A pointwise estimate for fractionary derivatives with applications to partial differential equations

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This article emphasizes the role played by a remarkable pointwise inequality satisfied by fractionary derivatives in order to obtain maximum principles and L^p-decay of solutions of several interesting partial differential equations. In particular, there are applications to quasigeostrophic flows, in two space variables with critical viscosity, that model the Eckman pumping [see Baroud, Ch. N., Plapp, B. B., She, Z. S. & Swinney, H. L. (2002) *Phys. Rev. Lett.* 88, 114501 and Constantin, P. (2002) *Phys. Rev. Lett.* 89, 184501].

The decay in time of the spatial L^p-norm, $1 \le p \le \infty$, is an important objective in order to understand the behavior of solutions of partial differential equations. The purpose of this article is to analyze the following pointwise inequality, $2\theta\Lambda^\alpha\theta(x) \ge \Lambda^\alpha\theta^2(x)$, valid for fractionary derivatives in R^n, $n \ge 1$, $0 \le \alpha \le 2$, together with its applications to several maximum principle and decay estimates. In particular, it is applied to the quasigeostrophic equation with critical viscosity

$$\theta_t + R(\theta)\cdot\nabla^\perp\theta = -\kappa\Lambda\theta,$$

where $\nabla^\perp\theta = (-\partial\theta/\partial x_2, \partial\theta/\partial x_1)$, $R(\theta) = (R_1(\theta), R_2(\theta))$, and R_j denotes the j^{th}-Riesz transform in R^2 (see refs. 1–6).

Given a weak solution $\theta(x, t)$ (obtained as limit of solutions of the equations

$$\theta_t^\varepsilon + R(\theta^\varepsilon)\cdot\nabla^\perp\theta^\varepsilon = -\kappa\Lambda\theta^\varepsilon + \varepsilon\Delta\theta^\varepsilon$$

with the same initial data θ_0, when the artificial viscosity ε tends to zero), it is proved that $\|\theta(\cdot, t)\|_{L^p}$, $1 \le p \le \infty$, decays and, furthermore, there is a time $T = T(\kappa, \theta_0) < \infty$ after which θ becomes regular.

In this article, we describe the main ideas of the proofs, together with some of the heuristic arguments. The complete details will appear elsewhere.

Pointwise Estimate

The nonlocal operator $\Lambda = (-\Delta)^{1/2}$ is defined with the Fourier transform by $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}$, where $\hat{f}$ is the Fourier transform of f.

Theorem 1. *Let* $0 \le \alpha \le 2$, $x \in R^n$, T^n $(n = 1, 2, 3\ldots)$ *and* $\theta \in C_0^2(R^n)$, $C^2(T^n)$. *Then the following inequality holds:*

$$2\theta\Lambda^\alpha\theta(x) \ge \Lambda^\alpha\theta^2(x). \qquad [1]$$

Proof: It is easy to check that the inequality is satisfied when $\alpha = 0$ and $\alpha = 2$. For $0 < \alpha < 2$ and $n \ge 2$, there are the formulas

$$\Lambda^\alpha\theta(x) = C_\alpha PV\int\frac{[\theta(x) - \theta(y)]}{|x-y|^{n+\alpha}}dy \quad x \in R^n \qquad [2]$$

$$\Lambda^\alpha\theta(x) = \tilde{C}_\alpha\sum_{\nu\in Z^n} PV\int_{T^n}\frac{[\theta(x) - \theta(y)]}{|x-y-\nu|^{n+\alpha}}dy \quad x \in T^n, \qquad [3]$$

where C_α, $\tilde{C}_\alpha > 0$.

With Eq. **2** (and Eq. **3** in the periodic case) inequality Eq. **1** is obtained easily:

$$\begin{aligned}\theta\Lambda^\alpha\theta(x) &= C_\alpha PV\int\frac{[\theta(x)^2 - \theta(y)\theta(x)]}{|x-y|^{n+\alpha}}dy \\ &= \frac{1}{2}C_\alpha PV\int\frac{[\theta(y) - \theta(x)]^2}{|x-y|^{n+\alpha}}dy \\ &\quad + \frac{1}{2}C_\alpha PV\int\frac{[\theta^2(x) - \theta^2(y)]}{|x-y|^{n+\alpha}}dy \\ &\ge \frac{1}{2}\Lambda^\alpha\theta^2(x).\end{aligned}$$

The proof of the remainder case $n = 1$ is as follows. Given $\psi(x_1)$ an application of the previous case, $n = 2$, to the function $\phi_\delta(x_1, x_2) = \psi(x_1)e^{-\pi\delta x_2^2}$ yields

$$\begin{aligned}2\psi(x_1)\Lambda^\alpha\psi(x_1) &= \lim_{\delta\to 0}2\phi_\delta(x_1, x_2)\Lambda^\alpha\phi_\delta(x_1, x_2) \\ &\ge \lim_{\delta\to 0}\Lambda^\alpha\phi_\delta^2(x_1, x_2) \\ &= \Lambda^\alpha\psi^2(x_1).\end{aligned}$$

Remark 1: The family of test functions $x^pe^{-\delta x^2}$, $\delta > 0$, shows that the condition $\alpha \le 2$ cannot be improved.

Also, the hypothesis $\theta \in C_0^2(R^n)$, $C^2(T^n)$ is not necessary. Inequality Eq. **1** holds when $\theta(x)$, $\Lambda^\alpha\theta(x)$, $\Lambda^\alpha\theta^2(x)$ are defined everywhere and are, respectively, the limits of the sequences $\theta_m(x)$, $\Lambda^\alpha\theta_m(x)$, $\Lambda^\alpha\theta_m^2(x)$, where $\theta_m \in C_0^2(R^n)$, $C^2(T^n)$ for each m.

Applications

L^p Decay. Let it be given the following scalar equation

$$(\partial_t + u\cdot\nabla)\theta = -\kappa\Lambda^\alpha\theta,$$

where the vector u satisfies either $\nabla\cdot u = 0$ or $u_i = G_i(\theta)$, together with the appropriate hypothesis about regularity and decay at infinity, which will be specified each time, in order to allow the integration by parts needed in the proofs.

Lemma 1. *If* $0 \le \alpha \le 2$ *and* $\theta \in C_0^2(R^n)(C^2(T^n))$, *it follows that*

$$\int|\theta|^{p-2}\theta\Lambda^\alpha\theta dx \ge \frac{1}{p}\int|\Lambda^{\alpha/2}\theta^{p/2}|^2dx, \qquad [4]$$

where $p = 2^j$ *and* j *is a positive integer.*

Proof: An iterated application of inequality Eq. **1** yields:

$$\begin{aligned}\int|\theta|^{p-2}\theta\Lambda^\alpha\theta dx &\ge \frac{1}{2}\int|\theta|^{p-2}\Lambda^\alpha\theta^2dx = \int|\theta|^{p-4}\theta^2\Lambda^\alpha\theta^2dx \\ &\ge \frac{1}{4}\int|\theta|^{p-4}\Lambda^\alpha\theta^4dx \ge \frac{1}{2^{k-1}}\int|\theta|^{p-2^k}\Lambda^\alpha\theta^{2^k}dx,\end{aligned}$$

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taking $k = j - 1$ and using Parseval's identity with the Fourier transform inequality Eq. **4** is obtained.

Remark 2: When $p = 2^j$ ($j \geq 1$) *Lemma 1* implies the following improved estimate:

$$\frac{d}{dt}\|\theta\|_{L^p}^p = -\kappa p \int |\theta|^{p-2}\theta\Lambda^\alpha\theta dx$$

$$\leq -\kappa \int |\Lambda^{\alpha/2}\theta^{p/2}|^2\, dx.$$

In the periodic case, this inequality yields an exponential decay of $\|\theta\|_{L^p}$, $1 \leq p < \infty$. For the nonperiodic case, Sobolev's embedding and interpolation produces

$$\frac{d}{dt}\|\theta\|_{L^p}^p \leq -\kappa\left(\int \theta^{2p/2-\alpha}\, dx\right)^{2-\alpha/2}$$

$$\leq -C(\|\theta\|_{L^p}^p)^{p-1+(\alpha/2)/p-1},$$

where $C = C(\kappa, \alpha, p, \|\theta_0\|_1)$ is a positive constant. It then follows

$$\|\theta(\cdot, t)\|_{L^p}^p \leq \frac{\|\theta_0\|_{L^p}^p}{(1 + \varepsilon Ct\|\theta_0\|_{L^p}^{p\varepsilon})^{1/\varepsilon}},$$

with $\varepsilon = \alpha/2(p - 1)$.

Remark 3: The decay for other L^p, $1 < p < \infty$, is obtained easily by interpolation. However, the L^∞ decay needs further arguments that will be presented in the next section.

Viscosity Solutions of the Quasigeostrophic Equation

A weak solution of

$$\theta_t + R(\theta)\cdot\nabla^\perp\theta = -\kappa\Lambda\theta$$

will be called a viscosity solution with initial data $\theta_0 \in H^s(R^2)(H^s(T^2))$, $s > 1$, if it is the weak limit of a sequence of solutions, as $\varepsilon \to 0$, of the problems

$$\theta_t^\varepsilon + R(\theta^\varepsilon)\cdot\nabla^\perp\theta^\varepsilon = -\kappa\Lambda\theta^\varepsilon + \varepsilon\Delta\theta^\varepsilon, \qquad [5]$$

with $\theta^\varepsilon(x, 0) = \theta_0$.

Theorem 2. *Let θ^ε, $\varepsilon > 0$, be a solution of Eq.* **5**, *then $\theta^\varepsilon(\cdot, t) \in H^s$ for each $t > 0$ and satisfies*

$$\|\theta^\varepsilon(\cdot, t)|_{L^\infty} \leq \frac{\|\theta_0\|_{L^\infty}}{1 + Ct\dfrac{\kappa\|\theta_0\|_{L^\infty}}{\|\theta_0\|_{L^2}}}$$

uniformly on $\varepsilon > 0$ for all time $t \geq 0$. Furthermore, for $s > 3/2$, there is a time $T_1 = T_1(\kappa, \|\theta_0\|_{H^s})$ such that $\|\Lambda^s\theta^\varepsilon(t)\|_{L^2} \leq 2\|\Lambda^s\theta_0\|_{L^2}$ for $0 \leq t < T_1$.

Theorem 3. *Let θ be a viscosity solution with initial data $\theta_0 \in H^s$, $s > 3/2$, of the equation $\theta_t + R(\theta)\cdot\nabla^\perp\theta = -\kappa\Lambda\theta$ ($\kappa > 0$). Then there exist two times $T_1 \leq T_2$ depending only on κ and the initial data θ_0 so that:*

(i) *If $t \leq T_1$ then $\theta(\cdot, t) \in C^1([0, T_1); H^s)$ is a classical solution of the equation satisfying*

$$\|\theta(\cdot, t)\|_{H^s} \ll \|\theta_0\|_{H^s}.$$

(ii) *If $t \geq T_2$ then $\theta(\cdot, t) \in C^1([T_2, \infty); H^s)$ is also a classical solution and $\|\theta(\cdot, t)\|_{H^s}$ is monotonically decreasing in t, bounded by $\|\theta_0\|_{H^s}$, and satisfying*

$$\int_{T_2}^\infty \|\theta\|_{H^s}^2\, dt < \infty.$$

In particular, this implies that

$$\|\theta(\cdot, t)\|_{H^s} = O(t^{-1/2})\ t \to \infty.$$

Sketch of the Proofs: For the L^∞-decay there is the following heuristic argument. Assuming that $\theta(\cdot, t)$ get its maximum value at the point x_t, depending smoothly on t, then the equation yields

$$\frac{\partial\theta}{\partial t}(x_t, t) = -\kappa\Lambda\theta(x_t, t) = -\kappa\cdot PV\int \frac{[\theta(x_t, t) - \theta(y, t)]}{|x_t - y|^3}\, dy.$$

And the decay is obtained because

$$PV\int \frac{[\theta(x_t, t) - \theta(y, t)]}{|x_t - y|^3}\, dy \geq C\,\frac{\theta^2(x_t, t)}{\|\theta(\cdot, t)\|_{L^2}}.$$

In the actual *Proof* the differentiability properties of Lipschitz functions are used in order to avoid the hypothesis about the existence of dx_t/dt.

The *Proof* of *Theorem 3* is based on both the L^∞-decay and a bootstrap mechanism associated with the evolution of different Sobolev norms. A crucial ingredient is the fact that $fR(f)$ belongs to Hardy's space $\mathcal{H}^1$ for each L^2-function f and every odd singular integral R. A typical example of that mechanism is the following chain of inequalities

$$\frac{d}{dt}\|\theta^\varepsilon\|_{L^2}^2 = 2\int \theta^\varepsilon R(\theta^\varepsilon)\cdot\nabla^\perp\theta^\varepsilon - 2\kappa\int\theta^\varepsilon\Lambda\theta^\varepsilon - 2\varepsilon\int|\Lambda\theta^\varepsilon|^2$$

$$= -2\kappa\|\Lambda^{1/2}\theta^\varepsilon\|_{L^2}^2 - 2\varepsilon\|\Lambda\theta^\varepsilon\|_{L^2}^2 \leq -2\kappa\|\Lambda^{1/2}\theta^\varepsilon\|_{L^2}^2$$

$$\frac{d}{dt}\|\Lambda^{1/2}\theta^\varepsilon\|_{L^2}^2 = 2\int\Lambda^{1/2}\theta^\varepsilon\Lambda^{1/2}(R(\theta^\varepsilon)\cdot\nabla^\perp\theta^\varepsilon) - 2\kappa\|\Lambda\theta^\varepsilon\|_{L^2}^2$$

$$- 2\varepsilon\|\Lambda^{3/2}\theta^\varepsilon\|_{L^2}^2$$

$$\leq (C\|\theta^\varepsilon(\cdot, t)\|_{L^\infty} - 2\kappa)\|\Lambda\theta^\varepsilon\|_{L^2}^2$$

$$\frac{d}{dt}\|\Lambda\theta^\varepsilon\|_{L^2}^2 \leq (C\|\Lambda\theta^\varepsilon\|_{L^2} - \kappa)\|\Lambda^{3/2}\theta^\varepsilon\|_{L^2}^2$$

$$\frac{d}{dt}\|\Lambda^{3/2}\theta^\varepsilon\|_{L^2}^2 \leq (C\|\theta^\varepsilon\|_{L^\infty} - \kappa)\|\Delta\theta^\varepsilon\|_{L^2}^2$$

valid for some universal constant C, uniformly with respect to the artificial viscosity ε.

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1. Chae, D. & Lee, J. (2003) *Commun. Math. Phys.* **233,** 297–311.
2. Constantin, P., Cordoba, D. & Wu, J. (2001) *Indiana Univ. Math. J.* **50,** 97–107.
3. Constantin, P. & Wu, J. (1999) *SIAM J. Math. Anal.* **30,** 937–948.
4. Resnick, S. (1995) Ph.D. thesis (University of Chicago, Chicago).
5. Wu, J. (2002) *Comm. Partial Differ. Eq.* **27,** 1161–1181.
6. Wu, J. (1997) *Indiana Univ. Math. J.* **46,** 1113–1124.

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