak type (1,1) with respect to ωdx then $\omega \in A_1$.

<u>Proof</u>.

Let $x \in Q_1 \subset Q_2$ where Q_1, Q_2 are cubes and x is a Lebesgue's point of ω, with $f = \chi_{Q_1}$ we have

$$f^*(z) \geq \frac{|Q_1|}{|Q_2|} \qquad \text{if} \qquad z \in Q_2,$$

therefore

$$\int_{Q_2} \omega \leq \int_{\left\{f^* \geq \frac{|Q1|}{|Q_2|}\right\}} \omega \leq c\ \frac{|Q_2|}{|Q_1|} \int_{Q_1} \omega$$

that is

$$\frac{1}{|Q_2|} \int_{Q_2} \omega \leq c\ \frac{1}{|Q_1|} \int_{Q_1} \omega$$

which implies (making $Q_1 \Longrightarrow x$)

$$\frac{1}{|Q_2|} \int_{Q_2} \omega \leq c\ \omega(x), \quad \text{that is} \quad \omega^*(x) \leq c\ \omega(x)$$

q.e.d.

Remark. Given a locally integrable ω and s, $0 \leq s < 1$, then $g = (\omega^*)^s$ satisfies A_1 with bounds independent of ω.

The next step was taken in [10] (weshall follow [11] very closely).

Definition. $\omega \in A_p$, $1 \leq p < \infty$, if there exists $c < \infty$ such that

$$\sup_Q \left|\frac{1}{|Q|} \int_Q \omega\right| \left|\frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}}\right|^{p-1} \leq c$$

Theorem 12. $f \longrightarrow f^*$ is bounded on $L^p(\omega\, dx)$ if and only if $\omega \in A_p$.

Proof.

(A) Assume that the max. function is bounded on $L^p(\omega\, dx)$. Given a dyadic cube Q and a locally integrable function f, we have

$$f^*(x) \geq \frac{1}{|Q|} \int_Q f(y)dy \quad \chi_Q(x)$$

therefore,

$$\left|\frac{1}{|Q|} \int_Q f\right|^p \int_Q \omega \leq c \int_Q |f(x)|^p \,\omega(x)dx$$

Taking $f = \omega^{-\frac{1}{p-1}}$ we get

$$\left|\frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}}\right|^p \int_Q \omega \leq c \int_Q \omega^{-\frac{1}{p-1}}$$

(B) Assume now that ω satisfies A_p. Given a locally integrable function f we have $(1/p + 1/q = 1)$,

$$\frac{1}{|Q|}\int_Q |f| \le \left|\frac{1}{|Q|}\int_Q |f|^p\,\omega\right|^{1/p}\left|\frac{1}{|Q|}\int_Q \omega^{-q/p}\right|^{1/q} =$$

$$= \left|\frac{1}{|Q|}\int_Q |f|^p\,\omega\right|^{1/p}\left|\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\right|^{\frac{p-1}{p}} \le$$

$$\le \left|\frac{\int_Q |f|^p\,\omega}{\int_Q \omega}\right|^{1/p} \qquad \text{that is}$$

$$f^*(x) \le c\,|M_\omega f^p(x)|^{1/p}$$

where $M_\omega g(x) = \sup\limits_{x\in Q}\frac{1}{\int_Q \omega}\int_Q |g(x)|\,\omega(x)dx$

The theorem is an inmediate consequence of the following lemmas:

Lemma 1. Assume that for each cube Q we have $\mu(Q^*) \le c\,\mu(Q)$ where C is a universal constant. Then $f \longrightarrow M_\mu f$ is bounded on $L^p(d\mu)$, $1 < p \le \infty$.

(The proof is the same as in the case μ = Lebesgue measure). Observe that lemma 1 added to the previous discussion yields the following partial result: $\omega \in A_p \implies f \longrightarrow f^*$ bounded on $L^r(\omega\,dx)$ for each $r > p$).

Lemma 2. If $\omega \in A_p$ then,

(a) $\omega \in A_\infty$ that is, there exists $\delta > 0$ so that if $E \subset Q$ (E measurable, Q cube) me have

$$\frac{\mu(E)}{\mu(Q)} \le C\left[\frac{|E|}{|Q|}\right]^\delta, \qquad \mu = \omega\,dx$$

(b) There exists $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}$.

This lemma is itself a consequence of the following.

<u>Lemma 3</u>. Given $\omega \in A_p$ there exists $\delta > 0$ and a finite constant C such that

$$\left[\frac{1}{|Q|}\int_Q \omega^{1+\delta}\right]^{\frac{1}{1+\delta}} \leq C\,\frac{1}{|Q|}\int_Q \omega$$

(reversed Holder's inequality).

<u>Proof</u>: <u>Lemma 3</u> $\Longrightarrow$ <u>Lemma 2</u>.

(a) We have $\mu(E) = \int_E \omega \leq \left[\frac{1}{|Q|}\int_Q \omega^{1+\delta}\right]^{1/1+\delta} |E|^{\delta/1+\delta}\,|Q|^{1/1+\delta} \leq$

$$\leq C\,\frac{1}{|Q|}\int_Q \omega\,|E|^{\delta/1+\delta}\,|Q|^{1/1+\delta}, \quad \text{that is}$$

$$\frac{\mu(E)}{\mu(Q)} \leq C\left[\frac{|E|}{|Q|}\right]^{\delta/1+\delta}$$

(b) Consider $v = \omega^{-\frac{1}{p-1}}$ then $\omega \in A_p$ implies $v \in A_q$ where $1/p + 1/q = 1$. Lemma 3 applied to v give us,

$$\left|\frac{1}{|Q|}\int_Q v^{1+\delta}\,dx\right|^{\frac{1}{1+\delta}} \leq C\,\frac{1}{|Q|}\int_Q v(x)\,dx$$

which means that $\omega \in A_{p-\varepsilon}$ with $\varepsilon = (p-1)\frac{\delta}{1+\delta}$, because

$$\left|\frac{1}{|Q|}\int_Q \omega\right|\left|\frac{1}{|Q|}\int_Q \omega^{-\frac{1+\delta}{p-1}}\right|^{\frac{p-1}{1+\delta}} \leq \left|\frac{1}{|Q|}\int_Q v^{-\frac{1}{q-1}}\right|^{q-1}.$$

$$\cdot\left|\frac{1}{|Q|}\int_Q v^{1+\delta}\right|^{\frac{1}{1+\delta}} \leq \tilde{c}$$

In order to prove <u>Lemma 3</u> let us make first the following observation: "Given ω in A_p there exists a constant $0 < c < 1$ such that if $A \subset Q$ and $|A| \leq \frac{1}{2}|Q|$ then $\mu(A) \leq c\,\mu(Q)$, where

$\mu = \omega\, dx$ and Q denotes a cube".

Proof.

Given $x \in Q$ we have $\chi^*_{Q-A}(x) \geq \frac{|Q-A|}{|Q|} \geq \frac{1}{2}$. Let us take $p_1 > p$, we have,

$$(\tfrac{1}{2})^{p_1} \int_Q \omega \leq \int \left[\chi^*_{Q-A}(x)\right]^{p_1} \omega(x)dx \leq C_{p_1} \int_Q \chi_{Q-A}(x)\, \omega(x)dx =$$

$$= C_1\, \mu(Q-A)$$

That is, $\mu(Q-A) \geq (\frac{1}{2})^{p_1} C^{-1}_{p_1}\, \mu(Q)$ i.e. $\mu(A) \leq \left[1 - (\frac{1}{2})^{p_1} C^{-1}_{p_1}\right] \mu(Q)$.

Proof of Lemma 3. Due to the invariance under dilations of $\mathbb{R}^n$ or multiplication of ω by a real number, we may assume that $|Q| = 1$ and $\int_Q \omega = 1$. For each $\lambda_k = 2^{(n+1).k}$ we apply the Calderón-Zygmund decomposition to obtain a family $\{Q^k_i\}$ of dyadic subcubes of Q so that:

(i) $\quad 2^{(n+1)k} < \omega_{Q^k_i} \leq 2^{(n+1)k+n}$ for each Q^k_i

(ii) $\quad \omega(x) \leq 2^{(n+1)k}$ on $D^c_k = Q - \bigcup_i Q^k_i$

It is easy to see that each cube in Q^{k+1}_j is a subcube of some cube Q^k_i and that $|Q^k_i \cap D_{k+1}| < \frac{1}{2} |Q^k_i|$ for each cube of the k-generation, which implies: $|D_k \cap D_{k+1}| < \frac{1}{2} |D_k|$.

By the previous observation we have $\int_{D_k} \omega \leq c \int_{D_{k-1}} \omega \leq c^k \int_{D_o} \omega$

Therefore

$$\int_Q \omega^{1+\delta} = \int_{Q-D_o} \omega^{1+\delta} + \sum_{k=o}^{\infty} \left| \int_{D_k - D_{k-1}} \omega(x)dx \right| 2^{(n+1)(k+1).\delta}$$

$$\leq 2^{\delta} \int_{Q-D_o} \omega + \sum_{k=o}^{\infty} c^k \, 2^{(n+1)(k+1)} \int_{D_o} \omega$$

$$\leq 2^{\delta} + \sum_{k=o}^{\infty} c^k \, 2^{(n+1)(k+1)\delta}$$

which converges for some $\delta > 0$.

We have finished the proof of Lemma 3 and Theorem 12.

Let T be a Calderón-Zygmund operator $Tf = K * f$, $\|\hat{K}\|_{\infty} \leq B$, $|\nabla K(x)| \leq B|x|^{-n-1}$ like in Theorem 6.

Theorem 13. For every $\omega \in A_p$, $1 < p < \infty$, the operator T is bounded on $L^p(\omega\, dx)$.

Proof.

(A) The Theorem follows from the following statements:

(a) For every $s > 1$ there exists $C(s,T) < \infty$ such that

$$Tf^{\#}(x) \leq C(s,T) \left[(f^s)^* (x)\right]^{1/s}$$

(b) If $\omega \in A_{\infty}$ and $f \in \bigcup_{p \geq 1} L^p(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} |f(x)|^p \, \omega(x) dx \leq c(p,\omega) \int_{\mathbb{R}^n} |f^{\#}(x)|^p \, \omega(x) dx$$

where

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy$$

We have:

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, \omega(x) dx \leq c(p,\omega) \int_{\mathbb{R}^n} |Tf^{\#}(x)|^p \, \omega(x) dx \leq$$

$$\leq c(p,\omega)\ C(s,T) \int_{\mathbb{R}^n} |(f^s)^*(x)|^{p/s}\ \omega(x)dx$$

Since $\omega \in A_p$ we know that $\omega \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. Taking $s > 1$ so that $p-\varepsilon \leq p/s$ we have

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega(x)dx \leq \tilde{c}(p,\omega)\ c(s,T) \int_{\mathbb{R}^n} |f(x)|^p\ \omega(x)dx$$

(B) Part (a) is a standard exercise. Given $F \in \bigcup_{p<\infty} L^p(\mathbb{R}^n)$ we shall prove the following inequality:

[1] $$\mu\{F^*(x) > \alpha,\quad F^{\#}(x) < \gamma\alpha\} \leq$$

$$\leq c\ \gamma^{\delta}\ \mu\{F^*(x) > \frac{\alpha}{2^{n+1}}\}, \quad \text{for every } \gamma,\alpha > 0$$

where C is a finite constant and $\mu = \omega\ dx$ satisfies:

$$E \subset Q, \qquad \mu(E)/\mu(Q) \leq c\ \left[|E|/|Q|\right]^{\delta}$$

From [1] we may get the theorem in the following way,

$$\int_{\mathbb{R}^n} |F^*(x)|^p\ \omega(x)dx = p \int_0^{\infty} \alpha^{p-1}\ \mu\ \{F^*(x) > \alpha\}d\alpha =$$

$$= c \int_0^{\infty} \alpha^{p-1}\ \mu\{F^*(x) > 2^{n+1}\ \alpha,\quad F^{\#}(x) < \gamma\, 2^{n+1}\alpha\}d\alpha +$$

$$+ c \int_0^{\infty} \alpha^{p-1}\ \mu\{F^{\#}(x) > \gamma\ 2^{n+1}\alpha\}d\alpha \leq$$

$$\leq c\ \gamma^{p-1} \int_0^{\infty} \alpha^{p-1}\ \mu\{x : F^{\#}(x) > \alpha\}d\alpha \ +$$

$$+ c\ \gamma^{\delta} \int_0^{\infty} \alpha^{p-1}\ \mu\{F^{\#}(x) > \alpha\}d\alpha$$ and we choose γ in such a way that $c\ \gamma^{\delta} < \frac{1}{2}$.

We shall present now a proof of inequality [1].

Let us consider the family $\{Q_j\}$ of dyadic cubes corresponding to the Calderón-Zygmund decomposition

$$\frac{\alpha}{2^{n+1}} \leq F_{Q_j} \leq 2^n \frac{\alpha}{2^{n+1}} = \frac{\alpha}{2}$$

and let us also consider the family $\{Q_\nu^1\}$ corresponding to $\alpha < F_{Q_\nu^{(1)}} \leq 2^n \alpha$. We know thath each $Q_\nu^{(1)}$ is a subcube of some Q_j. It is enough to prove, for each j, the following inequality:

$$\mu\{x \in Q_j : F^*(x) > \alpha, \quad F^{\#}(x) < \gamma\alpha\} \leq c\, \gamma^\delta \mu(Q_j)$$

Using condition A_∞ it is enough to prove

$$|\{x \in Q_j : F^*(x) > \alpha, \quad F^{\#}(x) < \gamma\alpha\}| \leq c\, \gamma\, |Q_j|$$

To see it, let us take $Q \in \{Q_j\}$, $\{Q_\mu\} \subset \{Q_\nu^1\}$ s.t. $Q_\mu \subset Q$, we have two cases:

(i) If $\frac{1}{|Q|} \int_Q |F-F_Q| > \gamma\alpha$, then $\{x \in Q;\ F^*(x) > \alpha, F^{\#}(x) < \gamma\alpha\}$ is empty and there is nothing to be proved.

(ii) Otherwise we have,

$$\gamma\alpha \geq \frac{1}{|Q|} \int_Q |F-F_Q| \geq \frac{1}{|Q|} \sum_\mu \int_{Q_\mu} |F(y) - F_Q|\, dy =$$

$$= \frac{1}{|Q|} \sum_\mu |Q_\mu|\, |F_{Q_\mu} - F_Q| \geq \frac{\alpha}{2} \sum_\mu \frac{|Q_\mu|}{|Q|}$$

which implies the desired estimate since

$$\bigcup_\mu Q_\mu \supset \{x \in Q : F^*(x) > \alpha\}$$

q.e.d.

We shall consider certain applications of the theory which follow from the following observation.

Lemma. For every $r > 1$ and locally integrable function g the function $A_r(g) = \left[(g^r)^*\right]^{1/r}$ satisfies condition A_1 with bounds independent of g. (Here $*$ denotes the dyadic Hardy-Littlewood maximal function).

Proof.

Let Q be a dyadic cube containing the point x and let us denote by $h \longrightarrow h^+$ the maximal function "restricted" to dyadic subcubes of Q. Then the mapping $f \longrightarrow f^+$ is bounded from $L^1(Q)$ to $L^p(Q)$ $0 < p < 1$.

We have a decomposition $Q = E \cup (Q - E)$ where

$$(g^r)^* / E = (g^r)^+ / E$$

$$(g^r)^* / Q - E = \text{constant}.$$

Therefore,

$$\frac{1}{|Q|}\int_Q |(g^r)^*(y)|^{1/r}\, dy = \frac{1}{|Q|}\int_E |(g^r)^+(y)|^{1/r}\, dy +$$

$$+ \frac{1}{|Q|}\int_{Q-E} |(g^r)^*(y)|^{1/r}\, dy \leq$$

$$\leq C_r \left|\frac{1}{|Q|}\int_Q g^r(y)\, dy\right|^{1/r} + \inf_{y \in Q-E} |(g^r)^*(y)|^{1/r}$$

But observe that: $\sup_{y \in Q-E} (g^r)^*(y) \leq \inf_{z \in E} (g^r)^*(z)$.

Thus

$$\frac{1}{|Q|}\int_Q |(g^r)^*(y)|^{1/r}\,dy \leq \tilde{C}_r \left|\frac{1}{|Q|}\int_Q g^r(y)dy\right|^{1/r} \leq$$

$$\leq \tilde{C}_r\ |(g^r)^*\ (x)|^{1/r}$$

And since this is true for every $Q \ni x$, property A_1 follows.

q.e.d.

Let $T_j f = K_j * f$ be a family of singular integrals with uniform bounds i.e. $|\hat{K}_j(\xi)| \leq c$, $|\nabla K_j(x)| \leq c\ |x|^{-n-1}$ for every j.

Theorem 14. The operator $\{f_j\} \xrightarrow{T} \{T_j\ f_j\}$ has a bounded extension to $L^p(l^r)$, $1 < p < \infty$, $1 < r < \infty$.

Proof. We have to show that

$$\|(\Sigma|T_jf_j|^r)^{1/r}\|_p \leq C_p\ \|(\Sigma|f_j|^r)^{1/r}\|_p$$

for every $\{f_j\}$.

A duality argument shows that if T is bounded on $L^p(l^r)$ then it must also be bounded on $L^q(l^s)$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$. Therefore it is enough to prove the theorem in the case $p \geq r$.

The case $p=r$ follows from the Calderón-Zygmund theorem. If $p>r$ we have,

$$\left\{\int_{\mathbb{R}^n}\left[\Sigma|T_jf_j\ (x)|^r\right]^{p/r} dx\right\}^{r/p} =$$

$$= \sup_{\|g\|_q = 1} \int_{\mathbb{R}^n} \Sigma\,|T_jf_j\ (x)|^r\ g(x)dx, \qquad \text{where } \frac{r}{p}+\frac{1}{q} = 1.$$

and

$$\sum_j \int_{\mathbb{R}^n} |T_j f_j(x)|^r \, g(x)dx \leq \sum_j \int_{\mathbb{R}^n} |T_j f_j(x)|^r \, A_s g(x)dx \leq$$

$$\leq C_{r,s} \int_{\mathbb{R}^n} \sum |f_j(x)|^r \, A_s g(x)dx \leq$$

$$\leq C_{r,s} \left\{ \int_{\mathbb{R}^n} \left| \sum |f_j(x)|^r \right|^{p/r} dx \right\}^{r/p} \left| \int_{\mathbb{R}^n} |A_s \, g(x)|^q \, dx \right|^{1/q}$$

Choosing $s > 1$ so that $\frac{q}{s} > 1$ we have

$$\left| \int_{\mathbb{R}^n} |A_s \, g(x)|^q \, dx \right|^{1/q} = \left| \int_{\mathbb{R}^n} |(g^s)^* \, (x)|^{q/s} \, dx \right|^{1/q} \leq$$

$$\leq c_s \, \|g\|_q = c_s$$

q.e.d.

[III] Beyond The Calderón-Zygmund Theory.

Our present knowledge of multipliers whose Kernels have more complicated sets of singularities than the Calderón-Zygmund's Kernels is rather rudimentary. Here we shall discuss two examples related with the spherical summation of multiple Fourier series one and with properties of trigonometric series with gaps the other.

[1] The Spherical Summation Multipliers.

In this section we are concerned with the family of multipliers T_λ, $\lambda \geq 0$, defined on functions of $\mathbb{R}^n$ by the formula

$$\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\hat{f}(\xi), \quad \text{where} \quad m_\lambda(\xi) = \begin{cases} (1-|\xi|^2)^\lambda & \text{if} \quad |\xi| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If λ is bigger than a critical exponent depending of the dimension $(\lambda > \frac{n-1}{2})$, then the Kernel of T_λ is integrable and therefore T_λ is bounded on each $L^p(\mathbb{R}^n)$. The problem arises when we consider $\lambda \leq \frac{n-1}{2}$. We can summarize the state of affairs in the following known facts:

1°) T_λ is unbounded outside the range

$$P(\lambda) = \frac{2n}{n+1+2\lambda} < p < \frac{2n}{n-1-2\lambda} = p'(\lambda)$$

2°) T_o is never bounded on $L^p(\mathbb{R}^n)$ except for the obvious cases $n=1$ or $p=2$.

3°) In $\mathbb{R}^2$, T_λ is bounded on $L^p(\mathbb{R}^2)$ whenever $\lambda > 0$ and $p(\lambda) < p < p'(\lambda)$.

4°) T_λ is bounded on $L^p(\mathbb{R}^n)$ provided that $p(\lambda) < p < p'(\lambda)$ and $\lambda > \frac{n-1}{2(n+1)}$.

To prove the first observation it is enough to compute the kernel of T_λ and this is done by using Bessel's functions (see Stein and Weiss [1], Theorem 4.15).

$$K_\lambda(x) = \hat{m}_\lambda(x) = \pi^{-\lambda}\,\Gamma(1+\lambda)\,|x|^{-\frac{n}{2}-\lambda}\,J_{\frac{n}{2}+\lambda}(2\pi|x|)$$

Therefore $K_\lambda \in L^p(\mathbb{R}^n)$ if and only if $p > \frac{2n}{n+1+2\lambda}$.

Let ϕ be a smooth function with support in $|\xi| \leq 2$ and such that $\phi(\xi) = 1$ for $|\xi| \leq 1$, then $T_\lambda\hat{\phi} = K_\lambda$, which implies that T_λ cannot be bounded outside $p(\lambda) < p < p'(\lambda)$.

Theorem 15. T_o is only bounded on $L^2(\mathbb{R}^n)$ $(n > 1)$.

It is enough to show that T_o is not bounded on $L^p(\mathbb{R}^2)$ $(p \neq 2)$, because L^p-boundedness of T_o on $\mathbb{R}^n$ implies boundedness on $\mathbb{R}^{n-1}$ by Theorem 4.

(The Proof. will follows by contradiction with the following chain of lemmas).

Assume that $\|Tf\|_p \leq C\,\|f\|_p$ where $p > 2$. From this we have,

Lemma 1. Let $\{v_j\}$ be a sequence of unit vectors in $\mathbb{R}^2$, and let H_j be the half-plane $\{x \in \mathbb{R}^2 \;\; x.v_j \geq 0\}$. Define a sequence of operators $\{T_j\}$ on $L^p(\mathbb{R}^2)$ by setting

$\widehat{T_j f}(\xi) = \chi_{H_j}(\xi)\hat{f}(\xi)$. Then for any sequence of functions $\{f_j\}$ the following inequality holds:

$$\|(\sum_j |T_j f_j|^2)^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p$$

<u>Proof</u>. It is enough to show the inequality when $\{f_j\}$ are smooth function with compact support.

Let $\widehat{T_j^r f}(\xi) = \chi_{D_j^r}(\xi)\, \hat{f}(\xi)$ where D_j^r is the disc of radius r and center rv_j, then $T_j f(x) = \lim_{r\to\infty} T_j^r f(x)$.

Therefore:

$$\|(\sum_j |T_j f_j|^2)^{1/2}\|_p \leq \liminf_{r\to\infty} \|(\sum_j |T_j^r f_j|^2)^{1/2}\|_p$$

So it is enough to prove that:

$$[1] \quad \|(\sum_j |T_j^r f_j|^2)^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p$$

with C independent of r.

Next we observe that it is enough to show [1] when $r=1$, and that

$$T_j^1 f_j(x) = e^{iv_j . x}\, T(e^{-iv_j . y} f_j(y))$$

So that the left-hand side of [1] is nothing but

$$\|(\sum_j |T(e^{iv_j . y} f_j(y))|^2)^{1/2}\|_p$$

But since T is bounded on $L^p(\mathbb{R}^n)$, then T is bounded in $L^p(l^2)$; that is

$$\|(\sum_j |T(e^{iv_j . y} f_j(y)|^2)^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p$$

Q.e.d.

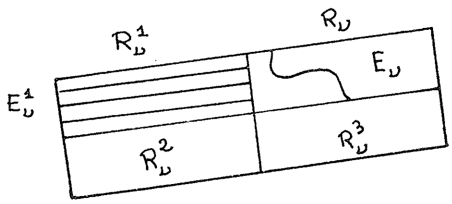
Suppose that $\{R_\nu\}$ is a sequence of rectangles whose directions lie in the set $\{v_j\}$ and satisfying the following property:

$$\forall \nu, \qquad |R_\nu \cap \bigcup_{\mu<\nu} R_\mu| \leq \frac{1}{2} |R_\nu|$$

then, under the hypothesis of the previous lemma, we have:

<u>Lemma 2</u>. $\|\sum \chi_{R_\nu}\|_{p/2} \leq \tilde{C} |\bigcup R_\nu|^{2/p}$, where $\tilde{C}$ is a constant depending only upon C $(\tilde{c} \leq 2c^4)$.

<u>Proof</u>. Consider $E_\nu = R_\nu - \bigcup_{\mu<\nu} R_\mu$, then $|E_\nu| \geq \frac{1}{2} |R_\nu|$, by hypothesis, which implies that



$|H_{i_\nu} \chi_{E_\nu}(x)| \geq \frac{1}{100}$ on E_ν^1 where $|E_\nu^1| \geq \frac{1}{2} |R_\nu^1|$ and E_ν^1 is a union of straight line segments (see the figure). If we denote $H_{i_\nu}^1$ the "Hilbert transform" in the perpendicular direction, we obtain,

$$\chi_{R_\nu}(x) \leq (100)^4 \left|H_{i_\nu}^1\{\chi_{R_\nu}^3 . H_{i_\nu}\left[\chi_{R_\nu}^2 . H_{i_\nu}^1 (\chi_{R_\nu}^1 . H_{i_\nu} \chi_{E_\nu})\right]\}\right|$$

Therefore we can invoque lemma 1 to conclude the proof.

<u>Lemma 3</u>. For each integer $N \geq 1$ there exists a family of rectangles $\{R_j\}_{j=1,...N}$ with the following properties:

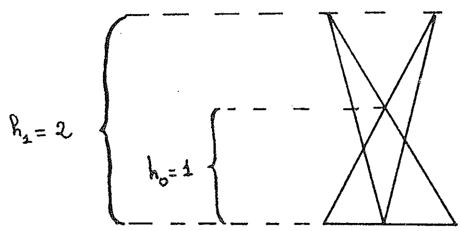
(1°) $\mu\{\bigcup_{s=1}^{N} R_j\} \leq \log N$

(2°) $\forall j$, $\mu\{R_j^* \cap \bigcup_{k\neq j} R_k^*\} \leq \frac{1}{2}\,\mu\{R_j^*\}$, where R_j^* is a rectangle which contanis R_j, and such that $\mu(R_j^*) = 2\mu(R_j)$ for each j. Furthermore $\mu\{\bigcup R_j^*\} \simeq N$.

Proof.

(a) We start with a triangle Δ_o of base with length equal to 1 and height $h_o = 1$, then we "Sprout" Δ_o to height $h_1 = 2$ to get the tree P_1 composed of two triangles Δ_1^1, Δ_1^2 (as in the figure). We have the estimate

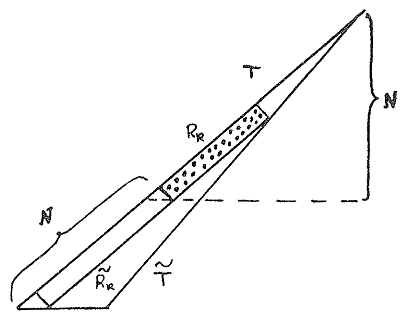
$$\mu\{P_1\} \leq \mu\{\Delta_o\} + 4 \,.\, \frac{1}{4}\,\mu\{\Delta_o\} = 2\mu\{\Delta_o\}$$



(b) By iteration of the process we can get a tree P_N composed of 2^N triangles of height $h_N = N$ and base 2^{-N}. Furthermore $\mu\{P_N\} \leq \mu\{\Delta_o\}\ \{1 + 2\ (\frac{1}{2} + \frac{1}{3} + \text{----} + \frac{1}{N}\} \cong \log N$.

(c) Now for every triangle T on the tree P_k we consider the region $\tilde{T}$ and the rectangles R_k and $\tilde{R}_k$ (see the figure),

$$R_k^* = R_k \cup \tilde{R}_k$$



And the point is that the regions T corresponding to the different triangles of the tree are pairwise disjoints.

Proof of the theorem.

Assuming that T is a bounded multiplier on $L^p(\mathbb{R}^2)$, $p > 2$, we have lemmas 1, and 2 for each family of direccions. In particular for the families $\{R_j^*\}$ of lemma 3.

Now,

$$N \simeq \mu\{\bigcup R_j^*\} \leq 2 \int \left[\sum \chi_{R_j^*}(x)\right] dx \leq$$

$$\leq C \quad \mu\{P_N\}^{1/q} \quad \mu\{\bigcup R_j^*\}^{2/p} \quad , \quad 1/q + 2/p = 1$$

Therefore.

$$N \leq C(\log N)^{1/q} \; N^{2/p}, \quad \text{(with a universal constant } C)$$

which it is false.

We shall discuss now the case of T_λ, $\lambda > 0$ on $L^p(\mathbb{R}^2)$.

The multiplier $m_\lambda(\xi) = (1 - |\xi|^2)_+^\lambda$ seems very complicated and one of our first tasks is to find out which are the basic blocks of the Calderón-Zygmund theory corresponding to m_λ. Since m_λ is radial and basically constant on thin annulae it seems reasonable to decompose

$$m_\lambda(\xi) = \sum_{o}^{\infty} (1 - |\xi|^2)_+^\lambda \quad \phi_k(|\xi|)$$

where ϕ_k, $k \geq 1$, is a smooth function supported in the interval $[1-2^{-k},\ 1-2^{-k-2}]$ such that $|D^\alpha \phi_k| \leq C_\alpha\, 2^{k\alpha}$, C_α independent of k, and $\sum_{k=1}^{\infty} \phi_k \equiv 1$ on $[1/2,\ 1]$, $\phi_o = 1 - \sum_{k=1}^{\infty} \phi_k$.

Then $\quad (1 - |\xi|^2)_+^\lambda\, \phi_k(|\xi|) \simeq 2^{-k\lambda}\, \phi_k(|\xi|)$.

and the problem is reduced to get good estimates for the growth, as $k \to \infty$, of the norm of the multipliers associated to the functions $\phi_k(|\xi|)$. For example, the result for T_λ, $\lambda > o$, will follows very easily if one can show that the operador

$$\widehat{T^k f}(\xi) = \phi_k(|\xi|)\ \hat{f}(\xi) \qquad \text{satisfies the}$$

inequality

$$\|T^k f\|_4 \leq C\ k^{1/4}\ \|f\|_4, \qquad \forall f \in \mathcal{S}(\mathbb{R}^2)$$

because then interpolation with the obvious estimate

$$\|T^k f\|_\infty \leq c\ 2^{k/2}\ \|f\|_\infty$$

yields, $\|T^k f\|_p \leq C\ k^{t/4}\ 2^{\frac{k(1-t)}{2}}\ \|f\|_p$

if $\frac{1}{p} = \frac{t}{4}$ and $4 < p < \frac{4}{1-2\lambda}$ then $1-t < 2\lambda$ and, therefore, the series $\sum_{k=1}^{\infty} 2^{-k\lambda}\ k^{1/4}\ 2^{\frac{k(1-t)}{2}}$ converges, proving the theorem.

Statement and Proofs of some Results

(a) Suppose that $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function supported in $[-1, +1]$ and consider the family of Fourier multipliers S_δ, where $\delta > 0$ is small, defined by the formula

$$\widehat{S_\delta f}(\xi) = \phi(\delta^{-1}(|\xi|-1)) \cdot \hat{f}(\xi)$$ for rapidly decreasing smooth functions f.

Theorem 16. There exists a constant C, independent of δ, such that

$$\|S_\delta f\|_4 \leq C (\log 1/\delta)^{1/4} \|f\|_4$$

for every $f \in \mathcal{S}(\mathbb{R}^2)$.

(b) Given real numbers $N \geq 1$ and $a > 0$, consider the family B of rectangles in $\mathbb{R}^2$ with dimensions a and Na but with arbitrary direction. For a locally integrable function f let us define the maximal function

$$Mf(x) = \sup_{x \in R \in B} \frac{1}{\mu(R)} \int_R |f(x)|\, d\mu(x)$$

where μ denotes Lebesgue measure in the plane.

Theorem 17. There exists a constant C, independent of a and N, such that

$$\|Mf\|_2 \leq C (\log 2N)^{1/2} \|f\|_2, \quad \text{for every } f \in L^2(\mathbb{R}^2)$$

(c) Given two positive real numbers N_1 and N_2 let us consider the family of intervals in the y-axis, $\{I_j\}_{-\infty<j\leq+\infty}$ whose length is equal to N_1 and such that the distance between two consequtive intervals is equal to N_2.

Denote by E_j the horizontal strip in $\mathbb{R}^2$ whose projection over the y-axis is the interval I_j.

Given $f \in \mathcal{S}(\mathbb{R}^2)$ we may define the g-function

$$g(f)(x) = \Big(\sum_j |P_j f(x)|^2\Big)^{1/2}, \quad \text{where} \quad \widehat{P_j f}(\xi) = \chi_{E_j}(\xi)\hat{f}(\xi).$$

Theorem 18. For every $p \geq 2$ there exists a constant C_p such that

$$\|g(f)\|_p \leq C_p \|f\|_p, \qquad \forall f \in \mathcal{S}(\mathbb{R}^2).$$

(d) Corollary. The operator T_λ, $1/2 \geq \lambda > 0$, defined by $T_\lambda f(\xi) = (1-|\xi|^2)_+^\lambda \hat{f}(\xi)$ for rapidly decreasing and smooth functions f, has a bounded extension to $L^p(\mathbb{R}^2)$ if and only if $\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda}$.

Proofs

(a) Using smooth cut off functions we may decompose

$$\phi = \sum_{j=o}^{3} \phi_j, \quad \text{where} \quad \operatorname{supp}(\phi_j) \quad \{z : -\frac{\pi}{3} + \frac{j\pi}{2} \leq \arg(z) < \frac{\pi}{3} + \frac{j\pi}{2}\}$$

Since we may consider ϕ_j, $j=1,2,3$, as a rotation of ϕ_o and the Fourier transform commutes with rotations, it is enough to prove that the multiplier ϕ_o satisfies the estimate of theorem 1.

Next we consider a smooth partition of unity $\{\Phi_j\}_{j=1,\dots,[\delta^{-1/2}]}$ in the unit circle, such that:

$$\Phi_j(\theta) = \Phi\Big(\delta^{-1/2}\Big(\theta - \frac{2\pi j}{|\delta^{-1/2}|}\Big)\Big) \quad \text{where} \quad \Phi \text{ is a smooth}$$

function, supported on the unit interval, with bounds independent of δ.

Therefore $\phi_o(\xi) = \phi_o(|\xi|\ e^{i\theta}) = \sum_j \phi_o(\xi)\ \Phi_j(\theta) = \sum_j m_j(\xi)$.

If we define $\widehat{T_j f}(\xi) = m_j(\xi)\ \hat{f}(\xi)$, then we have

$$\int_{\mathbb{R}^2} |\sum_j T_j f(x)|^4 dx = \int_{\mathbb{R}^2} |\sum_{jk} T_j f(x) . T_k f(x)|^2\ dx =$$

$$= \int_{\mathbb{R}^2} |\sum_{j,k} \widehat{T_j f} * \widehat{T_k f}(\xi)|^2\ d\xi \leq C \sum_{j,k} \int_{\mathbb{R}^2} |\widehat{T_j f} * \widehat{T_k f}(\xi)|^2\ d\xi$$

This last inequality holds because no point belongs to more that 64 sets of the family supp $(\widehat{T_j f})$ + supp $(\widehat{T_k f})$.

Therefore

$$\|\sum_j T_j f\|_4 \leq c\ \|(\sum_j |T_j f(x)|^2)^{1/2}\|_4$$

with c independent of δ.

Next we split the sum $(\sum_j |T_j f(x)|^2)^{1/2} \leq (\sum_j |T_{2j} f(x)|^2)^{1/2} +$ $+ (\sum_j |T_{2j+1} f(x)|^2)^{1/2}$ and we estimate each one of then.

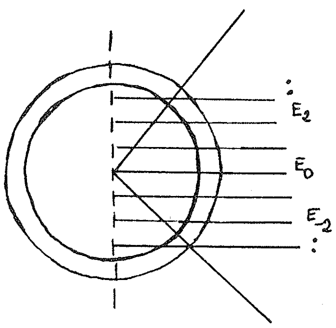


Fig. 1

Since the supports of the multipliers m_j are bassically perpendicular to the direction of the x-axis, we may find a configuration of horizontal strips, like in theorem 3, $\{E_{2j}\}$ with the property that

$$\chi_{E_{2j}}(\xi) \cdot m_{2j}(\xi) = m_{2j}(\xi)$$

Therefore $(\sum |T_{2j}f(x)|^2)^{1/2} = (\sum |T_{2j} P_{2j} f(x)|^2)^{1/2}$ where $\widehat{P_{2j}h}(\xi) = \chi_{E_{2j}}(\xi)\, \hat{h}(\xi)$.

An elementary integration by parts computation shows that the kernel K_o of T_o satisfies the following: for every pair of integers $p,q \geq 0$ there exists a finite constant $C_{p,q}$, independent of δ, such that

$$|K_o(x,y)| \leq C_{p,q}\, \delta^{3/2}\, |\delta x|^{-p}\, |\delta^{1/2}y|^{-q}$$

Therefore the operator T_o is majorized by the positive operator whose kernel is given by

$$c\left\{\sum_{n=o}^{\infty} 2^{-n} \frac{1}{\mu(R_n^o)} \chi_{R_n^o}(x,y)\right\}$$

where $R_n^o = \{(x,y) : |x| \leq 2^n \delta^{-1},\ |y| \leq 2^n \delta^{-1/2}\}$ for a suitable constant C independent of $\delta > 0$. By an appropiate rotation we may get an analogous majorization for every operator T_j.

Due to the exponential decay 2^{-n} it is enough to show that, for each n, the L^4-norm of the function

$$(\sum_j \left|\frac{1}{\mu(R_n^{2j})} \chi_{R_n^{2j}} * P_{2j}f(x)\right|^2)^{1/2} \quad \text{is dominated}$$

by $c(\log \delta)^{1/4}\ \|f\|_4$, for every $f \in L^4(\mathbb{R}^2)$.

Given $\omega \geq 0$ in $L^2(\mathbb{R}^2)$ we have

$$\sum_j \int_{\mathbb{R}^2} \left|\frac{1}{\mu(R_n^{2j})} \chi_{R_n^{2j}} * P_{2j}f(x)\right|^2 \omega(x)dx \leq$$

$$\leq \sum_j \int_{\mathbb{R}^2} |P_{2j}f(y)|^2 \frac{1}{\mu(R_n^{2j})} \chi_{R_n^{2j}} * \omega(y)dy \leq$$

$$\leq \sum_j \int_{\mathbb{R}^2} |P_{2j}f(y)|^2 \, M_n\omega(y)dy \leq \|(\sum_j |P_{2j}f|^2)^{1/2}\|_4^2 \cdot \|M_n\omega\|_2$$

where M_n is the maximal function of Theorem 2 with

$$N = \delta^{-1/2} \quad \text{and} \quad a = 2^n$$

Therefore $\|(\sum|T_{2j}f|^2)^{1/2}\|_4^2 \leq \sup\limits_{\|\omega\|_2 \leq 1} \int \sum\limits_j |T_{2j}f(x)|^2 \, \omega(x)dx$

$$\leq c \sum_n 2^{-n} (\log 1/\delta)^{1/2} \|(\sum_j |P_{2j}f|^2)^{1/2}\|_4^2 \leq \tilde{c} \, (\log 1/\delta)^{1/4} \, \|f\|_4^2$$

by Theorems 2 and 3.

An analogous argument works for the odd sum.

q.e.d.

(b) The exponent $1/2$ in Theorem 17 can not be improved as the case $a=1$, $f(x) = (1+|x|)^{-1}$ il $|x| \leq N$ and $f(x) = 0$ il $|x| > N$ shows.

Proof of Theorem 17.

First of all it is enough to prove the estimate for rectangles whose direction lies in the interval $[0, \pi/4]$. By using a convenient dilation we may also assume that $a=1$. We divide the plane, by vertical and horizontal lines, into a grid of squares of side N. The operador M acts "indepently" on the squares of the grid and therefore we can simplify the problem by considering only functions f supported on one of the squares of the grid. So let Q be a square with sides parallel to the coordinate axis and length = N and suppose that $f \in L^2(Q)$.

Then $Mf(x) = 0$ if $x \notin Q^{*}$ (1).

We decompose the square Q^* into $9N^2$ small squares $\{Q_{ip}\}$ of side = 1, by vertical and horizontal lines. The point is that for every square Q_{ip} one can find a rectangle R_{ip} (of direction in the interval $[0,\pi/4]$ and dimensions $1 \times N$) such that

$$Q_{ip} \cap R_{ip} \neq \emptyset \quad \text{and} \quad Mf(x) \leq 2 \frac{1}{|R_{ip}|} \int_{R_{ip}} |f(y)|\,dy \cdot \chi_{Q_{ip}}(x)$$

Therefore, if for f fixed we define the linear operator:

$$T_f(g)(x) = \sum_{i,p} \frac{1}{|R_{ip}|} \int_{R_{ip}} g(y) \cdot \chi_{Q_{ip}}(x)$$

we have $Mf(x) \leq 2\, T_f(|f|)(x)$ and, in order to prove the theorem, it is enough to prove that $\|T_f(g)\|_2 \leq c\,(\log 3N)^{1/2}\, \|g\|_2$ $\forall g \in L^2(Q^*)$, with C independent of f and N.

Thus we have linearized the problem and we can consider the adjoint of T_f, T_f^*, which is given by:

$$T_f^*(h)(x) = \sum_{i,p} \frac{1}{|R_{ip}|} \left(\int_{Q_{ip}} h(y)\,dy \right) \chi_{R_{ip}}(x)$$

Now given $h \in L^2(Q^*)$ we have the decomposition $h = h_1 + \ldots + h_{3N}$ where $h_i = h/E_i$ is the restriction of h to the vertical strip E_i of width 1. Then, in orden to prove that

$\|T_f^*(h)\|_2 \leq c\,(\log 3N)^{1/2}\, \|h\|_2$ it is enough to show that:

$$\|T_f^*(h_i)\|_2 \leq c\, N^{-1/2}\, (\log 3N)^{1/2}\, \|h_i\|_2, \quad i=1,\ldots,3N$$

because then

---

(1) Q^* denotes the expanded square by the factor 3.

$$\|T_f^*(h)\|_2 \leq \sum_i \|T_f^*(h_i)\|_2 \leq C\ N^{-1/2}\ (\log 3N)^{1/2} \sum_i \|h_i\|_2 \leq$$

$$\leq c\ (\log 3N)^{1/2}\ \|h\|_2 .$$

Suppose that the function h lies on the strip E_i. We decompose E_i into $3N$ squares $\{Q_{ip}\}_{p=1,\dots,3N}$ of side 1 and also we decompose the function $h = \Sigma\, h_p$ where $h_p = h/Q_{ip}$. We have

$$T_f^*(h)(x) = \sum_p T_f^*(h_p)(x) = \sum_p \frac{1}{|R_{ip}|} \int h_p(y)dy\ \chi_{R_{ip}}(x)$$

which implies

$$|T_f^*(h)(x)| \leq \frac{1}{N} \sum_{p=1}^{3N} \|h_p\|_2\ \chi_{R_{ip}}(x)$$

Therefore

$$\int |T_f^*(h)(x)|^2 dx \leq \frac{1}{N^2} \sum_{p,q} \|h_p\|_2\ \|h_p\|_2\ |R_{ip} \cap R_{iq}|$$

and an easy computation shows that $|R_{ip} \cap R_{iq}| \leq C \frac{N}{1+|p-q|}$

which implies

$$\|T_f^*(h)\|_2^2 \leq c\, \frac{1}{N} \sum_{p,q=1}^{3N} \frac{\|h_p\|_2\ \|h_q\|_2}{1+|p-q|} \leq C\ N^{-1} \log 3N .\ \|h\|_2^2$$

q.e.d.

(c) We have used a two dimensional version of Theorem 18 to prove Theorem 16 but that theorem is, bassically, one-dimensional. In the following we sketch the proof and, without loss of generality, we may assume that $N_1=2$, $N_2=N$ and the first interval I_o is centered at the origin: $I_j = (\omega_j-1,\ \omega_j+1)$, where $\omega_j = j(N+1)$.

Let ψ be a smooth function such that $\psi \equiv 1$ on I_o and $\psi \equiv 0$

outside the interval

$$(-1 - \frac{N}{4}, \quad 1 + \frac{N}{4})$$

and let $\psi_j(t) = \psi(t - \omega_j)$ and $\widehat{S_j f}(\xi) = \psi_j(\xi)\ \hat{f}(\xi)$.

<u>Lemma</u>. $\|(\sum_j \ S_j f(x)|^2)^{1/2}\|_p \leq C_p \ \|f\|_p$, for every $p \geq 2$.

<u>Proof</u>.

Consider for every θ, $0 \leq \theta \leq 2\pi$, the multiplier $\widehat{T_\theta f}(\xi) = \sum_j e^{i\theta j}\ \psi_j(\xi)\ \hat{f}(\xi)$ and observe that its kernel μ_θ is a measure of finite total variation, uniformly on θ. Therefore for every θ we have

$$\left(\int |T_\theta f(x)|^p dx\right)^{1/p} \leq C_p \ \|f\|_p$$

We integrate in θ and observe that if $p \geq 2$ then

$$C_p^p \int |f(x)|^p dx \geq \frac{1}{2\pi}\int_0^{2\pi} \left(\int |T_\theta f(x)|^p dx\right) d\theta \geq$$

$$= \int \frac{1}{2\pi}\int_0^{2\pi} |T_\theta f(x)|^p \ d\theta\ dx \geq$$

$$\geq \int \left(\frac{1}{2\pi}\int_0^{2\pi} |T_\theta f(x)|^2 \ d\theta\right)^{p/2} dx = \int \left(\sum_j |S_j f(x)|^2\right)^{p/2} dx$$

q.e.d.

The proof of Theorem 18 is now easy to obtain.

Let us consider now a sharper version theorem 16. Let χ_P denote the characteristic function of a regular polygon of N-sides in $\mathbb{R}^2$ and consider the Fourier multiplier defined by $\widehat{Tf}(\xi) = \chi_P(\xi)\ \hat{f}(\xi)$.

Theorem 19. For every p, $4/3 \leq p \leq 4$, there exist constants C_p, $a(p)$, independent of N, such that $\|Tf\|_p \leq$ $\leq C_p(\log N)^{a(p)} \|f\|_p$ for every $f \in \mathcal{S}(\mathbb{R}^2)$.

This theorem is a consequence of the following maximal function result: define

$$Mf(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)|\,dy$$

where the "sup" is taken over all rectangles in $\mathbb{R}^2$ having sides parallel to one of the sides of the polygon, then,

Theorem 20. There exists a constant C, independent of N, such that

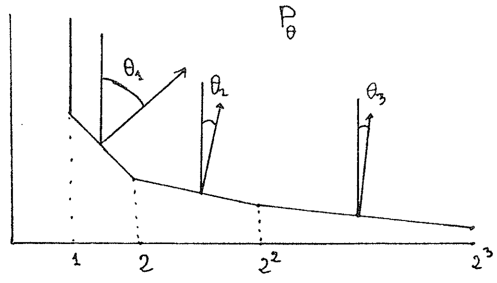
$$\mu\{x : Mf(x) > \alpha\} \leq C(\log N) \frac{\|f\|_2^2}{\alpha^2}$$

These results are sharp and explain the different behavior of S_o and S_λ, $\lambda > 0$, in theorems 15, 16.

Theorem 19 suggests, naturally, the following question: is there a polygonal region D, whose sides have infinitely many directions, such that the operator given by

$$\widehat{Tf}(\xi) = \chi_D(\xi)\,\hat{f}(\xi) \quad \text{is bounded on some} \quad L^p, \quad p \neq 2\ ?$$

Let $\theta_1 > \theta_2 > \dots$ be a decreasing sequence of angles, $0 < \theta_i < \frac{\pi}{2}$, and consider the region P_θ of the figure.



the multiplier $\widehat{T_\theta f}(\xi) = \chi_{P_\theta}(\xi)\,\hat{f}(\xi)$

and the maximal function

$$M_\theta f(x) = \sup_{x \in R \in B_\theta} \frac{1}{|R|}\int_R |f(y)|\,dy$$

Where B_θ = {rectangles of arbitrary eccentricity oriented in one of the directions θ_i}.

<u>Claim</u>. Boundedness properties of M_θ and T_θ are equivalent on a very precise sense.

(A) <u>The boundedness of the operator</u> T_θ.

We shall show now that under the assumption that M_θ is a bounded operator on $L^{(p/2)'}(\mathbb{R}^2)$, we have that T_θ is also a bounded operator on $L^q(\mathbb{R}^2)$, but on the range $q \in (p',p)$.

One of the main ideas here is to invoke the inequality

$$\int |\tilde{f}(x)|^2\,\omega(x)dx \le c_s \int |f(x)|^2\, A_s\,\omega(x)dx$$

where $A_s\,\omega(x) = \left[(\omega^s)^*(x)\right]^{1/s}$ and $*$ denotes the Hardy Littlewood maximal function.

With $E_k = \{(x,y) \in \mathbb{R}^2,\quad 2^k \leq x \leq 2^{k+1}\}$ let us consider the operators

$$\widehat{S_k f}(\xi) = \chi_{E_k}(\xi)\,\hat{f}(\xi)$$

Then we can use the Littlewood-Paley theory to obtain:

$$\|Tf\|_p \simeq \|(\sum |S_k\,Tf|^2)^{1/2}\|_p$$

But if H_k is the multiplier operator corresponding to the half-plane F_k tangent to P_θ along its kth side, we have,

$$S_k\,Tf = H_k\,Tf$$

Therefore

$$\|\sum |H_k\,S_k\,f|^2\|_{p/2}^{p/2} =$$

$$= \sup_{\|\omega\|_{L^{(p/2)'}} \leq 1} \sum_k \int |H_k S_k\,f(x)|^2\,\omega(x)dx \leq$$

$$\leq C \sup_{\omega} \sum_k \int |S_k f|^2\,\omega^*(x)dx \leq C_p\,\|f\|_p^p$$

where $\omega^* = \sup_k \left[m_k(\omega^{1+\varepsilon})\right]^{1/1+\varepsilon}$ and m_k denotes the Hardy-Littlewood maximal function in the direction θ_k.

q.e.d.

(B) We shall show now that if T_θ is bounded on L^p, $p > 2$, then M_θ is of weak type $((p/2)', (p/2)')$, modulo some tauberian condition.

First we show that assuming that T_θ is bounded on $L^p(\mathbb{R}^2)$ we have the following Meyer's Lemma.

$$\|(\sum_k |H_k f_k|^2)^{1/2}\|_p \leq C_p \|(\sum_k |f_k|^2)^{1/2}\|_p$$

where H_k represents the Hilbert transform in the direction θ_k.

To see that it is enough to work with finite collections of smooth functions $f_1,\dots,f_N$ and we may also assume that $\hat{f}_j$ have compact support. Then we look for estimates with constants C_p independent of these assumptions.

Let us expand P_θ by a convenient factor ρ so that

$$\sup_j (\text{diam}(\text{supp of } \hat{f}_j)) \leq \rho/2$$

Then for every $j=1,\dots,N$ there exists ω_j so that

$$\widehat{H_j f_j}(\xi) = \chi_{P_\theta^\rho}(\xi+\omega_j) \cdot \hat{f}_j(\xi) =$$

$$= \chi_{P_\theta^\rho}(\xi+\omega_j) \quad \widehat{e^{-i\omega_j \cdot x} f_j}(\xi+\omega_j) \implies$$

$$H_j f_j(x) = e^{i\omega_j \cdot x} \, T_\theta^\rho(e^{-i\omega_j \cdot x} f_j)$$

where T_θ^ρ is the multiplier associated to the characteristic function of P_θ^ρ.

Therefore

$$\|(\sum_k |H_k f_k|^2)^{1/2}\|_p = \|(\sum_j |T_\theta^\rho(e^{-i\omega_j \cdot x} f_j)|^2)^{1/2}\|_p \leq$$

$$\leq C_p \|(\sum_j |f_j|^2)^{1/2}\|_p$$

q.e.d.

Suppose now that $\{R_k\}$ is a collection of rectangles in B_θ satisfying the following property

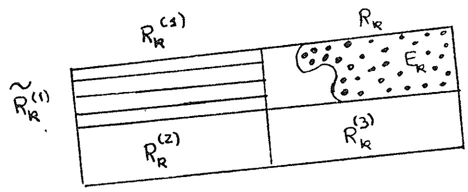
$$(P) \qquad \forall k, \qquad |R_k \cap \bigcup_{j<k} R_j| \leq \frac{1}{2} |R_k|$$

Then the following estimate must be true

$$[V_i] \qquad \|\sum \chi_{R_k}\|_{p/2} \leq C_p \quad |\bigcup R_k|^{2/p}$$

<u>Proof</u>. Consider $E_k = R_k - \bigcup_{j<k} R_j$ then $|E_k| \geq \frac{1}{2} |R_k|$ by hypothesis (P), which implies that

$$|H_{i_k} \chi_{E_k}(x)| \geq 1/100 \quad \text{on} \quad \tilde{R}_k^{(1)} \quad \text{(see figure)}$$



where H_{i_k} denotes the Hilbert transform in the direction of R_k.

If we denote by $H^1_{i_k}$ the Hilbert transform in the perpendicular direction, we obtain

$$\chi_{R_k}(x) \leq (100)^4 \{H^1_{i_k} \chi_{R_k^{(3)}} [H_{i_k} \chi^2_{R_k} (H^1_{i_k} \chi^1_{R_k} (H_{i_k} \chi_{E_k}))]\}(x)$$

Therefore we can invoke the Meyer's lemma to conclude the proof of the covering lemma $[Vi]$.

Suppose, for example, that we know that M_θ verifies the following estimate: $|\{M_\theta \chi_E(x) > 1/2\}| \leq c\, |E|$ for every open set E. Then property $[Vi]$ will imply that M_θ is of weak type $(p/2)'$ automatically.

Therefore, it is an interesting question to decide for which families $\{\theta_k\}$ the operators M_θ and T_θ are bounded on some L^p-spaces. Is there any geometric characterization of good sets of directions?. Bassically, our present knowledge is contained in the following examples:

(1°) If the sequence $\{\theta_k\}$ is lacunar then T_θ is bounded on every L^p-space, $1 < p < \infty$, and M_θ on every L^p, $p > 1$.

(2°) If $\theta_k \simeq k^{-n}$, $(k=1,2,....)$ then T_θ is bounded only on $L^2(\mathbb{R}^2)$ and M_θ in $L^\infty(\mathbb{R}^2)$. Here the ennemy is again the Kakeya set.

Theorem 20. If $\{\theta_k\}$ is lacunar $\left(\frac{\theta_k}{\theta_{k+1}} > \rho > 1\right)$ then T_θ is bounded on every L^p, $1 < p < \infty$ and M_θ is bounded on L^p, $p > 1$. (see [14]).

This theorem can be used to prove the following version of the Littlewood-Paley Theorem 9:

Let $\theta_1 > \theta_2 > \ldots > \theta_k > \ldots.$ be a lacunar sequence with $\lim_{k\to\infty} \theta_k = 0$. Consider σ_k to be the characteristic function of the angle (θ_k, θ_{k-1}) in $\mathbb{R}^2$ and $\sigma_o = 1 - \sum \sigma_k$.

Define $\widehat{S_k f}(\xi) = \sigma_k(\xi)\ \hat{f}(\xi)$.

Theorem 21. For each p, $1 < p < \infty$, there exist finite constants $0 < A_p$, $B_p < \infty$ such that

$$A_p \|f\|_p \leq \|(\sum |S_k f|^2)^{1/2}\|_p \leq B_p \|f\|_p,$$

for every $f \in \mathcal{S}(\mathbb{R}^2)$.

Proof.

For each k let $\tilde{\sigma}_k$ be a smooth function, homogeneous of degree zero and such that $\tilde{\sigma}_k . \sigma_k = \sigma_k$,

$$\operatorname{supp}(\tilde{\sigma}_k) \subset \operatorname{supp}(\sigma_{k-1}) \cup \operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1})$$

$\left| \frac{\partial}{\partial \theta} (\tilde{\sigma}_k)(e^{i\theta}) \right| \leq 2^{k+2}$. Let us denote by $\{r_n(t)\}$ the orthonormal system of Rademacher's functions on the interval $[0,1]$. For each $t \in [0,1]$ the multiplier $m_t(\xi) = \sum_{n=1}^{\infty} r_n(t) \tilde{\sigma}_n(\xi)$ satisfies the hypothesis of Theorem 10 uniformly on t. Therefore if

$$\widehat{T_t f}(\xi) = m_t(\xi)\, \hat{f}(\xi), \qquad \widehat{\tilde{S}_k f}(\xi) = \tilde{\sigma}_k(\xi)\, \hat{f}(\xi),$$

we have:

$$\int |T_t f(x)|^p\, dx \leq C \;\|f\|_p^p$$

integrating on t we get

$$\int \left(\sum |\tilde{S}_k f(x)|^2\right)^{p/2} dx = \int_0^1 \int |T_t f(x)|^p\, dx\, dt \leq C \;\|f\|_p^p$$

Finally, let us observe that

$$\left(\sum |S_k f(x)|^2\right)^{1/2} = \left(\sum |S_k \tilde{S}_k f(x)|^2\right)^{1/2} \quad \text{and}$$

$$\int \sum_k |S_k f(x)|^2\, \omega(x)dx = \sum_k \int |S_k \tilde{S}_k f(x)|^2\, \omega(x) dx \leq$$

$$\leq C_s \sum_k \int |\tilde{S}_k f(x)|^2 \;|M_\theta(\omega^s)(x)|^{1/s}\, dx.$$

And we are in conditions to apply the previous observation together with theorem 20.

Remarks. Recently A. Ruiz [21] has obtained versions of Theorems 15 and 16 for more general types of curves enabling him to give a negative answer to a problem of N. Riviere: The fundamental solution, given by the multiplier $m(x,y) = \frac{1}{x^2-y+i}$, of the Schrödinger operator is not a Fourier multiplier of $L^p(\mathbb{R}^2)$, $p \neq 2$. This result have been obtained indepently by C. Kenig and P. Tomas [20] by a sligthly different method.

The best estimates known for the Bochner-Riesz operators in higher dimensions follow fron the (L^2,L^p) restriction theorem proven by P. Tomas [19].

2 Trigonometric Series with gaps.

The history of lacunary Fourier series goes back to Weierstrass and Hadamard if not to Riemann. Here we shall be interested in the following type of statement: suppose that $f \in L^1(0,1)$ has a Fourier series of the form $f \sim \Sigma\, a_\nu\, e^{2\pi i\, n_\nu . x}$ where $n_\nu / n_{\nu-1} \geq \rho > 1$, then $f \in L^p(0,1)$ for every $p < \infty$ (In fact more is true: $\exp(f^2)$ is integrable and, of course, $f \in$ B.M.O.). This phenomenon may have different explanations: the arithmetical properties of such sequences $\{n_\nu\}$; the fact that the functions $\exp(2\pi i\, n_\nu . x)$ behave very similarly to independent random variables, or from the point of view of the Calderón-Zygmund theory.

Consider the multiplier defined by $m(k) = 1$ if $k \in \{n_\nu\}$ and $m(k) = 0$ otherwise: $Tg \sim \Sigma\, m(k)\, \hat{g}(k)\, e^{2\pi ikx}$. Then the preceding statement is equivalent to the following multiplier theorem: T is a bounded operator from L^2 to B.M.O.

Given an integer $k \geq 1$ let us consider Fourier series of the form:

$$f \sim \sum a_n\, e^{2\pi i n^k . x}$$

It is easy to see that f does not have to be, in general, in BMO if $f \in L^1$. In fact for each $p > 2k$ there is a sequence $\{a_n\}$ such that $\sum |a_n|^2 < \infty$ and $\sum a_n\, e^{2\pi i n^k x}$ is not in $L^p(0,1)$.

To see this let us consider the multiplier $Tf \sim \Sigma\, m(\nu)\hat{f}(\nu) e^{2\pi i\nu x}$, where $m(\nu) = 1$ if $\nu \in \{n^k\}$ and $m(\nu) = 0$ otherwise. We want to show that T is not bounded from L^2 to L^p, $p > 2k$. By duality it is enough to show that T is unbounded from $L^{p'}$ to L^2, $1/p + 1/p' = 1$.

Consider the series $f_\beta(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi inx}}{n^\beta}$, $0 < \beta < 1$. By the Poisson's summation formula (see [15]) we know that $f_\beta(x) \simeq C_\beta\, x^{\beta-1}$, $x \to 0$. If we take $\beta = \frac{1}{2k}$ we see that $f_\beta \in L^r[0, 1]$ for every $r < (1 - \frac{1}{2k})^{-1} = \frac{2k}{2k-1}$. Nevertheles $Tf_\beta \sim \sum_{n=1}^{\infty} \frac{2^{2\pi in^k x}}{n^{1/2}}$ is not in $L^2(0, 1)$.

In the case $k=2$ we can see that even the end point $L^4(0,2\pi)$ is not allowed by observing (see [16]) that $\sum_1^N r(k)^2 = C\, N^2 \log N + o(N^2 \log N)$, where $r(k)$ denotes the number of representation of the integer k as sume of two squares. The following conjecture had been made long ago:

<u>Conjecture</u>. If f has Fourier series of the form $f \sim \sum a_n\, e^{2\pi in^k x}$ then $\|f\|_p \sim (\sum |a_n|^2)^{1/2}$ for $p < 2k$.

Here we shall prove the following result.

<u>Theorem</u>. Suppose that $\{a_n\}$ is monotonically decreasing and $f \sim \sum a_n\, e^{2\pi in^2 x}$ is in $L^1(0, 1)$, then it is in $L^p(0, 1)$ for each $p < 4$, and $\|f\|_p \simeq \|f\|_2$.

In particular we have,

$$W_\varepsilon(x) = \sum_{n\geq 2} \frac{e^{2\pi in^2 . x}}{n^{1/2}(\log n)^{1/2+\varepsilon}}, \quad \varepsilon > 0, \text{ is in } L^p(0, 1)$$

for each $p < 4$.

The proof will be based on the following lemmas:

<u>Lemma 1</u>. Let f be continuous on $[0,1]$ and satisfying

(i) $\|f\|_\infty \leq N, \quad \|f\|_2 \leq N^{1/2}$.

(ii) If $|x - \frac{p}{q}| \leq \frac{1}{q^2}$ then $|f(x)| \leq q^{1/k} + \frac{N}{q^{1/k}}$

Then f is in Weak (L^{2k}) and $\|f\|_{WL^{2k}} \leq C\, N^{1/2}$ for some universal constant C.

Proof. Want to show that $\forall\alpha < 0$

$$\mu\{E_\alpha\} = \mu\{|f(x)| \geq 2N^{1/2}\alpha\} \leq C^2 \frac{1}{\alpha^{2k}}$$

with C independent of N.

* Observe that it is enough to show it for $\frac{N^{1/2}}{2} \geq \alpha \geq 1$

Given $x \in [0,1)$ let

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

be its continuous fraction development and denote by $\{\frac{P_\nu}{Q_\nu}\}$ the sequence of convergents.

If $x \in E_\alpha$ is irrational the following is true for every $\frac{P_\nu}{Q_\nu}$:

$$Q_\nu^{1/k} + \frac{N}{Q^{1/k}} \geq 2\, N^{1/2}\alpha$$

that is, for every ν

$$\text{either} \quad Q_\nu^{1/k} \geq N^{1/2}\alpha$$

$$\text{or} \quad \frac{N}{Q_\nu^{1/k}} \geq N^{1/2}\alpha$$

(1°) Case. Suppose that

$$Q_1^{1/k} \geq N^{1/2} \implies Q_1 \geq N^{k/2}\,\alpha^k$$

and $$x \leq \frac{1}{Q_1} \implies x \in I_o = \left[0\ ,\ \frac{1}{N^{k/2}\,\alpha^k}\right)$$

and $$\mu(I_o) = \frac{1}{N^{k/2}\,\alpha^k} \leq \frac{2^k}{\alpha^{2k}}$$

(2°) Case. In general $\exists\,\nu$ such that

$$\begin{cases} \dfrac{N}{Q_\nu^{1/k}} \geq N^{1/2}\,\alpha & \text{and} \\ Q_{\nu+1}^{1/k} \geq N^{1/2}\,\alpha & \end{cases}$$

that is

$$\begin{cases} Q_\nu \leq \dfrac{N^{k/2}}{\alpha^k} \\ Q_{\nu+1} \geq N^{k/2}\,\alpha^k \end{cases}$$

But then

$$\left|x - \frac{P_\nu}{Q_\nu}\right| \leq \frac{1}{Q_\nu\,Q_{\nu+1}} \leq \frac{1}{Q_\nu}\cdot\frac{1}{N^{k/2}\,\alpha^k}$$

Therefore

$$x \in \left(\frac{P_\nu}{Q_\nu} - \frac{1}{Q_\nu}\cdot\frac{1}{N^{k/2}\,\alpha^k}\ ,\ \frac{P_\nu}{Q_\nu} + \frac{1}{Q_\nu}\cdot\frac{1}{N^{k/2}\,\alpha^k}\right)$$

Given integers r,s s.t. $r < s$, $(r,s) = 1$

$$s \leq \frac{N^{k/2}}{\alpha^k}$$

denote

$$I_{r,s} = \left(\frac{r}{s} - \frac{1}{s}\ \frac{1}{N^{k/2}\,\alpha^k}\ ,\ \frac{r}{s} + \frac{1}{s}\ \frac{1}{N^{k/2}\,\alpha^k}\right)$$

The previous observation shows that

$$E_\alpha - \{\text{rationals}\} \subset I_o \cup \bigcup_{r,s} I_{r,s}$$

therefore

$$\mu\{E_\alpha\} \le \mu(I_o) + \sum_{\substack{r=1 \\ (r,s)=1}}^{s-1} \sum_{s=1}^{\left[\frac{N^{k/2}}{\alpha^k}\right]} \mu\{I_{r,s}\} \le$$

$$\le \frac{2^k}{\alpha^{2k}} + \sum_{s=1}^{\left[\frac{N^{k/2}}{\alpha^k}\right]} \sum_{r=1}^{s} \frac{2}{s} \frac{1}{N^{k/2}\alpha^k} \le$$

$$\le \frac{2+2^k}{\alpha^{2k}} \qquad \text{q.e.d.}$$

Given the trigonometric series $S_N(x,y) = \sum_1^N e^{2\pi i(n^2x+ny)}$ let us consider,

$$S_N^*(x,y) = \max_{1 \le M \le N} |S_N(x,y)|$$

Lemma 2. Let x have a rational approximation of the form $|x - p/q| \le \frac{1}{q^2}$, $1 \le q \le N^2$, $(p,q) = 1$.

Then we have

$$S_N^*(x,y) \le C \{q^{1/2} + \frac{N}{q^{1/2}}\}$$

for some universal constant C.

This result was known to Hardy and Littlewood who proved it using the approximate functional equation of the θ-function. Here we will follow [18] to emphasize the relationship between number theory and Fourier analysis via the use of the Carleson's theorem.

Proof.

1^a Fact. Let $|x-x_o| \leq \frac{1}{4N^2}$ and $|y-y_o| \leq \frac{1}{4N}$ then we have

$$S_N^*(x, y) \leq 100\, S_N^*(x_o\ y_o)$$

2^a Fact. If $1 \leq M \leq N$ then we have

$$S_N^*(x,y) \leq 2\, S_{2N}^*(x,\ y - 2Mx)$$

Proof. Clearly

$$\left|\sum_1^{M'} e^{2\pi i(n^2x+ny)}\right| = \left|\sum_{n=M+1}^{M'+M} e^{2\pi i((n-M)^2x + (n-M)y)}\right|$$

$$= \left|\sum_{n=M+1}^{M+M'} e^{2\pi i(n^2x + (y - 2\ x)n)}\right| \leq 2\, S_{2N}^*(x,\ y - 2Mx)$$

Therefore we have

$$|S_N^*(x,y)|^2 \leq \frac{4}{N} \sum_{M=1}^{N} |S_{2N}^*(x,\ y - 2Mx)|^2 \qquad [\text{Fact } 2]$$

But by [Fact 1]

$$|S_{2N}^*(x,y - 2Mx)|^2 \leq (100)^2(2N) \int_{I(y-2Mx)} |S_{2N}^*(x,\ y_o)|^2\, dy_o$$

where $I(z) = \{w : |z-w| \leq \frac{1}{8N}\}$

Therefore

$$|S_N^*(x,y)|^2 \leq C \sum_{M=1}^{N} \int_{I(y-2Mx)} |S_{2N}^*(x,y_o)|^2\, dy_o$$

Observe that the overlapping of the family of rectangles is given by the number P of solutions of the system

$$|2Mx - y - z| \leq \frac{1}{8N} \qquad 1 \leq M \leq N$$

Therefore:

$$|S_N^*(x,y)|^2 \leq C \ P \int_0^1 |S_{2N}^*(x,y_o)|^2 \, dy_o$$

$$\leq C \ P \ N \qquad \text{\underline{by Carleson's theorem}}.$$

let us denote by $\|x\|$ = distance from x to the integers.

Lemma. Suppose that $C \geq 1$ and $|\alpha - \frac{p}{q}| < \frac{c}{q^2}$ $(p,q) = 1$, $\beta \in R$ and $m \geq 1$ is an integer. Then the number of solutions of the system

$$\begin{cases} \|m \ \alpha \ x + \beta\| \leq \frac{1}{Y} \\ |x| \leq X \end{cases}$$

is $\leq 16 \ X \{\frac{mC}{Y} + \frac{mC}{q} + \frac{1}{X} + \frac{q}{X.Y}\}$

(The proof is an easy exercise with the well-known Dirichlet's pigeon hole principle).

If we apply the lemma with $m=2$, $X=N$, $Y = 8N$, we obtain:

$$|S_N^*(x,y)|^2 \leq C \ N^2 \{\frac{1}{q} + \frac{q}{N^2}\}$$

q.e.d.

Corollary. $\|\sum_1^N e^{2\pi i n^2 x}\|_p \simeq N^{1/2}$, $p < 4$.

Proof of Theorem.

Let us assume

$$f \sim \sum a_k \ e^{2\pi i k^2 x} \qquad a_k \searrow 0$$

$$f = \sum_{n=1}^{\infty} \Delta_n(x)$$

$$\Delta_n(x) = \sum_{k^2=2^n+1}^{2^{n+1}} a_k \; e^{2\pi i k^2 x} \qquad |p \geq 2|$$

$$\|f\|_p \sim \|\,(\sum_{n=1}^{\infty} |\Delta_n(x)|^2)^{1/2}\|_p =$$

$$= \left| \left|\int_0^1 (\sum_{n=1}^{\infty} |\Delta_n(x)|^2)^{p/2} \, dx\right|^{2/p} \right|^{1/2} \leq$$

$$\leq \left| \sum_{n=1}^{\infty} (\int_0^1 |\Delta_n(x)|^p \, dx)^{2/p} \right|^{1/2}$$

$$= \left[\sum_{n=1}^{\infty} \|\Delta_n\|_p^2 \right]^{1/2}$$

Enough to show that $\sum_{n=1}^{\infty} \|\Delta_n\|_p^2 < \infty$

$$\Delta_n(x) = \sum_{2^n \leq k^2 < 2^{n+1}} a_k \; e^{2\pi i k^2 x} =$$

$$= \sum_{k \in I_n} a_k \, |S_k(x) - S_{k-1}(x)|$$

$$I_n = (\nu_o, \nu_1)$$

where $S_j(x) = \sum_{\nu=1}^{j} e^{2\pi i \nu^2 x}$

Therefore

$$\Delta_n(x) = \sum_{k=\nu_o}^{\nu_1} a_k |S_k(x) - S_{k-1}(x)| =$$

$$= \sum_{k=\nu_o}^{\nu_1} a_k \; S_k(x) - \sum_{k=\nu_o}^{\nu_1} a_k \; S_{k-1}(x) =$$

$$= \sum_{k=\nu_o}^{\nu_1} a_k\ S_k(x) - \sum_{\nu_o-1}^{\nu_1-1} a_{k+1}\ S_k(x)$$

$$= \sum_{\nu_o}^{\nu_1-1} (a_k-a_{k+1})\ S_k(x) + a_{\nu_1}\ S_{\nu_1}(x) - a_{\nu_o}\ S_{\nu_o-1}(x)$$

$$\leq (a_{\nu_o}-a_{\nu_1})\ S^*_{\nu_1}(x,0) + a_{\nu_1}\ S^*_{\nu_1}(x,0) = a_{\nu_o}\ S^*_{\nu_1}(x,0)$$

$$= a_{[2^{n/2}]}\ S^*_{[2^{n+1/2}]}(x,0)$$

Which implies

$$||\Delta_n||_p \leq \sum_{\nu_o}^{\nu_1-1} (a_k-a_{k+1})\ ||S_k||_p + a_{\nu_1}\ ||S_{\nu_1}||_p +$$

$$+ a_{\nu_o}\ ||S_{\nu_o-1}||_p$$

$$\leq C_p\ 2^{n/4} \sum_{\nu_o}^{\nu_1-1} (a_k-a_{k+1}) + C_p\ 2^{n/4}\ (a_{\nu_1}+a_{\nu_0})$$

$$\leq \tilde{C}_p\ 2^{n/4}\ a_{\nu_0}$$

Finally observe that $a_k \searrow 0$, $\Sigma\ |a_k|^2 < \infty$

$$\Longrightarrow \sum_n a^2_{[2^{n/2}]} \cdot 2^{n/2} < \infty$$

q.e.d.

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