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Inventiones mathematicae

L^p bounds for Hilbert transforms along convex curves

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§ 1. Introduction

Let $\Gamma: \mathbb{R} \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$ with $\Gamma(0)=0$, $n \geqq 2$. To Γ we associate the Hilbert transform operator $\mathscr{H}$ defined by the principal-value integral

$$\mathscr{H} f(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x-\Gamma(t)) \frac{dt}{t} \qquad (x \in \mathbb{R}^n). \tag{1}$$

(f denotes an arbitrary function in an appropriate class; say, $f \in C_c^\infty(\mathbb{R}^n)$.) It is of substantial interest to determine for which curves Γ, and which indices p, one has the L^p bound

$$\|\mathscr{H} f\|_p \leqq A_p \|f\|_p \tag{2}$$

for a constant A_p depending only on Γ and p, not f. See [SW] for a survey of this problem's history through 1977. More recent results are found in [Ne, Wn, NSW2, NVWW1, NVWW2, Ch].

To clarify the purpose of this paper, let us recall the main strategy up until this time for obtaining estimates of the form

$$\|\mathscr{H} f\|_p \leqq A_p \|f\|_p. \tag{2}$$

First, since $\mathscr{H}$ is a convolution operator, it follows easily that

$$\widehat{\mathscr{H} f} = m \cdot \hat{f} \tag{3}$$

where $\hat{}$ denotes the Fourier transform and the "Fourier multiplier" m is the function

$$m(\xi) = \text{p.v.} \int_{-\infty}^{\infty} \exp(i\xi \cdot \Gamma(t)) \frac{dt}{t} \qquad (\xi \in \mathbb{R}^n). \tag{4}$$

So, to prove the estimate (2) for $p=2$, it suffices to show that $m(\xi)$ is a bounded function on $\mathbb{R}^n$.

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To prove the estimate (2) for $p \neq 2$, one has had to take advantage of "good properties" of the Fourier transform of a measure or distribution supported on the curve Γ. These "good" properties might be expressed in terms of certain smoothness of the function $m(\xi)$ in (4), (derivatives of m must decay at ∞), or the decay of the Fourier transform of measures supported on Γ, or the boundedness of certain "worse" functions - i.e. worse than $m(\xi)$. For example, in the case of a plane curve (i.e. $n=2$) $\Gamma(t)=(t, \gamma(t))$, one might try to prove that

$$m_\varepsilon(\xi, \eta) = \text{p.v.} \int_{-\infty}^{\infty} (1+\eta^2\gamma^2(t))^\varepsilon \cdot \exp(i\xi t + i\eta\gamma(t)) \frac{dt}{t} \tag{5}$$

is a bounded function on $\mathbb{R}^2$ for some positive ε. The idea is then to show that an "improved" operator such as that corresponding to the multiplier $m_{-\delta}(\xi, \eta)$ is bounded on $L^p(\mathbb{R}^2)$, $1<p<\infty$, for some positive δ, and then finish by applying Stein's analytic interpolation theorem [SWe], p. 205.

The results of [NVWW1] show that estimate (2) holds for $p=2$ for curves on which the "good" properties described above fail. For example, $\mathscr{H}$ is bounded on $L^2(\mathbb{R}^2)$ for certain plane curves $\Gamma(t) + (t, \gamma(t))$ where $\gamma(t)$ is linear on intervals $a_j \leqq t \leqq b_j$ with $b_j = 2a_j$ and $a_j \to 0$ as $j \to \infty$. By considering points (ξ, η) orthogonal to these straight line portions of Γ, one can see that the "good" properties alluded to above fail.

In this paper, we see that the above method of obtaining L^p estimates can still be used for these curves if the "bad parts" of the curve are "cut out" with appropriate Paley-Littlewood decompositions. (The "bad parts" of the curve are treated separately via a maximal function.) We shall use the lacunary Faley-Littlewood decomposition of [CF] and [NSW1]. See §3.1 for details of this decomposition. A similar idea of cutting out "bad" directions was previously used in the estimation of a certain maximal function; see [NSW1].

§2. Statement of results

Theorem. *Suppose $\Gamma(t) = (t, \gamma(t))$ $(t \in \mathbb{R})$ is a continuous plane curve with $\gamma: \mathbb{R} \to \mathbb{R}$ convex for $t \geqq 0$, $\gamma(0) = 0$, $\gamma'(0)^+ = 0$, and γ either even or odd. Suppose also that γ' has bounded doubling time: there exists a constant $C > 1$ with*

$$\gamma'(Ct)^+ \geqq 2\gamma'(t)^- \quad \textit{for } t \geqq 0. \tag{6}$$

Then, $\mathscr{H}$ is bounded on $L^p(\mathbb{R}^2)$ (2) for $4/3 < q < 4$. [The convexity hypothesis means that $[\gamma(C)-\gamma(B)]/(C-B) \geqq [\gamma(B)-\gamma(A)]/(B-A)$ for $0<A<B<C$.]

Our theorem not only broadens the class of curves for which L^p estimates $(p \neq 2)$ are known, but it also extends the range of p obtained for example in [NW]. Moreover, if our theorem is combined with the results of [NVWW1], we see that for any p with $4/3 < p < 4$, the problem of L^p boundedness of the Hilbert transform for even convex plane curve (with $\gamma(0) = \gamma'(0)^+ = 0$) is completely solved:

Corollary. *Let $\Gamma(t)=(t,\gamma(t))$ be a continuous plane curve with $\gamma\colon \mathbb{R}\to\mathbb{R}$ convex for $t\geqq 0$, $\gamma(0)=\gamma'(0)^+=0$, and γ an even function. Let $4/3<p<4$. Then, $\mathscr{H}$ is bounded on $L^p(\mathbb{R}^2)$ (2) if and only if γ' has bounded doubling time (6).*

§ 3. Proof of the theorem

We give the proof in the case that γ is an even function; the odd case is somewhat easier. Let us also assume that $\gamma\in C^2(0,\infty)$. Thus, $\gamma''(t)\geqq 0$ for $t\neq 0$. The treatment of a general convex γ, which contains no substantive additional ideas, will be indicated briefly in § 3.5. Finally, a note about rigor: one should focus upon truncated operators

$$H_{\varepsilon,N}f(x,y)=\int_{\varepsilon\leqq|t|\leqq N} f(x-t,y-\gamma(t))t^{-1}\,dt;$$

however, this entails so much additional notation, etc., that we choose to ignore the truncation and proceed instead in the simpler "limit operator" setting.

Throughout our proof, $C_0>1$ will be a constant so that

$$\gamma'(C_0t)\geqq 8\gamma'(t) \quad \text{for } t\geqq 0 \tag{7}$$

e.g. $C_0=C^3$, C as in (6).

Some of our estimates will require the following lemma.

Lemma A. *Van der Corput Lemma.* (See [Z], p. 197.) *Suppose $\phi\colon [a,b]\to\mathbb{R}$ is in $C^1[a,b]$, ϕ' is monotone, and there is a $\lambda>0$ with $|\phi'(t)|\geqq\lambda$ for $a\leqq t\leqq b$. Then $\left|\int_a^b \exp(i\phi(t))\,dt\right|\leqq B/\lambda$ for a constant B independent of a, b, f, and λ.*

§ 3.2. A Paley-Littlewood decomposition

Partition the plane into sectors R_k ($k=\pm 1,\pm 2,\dots$), each sector symmetric about both coordinate axes, as shown below.

Specifically, let

$$R_k=\{(\xi,\eta)\in\mathbb{R}^2\colon \tan(2^{-k-2}\pi)\leqq|\eta/\xi|\leqq\tan(2^{-k-1}\pi)\} \quad \text{for } k=1,2,\dots,$$

and

$$\begin{aligned} R_k&=\{(\xi,\eta)\in\mathbb{R}^2\colon (\eta,\xi)\in R_{-k}\}\\ &=\{(\xi,\eta)\in\mathbb{R}^2\colon \tan(2^{k-2}\pi)\leqq|\xi/\eta|\leqq\tan(2^{k-1}\pi)\} \quad \text{for } k=-1,-2,\dots. \end{aligned} \tag{8}$$

Write $(T_kf)\hat{}=X_{R_k}\cdot\hat{f}$. ($X_S$ denotes the characteristic function of the set S.) We have

Theorem B. (See [NSW1], Corollary 2.) *For $1<p<\infty$ there exists positive constants A_p and B_p with $A_p\|f\|_p\leqq\|(\sum_k|T_kf|^2)^{1/2}\|_p\leqq B_p\|f\|_p$ for each $f\in L^p(\mathbb{R}^2)$.*

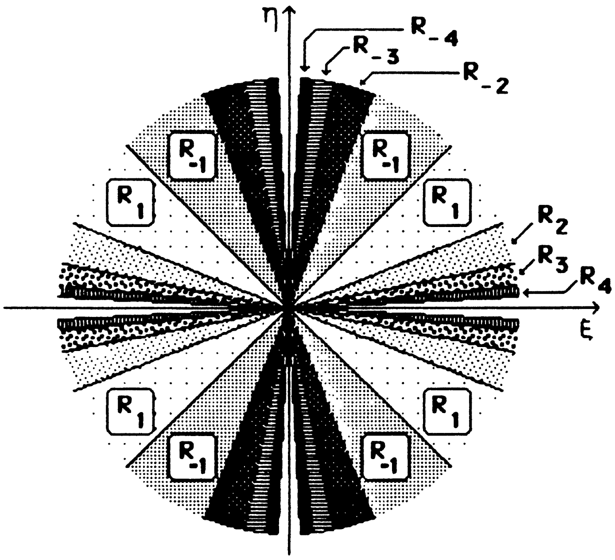

Fig. 1. The sectors R_k of the Paley-Littlewood decomposition

§ 3.2. The splitting of $\mathscr{H}$

For $k=\pm 1, \pm 2, \ldots$ we write $\mathscr{H}=H_k+L_k$ where

$$\underset{k}{H}f(x,y)=\int_{|t|\in I_k} f(x-t, y-\gamma(t))\,\frac{dt}{t}, \qquad (x,y)\in\mathbb{R}^2. \tag{9}$$

$I_k \Subset (0,\infty)$ – to which corresponds the part of the curve "bad" for (ξ,η) in sector R_k – is an interval of the form $I_k=[\alpha_k/C_0, \alpha_k\cdot C_0]$, and α_k is chosen as follows. Suppose $R_k=\{(\xi,\eta)\in\mathbb{R}^2\colon r_k\leqq|\xi/\eta|\leqq\rho_k\}$; then, let α_k be any positive number with $\gamma'(\alpha_k)=r_k$.

If we use Theorem B and then the triangle inequality, we obtain

$$A_p\|\mathscr{H}f\|_p\leqq\|\sum_k(|T_kH_kf|^2)^{1/2}\|_p+\|(\sum_k|T_kL_kf|^2)^{1/2}\|_p \tag{10}$$

for $1<p<\infty$. We therefore need to dominate each of the above terms by a constant multiple of $\|f\|_p$. For the second term, we will see in § 3.4 that the bounded doubling time of γ' guarantees oscillation in multiplier integrals for T_kL_k; this facilitates an argument via the Marcinkiewicz multiplier theorem [S], p. 109, and analytic interpolation. The first term in (10) we control by appropriate maximal functions in § 3.3.

§ 3.3. Estimate for $\|(\sum_k |T_k H_k f|^2)^{1/2}\|_p$

For each k, T_k and H_k commute since they are both multiplier operators. So, if we can prove that

$$\|(\sum_k |H_k g_k|^2)^{1/2}\|_p \leqq C_p \|(\sum_k |g_k|^2)^{1/2}\|_p \quad (4/3<p<4) \tag{11}$$

for a constant C_p independent of the arbitrary measurable functions q_k ($k=\pm 1, \pm 2, \dots$), then we can apply (11) and Theorem B to obtain

$$\|(\sum_k |T_k H_k f|^2)^{1/2}\|_p \leqq B_p \cdot C_p \|f\|_p \quad (4/3<p<4).$$

Toward a proof of (11), we define a maximal operator M by

$$Mf(x,y)=\sup_{j\in\mathbb{Z}} |M_j f(x,y)|,$$

$$M_j f(x,y)=[2(C_0^{j+1}-C_0^j)]^{-1}\int_{J_j} f(x-t, y-\gamma(t))\,dt,$$

$$J_j=\{t\in\mathbb{R}:\ C_0^j \leqq |t| < C_0^{j+1}\}.$$

We see easily that $|H_k f(x,y)| \leqq C\cdot M(|f|)(x,y)$ for a constant C independent of k, f, and (x,y). Moreover, comparison via a g-function to (essentially) the strong maximal function shows that M is bounded on $L^2(\mathbb{R}^2)$; see Lemma C below. Now, we consider the inequalities

$$\|(\sum_k |H_k g_k|^q)^{1/q}\|_p \leqq C_p \|(\sum_k |g_k|^g)^{1/q}\|_p. \tag{13}$$

For $p=2$, $q=\infty$, we verify (13) using the positivity and L^2-boundedness of M. For $p=q>1$, (13) holds simply because each H_k is concolution with a measure of mass $4\ln(C_0)$. (13) follows by interpolation (see [BP], Theorem 2) for $q=2$, $4/3<p\leqq 2$, and by duality for $q=2$, $2\leqq p<4$. This proves (11).

Let us now prove that M is bounded on $L^2(\mathbb{R}^2)$. For this purpose we consider for each of the average $M_j f$ a companion average $N_j f$, defined as follows:

$$N_j f(x,y)=[2(C_0^{j+1}-C_0^j)]^{-1}\int_{J_j} [\gamma(C_0^j)]^{-1}\left[\int_0^{\gamma(C_0^j)} f(x-t, y-s)\,ds\right]dt, \tag{14}$$

i.e. N_j is convolution with the (normalized) characteristic function of the shaded set in Fig. 2.

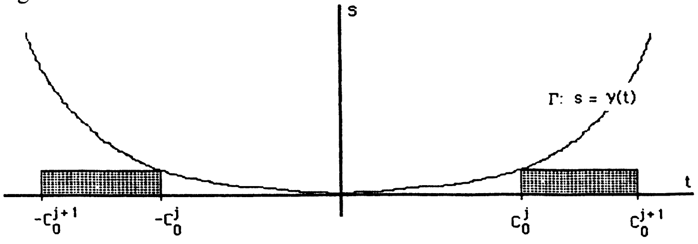

Fig. 2

Evidently, $\sup_j |N_j f|$ is dominated by a constant multiple of the strong maximal function (the operator f^* of [Z] p. 306, e.g.). Thus, the L^2 bound for M will from

Lemma C. *Define*

$$g(f)(x,y)=\left[\sum_{j=-\infty}^{\infty} |M_j f x,y)-N_j f x,y)|^2\right]^{1/2}.$$

Then $\|g(f)\|_2 \leqq C\|f\|_2$ *for a constant* C *independent of* f.

Proof. By the Plancherel theorem, it suffices to show that the function

$$\sigma(\xi,\eta)=\sum_{j=-\infty}^{\infty} |m_j(\xi,\eta)-n_j(\xi,\eta)|^2, \qquad (\xi,\eta)\in \mathbb{R}^2 \tag{15}$$

is (essentially) bounded on $\mathbb{R}^2$, where

$$n_j(\xi,\eta)=[2(C_0^{j+1}-C_0^j)]^{-1}\int_{J_j} \exp(i\xi t)\cdot\left[\frac{\exp(i\eta\gamma(C_0^j))-1}{i\eta\gamma(C_0^j)}\right]dt \tag{16}$$

and

$$m_j(\xi,\eta)=[2(C_0^{j+1}-C_0^j)]^{-1}\int_{J_j} \exp(i\xi t)[\exp(i\eta\gamma(t))]\,dt \tag{17}$$

are the Fourier multipliers for the operators N_j and M_j. (J_j is defined in (12).)

Given $(\xi,\eta)\in\mathbb{R}^2$, consider first those integers j with $|\eta|\gamma(C_0^{j+1})\leqq 1$; let us write $j\in I$. If $j\in I$ and $t\in J_j$, we see easily that for the quantities $\alpha=[\exp(i\eta\gamma(C_0^j))-1]/[i\eta\gamma(C_0^j)]$ and $\beta=\exp(i\eta\gamma(t))$ appearing in (16) and (17), we have $|\alpha-\beta|<|\alpha-1|+|\beta-1|\leqq 3|\eta|\gamma(C_0^{j+1})$. Thus for $j\in I$, $|n_j(\xi,\eta)-m_j(\xi,\eta)|^2\leqq 9|\eta\gamma(C_0^{j+1})|^2$. Since γ is convex, this latter quantity decreases geometrically as $j\to-\infty$. Hence $\sum_{j\in I}|m_j(\xi,\eta)-n_j(\xi,\eta)|^2$ is bounded by a constant independent of (ξ,η).

Next we consider those integers j for which $|\eta|\gamma(C_0^j)>1$; let us write $j\in\mathrm{II}$. But $|n_j|\leqq 2/|\eta\gamma(C_0^j)|$ for all j, so by again comparing with a geometric series we see that $\sum_{j\in\mathrm{II}}|n_j(\xi,\eta)|^2\leqq C$, C independent of (ξ,η). As for m_j, we consider those $j\in\mathrm{II}$ for which $|\xi|\geqq 2|\eta\gamma'(t)|$ whenever $t\in J_j$ – we write $j\in\mathrm{II}A$ – and those $j\in\mathrm{II}$ for which $|\eta\gamma'(t)|>2|\xi|$ whenever $t\in J_j$ – we write $j\in\mathrm{II}B$. Put $\phi(t)=\xi t+\eta\gamma(t)$. For $t\in J_j$ and $j\in\mathrm{II}A$ we have $|\phi'(t)|>|\xi|/2\geqq|\eta\gamma'(C_0^j)|$. For $t\in J_j$ and $j\in\mathrm{II}B$ we have $|\phi'(t)|\geqq|\eta\gamma'(t)|/2\geqq|\eta\gamma'(C_0^j)|/2$. Thus, Lemma A plus convexity of γ give

$$|m_j(\xi,\eta)|<\frac{a}{(C_0^{j+1}-C_0^j)|\eta\gamma'(C_0^j)|}<\frac{a}{(C_0-1)|\eta\gamma(C_0^j)|} \tag{18}$$

for $j\in\mathrm{II}A\cup\mathrm{II}B$, a independent of j and (β,η). Therefore both

$$\sum_{j\in\mathrm{II}A}|m_j(\xi,\eta)|^2 \quad\text{and}\quad \sum_{j\in\mathrm{II}B}|m_j(\xi,\eta)|^2$$

are suitably bounded.

Finally, there is only one j in $\mathbb{Z}\setminus(\mathrm{I}\cup\mathrm{II})$, and (7) shows that there are at most three j in $\mathrm{II}\setminus(\mathrm{II}A\cup\mathrm{II}B)$. This is satisfactory since $|m_j(\xi,\eta)|\leqq 1$ and $|n_j(\xi,\eta)|\leqq 1$ for all j and (ξ,η).

§ 3.4. Estimate for $\|(\sum_k |T_k L_k f|^2)^{1/2}\|_p$

For this estimate, we intend to use the Marcinkiewicz multiplier theorem [S], p. 109. However, the multiplier χ_{R_k} of the operator T_k is not smooth. Our first step is therefore

§ 3.4.1 Replacing T_k by a "smooth" operator S_k. We shall presently define operators S_k so that $T_k = T_k S_k$. Then we will be able to "replace" T_k by S_k:

$$\begin{aligned}\|(\sum_k |T_k L_k f|^2)^{1/2}\|_p &= \|(\sum_k |T_k S_k L_k f|^2)^{1/2}\|_p \\ &\leqq C_p \|(\sum_k |S_k L_k f|^2)^{1/2}\|_p \quad \text{for } 4/3<p<4 \end{aligned} \tag{19}$$

where C_p is independent of f. The inequality in (19) follows from [CF], p. 425, specifically from the estimate $\|(\sum_j |P_j f_j|^2)^{1/2}\|_p \leqq C_p \|(\sum_j |f_j|^2)^{1/2}\|_p$ in which P_j is the operator whose multiplier is the characteristic function of a half-plane with boundary line $\eta = \tan(2^{-j})\xi$.

Now, we choose S_k to be the operator such that

$$(S_k f)^\wedge = \omega_k \hat{f} \tag{20}$$

where the Fourier multiplier ω_k is defined as follows. Let $\omega\colon \mathbb{R}^2\setminus(0,0)\to[0,1]$ be a C^∞ function, homogeneous of degree 0, even in each variable, $\omega(\xi,\eta)=1$ if $\pi/4 \leqq |\eta/\xi| \leqq 2$, $\omega(\xi,\eta)=0$ if $|\eta/\xi|\geqq\pi$ or $|\eta/\xi|\leqq 1/2$; then, for $k=1,2,3,\dots$ put $\omega_k(\xi,\eta)=\omega(\xi,2^k\eta)$, and for $k=-1,-2,-3,\dots$ put $\omega_k(\xi,\eta)=\omega_{-k}(\eta,\xi)$. We verify that

$$\omega_k \equiv 1 \quad \text{on} \quad R_k \tag{21}$$

so that $T_k = T_k S_k$ and (19) holds.

Two other properties of ω_k will be important. First, their support are nearly disjoint in that

$$\begin{gathered}\text{each } (\xi,\eta)\in\mathbb{R}^2\setminus(0,0) \text{ is in the support} \\ \text{of at most ten of the functions } \omega_k. \end{gathered} \tag{22}$$

Second, for $k=\pm 1, \pm 2, \dots$ and $(\xi,\eta)\in\operatorname{supp}(\omega_k)$, if $|t|\leqq \alpha_k/C_0$ then $|\xi|>2|\eta\gamma'(t)|$ and if $|t|>\alpha_k\cdot C_0$ then $|\eta\gamma'(t)|>2|\xi|$. (α_k is as defined in § 3.2.) Thus,

$$\begin{gathered}(\xi,\eta)\in\operatorname{supp}(\omega_k) \\ \text{and} \\ t\notin I_k=[\alpha_k/C_0, \alpha_k\cdot C_0] \Rightarrow |\xi+\eta\gamma'(t)| \geqq \tfrac{1}{2}\max(|\xi|, |\eta\gamma'(t)|). \end{gathered} \tag{23}$$

§ 3.4.2. An analytic family of operators T_z. By (19), it remains to dominate $\|(\sum_k |S_k L_k f|^2)^{1/2}\|_p$ by a constant multiple of $\|f\|_p$, $4/3<p<4$. An application

of the Rademacher functions (e.g., see [S], p. 104) shows that it is enough to prove that the operator T

$$Tf=\sum_k \pm S_k L_k f \tag{24}$$

is bounded on $L^p(\mathbb{R}^2)$, $4/3<p<4$, with a bound independent of the choice of $\pm$ signs. To do this, we introduce a complex parameter z as follows:

$$\begin{aligned}
&(L_{k,z} f)^\wedge = m_{k,z}\cdot \hat{f}\\
&m_{k,z}(\xi,\eta)=\text{p.v.} \int_{|t|\notin I_k} \exp(i\xi t+i\eta\gamma(t))\cdot[1+|\eta\gamma(t)|]^z \frac{dt}{t}\\
&T_z f=\sum_k \pm S_k L_{k,z} f
\end{aligned} \tag{25}$$

so that $T=T_0$ and $L_k=L_{k,0}$.

Now, fix an arbitrarily small $\varepsilon>0$. In § 3.4.3 we shall show that T_z is bounded in $L^2(\mathbb{R}^2)$ for $\mathrm{Re}(z)=1-\varepsilon$, and in § 3.4.4 that T_z is bounded on $L^p(\mathbb{R}^2)$ for $\mathrm{Re}(z)=-1-\varepsilon$ and $1<p<\infty$. Our estimates will be independent of the choice of $\pm$ signs and will grow at most polynomially in $|z|$. The required estimate for T will then follow by analytic interpolation [SWe], p. 205, and the proof of our theorem will be complete.

§ 3.4.3. L^2-boundedness of T_z, $\mathrm{Re}(z)=1-\varepsilon$. We will show that the multiplier for T_z, $\sum_k \pm\omega_k(\xi,\eta)\cdot m_{k,z}(\xi,\eta)$, is (essentially) bounded on $\mathbb{R}^2$. Since $|\omega_k|\leqq 1$ and in view of (22), it suffices to show that $m_{k,z}$ is bounded (independent of k) on the support of ω_k. So, we fix $(\xi,\eta)\in\mathrm{supp}(\omega_k)$ and write $\phi(t)=\xi t+\eta\gamma(t)$.

Estimate 1. Suppose $0<a<b$ and $|\eta\gamma(t)|\leqq 1$ for $a\leqq|t|\leqq b$. Then

$$\Big|\int_{a\leqq|t|\leqq b} \exp(i\phi(t))\cdot[1+|\eta\gamma(t)|]^z t^{-1}\,dt\Big|\leqq C\cdot(1+|z|). \tag{26}$$

Estimate 2. Suppose that for $a\leqq t\leqq b$ we have

$$|\eta\gamma(t)|\geqq 1 \quad\text{and}\quad |\phi'(t)|>|\eta\gamma'(t)|/2. \tag{27}$$

Then

$$\left|\int_a^b \exp(i\phi(t))\cdot[1+|\eta\gamma(t)|]^z t^{-1}\,dt\right|\leqq C(1+|z|). \tag{28}$$

(In both (26) and (28), C is independent of a, b, ξ, η, and $\mathrm{Im}(z)$.)

Proof of Estimate 1. Let $\Phi(t)=\int_a^t \exp(i\phi(s)-i\phi(-s))s^{-1}\,ds$. Then, since γ is even, the integral to be estimated equals

$$\begin{aligned}
&\int_a^b \phi'(t)\cdot[1+|\eta\gamma(t)|]^z\,dt\\
&\qquad=\Phi(t)\cdot[1+|\eta\gamma(t)|]^z\big|_a^b - z\int_a^b \Phi(t)\cdot[1+|\eta\gamma(t)|]^{z-1}|\eta|\gamma'(t)\,dt.
\end{aligned}$$

By the L^2 theory of [NVWW1], Theorem 2, $|\Phi(t)| \leqq K$ for a constant K depending only on the curve γ. Thus, each boundary term in the above integration by parts is at most $2K$, and the integrated term is at most

$$K|z| \int_a^b [1+|\eta\gamma(t)|]^{-\varepsilon} |\eta| \gamma'(t)\, dt \leqq K|z| \cdot |\eta| (\gamma(b)-\gamma(a)) \leqq K|z|.$$

Proof of Estimate 2. We use the inequalities

$$2|t\phi'(t)| \geqq |\eta t \gamma'(t)| \geqq |\eta\gamma(t)| \geqq 1 \qquad \text{for } a \leqq t \leqq b, \tag{29}$$

and

$$|[1+|\eta\gamma(t)|]^z| < 2|\eta\gamma(t)|^{1-\varepsilon} \qquad \text{for } a \leqq t \leqq b, \tag{30}$$

which follow from (27) and the convexity of γ. The integral to be estimated is $\int_a^b [1+|\eta\gamma(t)|]^z \cdot [t\phi'(t)]^{-1} \cdot d[\exp(i\phi(t))]$. We integrate by parts in the indicated way. For the boundary terms, we have

$$|\exp[i\phi(t)] \cdot [1+|\eta\gamma(t)|]^z / [t\phi'(t)]| \leqq 4|\eta\gamma(t)|^{1-\varepsilon} / |\eta\gamma(t)| \leqq 4.$$

Two of the three integrated terms are

$$\int_a^b \exp(i\phi(t)) \cdot [1+|\eta\gamma(t)|]^z \cdot [t\phi'(t)]^{-1} \cdot [t\eta\gamma'(t)]^{-1} \cdot \eta\gamma'(t)\, dt$$

and

$$z\int_a^b \exp(i\phi(t)) \cdot [1+|\eta\gamma(t)|]^{z-1} \cdot [t\phi'(t)]^{-1} \cdot \eta\gamma'(t)\, dt.$$

Both are dominated by

$$4(1+|z|) \int_a^b |\eta\gamma(t)|^{-1-\varepsilon} |\eta\gamma'(t)|\, dt \leqq 4(1+z) \int_1^\infty u^{-1-\varepsilon}\, du.$$

For the third integrated term,

$$\int_a^b \exp(i\phi(t)) \cdot [1+|\eta\gamma(t)|]^z \cdot t^{-1} \cdot [\phi'(t)]^{-2} \cdot \eta\gamma''(t)\, dt,$$

we assume with no loss of generality that $0<a<b$. This term is dominated by

$$\begin{aligned}
&8\int_a^b |\eta\gamma(t)|^{1-\varepsilon} \cdot t^{-1} \cdot |\eta\gamma'(t)|^{-2} \cdot |\eta| \gamma''(t)\, dt \\
&\qquad \leqq 8\int_a^b |\eta t \gamma'(t)|^{1-\varepsilon} \cdot t^{-1} \cdot |\eta\gamma'(t)|^{-2} |\eta| \gamma''(t)\, dt \\
&\qquad \leqq 8|\eta|^{-\varepsilon} a^{-\varepsilon} \int_a^b |\gamma'(t)|^{-1-\varepsilon} \gamma''(t)\, dt \leqq 8\varepsilon^{-1}(|\eta| a\gamma'(a))^{-\varepsilon} \leqq 8\varepsilon^{-1}(|\eta|\gamma(a))^{-\varepsilon} \leqq 8\varepsilon^{-1}.
\end{aligned}$$

Thus Estimate 2 holds.

Property (23) and Estimates 1 and 2 imply the required bound for $m_{k,z}$ on $\operatorname{supp}(\omega_k)$. Specifically, define $\tau>0$ by $|\eta|\gamma(\tau)=1$. If $\tau<\alpha_k/C_0$, we treat $\int_{-\tau}^{\tau}\exp(i\phi(t))\cdot[1+|\eta\gamma(t)|]^z t^{-1}\,dt$ with Estimate 1 and $\int_{\tau}^{\alpha_k/C_0}$, $\int_{\alpha_k C_0}^{\infty}$, $\int_{-\alpha_k/C_0}^{-\tau}$, and $\int_{-\infty}^{-\alpha_k C_0}$ with Estimate 2. If $\tau\geqq\alpha_k/C_0$, a singular splitting of the integral for $m_{k,z}$ suffices.

§ 3.4.4. L^p-boundedness of T_z, $\operatorname{Re}(z)=-1-\varepsilon$, $1<p<\infty$. We will apply the Marcinkiewicz multiplier theorem [S], p. 109. For the multiplier m of T_z, $m=\sum_k \pm\omega_k(\xi,\eta)\cdot m_{k,z}(\xi,\eta)$, we need to show that the functions m, $\xi\frac{\partial m}{\partial\xi}$, $\eta\frac{\partial m}{\partial\eta}$, and $\xi\eta\frac{\partial^2 m}{\partial\xi\,\partial\eta}$ are bounded. By (22), it suffices to verify these four estimates for $\omega_k\cdot m_{k,z}$, with bounds independent of k. The form of the functions $\omega_k(\xi,\eta)$ – all dilates of $\omega(\xi,\eta)$ or of $\omega(\eta,\xi)$, ω homogeneous of degree 0 and smooth on $\mathbb{R}^2\backslash(0,0)$ – shows that we need only check the four estimates for $m_{k,z}(\xi,\eta)$ with $(\xi,\eta)\in\operatorname{supp}(\omega_k)$. So, we again fix $(\xi,\eta)\in\operatorname{supp}(\omega_k)$ and write $\phi(t)=\xi t+\eta\gamma(t)$.

Bound for $\xi\eta\dfrac{\partial^2 m_{k,z}}{\partial\xi\,\partial\eta}$

There are two terms,

$$\alpha=\int_{|t|\notin I_k}\exp(i\phi(t))\cdot[1+|\eta\gamma(t)|]^z\,\xi\eta\gamma(t)\,dt$$

and

$$\beta=z\int_{|t|\notin I_k}\exp(i\phi(t))\cdot[1+|\eta\gamma(t)|]^{z-1}\,\xi\eta\gamma(t)\,dt.$$

Evidently, we need only estimate α.

Write

$$\alpha=\int_0^{\alpha_k/C_0}+\int_{-\alpha_k/C_0}^{0}+\int_{\alpha_k C_0}^{\infty}+\int_{-\infty}^{-\alpha_k C_0}.$$

For the first of these, let $\Phi(t)=\int_0^t\exp(i\phi(s))\,ds$. By the convexity of γ, (23) and Lemma A, $|\Phi(t)|\leqq 2B/|\xi|$ for $0\leqq t\leqq\alpha_k/C_0$. Integration by parts gives

$$\begin{aligned}&\int_0^{\alpha_k/C_0}\exp(i\phi(t))\cdot[1+|\eta\gamma(t)|]^z\,\xi\eta\gamma(t)\,dt\\&\qquad=\xi\Phi(t)\cdot[1+|\eta\gamma(t)|]^z\,\eta\gamma(t)\big|_0^{\alpha_k/C_0}+\text{two integrated terms.}\end{aligned}$$

The boundary terms are suitably bounded since $\operatorname{Re}(z)\leqq-1$. For one integrated term, we have

$$\begin{aligned}&\left|z\int_0^{\alpha_k/C_0}\xi\Phi(t)[1+|\eta\gamma(t)|]^{z-1}\eta\gamma(t)|\eta|\gamma'(t)\,dt\right|\\&\qquad\leqq 2B|z|\int_0^{\alpha_k/C_0}|\eta|\gamma(t)\cdot[1+|\eta\gamma(t)|]^{-2-\varepsilon}|\eta|\gamma'(t)\,dt\end{aligned}$$

which is satisfactory by comparison to $\int_0^\infty u\cdot[1+u]^{-2-\varepsilon}du$. The other integrated term,

$$\int_0^{\alpha_k/C_0} \xi\Phi(t)\cdot[1+|\eta\gamma(t)|]^z\cdot\eta\gamma'(t)\,dt,$$

is likewise compared to $\int_0^\infty [1+u]^{-1-\varepsilon}du$. Thus, $\int_0^{a_k/C_0} dt$ is appropriately bounded, as is $\int_{-\alpha_k/C_0}^0$ by the same reasoning.

As for $\int_{\alpha_k\cdot C_0}^\infty$ $\left(\text{and } \int_{-\infty}^{-\alpha_k\cdot C_0}\right)$, we begin by writing

$$\Phi(t)=\int_{\alpha_k\cdot C_0}^t \exp(i\Phi(s))\,ds, \qquad |\Phi(t)|\leqq 2B/|\xi|,$$

and proceed in the same way.

Bound for $\xi\dfrac{\partial m_{k,z}}{\partial\xi}$

This equals $\int_{|t|\notin I_k} \exp(i\Phi(t))[1+|\eta\gamma(t)|]^z\xi\,dt$, which we split into four integrals and estimate exactly as above.

Bound for $\eta\dfrac{\partial m_{k,z}}{\partial\eta}$

There are two terms,

$$\alpha=\int_{|t|\notin I_k} \exp(i\Phi(t))[1+1|\eta\gamma(t)|]^z\eta\gamma(t)t^{-1}\,dt$$

and

$$\beta=z\int_{|t|\notin I_k} \exp(i\Phi(t))\cdot[1+|\eta\gamma(t)|]^{z-1}\eta\gamma(t)t^{-1}\,dt.$$

By the evenness and convexity of γ, we have

$$|\alpha|\leqq\int_{|t|\notin I_k} [1+|\eta\gamma(t)|]^{-1-\varepsilon}|\eta\gamma'(t)|\,dt\leqq\int_{-\infty}^\infty [1+|u|]^{-1-\varepsilon}du.$$

β is evidently easier.

Bound for $m_k(\xi,\eta)$

This is

$$\begin{aligned}&\text{p.v.}\int_{|t|\notin I_k} \exp(i\Phi(t))[1+|\eta\gamma(t)|]^z t^{-1}\,dt\\ &=\left(\int_0^{\alpha_k/C_0}+\int_{\alpha_k\cdot C_0}^\infty\right)[\exp(i\phi(t))-\exp(i\phi(-t))]\cdot[1+|\eta\gamma(t)|]^z t^{-1}\,dt\end{aligned}$$

by the evenness of γ.

For $\int_0^{\alpha_k/C_0} \cdot\cdot\, dt$, we write $\Phi(t)=\int_0^t [\exp(i\phi(s))-\exp(i\phi(-s)]s^{-1}ds$. By the L^2 theory of [NVWW1], Theorem 2, $|\Phi(t)|\leqq C=$a constant depending only on the curve γ. We now rewrite $\int_0^{\alpha_k/C_0} \cdot\cdot\, dt$ using integration by parts in the way indicated by our choice of Φ. The resulting boundary terms are bounded because $\mathrm{Re}(z)<0$, and the integrated term is dominated by $C|z|\int_0^\infty [1+u]^{-2-\varepsilon}du$.

For $\int_{\alpha_k\cdot C_0}^{\infty} \cdot\cdot\, dt$, we write

$$\Phi(t)=\int_{\alpha_k\cdot C_0}^{t} [\exp(i\phi(s))-\exp(i\phi(-s))]s^{-1}ds$$

and proceed in the same way.

§ 3.5. Extension to a general convex curve

Nearly all of our proof stands as is in the case of a general convex curve, provided one indicates one-sided derivatives, etc., as necessary. Lemma A holds for a convex or concave ϕ, rather than $\phi\in C^1$, with the hypothesis $|\phi'(t)|\geqq\lambda$ replaced by $|\phi(s)-\phi(t)|/|s-t|\geqq\lambda$. Also, the various integrations by parts are valid, since one can check that appropriate functions are absolutely continuous. There is one exception to this last comment: the proof of *Estimate 2* in § 3.4.3, in which γ'' appears. There, one approximates the convex curve $\gamma(t)$ by $\gamma_j(t)\equiv\psi_j*\gamma(t)-\psi_j*\gamma(0)$, where $\{\psi_j\}_{j=1}^\infty$ is an appropriate smooth approximate identity, each ψ_j an even function with support in the interval $[-1/j, 1/j]$. The proof given for *Estimate 2* applies to γ_j for $\int_{a+1/j}^{b-1/j} \cdot\cdot\, dt$. Finish by letting $j\to\infty$.

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