Phase transitions: Uniform regularity of the intermediate layers

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3rd September 2003

1 Introduction

In this work we continue the study, initiated in [C.C], about the geometry of the transition layers for minimizers of the functionals

$$J_\epsilon(u) = \int_\Omega \{\epsilon^2 \|\nabla u\|^2 + F(x,u)\}\mathrm{d}x.$$

Here $\epsilon > 0$ is a (small) parameter, say $0 < \epsilon < 1$, and $\Omega \subset \Re^n$ is an open bounded domain.

The nonnegative function F is a double well potential vanishing only for two values of u, say $u = -1$ and $u = +1$, and the minimizers u under consideration will take values precissely at that interval $(-1 \leq u \leq +1)$.

A description of the physical model can be found in [C.H], [W]. The main idea is that u represents the state of a parameter (for instance a concentration of one component of a binary alloy), and the function $F(x,u)$ the energy density of the configuration $u(x)$.

In general, this density is indexed by another parameter, say the temperature T:

$$F = F_T(x,u).$$

For large values of T, F_T is convex and we are then in the classical theory of minimizers of convex functionals in the calculus of variations. However, when T becomes small, the free energy density ceases to be convex and it has a local minimum at two states (say $u = -1$, and $u = +1$). In this last case, any configuration that only takes the states $u = -1$ and $u = +1$, would be an energy minimizer.

The term $\epsilon^2\|\nabla u\|^2$ was then added by many authors (Van der Waals, Cahn-Hilliard and Ginzburg-Landau) to take into account the "surface energy" separating both phases.

As ϵ goes to zero, it is well known that sequences of minimizers converging in $L^1_{loc}(\Omega)$ have limits which are configurations

$$u(x) = \chi_A(x) - \chi_{A^c}(x)$$

where $\partial A \cap \Omega$ is a minimal surface.

More precissely, we have the following result:

Theorem (Modica[M]) *Fix $M \in \Re$ so that $\mid M \mid < vol(\Omega)$ and suppose that u_ϵ, for every $\epsilon > 0$, is a solution of the variational problem:*

$$J_\epsilon(u) = \min\left\{J_\epsilon(v) \mid v \in L^1(\Omega), -1 \leq v \leq +1, \int_\Omega v(x)\mathrm{d}x = M\right\}$$

If $\{\epsilon_h\}$ is a sequence such that $\epsilon_h \to 0$ and u_{ϵ_h} converges in $L^1_{loc}(\Omega)$ to a function u_0, then we have:

i) $F(x, u_0(x)) = 0$, *a.e. in* Ω, *i.e.* $u_0(x) = -1$ *or* $u_0(x) = +1$ *a.e. in* Ω.

ii) the set $A = \{x \in \Omega \mid u_0(x) = -1\}$ *is a solution of the variational problem*

$$P_\Omega(A) = (Perimeter\ of\ A\ in\ \Omega) =$$
$$= \min\left\{P_\Omega(B) \mid B \subset \Omega, vol(B) = \frac{vol(\Omega) - M}{2}\right\}.$$

iii)

$$\lim_{\epsilon_h \to \infty} \epsilon_h^{-1} J_{\epsilon_h}(u_{\epsilon_h}) = 2C_0 P_\Omega(A).$$

$$where\ C_0 = \int_{-1}^{+1} \sqrt{F(s)}\mathrm{d}s.$$

Let us mention that in our previous work [C.C] we proved the following:

Theorem [C.C] *If the minimizers of u_ϵ converge in L^1_{loc}, when ϵ goes to zero, and the limit is given by $-\chi_{\Omega_1} + \chi_{\Omega - \Omega_1}$, then the level surfaces $S^\epsilon_\mu = \{u_\epsilon = t\}$, $-1 < t < +1$, converge uniformly to $\partial\Omega_1 \cap \Omega$ on any compact subset $K \subset \Omega$.*

This theorem was obtained as a consequence of the following density result, also proved in [C.C] and that will be used in this paper.

Theorem [C.C] *Let u, $-1 \leq u \leq +1$, be a local minimizer of J_1 in the ball $B_R(0)$. Asume that $\mu\{B_1(0) \cap \{u > t\}\} \geq \lambda_0$ (for a certain positive constant λ_0 independent of R and t, $-1 < t < +1$). Then there exists a positive constant C such that:*

$$\mu\Big\{B_\rho(0) \cap \{u > t\}\Big\} \geq C\rho^n$$

for any ρ, $1 \leq \rho \leq R$.

As in our paper [C.C] we shall consider here a family of potentials F representing different types of regularity. Namely:

$$F_0(u) = \chi_{\{|u| \leq 1\}},\ F_\delta(u) = (1-u^2)_+^\delta\ ,\ 0 < \delta \leq 2.$$

While the cases $0 \leq \delta < 2$ generate free-boundary problems (transition strips), the solution to the last one has exponential convergence to the states ± 1. Now we can present our main result.

Theorem 1 *If u is a solution to the J_ϵ problem in the ball $B_R(0)$, for which the level set $\{u = 0\}$ is a Lipschitz graph, $x_n = f(x_1, \cdots, x_{n-1})$, then for every t_0, $0 < t_0 < 1$, there exist positive numbers ϵ_0, α, $0 < \tau < 1$, (depending only upon the Lipschitz norm of f and t_0), such that, for $\epsilon \leq \epsilon_0$, all level surfaces $u = t, |t| < t_0$, are uniformly $C^{1,\alpha}$ inside the ball $B_{\tau R}(0)$. Furthermore, for $\delta < 2$ one can take $t_0 = 1$.*

This result is closely related to a conjecture of de Giorgi [G]. It concerns a Liouville type theorem: when are global minimizers one-dimensional?

E. de Giorgi conjectured that if $\lim_{x_n \to \pm\infty} u = \pm 1$, then the solution must be one-dimensional. The theorems of Modica [M] and [C.C] imply that the transition regions (say $-1/2 < u < +1/2$) look at infinity like a minimal surface, therefore the conjecture is very plausible when such a surface is a hyperplane, for instance in dimension smaller than eigth. But, since in higher dimensions there are global minimal graphs which are not planes, the conjecture becomes more difficult. In any case, it has received a lot of attention recently (see references [B.C.N], [B.B.G], [A.Cb]). A corollary of our theorem is:

Corollary 1 *Let u be a local minimizer of the functional J_1 in the whole space $\Re^n$. Assume that $\partial\{u > 0\}$ is a Lipschitz graph. Then u is one-dimensional i.e. there exists $v \in \Re^n$ such that $u(x) = \varphi(x \cdot v)$, for a suitable $\varphi : \Re \to \Re$.*

In the proof of our main Theorem 1 we shall consider first the case where all level sets $\{u = t\}, |t| < 1$, are uniformly Lipschitz surfaces inside a ball. But, at the end of the paper, we shall present an argument to conclude that if $\{u = 0\}$ is a Lipschitz surface inside B_r, then, in a smaller ball, say $B_{r/2}$, all the other level sets are also uniformly Lipschitz surfaces.

Theorem 2 *Let u be a minimizer of $J_\epsilon(u) = \int_{B_R(0)} \{\epsilon^2 \|\nabla u\|^2 + (1-u^2)^\delta\} \mathrm{d}x$ such that all level sets $\{u = t\}, |t| < 1$, are uniformly Lipschitz graphs $x_n = f^t(x_1, \cdots, x_{n-1}), \sup_{|t|<1} \|f^t\|_{Lip} = K < \infty$. Then there exists positive $\epsilon_0, t_0, \tau (0 < \tau < 1)$ and $\alpha > 0$ (depending only upon the Lipschitz norm K), such that, for $\epsilon \leq \epsilon_0$, the level surfaces $\{u = t\}, |t| < t_0$, are uniformly $C^{1,\alpha}$ inside the ball $B_{\tau R}(0)$. Moreover, in the case $\delta < 2$ one can take $t_0 = 1$.*

Remark. Our Theorem implies, in particular, the regularity of the free boundary under the Lipschitz assumption about the level surfaces. Therefore one obtains an alternative "penalization method approach" to the results presented in [C], [C'], [A.C] and [P], corresponding to the cases $\delta = 0, \delta = 1$ and $1 < \delta < 2$. However our method covers also the cases $0 < \delta < 1$.

2 Description of the potentials. One dimensional behavior, optimal regularity and decay

In our previous work [C.C] we did consider a class $S_{\gamma,C}(\delta)$ of admissible potentials F, for which we required several properties making precisse the statement that they behave assymptotically, when u goes to ± 1, like one of the models $F(u) = (1-u^2)_+^\delta$, $0 \leq \delta \leq 2$. These properties are the following:

i) $0 \leq F \leq 1$

ii) $F(x, -1) = F(x, +1) = 0$, for every $x \in \Omega$.

iii) $\inf_{|t|<\lambda, x\in\Omega} F(x,t) \geq \gamma(\lambda)$, where γ is a decreasing and strictly positive function in the interval $(0, 1)$.

iv) $F(x, u) \geq C(1- \mid u \mid)^\delta$, if $1 >\mid u \mid> \lambda$, $0 \leq \delta \leq 2$.

v) in the case $0 < \delta < 2$, we assume also that $F_u(x, u)$ is continuous in $\Omega \times (-1, +1)$ and satisfies the estimate

$$F_u(x, -1 + s) \geq Cs^{\delta-1}, F_u(x, 1 - s) \leq Cs^{\delta-1} \quad if \quad 0 < s < \lambda.$$

vi) In the case $\delta = 2$ we will assume the continuity of F_{uu} and the existence of a region of positivity: if u is near 1, then $F_{uu} = a - b(x)(1-u) + o(1-u)$, for some positive $a > 0$, where $b(x)$ is bounded below from zero. Similarly $F_{uu} = a - b(x)(1+u) + o(1+u)$, near $u = -1$.

However, some of the previous results that are needed in the proof of our main theorem, althought true for the general class $S_{\gamma,C}(\delta)$, are only stated in the literature for $F(u) = (1-u^2)_+^\delta$, $0 \le \delta \le 2$. Following that tradition, and in order to limit the length of this paper, we shall restrict our attention mainly to these model cases. Taking variations we obtain the following local problems:

Case(a): $\delta = 0$.

In the unit ball B_1 we are given a bounded Lipschitz function u_ϵ such that

$$\begin{cases} \Delta u_\epsilon = 0 & \text{in the region } \mid u_\epsilon \mid < 1 \\ \mid \nabla u_\epsilon \mid = \frac{1}{\epsilon} & \text{on } (\partial\{u_\epsilon = +1\} \cup \partial\{u_\epsilon = -1\}) \cap B_1 \\ u_\epsilon(0) = 0, & \mid u_\epsilon \mid \le 1. \end{cases}$$

Case(b): $0 < \delta < 2$.

in B_1 we are given a $C^{1,\beta-1}$ function u_ϵ, $\beta = \frac{2}{2-\delta}$, such that

$$\begin{cases} \Delta u_\epsilon = -\frac{1}{\epsilon^2} 2\delta u_\epsilon (1-u_\epsilon^2)^{\delta-1}, & \text{if } \mid u_\epsilon \mid < 1 \\ \mid \nabla u_\epsilon \mid = 0 & \text{on } (\partial\{u_\epsilon = +1\} \cup \partial\{u_\epsilon = -1\}) \cap B_1 \\ u_\epsilon(0) = 0, & \mid u_\epsilon \mid \le 1 \end{cases}$$

When $n = 1$, $\delta = 1$ we get the following solution:

$$\begin{cases} u_\epsilon(x) = \sin\frac{x}{\epsilon}, & \text{if } \mid x \mid \le \frac{\pi}{2}\epsilon. \\ u_\epsilon(x) = +1, & \text{if } x > \frac{\pi}{2}\epsilon \\ u_\epsilon(x) = -1 & \text{if } x < -\frac{\pi}{2}\epsilon \end{cases}$$

Case(c): $\boldsymbol{\delta}$**=2** .

Here we have a C^∞ solution of the equation

$$\begin{cases} \Delta u_\epsilon = -\frac{2}{\epsilon^2} u_\epsilon (1-u_\epsilon^2) \\ u_\epsilon(0) = 0, & \mid u_\epsilon \mid \le 1 \end{cases}$$

The one-dimensional solutions are given by

$$u_\epsilon(x) = \mathrm{th}\Big(e^{(\sqrt{2}/2\epsilon)x}\Big) = \frac{\exp\left((\sqrt{2}\epsilon)x\right) - 1}{\exp\left((\sqrt{2}\epsilon)x\right) + 1}$$

It is interesting also to consider the following variation of this case (d):

$$\begin{cases} \Delta u_\epsilon = \frac{1}{\epsilon^2} F'(u), & \text{in } \mid u \mid < 1 \\ u_\epsilon(0) = 0, & \mid u_\epsilon \mid \leq 1 \end{cases}$$

Where

$$F(u) = \begin{cases} \frac{1}{2}(1+u)^2, & \text{if } -1 \leq u \leq -\frac{1}{2} \\ \frac{1}{4} - \frac{1}{2}u^2, & \text{if } \mid u \mid \leq \frac{1}{2} \\ \frac{1}{2}(-1+u)^2, & \text{if } \frac{1}{2} \leq u \leq 1 \end{cases}$$

Then we obtain the one-dimensional solution:

$$u_\epsilon(x) = \begin{cases} -1 + \frac{1}{2}\exp\left(-\frac{x}{\epsilon} - \frac{\pi}{4}\right), & \text{if } x \leq -\frac{\pi}{4}\epsilon \\ \frac{1}{\sqrt{2}} \sin\frac{x}{\epsilon}, & \text{if } \mid x \mid \leq \frac{\pi}{4}\epsilon \\ 1 - \frac{1}{2}\exp\left(-\frac{x}{\epsilon} + \frac{\pi}{4}\right), & \text{if } x \geq \frac{\pi}{4}\epsilon \end{cases}$$

We are interested in the transition layers. That is the reason to impose the condition $u(0) = 0$ in the statement of our local problems. The regularity assumption $u \in C^{1,\frac{2}{2-\delta}}$ is optimal as it was shown in references [C], [A.P], [A.C] and [P]. Observe also that in the last two examples, we do not obtain a strip connecting both states ± 1, but an exponential decay outside a narrow transition layer. As we have mentioned before, our previous publication [C.C] contains estimates about the size of the transition layers (relatively to ϵ), and the uniform convergence of the level sets, $u_\epsilon = \lambda$, to the limiting minimal surface.

There exists a natural scaling which plays an important role in the forthcoming analysis. A local minimizer of J_ϵ in the ball B_r, produces a local minimizer of J_1 in the ball $B_{r/\epsilon}$ by means of the change $\tilde{u}(x) = u(\epsilon x)$.

In the task of analyzing the level surfaces of u_ϵ, let us point that their smoothness, up to size ϵ, is a consequence of free boundary regularity in the cases $\delta < 2$. If $\delta = 2$ the regularity follows easily, for each ϵ, from the standard elliptic theory. Our Lipschitz assumption gives us regularity up to the scale 1. In this setting, the uniform $C^{1,\alpha}$ character of the level sets is a question about all the intermediate scales, i.e., those between ϵ and 1.

Exponential decay in the case $\delta = 2$

Suppose the ω satisfies the equation $\Delta\omega = \omega$ inside $\Omega \supset B_R(x)$. Then we have the representation:

$$\omega(x) = \frac{1}{\phi(R)} \int_{S^{n-1}} \omega(x + R\bar{y}) \mathrm{d}\sigma(\bar{y})$$

where $\phi(r) = \int_{S^{n-1}} e^{rt_1} \mathrm{d}\sigma(t)$, and d$\sigma$ denotes uniform measure in S^{n-1}.

The proof is an inmediate consequence of the uniqueness theorem for O.D.E.'s. Observe that the following two functions of the variables r given by

$$v_1(r) = \omega(x)\phi(r) \text{ and } v_2(r) = \int_{S^{n-1}} \omega(x + r\bar{y})\mathrm{d}\sigma(\bar{y})$$

satisfy the following problem:

$$v_i''(r) + \frac{n-1}{r} v_i'(r) = v_i(r)$$

$$v_i(0) = \omega_{n-1}\omega(x)$$

$$v_i'(0) = 0$$

where ω_{n-1} = area of S^{n-1}.

The representation formula yields the exponential decay of ω when $\mid \omega \mid$ is bounded. This happens in our second example (d) taking $\omega = 1 - u$ in the region $\frac{1}{2} < u < 1$ (respectively to $\omega = 1 + u$, when $-1 < u < -\frac{1}{2}$).

A similar result holds if the bounded function ω satisfies $\Delta\omega \geq \omega \geq 0$ in Ω. This is because we can compare ω with the solution of the problem:

$$\left.\begin{array}{l} \Delta u = u \text{ in } \Omega \\ u/\partial\Omega = \omega \end{array}\right\}$$

In the first example (c) $\Delta u = -u(1 - u^2)$, we obtain for $\omega = 1 + u$ the inequality $\Delta\omega = \omega(2 - e\omega + \omega^2) \geq \omega$ if $-1 < u < -\frac{1}{2}$. Similarly for $\omega = 1 - u$ inside $\frac{1}{2} \leq u < 1$.

Gradient estimates

In this section we consider the case where all level sets $\{u = t\}, \mid t \mid \leq 1$, are uniformly Lipschitz surfaces inside a ball. At the end of the paper we present an argument to deduce that assumption from the Lipschitz character of a single level, namely $\{u = 0\}$.

These problems are on the borderline where the implicit function theorem applies. Therefore it is very important to have a precise control of ∇u where $\mid u \mid$ is close to the value 1. The uniform Lipschitz assumption allows us (renormalizing by a convenient change of scale, if necessary), to consider the following scenario.

We have a function u defined inside a cylinder $B_R \times [-T, T] = \{(x', x_n) \mid x' \in B_R, -T \leq x_n \leq T\}$ with T a large multiple of R. Furthermore we will assume that $\mid u \mid \leq 1$ is monotonic in the x_n direction and satisfies

$$\|\nabla u\| \leq K u_{x_n}, \quad u(0) = 0.$$

Therefore, the level sets given by the implicit functions $x_n = \varphi(x'; \lambda), u(x', \varphi(x'; \lambda)) = \lambda, -1 < \lambda < +1$, are Lipschitz surfaces with constant $K < \infty$.

Proposition 1 *Under the hypothesis stated above and under the asumption that u has optimal regularity, $u \in C^{1, \frac{\delta}{2-\delta}}$, we have $\|\nabla u\| \sim u_n \sim \sqrt{F(u)}$, uniformly inside the region $\mid u \mid < 1$.*

In the following we shall use the standard notation $f \ll g$ if there exists a final constant C so that $f(x) \leq C g(x)$ for every x. In case that we have both $f \ll g$ and $g \ll f$ we shall write $f \sim g$.

The proof of Proposition 1 will be based on the following lemmas.

Lemma 2 *Suppose that the non-negative function ω satisfies $\Delta\omega = \omega$ in a domain Ω containing the ball $B_{4R}(x_0)$. Then we have:*

$$\sup_{x \in B_R(x_0)} \omega(x) \leq C(R) \inf_{y \in B_{R/2}(x_0)} \omega(y)$$

Proof. It follows from the representation formula

$$\omega(x) = \frac{1}{\phi(r)} \int_{S^{n-1}} \omega(x + r\bar{y}) \mathrm{d}\sigma(\bar{y})$$

$$\phi(r) = \int_{S^{n-1}} \exp(r t_1) \mathrm{d}\sigma(t).$$

Then if $B_\lambda(x) \subset \Omega$ we get for each $\bar{\lambda} < \lambda$:

$$\omega(x) = \frac{1}{\lambda - \bar{\lambda}} \int_{B_\lambda - B_{\bar{\lambda}}} \frac{\omega(x + y)}{\phi(\mid y \mid) \mid y \mid^{n-1}} \mathrm{d}y, \quad B_\lambda = B_\lambda(0).$$

Let us observe that if $d = \mid x_1 - x_2 \mid$ then

$$B_{2d}(x_1) - B_{\frac{3}{2}d}(x_1) \subset B_{3d}(x_2) - B_{\frac{1}{2}d}(x_2),$$

which yields our result with the estimate

$$C(R) \leq \exp(3R).$$

With the help of the maximum principle we can extend lemma 2 to the case of non-negative functions u satisfying $\lambda^2 u \leq \Delta u \leq 4\lambda^2 u$.

Lemma 3 *Suppose that $\omega \geq 0$ satisfies $\lambda^2\omega \leq \Delta\omega \leq 4\lambda^2\omega$ in a domain $\Omega \in \Re^n$ containig the ball $B_2(x)$. Then we have $\|\nabla\omega(x)\| \leq C\omega(x)$, for some finite constant $C = C(\lambda; n)$.*

Proof. It follows from the well-known estimate

$$\sup_{\Omega} \{\mathrm{d}(y)\|\nabla u(y)\|\} \leq C_n \{\sup_{\Omega} |u| + \sup_{\Omega} \mathrm{d}^2(y)\, |f(y)|\}$$

for solutions of the equation $\Delta u = f$ in Ω, where $\mathrm{d}(y)$ denotes distance to the boundary $\partial\Omega$.

In our case

$$\|\nabla\omega(x)\| \leq C_n \sup_{B_2(x)} \omega \leq \tilde{C}_n \inf_{B_1(x)} \omega \leq \tilde{C}_n \omega(x).$$

Let us consider in $\Re^n$ the following domain

$$Q(t) = \{(x, x_n) \in \Re^n |\, |x_j| \leq t, j = i, \cdots, n-1,\ \varphi(x) \leq x_n \leq Mt\}$$

where $\varphi : \Re^{n-1} \to \Re$ is a Lipschitz function with Lipschitz constant less than M and such that $\varphi(0) = 0$.

Define $S = \{(x, \varphi(x))|\, |x_j| \leq t\}$ and suppose that ω_1, ω_2 are two functions satisfying $\Delta\omega_1 \leq \lambda\omega_1$, $\Delta\omega_2 \geq \lambda\omega_2$, $0 \leq \omega_j \leq 1$, in $Q(t)$ for some positive number λ.

Assume furthermore that there exists a positive number $\delta > 0$ such that

$$\omega_1 \geq \delta \text{ on } S.$$

Lemma 4 *Under the assumptions stated above, there exists a positive constant $0 < c = c(M, n) < 1$ and a positive number $N = N(M, n)$ so that*

$$\delta^{-1} N\omega_1(x) \geq \omega_2(x), \quad \textit{for every } x \in Q(ct).$$

Proof. We shall use the following fact of elementary geometry: if $x \in Q(ct)$, for a small positive number c, then there exist a finite number of rotations centered at x and such that the corresponding images of the surface S limit a bounded Lipschitz domain Ω containing the point x in its interior. (In fact it is enough to show it for S a cone $x_n = M|x|$.)

Denote by $\rho_0 = id, \rho_1, \cdots, \rho_{N-1}$ the inverses of those rotations. Then the function

$$\omega(y) = \omega_1(\rho_0 y) + \cdots + \omega_1(\rho_{N-1} y)$$

satisfies $\Delta\omega \leq \lambda\omega$ in Ω, taking boundary values $\omega/\partial\Omega \geq \delta > 0$. Since $\omega_2/\partial\Omega \leq 1$, we can apply the maximum principle to conclude that

$$\delta^{-1}\sum_{j=0}^{N-1}\omega_1(\rho_1 y) \geq \omega_2(y) \text{ in } \Omega$$

In particular, if we take $y = x$ we get

$$\delta^{-1}N\omega_1(x) \geq \omega_2(x).$$

Proof of Proposition 1. The part of this proposition correponding to the cases $\delta < 2$ are already known (see [A.C], [C], [C'], [P]). Therefore, in the following we will present the proof when $\delta = 2$ (Examples (c) and (d)).

The main idea is to apply the lemmas to the functions $\omega_1 = u_{x_n}$ and $\omega_2 = 1 - u$ when u is close to $+1$ (resp. to $\omega_2 = 1 + u$ when u is near -1). An important ingredient will be the band and density estimates obtained in [C.C].

First, let us observe that an straightforward application of Lemma 3 yields the estimate

$$\omega_1 = u_{x_n} \leq \|\nabla u\| \ll \omega_2.$$

To get the reversed inequality we observe that u_{x_n} is bounded below away from zero inside the region $|u| \leq \lambda_0 < 1$,i.e., there exists a constant $\delta > 0$ such that $u_{x_n} \geq \delta$ if $|u| \leq \lambda_0 < 1$.

This is true because the density estimate (Theorem 1 of [C.C]) implies that

$$osc_{B_r(x)}(u) \geq \frac{1}{4}$$

for some large r and x inside the region $|u(\lambda)| \leq \lambda_0 < 1$.

Therefore, there exists $\delta = \delta(r, \lambda_0) > 0$ such that

$$\delta \leq \sup_{B_r}\|\nabla u\| \ll \sup_{B_r} u_{x_n}.$$

Then by Harnack's inequality we may conclude that $u_{x_n} \geq \bar{\delta} > 0$ for another constant $\bar{\delta}$. To finish the proof in our second example of case $\delta = 2$, we apply Lemma 4 to $\omega_1 = u_{x_n}$, $\omega_2 = 1 - u$.

In general, we have:

$$\Delta u_{x_n} = f_u \cdot u_{x_n}$$

$$\Delta(1-u) = -f(u) = \int_u^1 f_u(s)\mathrm{d}s \geq f_u(1-u)$$

Taking into account the structural hypothesis made about $f(= F_u)$, in the region where $1-u$ is small we have the following expansion:

$$f_u = a - b(x)(1-u) + o(1-u)$$

$$-f(u) = \Big[a - \frac{1}{2}b(x)(1-u) + o(1-u)\Big](1-u)$$

where $a > 0$ and $b(x)$ is a strictly positive bounded function.

Therefore the function $\omega_2 = (1-u) + C(1-u)^2$, $C > 0$, satisfies

$$\Delta\omega_2 \geq a\{(1-u) + C(1-u)^2\} - \frac{1}{2}b(1-u)^2 + aC(1-u)^2 \geq a\omega_2$$

So long as $aC > \frac{1}{2}\sup b(x)$ and $1-u$ is small enough.

Therefore the two functions $\omega_1 = u_{x_n}$ and $\omega_2 = (1-u) + C(1-u)^2$ satisfy $\Delta\omega_1 \leq a\omega_1$, $\Delta\omega_2 \geq a\omega_2$, $0 \leq \omega_j \leq 1$.

We are then in conditions to apply Lemma 4 to conclude that $\omega_2 \ll \omega_1$, which implies the desired estimate $1-u \ll u_{x_n}$.

The case when u is close to the value -1 can be treated with the same method.

3 Energy Estimates

We have normalized local problems $\Delta u = \frac{1}{\epsilon^2}F'(u)$ $(= \frac{1}{\epsilon^2}f(u))$ in the unit ball B_1 or, equivalently, $\Delta u = f$ inside the ball $B_{1/\epsilon}$. We will assume that $y = x_n$ is a direction of monotonicity and that $\|\nabla u\| \leq K u_y$ for some finite constant K, uniformly on $\epsilon > 0$.

Therefore one may consider the level surfaces $y = \varphi(x, \lambda)$, $(x = (x_1, ..., x_{n-1})$, $0 \leq \lambda \leq 1)$, $u(x, \varphi(x,\lambda)) = \lambda$ which are graphs of Lipschitz functions.

Let u_x be a derivate of u in the "horizontal" x-plane. It is easy to check that the function $\psi = u_x/u_y$ satisfies the (non-uniformly) elliptic equation

$$div(u_y^2 \nabla\psi) = 0.$$

Again, since we are mainly interested in the analysis of the transition layer, for ϵ small, we may assume that $u(0) = 0$. In the following we shall consider separately the cases $0 \leq \delta < 2$ and $\delta = 2$.

Case $0 \leq \delta < 2$**.** In this case $u(= u_\epsilon)$ must reach both extreme values -1, $+1$ inside the cylinder $B_1 \times [-K, +K]$ (or $B_{1/\epsilon} \times [-K/\epsilon, +K/\epsilon]$). We will consider truncations

$$\bar{\psi} = \Big(\frac{u_x}{u_y} - c\Big)_+ \quad \bigg(\text{or } \bar{\psi} = \Big(-\frac{u_x}{u_y} - c\Big)_+\bigg),\ c > 0,$$

and cut-off functions $\eta(x)$ depending only upon the horizontal variables x.

We have

$$\int \eta^2(x)\bar{\psi}(x,y)div(u_y^2\nabla\bar{\psi})\mathrm{d}x\mathrm{d}y = 0.$$

Integration by parts yields

$$\int \|\nabla(\eta\bar{\psi})\|^2 u_y^2 \mathrm{d}x\mathrm{d}y \ll \int |\bar{\psi}|^2 \|\nabla_x \eta\|^2 u_y^2 \mathrm{d}x\mathrm{d}y. \quad (*)$$

In the following, it will be important to write $(*)$ in (x,λ)-coordinates:

$$\begin{aligned} &u_{x_j} + u_y\varphi_{x_j} = 0\\ &u_y \cdot \varphi_\lambda = 1\\ &\mathrm{d}x\mathrm{d}y = \varphi_\lambda \mathrm{d}x\mathrm{d}\lambda \end{aligned}$$

Define $\tilde{\psi}(x,\lambda) = \tilde{\psi}(x,\varphi(x,\lambda))$.

We have:

Proposition 2 *Under the assumptions stated above the following estimate holds*

$$\int_{-1}^{+1}\int_{B_{1/\varepsilon}} \frac{1}{\varphi_\lambda}\left\{|\nabla_x(\eta\tilde{\psi})|^2 + \frac{1}{\varphi_\lambda^2}(\eta\tilde{\psi})_\lambda^2\right\} dx d\lambda \ll \int_{-1}^{+1}\int_{B_{1/\varepsilon}} \frac{1}{\varphi_\lambda}|\tilde{\psi}|^2|\nabla_x\eta|^2 dx d\lambda\ .$$

Proof. Consider

$$\begin{aligned} \phi(x,y) &= \eta(x)\tilde{\psi}(x,y),\\ \tilde{\phi}(x,\lambda) &= \eta(x)\tilde{\psi}(x,\lambda),\\ \phi(x,y) &= \tilde{\phi}(x,\varphi^{-1}(x,\lambda) = \tilde{\phi}(x,u(x,y)). \end{aligned}$$

Therefore

$$\begin{aligned} \nabla_x\phi &= \nabla_x\tilde{\phi} + \tfrac{\partial\tilde{\phi}}{\partial\lambda}\nabla_x u\\ \tfrac{\partial\phi}{\partial y} &= \tfrac{\partial\tilde{\phi}}{\partial\lambda}u_y \end{aligned}$$

If we substitute these formulas in the left hand side of the energy inequality $(*)$ we get:

$$\int\int\left\{|\nabla_x\tilde{\phi}|^2 + \left|\frac{\partial\tilde{\phi}}{\partial\lambda}\right|^2|\nabla_x u|^2 + 2\frac{\partial\tilde{\phi}}{\partial\lambda}\nabla_x\tilde{\phi}\cdot\nabla_x u + \right.$$
$$\left. + \left(\frac{\partial\tilde{\phi}}{\partial\lambda}\right)^2 u_y^2\right\} u_y^2 dx dy = \int\int Q(x,y)u_y^2\ dx\ dy$$

The Lipschitz hypothesis yields $|\nabla u|^2 \leq K^2 u_y^2$ and, therefore, the term $Q(x,y)$ in the integrand must be greater than

$$\frac{C}{K}\left\{|\nabla_x \tilde{\psi}|^2 + u_y^2 \left(\frac{\partial \tilde{\phi}}{\partial \lambda}\right)^2\right\}$$

for some strictly positive constant C. This proves our inequality.

Remark. We have used the fact that $\tilde{\psi}$ vanishes when $u = +1, -1$, allowing us to take cut-off $\eta(x)$ depending only upon the horizontal variables x. If we re-scale back to the unit ball, then we get:

$$\varepsilon = \frac{1}{R}, \ \frac{1}{\varphi_\lambda} = u_y \sim \frac{1}{\varepsilon}(1-|\lambda|)^{\delta/2}$$

i.e.,

$$\int_{-1}^{+1}\int_{B_1}(1-|\lambda|)^{\delta/2}\left\{|\nabla_x(\eta\tilde{\psi})|^2 + \frac{1}{\varepsilon^2}(1-|\lambda|)^{\delta}(\eta\tilde{\psi})_\lambda^2\right\}\,dx\,d\lambda$$
$$\ll \int_{-1}^{+1}\int_{B_1}(1-|\lambda|)^{\delta/2}|\tilde{\psi}|^2|\nabla_x\eta|^2\,dx\,d\lambda\ .$$

Case $\delta = 2$**.**

The main difference with respect to the previous case lies in the fact that now both states ± 1 are not reached inside the ball, instead we only have an exponential decay.

On the other hand, if we take cut-off $\eta = \eta(x,\lambda)$, depending upon λ, then the energy estimate becomes:

$$\int_{-1}^{+1}\int_{B_1}(1-|\lambda|)\left\{|\nabla_x(\eta\tilde{\psi})|^2 + \frac{1}{\varepsilon^2}(1-|\lambda|)^2(\eta\tilde{\psi})_\lambda^2\right\}\,dx\,d\lambda$$
$$\ll \int_{-1}^{+1}\int_{B_1}(1-|\lambda|)|\tilde{\psi}|^2\left\{|\nabla_x\eta|^2 + \frac{1}{\varepsilon^2}(1-|\lambda|)^2\eta_\lambda^2\right\}\,dx\,d\lambda$$

and the term $\frac{1}{\varepsilon^2}(1-|\lambda|)^2\eta_\lambda^2$ gives us problems.

To avoid that term, we shall consider now truncations of the following form

$$\tilde{\psi} = \left[\left(\psi - \frac{\varepsilon^m}{u_y}\right) - C\right]_+ \quad \left(\text{or } \left[\left(-\psi - \frac{\varepsilon^m}{u_y}\right) - C\right]_+\right)\ .$$

Let us recall that in this case we have $u_y \sim \frac{1}{\varepsilon}(1-|\lambda|), \varphi_\lambda \sim \frac{\varepsilon}{1-|\lambda|}, \frac{\varepsilon^m}{u_y} \gg \frac{\varepsilon^{m+1}}{1-|\lambda|}$ which will be greater than K so long as $|\lambda| \geq 1 - C\varepsilon^{m+1}$ for a conveniently fixed constant $C > 0$.

Therefore $\tilde{\psi}$ has compact support on λ and satisfies the equation

$$\operatorname{div}(u_y^2 \nabla\tilde{\psi}) = -\varepsilon^m \Delta u_y \cdot \chi_{\tilde{\psi}>0} = -\varepsilon^{m-2} f(x,u) u_y \chi_{\tilde{\psi}>0} \ .$$

If we use $\eta^2\tilde{\psi}$ as test function then, after integration by parts, we obtain the inequality

$$\int\int |\nabla(\eta\tilde{\psi})|^2 u_y^2 \, dx \, dy \ll \int\int |\tilde{\psi}|^2 |\nabla_x \eta|^2 \, dx \, dy + \varepsilon^{m-2} \int\int \eta^2 \tilde{\psi} u_y \chi_{\tilde{\psi}>0} \, dx \, dy \ .$$

That is

$$\int\int |\nabla(\eta\tilde{\psi})|^2 u_y^2 \ dx \ dy \ll \int\int |\tilde{\psi}|^2 |\nabla_x \eta|^2 \ dx \ dy + O(\varepsilon^{m-2})$$

where $m \geq 3$ will be fixed later.

Writing this inequality in the (x,λ) coordinates, we obtain

Proposition 3 *Under the assumptions stated above, the following estimate holds:*

$$\int_{-1}^{+1} \int_{B_1} (1-|\lambda|) \left\{ |\nabla_x(\eta\tilde{\psi})|^2 + \frac{1}{\varepsilon^2}(1-|\lambda|)^2 (\eta\tilde{\psi})_\lambda^2 \right\} \ dx \ d\lambda$$
$$\ll \int_{-1}^{+1} \int_{B_1} (1-|\lambda|) |\tilde{\psi}|^2 |\nabla_x \eta|^2 \ dx \ d\lambda + O(\varepsilon^{m-2}) \ .$$

Our next step will be to use these energy estimates to produce (L^2, L^p) embeddings for those truncations. As before, we shall consider separately the cases $0 \leq \delta < 2$ and $\delta = 2$.

Proposition 4 *For each $\delta, 0 \leq \delta < 2$, there exists $p = p(\delta) > 2$ such that if $\phi(x,\lambda)$ has compact support we have*

$$\left[\int_{-1}^{+1} \int_B (1-|\lambda|)^{\delta/2} |\phi(x,\lambda)|^p \ dx \ d\lambda \right]^{2/p}$$
$$\ll \int_{-1}^{+1} \int_B (1-|\lambda|)^{\delta/2} \left\{ |\nabla_x \phi|^2 + (1-|\lambda|)^\delta (\phi_\lambda)^2 \right\} \ dx \ d\lambda \ .$$

Proof. a) The case $\delta = 0$ is just a consequence of the standard Sobolev's embedding and we can take $p \leq \frac{2n}{n-2}$.

b) Let us assume now that $0 < \delta < 2$. We shall consider separately the domain $0 \leq u < 1$ (resp. $-1 < u \leq 0$).

In the region $0 \leq \lambda < 1$ we make a change of variables:

$$(1-\lambda)^{\delta/2}\ d\lambda = dt \text{ i.e., } t = \frac{2}{2+\delta}(1-\lambda)^{\frac{2+\delta}{2}} .$$

Therefore if $\tilde{\phi}(x,t) = \phi(x,\lambda)$ we have

$$\nabla_x \tilde{\phi} = \nabla_x \phi \ , \ \frac{\partial \phi}{\partial \lambda} = C_\delta t^{\frac{\delta}{2+\delta}} \frac{\partial \tilde{\phi}}{\partial t}$$

and our proposition asks for the estimation of

$$\left[\int_{-1}^{+1} \int (\tilde{\phi})^p \ dx \ dt\right]^{2/p}$$

in terms of the energy

$$\int_{-1}^{+1} \int \left\{ |\nabla_x \tilde{\phi}|^2 + t^{\frac{4\delta}{2+\delta}} (\tilde{\phi}_t)^2 \right\} dx \ dt \ .$$

Let us recall that since $\tilde{\phi}(x,0) = 0$ we have

$$|\tilde{\phi}(x,t)| \ll \int \int_{\Gamma_{x,t}} \frac{|\nabla_x \tilde{\phi}(z,s)| + |\tilde{\phi}_t(z,s)|}{(|x-z| + |t-s|)^{n-1}} \ dz \ ds$$

where $\Gamma_{x,t} = \{(z,s) | |x-z| \leq t-s\}$. Therefore we get

$$|\tilde{\phi}(x,t)| \ll \int \int \frac{|\nabla_x \tilde{\phi}(x-z,t-s)| + |t-s|^{\frac{2\delta}{2+\delta}} |\tilde{\phi}_t(x-z,t-s)|}{[|x-z|^a \cdot |t-s|^b]^{\frac{2\delta}{2+\delta}} [|z| + |s|]^{n-1}} \ dz \ ds$$

where a, b are positive numbers to be chosen later under the condition $a + b = 1$.

Let us denote

$$F(x,t) = \frac{|\nabla_x \tilde{\phi}(x,t)| + t^{\frac{2\delta}{2+\delta}} |\tilde{\phi}_t|}{[|x|^a t^b]^{\frac{2\delta}{2+\delta}}}$$

Since convolution with the kernel $\frac{1}{[|z|+|s|]^{n-1}}$ maps L^r to L^p if $\frac{1}{p} > \frac{n-1}{n} + \frac{1}{r} - 1$ and we want to have $p > 2$, we need $F \in L^r$ for $\frac{1}{r} < \frac{1}{2} + \frac{1}{n}$. Furthermore Hölder's inequality yields

$$||F||_r \ll \left(\int_0^1 \int \left\{|\nabla_x(\tilde{\phi})|^2 + t^{\frac{4\delta}{2+\delta}}(\tilde{\phi}_t)^2\right\} dx\ dt\right)^{1/2} \left[\int_0^1 \int [|x|^a t^b]^{-\frac{2\delta r}{2+\delta}\frac{2}{2-r}}\right]^{\frac{2-r}{2}}$$

Therefore our proposition follows because if $0 \leq \delta < 2$ we can always choose r so that

$$\frac{2n}{n+2} < r < \frac{2n}{n + \frac{4\delta}{2+\delta}}$$

and then take $b = \frac{1}{n}, a = 1 - \frac{1}{n}$.

Case $\delta = 2$**.**

Recall that in the (x, λ) coordinate system the energy inequality reads

$$\int\int(1-|\lambda|)\left\{|\nabla_x(\eta\tilde{\psi})|^2 + \frac{1}{\varepsilon^2}(1-|\lambda|)^2(\eta\tilde{\psi})^2_\lambda\right\} dx\ d\lambda$$
$$\ll \int\int(1-|\lambda|)|\tilde{\psi}|^2|\nabla_x\eta|^2\ dx\ d\lambda + O(\varepsilon^{m-2})$$

where $\tilde{\psi} = [(\frac{u_x}{u_y}) - \frac{\varepsilon^m}{u_y} - c]_+, c \geq 0$, (or $\tilde{\psi} = [(-\frac{u_x}{u_y} - \frac{\varepsilon^m}{u_y}) - c]_+$) and $\tilde{\psi}$ vanishes when $|\lambda| \gg 1 - \varepsilon^{m+1}$. Therefore we used the following embedding:

Proposition 5 *If $\phi(x, \lambda)$ has compact support inside the region $|x| \leq 1, |\lambda| < 1 - \varepsilon^{m+1}$ then it satisfies*

$$\left[\int\int(1-|\lambda|)|\phi(x,\lambda)|^p\ dx\ d\lambda\right]^{2/p} \ll \int(1-|\lambda|)\left\{|\nabla_x\phi|^2 + \frac{(1-|\lambda|)^2}{\varepsilon^2}\phi^2_\lambda\right\} dx\ d\lambda$$

for a certain $p > 2$.

Proof. Let us work first inside the region $0 \leq \lambda < 1 - \varepsilon^{m+1}$. The change of variables

$$t = (1-\lambda)^2, \quad dt = -2(1-\lambda)d\lambda$$

leads us to estimate $[\int\int |\phi(x,t)|^p\ dx\ dt]^{2/p}$ in terms of the energy

$$\int\int\left\{|\nabla_x\phi|^2 + \frac{t^2}{\varepsilon^2}(\phi_t)^2\right\} dx\ dt\ .$$

For functions ϕ compactly supported in x and vanishing at $t \le \varepsilon^M$, $M = 2(m+1)$.

We have:

$$|\phi(x,t)| \ll \int\int_\Gamma \frac{|\nabla_x \phi(x-z,t-s)| + |\phi_t(x-z,t-s)|}{(|z|+|s|)^{n-1}} \; dz \; ds$$

where $\Gamma = \{|(z,s)| \; |z| \le s, s \ge \varepsilon^M\}$. That is

$$\begin{aligned} |\phi(x,t)| &\ll \int\int \frac{|\nabla_x \phi(x-z,t-s)|}{(|z|+|s|)^{n-1}} \; dz \; ds \; + \\ &+ \int\int \frac{\frac{\varepsilon^M + |t-s|}{\varepsilon} |\phi_t(x-z,t-s)|}{\left[\frac{\varepsilon^M + |x-z|}{\varepsilon}\right]^a \left[\frac{\varepsilon^M + |t-s|}{\varepsilon}\right]^b} \frac{1}{(|z|+|s|)^{n-1}} \; dz \; ds \\ &= \phi_1 + \phi_2 \end{aligned}$$

where, as before, we have taken positive numbers a, b to be fixed later, satisfying the condition $a + b = 1$.

Clearly for ϕ_1 we have the inequality

$$\left[\int\int |\phi_1(x,t)|^{p_1} \; dx \; dt\right]^{2/p_1} \ll \int\int |\nabla_x \phi(x,t)|^2 \; dx \; dt$$
$$\text{if } \frac{1}{p_1} > \frac{1}{2} - \frac{1}{n} \text{ i.e., } 2 \le p_1 < \frac{2n}{n-2}$$

To estimate ϕ_2 we consider the convolution

$$\phi_2 = F * \frac{1}{(|z|+|t|)^{n-1}} \; .$$

Therefore we have

$$||\phi_2||_p \le ||F||_q \text{ if } \frac{1}{p} \ge \frac{1}{q} - \frac{1}{n} \; .$$

Since

$$F(x,t) = \frac{\frac{\varepsilon^M + |x|}{\varepsilon} |\psi_t|}{\left[\frac{\varepsilon^M + |x|}{\varepsilon}\right]^a \left[\frac{\varepsilon^M + |t|}{\varepsilon}\right]^b}$$

we want

$$||F||_q \ll \left|\left| \frac{\varepsilon^M + |t|}{\varepsilon} \psi_t \right|\right|_2 \cdot \left|\left| \left[\frac{\varepsilon^M + |x|}{\varepsilon}\right]^{-a} \left[\frac{\varepsilon^M + |t|}{\varepsilon}\right]^{-b} \right|\right|_r .$$

Let us take $\frac{q}{2} + \frac{1}{s} = 1, r = \frac{2q}{2-q}$ and observe that

$$\int_{|x|\le 1} \left[\frac{\varepsilon^M + |x|}{\varepsilon}\right]^{-a\frac{2q}{2-q}} dx \ll \int_0^\infty \frac{t^{n-2}\ dt}{(1+t)^{\frac{2aq}{2-q}}} \cdot \varepsilon^{M(n-1)-(M-1)\frac{2aq}{2-q}}$$

We want:

$$\begin{cases} M(n-1) = (M-1)\frac{2aq}{2-q} \\ a\frac{2q}{2-q} > n-1 \\ M = (M-1)\frac{2bq}{2-q} \\ b\frac{2q}{2-q} > 1 \end{cases}$$

That is, we need to choose

$$\frac{1}{p} > \frac{1}{q} - \frac{1}{n}\ , \quad q < 2, p > 2$$

$$Mn = (M-1)\frac{2q}{2-q}$$

and also $a + b = 1$ in such a way that $a\frac{2q}{2-q} > n-1, b\frac{2q}{2-q} > 1$.

If q_0 is given by $\frac{2q_0}{2-q_0} = n$ then we have $\frac{M}{M-1}n > \frac{2q_0}{2-q_0}$. Therefore, the solution q to the equation $\frac{2q}{2-q} = \frac{M}{M-1}n$ must satisfy the inequality $q_0 < q < 2$. To finish we choose p by the formula

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{n} < \frac{1}{2}$$

and take $a = 1 - \frac{1}{n}, b = \frac{1}{n}$, which satisfy all the requirements.

4 Proof of Theorem 2

Our proof of the uniform smoothness of the level sets, follows closely the technique introduced by E. de Giorgi to study the regularity of solutions of uniformly elliptic, second order equations, given in divergence form. We shall apply it to the functions $\psi = \varphi_x(x;\lambda)$ i.e., to first derivatives of φ on any horizontal direction. Here we will also assume that $y = x_n$ is a direction of monotonicity and that the level surfaces $y = \varphi(x;\lambda)$ are uniformly Lipschitz.

The argument consists mainly in two steps:

Step 1. An $L^2 \to L^\infty$ inequality, i.e., a control in measure which produces a uniform estimate.

Step 2. An oscillation decrease: There exist constants $\beta > 0, \rho < 1, \tau(\beta) > 0$ such that if $\tilde{\psi}$ (a truncation of ψ) is smaller than 1 in the ball B_r, and if the following quotient of Lebesgue measures is bounded below by β:

$$\mu\{(\bar{\psi} = 0) \cap B_r\}/\mu(B_r) \geq \beta \ .$$

Then, in the smaller ball $B_{\rho r}$, we have the estimate

$$\sup_{B_{\rho r}} \bar{\psi} \leq 1 - \tau(\beta) \ .$$

This second step when applied conveniently either to the positive or the negative part of ψ produces the oscillation decrease

$$\mathrm{osc}_{B_{\rho r}} \ (\psi) \leq (1 - \tau(\beta)) \ \mathrm{osc}_{B_r} \ (\psi)$$

whose iteration yields the desired Hölder continuity of $\psi = \varphi_x$.

To simplify the notation, in the following we will assume, without any loss of generality, that the Lipschitz constant is $K = 1$. Also with a convenient change of scale we can renormalize always to the unit ball B_1 .

We will consider functions $\psi \in H_0^1(\Omega \times (-1, 1))$ whose truncations $\phi = (\psi - c)_+$ or $(-\psi - c)_+$, satisfy the energy estimate as well as the (L^2, L^p) embedding, given by propositions $2, 3, 4$ and 5. And we shall assume also that $\overline{B_1(0)} \subset \Omega$.

Proposition 6 *For a given a truncation ϕ let us consider the integral*

$$b = \int_{-1}^{+1} \int_{B_1} (1 - |\lambda|)^{\delta/2} (\phi(x, \lambda))^2 \ dx \ d\lambda$$

then we have:

a) Case $0 \leq \delta < 2$**:** *There exists a positive number $b_0 > 0$ such that if $b \leq b_0$ then*

$$\sup_{\substack{x \in B_{1/2} \\ |\lambda| < 1}} \phi(x, \lambda) \leq 1$$

b) Case $\delta = 2$: *For each $\gamma < 1$ there exists $b_0(\gamma) > 0$ such that $b \le b_0(\gamma)$ implies*

$$\sup_{\substack{x \in B_{1/2} \\ |\lambda| < \gamma}} \phi(x, \lambda) \le 1$$

Proof. Let η_k, for each $k = 1, 2, \dots$, be a cut-off satisfying:

i) $\eta_k \equiv 1$ inside $B_{\frac{1}{2}+2^{-k}}(0) = B_k$

ii) $\eta_k \equiv 0$ outside $B_{\frac{1}{2}+2^{-(k-1)}}(0) = B_{k-1}$

iii) $|\nabla_{\eta_k}| \le C2^k$, for some fixed constant C.

a) For $0 \le \delta < 2$ we define the family of truncations

$$\phi_k(x, \lambda) = \left[\phi(x, \lambda) - \left(\frac{1}{2} - 2^{-k}\right)\right]_+$$

Let $p = p_\delta > 2$ be given by Proposition 5 and take $r : \frac{1}{r} + \frac{2}{p} = 1$. We obtain:

$$\int\int (1-|\lambda|)^{\delta/2}[\eta_k(x)\phi_k(x,\lambda)]^2 \, dx \, d\lambda \ll \left[\int_{-1}^{+1}\int_{\{\eta_k\phi_k>0\}} (1-|\lambda|)^{\delta/2}\eta_{k-1}^2(x) \, dx \, d\lambda\right]^{1/r} \cdot$$
$$\cdot \left[\int\int (1-|\lambda|)^{\delta/2}(\eta_k\phi_k)^p \, dx \, d\lambda\right]^{2/p}$$

Observe that $\phi_k > 0 \Leftrightarrow \phi_{k-1} > 2^{-k}$ which produces,

$$\int\int_{\{\eta_k\phi_k>0\}} (1-|\lambda|)^{\delta/2}\eta_{k-1}^2(x) dx \, d\lambda \ll 2^{2k}\int\int (1-|\lambda|)^{\delta/2}[\eta_{k-1}(x)\phi_{k-1}(x,\lambda)]^2 \, dx \, d\lambda \, .$$

On the other hand, we have:

$$\left[\int\int (1-|\lambda|)^{\delta/2}(\eta_k\phi_k)^p \, dx \, d\lambda\right]^{1/p} \ll$$
$$\ll \left[\int\int (1-|\lambda|)^{\delta/2}(\phi_k)^2|\nabla_x\eta_k|^2 \, dx \, d\lambda\right]^{1/2}$$
$$\ll 2^k\left[\int\int (1-|\lambda|)^{\delta/2}(\eta_{k-1}\phi_{k-1})^2 \, dx \, d\lambda\right]^{1/2} \, .$$

Therefore, if we define the sequence

$$b_k = \int\int (1-|\lambda|)^{\delta/2} (\eta_k \phi_k)^2 \ dx \ d\lambda$$

we have obtained the following recursion

$$b_k \leq C 2^{k(1+\alpha)} b_{k-1}^{1+\alpha}$$

with $\alpha = \frac{2}{p} > 0$.

It is well known that such a recurrence produces

$$\lim_{k\to\infty} b_k = 0$$

so long as b_0 is small enough.

b) In the case $\delta = 2$ we to take the following truncations

$$\phi_k = \left[\left(\frac{u_x}{u_y} - \frac{\epsilon^m}{u_y} \right) - \left(\frac{1}{2} - 2^{-k} \right) \right]_+, \text{ or } \left[\left(-\frac{u_x}{u_y} - \frac{\epsilon^m}{u_y} \right) - \left(\frac{1}{2} - 2^{-k} \right) \right]_+.$$

Similarly to the preceeding case we get recurrence for the sequence

$$b_k = \int\int (1-|\lambda|)[\eta_k(x)\phi_k(x,\lambda)]^2 dx d\lambda.$$

Namely,

$$b_k \leq C 2^{2k} b_{k-1} [2^{2k} b_{k-1} + \epsilon^{m-2}]^\alpha$$

which leads to

$$\limsup b_k \ll \epsilon^{(m-2)(1+\alpha)}$$

if b_0 is small enough (but independently of ϵ).

More precisely, if b_0 is small enough, then there exists k_0 such that for $k \geq k_0$ we have $b_k \leq \epsilon^{(m-2)(1+\alpha)}$. Since $|\nabla \phi| = O(1/\epsilon)$ uniformily inside the region $|\lambda| \leq \eta < 1$, the estimate above implies

$$\sup_{\substack{x\in B_{1/2} \\ |\lambda|<\eta}} \phi(x,\lambda) \leq 1$$

if m is big enough, *q.e.d.*

Next, given $0 \leq \phi \leq 1$ being one of the truncations considered in Proposition 6, let us define the sets:

$$N(\lambda) = \{|x| \leq r | \phi(x, \lambda) = 0\}.$$

Proposition 7 *For each $\gamma, 0 \leq \gamma < 1$, there exists a strictly positive constant $C(\gamma)$ such that if*

$$\int_{-\gamma}^{+\gamma} \mu\{N(\lambda)\} d\lambda \geq \gamma \mu(B_r)$$

then

$$\sup_{\substack{x \in B_{r/2} \\ |\lambda| \leq \gamma}} \phi(x, \lambda) \leq 1 - C(\gamma).$$

Corollary 8. *Applying Proposition 7 either to ϕ_+ or ϕ_-, as many times as necessary, we obtain*

$$osc_{\substack{B_{(1/2)^k} \\ |\lambda| \leq \gamma}} (\phi) \leq (1 - C(\gamma))^k.$$

Therefore

$$|\phi(x, \lambda) - \phi(z, \lambda)| \ll |x - z|^{\alpha}$$

with $\alpha = -\log(1 - C(\gamma))/\log 2$.

Proof of Proposition 7. We shall work in the normalized case $r = 1$. The proof for a general radius will then follows by a change of scale. Obviously, if

$$\mu(N) = \int_{-\gamma}^{+\gamma} \mu(N(\lambda)) d\lambda$$

is big enough, then we can apply Proposition 6 to finish the proof.

Let us take the following family of scaled truncations:

$$\phi_k(x, \lambda) = 2^k [\phi(x, \lambda) - (1 - 2^{-k})]_+ \eta(x)$$

with a cut-off $\eta \equiv 1$ in $B_{3/4}, \eta = 0$ outside B_1. We have the sets:

$$N_k(\lambda) = \{x \in B_{3/4} | \phi_k(x, \lambda) = 0\}$$

$$G_k(\lambda) = B_{3/4} - N_k(\lambda) = \left\{ x \in B_{3/4} | \phi_{k-1}(x, \lambda) > \frac{1}{2} \right\}$$

and the inclusions:

$$N_k(\lambda) \subset N_{k+1}(\lambda), k = 1, 2, \ldots$$

$$G_k(\lambda) \supset G_{k+1}(\lambda).$$

Our goal is to show that $\lim_{k\to\infty} \mu(G_k(\lambda)) = 0$, i.e., after a finite number of steps we will be in position to apply Proposition 6 to the function ϕ_k and finish the proof.

First, let us show that

$$(*) \qquad \sup_{|\lambda|\leq\gamma} \mu(G_k(\lambda)) \leq \inf_{|\lambda|\leq\gamma} \mu(G_{k-1}(\lambda)) + O(\epsilon).$$

This is true because given λ_1, λ_2 in $[-\gamma, +\gamma]$ we have:

$$\int_{N_k(\lambda_1)} \phi_k(x, \lambda_2)dx = \int_{N_k(\lambda_1)} [\phi_k(x, \lambda_2) - \phi_k(x, \lambda_1)]dx$$

$$\ll \left[\int_{N_k(\lambda_1)} \int_{-\gamma}^{+\gamma} (\phi_k)_\lambda^2 dx d\lambda\right]^{1/2}$$

$$\ll \left[\int\int (1 - |\lambda|)^{\frac{3\delta}{2}} (\phi_k)_\lambda^2 dx d\lambda\right] (1 - \gamma)^{-\frac{3\delta}{4}}$$

$$\ll \epsilon \text{ by the energy inequality.}$$

Therefore, since $N_{k-1}(\lambda_1) = G_{k-1}^c(\lambda_1)$ we get

$$\frac{1}{2}\mu(G_k(\lambda_2)) \leq \int_{G_k(\lambda_2)} \phi_{k-1}(x, \lambda_2)dx$$

$$= \int_{G_{k-1}(\lambda_1)\cap G_k(\lambda_2)} \phi_{k-1}(x, \lambda_2)dx + \int_{G_k(\lambda_2)\cap N_{k-1}(\lambda_1)} \phi_{k-1}(x, \lambda_2)dx$$

$$\leq \mu(G_{k-1}(\lambda_1)) + O(\epsilon).$$

This proves $(*)$.

Next, let us define the maximal functions

$$\overline{\phi}_k(x) = \sup_{|\lambda|\leq\gamma} \phi_k(x, \lambda)$$

we choose $\lambda_1 (|\lambda_1| \leq \gamma)$ satisfying

$$\mu(N_k(\lambda_1)) \geq \frac{1}{2}\mu(B_{3/4}).$$

Then

$$\overline{\phi}_k(x) - \phi_k(x, \lambda_1) \leq \left[\int_{-\gamma}^{+\gamma} (\phi_k)_\lambda^2\right]^{1/2}$$

that is,

$$\int_{N_k} \overline{\phi}_k(x)dx \ll \left[\int_{-\gamma}^{+\gamma} \int_{N_k(\lambda_1)} (\phi_k)_\lambda^2 dx d\lambda\right]^{1/2}$$

$$\ll (1-\gamma)^{-\frac{3\delta}{4}} \left[\int_{-\gamma}^{+\gamma} \int_B (1-|\lambda|)^{\frac{3\delta}{2}} (\phi_k)_\lambda^2 dx d\lambda\right]_+^{1/2}$$

$$= O(\epsilon), \text{again by the energy inequality.}$$

Therefore $\mu\{x \in N_k(\lambda_1) | \overline{\phi}_k(x) \geq \frac{1}{2}\} = O(\epsilon)$, i.e., for each $|\lambda| \leq \gamma$ we have

$$\mu\left\{x | \phi_k(x, \lambda) < \frac{1}{2}\right\} \geq \mu(N_k(\lambda_1)) - C\epsilon \geq \frac{1}{2}\mu(B_{3/4}) - C\epsilon.$$

Since $\phi_k(x, \lambda) < \frac{1}{2} \iff \phi_{k+1} = 0$ we obtain

$$\mu\left\{N_{k+1}(\lambda)\right\} \geq \frac{1}{2}\mu(B_{3/4}) - C\epsilon.$$

Given $x \in G_k(\lambda)$ we have the estimate

$$\frac{1}{2} \leq \int_{\sigma_1(x,\lambda;\xi)}^{\sigma_2(x,\lambda;\xi)} \nabla\phi_{k-1}(x - t\xi, \lambda) \cdot \xi dt$$

where $\xi \in \Sigma_x = \{\xi \in S^{n-2} | \exists t > 0, x - t\xi \in N_{k-1}(\lambda)\}$ and $\sigma_1(x, \lambda; \xi), \sigma_2(x, \lambda; \xi)$ are chosen in such a way that

$$x - \sigma_1(x, \lambda; \xi) \cdot \xi \in \overline{G}_k(\lambda)$$

$$x - \sigma_2(x, \lambda; \xi) \cdot \xi \in \overline{N}_k(\lambda)$$

$$\{x - t\xi | \sigma_1(x, \lambda; \xi) < t < \sigma_2(x, \lambda; \xi)\} \subset G_{k-1}(\lambda) - G_k(\lambda).$$

The lower bound estimate for the measures of the sets $N_k(\lambda)$ yields a lower bound for the area of the spherical caps:

$$\sigma(\Sigma_x) \geq C > 0.$$

Therefore, we obtain:

$$0 < \frac{C}{2} \leq \int_{S^{n-2}} \int_{\sigma_1(x,\lambda;\xi)}^{\sigma_2(x,\lambda;\xi)} |\nabla\phi_{k-1}(x - t \cdot \xi, \lambda)| dt d\lambda$$

$$\leq \int_{G_{k-1}(\lambda)-G_k(\lambda)} \frac{|\nabla\phi_{k-1}(x-z,\lambda)|}{|z|^{n-2}} dy$$

Next, we integrate over the set $G_k(\lambda)$:

$$\frac{C}{2}\mu(G_k(\lambda)) \leq \int\int \chi_{G_k^*(\lambda)}(x)\chi_{D_k^*(\lambda)}(z)\frac{|\nabla\phi_{k-1}(x-z,\lambda)|^*}{|z|^{n-2}}\mathrm{d}x\mathrm{d}z$$

where $D_k(\lambda) = G_{k-1}(\lambda) - G_k(\lambda)$ and *-denotes non increasing rearrangement.
We get

$$\mu(G_k(\lambda)) \ll \Big[\int |\nabla\phi_{k-1}(x,\lambda)|^*\mathrm{d}x\Big]\cdot\sup_x\Big[\int \chi_{G_k^*(\lambda)}(x-y)\chi_{D_k^*(\lambda)}(y)\frac{1}{|y|^{n-2}}\mathrm{d}y\Big] \ll$$

$$\ll \Big[\mu(G_{k-1}(\lambda)) - \mu(G_k(\lambda))\Big]^{\frac{1}{n-1}}$$

which implies that $\lim_{k\to\infty}(G_k(\lambda)) = 0$.

5 Proof of Theorem 1

To finish the reduction of Theorem 1 to Theorem 2 we have to show that if one level set, says u=0, is Lipschitz then all the others above it (respectively below it) are also Lipschitz. From now on we will separate both cases: Above the level u=0 it is convenient to change u to 1-u (respect. to 1+u below). The transformed functionals are now

$$J(u) = \int \{\|\nabla u\|^2 + F_\delta(u)\}\mathrm{d}x, \quad F_\delta(u) \sim u^\delta$$

with $0 \leq \delta \leq 2$ and $0 \leq u \leq 1$. Here we have a Lipschitz level set, says at u=1. In our discussion we will use the following series of papers: [B.C.N], [C], [A.C], [P], [A.P].

Theorem 3 *Given $0 < t_0 \leq 1$ and $K < \infty$ there exists a finite $R(t_0, K)$ so that a non-negative minimizer of*

$$J(u) = \int_{\Omega \cap B_R(0)} \{\|\nabla u\|^2 + F_\delta(u)\}\mathrm{d}x$$

where $R \geq R(t_0, K)$, $\Omega = \{x_n > f(x)'\}, \|f\|_{Lip} \leq K, u|_{\partial\Omega} = 1$, must satisfy:

a) If $t_0 > 0$, then u is monotone respect to the direction x_n in the domain $\{u > t_0\}$ and, for any $t > t_0$, all level surfaces $\{u = t\} = \{x_n = f^t(x')\}$ are Lipschitz graphs with norm $\|f^t\|_{Lip} \leq 2K$.

b) In the case $\delta < 2$, t_0 can be taken equal to 0.

Corollary 9 *u is monotonic in any direction going "inwards" of Ω and all level surfaces of u are Lipschitz uniformly.*

Proof.

a) Here t_0 is strictly bigger than zero. This part follows by compactness, suppose that the statement is false: Then there is, for $R_k \uparrow \infty$, a sequence of non-negative solutions u_k and a sequence of points P_k inside the ball $B_{\frac{R_k}{2}}(0)$ so that

$$u_k(P_k) \geq \bar{t_0} > t_0$$

and

$$-\frac{\|\nabla_{x'} u_k(P_k)\|}{(u_k)_{x_n}(P_k)} > 2K$$

We can translate each solution u_k in such a way that P_k becomes the new origin. Taking a subsequence, if necessary, we can assume that those translates, which we shall denote also u_k, will converge uniformly on compact sets to a global solution u^∞, inside a limiting domain belonging also to the same Lipschitz class. Since $t_0 > 0$ we have from the gradient estimate that $(u^\infty)_{x_n} \neq 0$ and ∇u_k converges to ∇u^∞. But this contradicts the fact that for global solutions in a K-Lipschitz domain, we must have $-\|\nabla_{x'} u^\infty\|/(u^\infty)_{x_n} \leq 2K$ (see ref [B.C.N]).

b) In the case $\delta < 2$ we can apply the free boundary regularity theory. For different values of δ there are different results (see previous quoted references) but a common one says that flatness implies $C^{1,\alpha}$. That is, if the free boundary is trapped between two closed enough hyperplanes, inside the ball of radius r, then it is $C^{1,\alpha}$, for some $\alpha > 0$, in the ball of radius $\frac{r}{2}$.

Now, again by compactness, we can show that for R large enough every point in the free boundary becomes sufficently flat.

Let θ be the necessary flatness for the free boundary to be regular. That is θ is a universal constant such that, if the free boundary inside a ball of radius r is contained between two hyperplanes at distance θr, then all level surfaces of u in the ball of radius $\frac{r}{2}$ are uniformly $C^{1,\alpha}$.

Let us recall that solutions about a Lipschitz graph having all level surfaces Lipschitz, are uniformly $C^{1,\alpha}$ from our previous theorem 2. Therefore we can choose ρ_0 small enough so that the free boundary of any global solution is $\frac{\theta}{2}$ flat in $B_{\rho_0}(x)$ for any free boundary point x.

Fix now $t_0 > 0$ so that, from non-degeneracy, the set $\{u < t_0\}$ is contained in a $\frac{\rho_0}{4}$ neighborhood of the free boundary.

Claim: For $\delta < 2$ there exits $R_0 = R_0(\delta, K)$ such that if u is defined in B_R, for $R > R_0$, then all conclusions a) holds for any $t > 0$.

To prove the claim it is sufficient to show that for R large enough the free boundary of u becomes θ-flat.

If not, as before one can find a sequence of minimizers u_k and points x_k in the free boundary of u_k which are not θ-flat inside the ball B_{ρ_0}. But u_k converge uniformly to u^∞, and the non-degeneracy property implies that the free boundary of u_k also converge uniformly to the free of u^∞. But this is a contradiction because the last free boundary has to be $\frac{\theta}{2}$-flat in B_{ρ_0}.

Appendix I

In this appendix we show how to extend the results on [B.C.N] to solutions of free boundary problems. That is, we prove that global solutions of the free boundary problem above a Lipschitz graph have Lipschitz level sets.

More precisely we prove.

Theorem 4 *Let Ω be a domaion of the form $\Omega = \{x \, : \, x_n > f(x'), \; f \; Lipschitz\}$. Let u be a Non-negative solution in Ω of the δ-problem with $0 \leq \delta < 2$ and $u|_{\partial\Omega} = 1$.*

Then u is unique and monotone decreasing in the x_n-direction.

Corollary *u is monotone in any direction going "inwards" of Ω, and all level surfaces of u are uniformly Lipschitz with the same norm than f.*

In order to prove the theorem we shall need some properties of u and some comparison techniques. Namely, a comparison argument with the first non-negative eigenfunction of the Laplacian in a ball, as was done in reference [B.C.N]. Using the fact that u is weakly superharmonic we obtain:

1) Near the fixed boundary u is a C^α superharmonic function satisfying:

$$u \leq 1 - d(x, \partial\Omega)^\theta$$

for some $0 < \theta = \theta(\delta, \|f\|_{Lip})$, $0 \leq \delta < 2$.

2) The free boundary $F = \partial\{u > 0\}$ is at a finite positive distance from $\partial\Omega$:

$$0 < C_0 \leq d(F, \partial\Omega) \leq C_1 < \infty$$

where C_0, C_1 depend only on δ and the Lipschitz norm of f.

3) Let v_1 (respectively v_2) be a non-negative subsolution (respectively supersolution) in the upper half-plane of $\Delta u = u^\alpha$, $0 \leq \alpha < 1$. Then we have the exact growth:

i) If v_1 is not identically equal to zero near the origin and $v_1|_{\{x_n=0\}} \equiv 0$ then:

$$v^\lambda = \sup_{\{x_n=0\}} v_1 \geq C(\alpha)\lambda^\beta$$

where $4\beta = 2/(1-\alpha)$, $C(\alpha) = [(1-\alpha)^2/(2(1+\alpha)]^{1/(1-\alpha)}$.

ii) If v_2 vanishes at the origin then:

$$v_\lambda = \inf_{\{x_n=0\}} v_2 \leq C(\alpha)\lambda^\beta.$$

The proof follows from the observations

$$(v^\lambda)'' \geq (v^\lambda)^\alpha, \qquad (v_\lambda)'' \leq (v_\lambda)^\alpha$$

$$v^\lambda(0)=0,\quad (v^\lambda)'(0)\geq 0,\qquad v_\lambda(0)=0,\quad (v_\lambda)'(0)=0$$

Theorem 4 will be a consequence of the following:

Let u_1, u_2 be two (possibly different) solutions of our problem ($0\leq\delta<2$) in the same Lipschitz domain Ω. Given $\lambda>0$ let us consider $u_1^\lambda(x)=u_1(x+\lambda e_n)$, $e_n=(0,\ldots,0,1)$. Then we have $u_1^\lambda\leq u_2$.

The proof relies on the translation comparison method:

Let us consider

$$w_1(x)=\inf_{B_\sigma(x)} u_1,\qquad w_2(x)=\sup_{B_\sigma(x)} u_2.$$

Then w_1 is a supersolution of the δ-problem and w_2 a subsolution. Furthermore, the level surfaces of w_1: $\partial S_t=\partial\{w>t\}$ have a tangent ball from "inside", while those of w_2 a tangent ball from "outside".

For large value of λ, $w_1(x+\lambda e_n)\leq w_2(x)$ and all the level surfaces up to the free boundary are at positive distance from each other. Let λ_0 be the smallest λ for which the free boundary remains at positive distance, and assume that $\lambda_0\gg\sigma$.

After a sequence of translations we suppose that the free boundaries of $w_1(x+\lambda_0 e_n)$ and w_2 "touch" at a point, say 0, placed at finite distance from $\partial\Omega$, and with a well defined normal. Since ($\lambda_0\gg\sigma$) $w_1(x+\lambda_0 e_n)|_{\partial\Omega}$ is strictly less than one, while $w_2|_{\partial\Omega}\equiv 1$.

Therefore one can multiply our original u_1 by $1+\epsilon$, for certain $\epsilon>0$, and construct the corresponding w_1 for the new supersolution. Observe that

$$\Delta((1+\epsilon)u_1)=(1+\epsilon)^{1-\alpha}((1+\epsilon)u_1)^\alpha\geq((1+\epsilon)u_1)^\alpha.$$

But then if w_1,w_2 "touch" at the free boundary it cannot happens that $w_2\geq w_1$ at the fixed boundary $\partial\Omega$.

Appendix II

We shall consider here a special case of de Giorgi's conjecture, namely the following: Let $-1\leq u\leq 1$ be a local minimizer of $J(u)=\int_{\Re^n}(|\nabla u|^2+\chi_{|u|<1})dx$ in the whole space $\Re^n$. Let us also assume that $\lim_{x_n\to\pm\infty}u=\pm 1$ and that $n\leq 7$.

Theorem 5 *Under the hypothesis stated above u is one-dimensional (i.e. its level sets are hyperplanes).*

<u>Sketch of the proof</u>

It is well known that in this case we have the equivalent free boundary problem:

$$\begin{cases} \Delta u & = 0 \quad \text{inside} \quad |u| < 1 \\ |\nabla u| & = 1 \text{ on } \Sigma_+ \cup \Sigma_-, \ (\Sigma_\pm = \{u = \pm 1\}) \end{cases}$$

We will show that flatness implies Lipschitz and therefore one reaches the position to apply theorem 2.

Since u is a local minimizer on $B_R(0)$, for each $R \uparrow \infty$, we can scale everything to the unit ball $B_1(0)$, obtaining local minimizer of

$$J_\epsilon(v_\epsilon) = \int_{B_1} (\epsilon^2 |\nabla v_\epsilon|^2 + \chi_{|v_\epsilon|<1}) dx$$

$v_\epsilon(x) = u(\frac{x}{\epsilon})$, $\epsilon = \frac{1}{R}$.

By our main theorem in [C.C] we know that the level sets of v_ϵ converge uniformly to a limiting minimal surface S in $B_{\frac{1}{2}}(0)$. Since we are in dimension $n \leq 7$, S is a smooth surface and, therefore, the level sets v_ϵ will be as flat as needed for ϵ small enough (R big enough).

Inside the region $|u| < 1$ we have $\Delta(|\nabla u|^2) = 2 \sum (u_{j,k})^2 \geq 0$. Therefore $|\nabla u|^2 - 1$ is subharmonic and vanishes on the boundary of the strip $|u| < 1$ in $B_R(0)$. By the maximum principle we get that

$$|\nabla u(x)| \leq 1 + O(e^{-cR}),$$

inside $B_{\frac{R}{2}}(0)$ for some absolute constant $c > 0$.

<u>Claim:</u> In the viscosity sense $\sum_+$ has positive mean curvature (respect. $\sum_-$ has negative mean curvature) from the point of view of the domain $|u| < 1$.

Suppose not: Mean curvature $(\sum_+) \leq -\beta < 0$ in the viscosity sense at some point, that we can take to be the origin. Then one can find a quadratic comparison surface $\Lambda : x_n = P(x)$, $\Delta P = -\beta$, so that Λ and $\sum_+$ are tangent at the origin and Λ is above $\sum_+$ there.

With the help of a small translation down in the vertical direction, we will get a domain $D \subset \{|u| < 1\}$ such that ∂D consists of two parts, one of them will be contained in $\sum_+$ while the other will be in the translated surface $\Lambda^* = \Lambda - \tau e_n$, $\tau > 0$ small. Let us denote by $\partial D = (\partial D)_+ \cup (\partial D)_*$

such decomposition and let d(x) be the distance function to the surface Λ^*. We have:

$$\Delta d(x) = -\sum \frac{\kappa_j}{1-\kappa_j d(x)} \geq -\sum \kappa_j \geq \beta > 0,$$

where κ_j denotes the corresponding curvature of Λ^* at the point where x realizes its distance. Therefore

$$0 = \int_D \Delta u = \int_{\partial D} u_\nu dH_{n-1} = \int_{(\partial D)_+} dH_{n-1} - \int_{(\partial D)_*} u_\nu dH_{n-1}$$

which implies:

$$area((\partial D)_+) \leq (1 + O(e^{-cR}))area((\partial D)_*)$$

On the other hand we have:

$$\beta vol(D) \leq \int_D \Delta d \leq \int_{\partial D} d_\nu dH_{n-1} = \int_{(\partial D)_+} d_\nu dH_{n-1} - \int_{(\partial D)_*} dH_{n-1}$$

Thus:

$$\beta vol(D) + area((\partial D)_*) \leq area((\partial D)_+)$$

And:

$$\beta vol(D) + area((\partial D)_*) \leq (1 + O(e^{-cR})area((\partial D)_*)$$

which will yield a contradiction if R is big enough. This proves the claim.

Given $\delta > 0$ we know from our quoted result [C.C] that $\{|u| < 1\}$ is δ-flat for R big enough. That is: $\{|u| < 1\} \cap B_R(0) \subset \{|x^{'}| \leq R\} \times [-\delta R, \delta R]$.

Let S_+ denotes the envelope of paraboloids, $x_n = \frac{1}{\delta R}|x^{'} - a|^2 + b$, tangent to $\sum_+$ from above (respect. S_- is the envelope of paraboloids, $x_n = -\frac{1}{\delta R}|x^{'} - a|^2 + b$, tangent to $\sum_-$ from below). Then $\sum_\pm$ are, respectively, the graphs of two continuous Lipschitz function $f_\pm$, with Lipschitz constant ~ 1. Also in the viscosity sense we have $\Delta f_+ \geq 0$, $\Delta f_- \leq 0$, see reference [C.C*].

Let us recall that the band estimate of reference [C.C] yields vol($\{|u| < 1\} \cap B_R) \leq cR^{n-1}$. Since the distance function to $\sum_\pm$ is superharmonic in the viscosity sense, an energy comparison with the natural test functions associated to the enveloping surfaces, allows us to conclude that the set of contact points is big enough. Therefore, the two surfaces $S_\pm$ are, for R big enough, at distance $\leq \delta R^\epsilon$, for every $\epsilon > 0$, except for a set of small measure, $O(\delta^{-1}R^{n-1-\epsilon})$, in B_R.

Scaling to the unit ball $B_1(0)$ we get two continuous Lipschitz function $\phi_\pm(x) = \frac{1}{R} f_\pm(Rx)$, satisfying $\Delta\phi_+ \geq 0$, $\Delta\phi_- \leq 0$, $0 \leq \phi_- \leq \phi_+ \leq \delta$, $\mu\{\phi_+(x) - \phi_-(x) \geq \delta R^{\epsilon-1}\} = O(\delta^{-1} R^{-1-\epsilon})$.

Then, again for R big enough, one can apply lemma 7 of reference [C.C*] to conclude a flatness improvement. Rescaling back to B_R we obtain that inside a smaller ball, say $B_{\frac{R}{2}}$, our set $(\{|u| < 1\} \cap B_{\frac{R}{2}})$ is $\tau\delta$-flat for some universal $\tau < 1$. And this finishes the proof.

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