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Lattice Points

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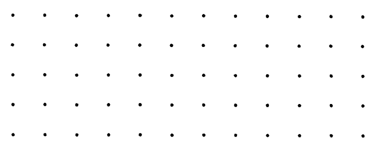
1. Andante: Small Arcs, But Not Too Small

At the beginning there were lattice points



and arithmetical functions like the following:

$r(n) = \#\{n = a^2 + b^2/a, b \in \mathbf{Z}\}$ = number of lattice points on the circle $x^2 + y^2 = n$.

It is well known that

$$\begin{aligned} r(n) &= O(n^{\varepsilon}), \quad \text{for every } \varepsilon > 0 . \\ r(n) &\neq O((\log n)^{\alpha}), \quad \text{for any } \alpha . \end{aligned}$$

On the other hand, let $\{v_i\}_{i=1,2,3}$ be three lattice points on the circle of radius R, centered at the origin. The theorem of Hero of Alexandria gives us: $\|v_1 - v_2\| \cdot \|v_2 - v_3\| \cdot \|v_1 - v_3\| = 4R$ area $(T) \geq 2R$. Therefore, length of the arc $\widehat{v_1 v_3} > (2R)^{1/3}$. That is, an arc whose length is $(2R)^{1/3}$ contains, at most, two lattice points. We can state our first result.

Theorem 1. (J. Cilleruelo, A. Córdoba)

For every α, $1/3 \leq \alpha < 1/2$, there exists a finite constant c_α, so that an arc of length R^α contains, at most, c_α lattice points.

What happens when $\frac{1}{2} \leq \alpha < 1$? This is an interesting open problem.

This theorem is of an arithmetical nature. The function $r(n)$ is better understood in the ring of gaussian integers and there is an extension to $Q(\sqrt{-d})$, $d > 0$ square free, where we consider now $r_d(n) = \#\{n = a^2 + db^2/a, b \in \mathbf{Z}\}$, i.e., lattice points on arcs of ellipses [2]. In his Ph.D. thesis [3] Jiménez has also obtained the corresponding version for real quadratic fields, that is, lattice points on arcs of hyperbolas.

The following is a list of closely related questions:

i. Restriction lemmas for Fourier Series and integrals.

Math Subject Classifications.
Keywords and Phrases.
Acknowledgements and Notes. Para Miguel en sus sesenta años.

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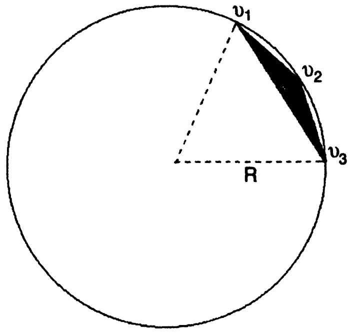
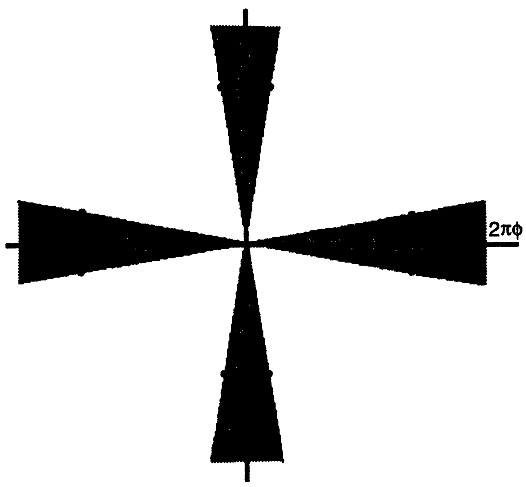


ii. Estimates for Gauss sums.
iii. Existence of infinitely many primes in the sequence $\{n^2 + 1\}$.
iv. Problems in the theory of Quantum chaos: Characterization of weak limits of sequences $|\psi_k(x)|^2 dx$, where ψ_k is a normalized eigenfunction of the Laplacian [4].

Sketch of the Proof [1].

Let p be a prime so that $p \equiv 1(4)$. It can be represented in eight manners as a sum of two squares $p = a^2 + b^2$. They correspond to the gaussian integers

$$a + bi = \sqrt{p}e^{2\pi i\{\pm\phi + \frac{t}{4}\}} \qquad t = 0, 1, 2, 3 .$$



We assign to such a prime $p \equiv 1(4)$ an angle ϕ. An important fact about those angles, corresponding to different primes, is that they are linearly independent over the rationals.

Assume that $r(n) > 0$ and that

$$n = 2^{\nu} \prod_{q_k \equiv 3(4)} q_k^{2\beta_k} \prod_{p_j \equiv 1(4)} p_j^{\alpha_j}$$

Then $r(n) = 4\prod(1+\alpha_j)$ and each representation $n = a^2 + b^2$ yields the gaussian integer

$$a + bi = \sqrt{n}e^{2\pi i\{\phi_0 + \Sigma\gamma_j\phi_j + \frac{t}{4}\}}$$

where:

$$\begin{cases} \phi_j \ \text{ is the angle assigned to } \ p_j \\ |\gamma_j| \le \alpha_j \ \text{ and } \ \gamma_j \equiv \alpha_j \mod (2) \\ t = 0, 1, 2, 3 \\ \phi_0 = \begin{cases} 0 & \text{if } \nu \text{ is even} \\ \frac{1}{8} & \text{if } \nu \text{ is odd} \end{cases} \end{cases}$$

On the circle of radius $R = \sqrt{n}$ let us assume that an arc of length $\sqrt{2}R^{\alpha}$ contains $m+1$ lattice points:

We use $s = 1, \ldots, m+1$ to label those lattice points and angles:

$$\sum \gamma_j^s \phi_j + \frac{t^s}{4}$$

Given two different points $s \neq s'$, we consider the angle:

$$\psi^{s,s'} = \sum_j \frac{\gamma_j^s - \gamma_j^{s'}}{2}\phi_j + \frac{t^s - t^{s'}}{2}$$

we have:

a) If $t^s \equiv t^{s'} \mod (2)$ then $\psi^{s,s'}$ is the angle corresponding to a representation of the number

$$\prod_j p_j^{\frac{|\gamma_j^s - \gamma_j^{s'}|}{2}}$$

as a sum of two squares.

b) If $t^s \not\equiv t^{s'} \mod (2)$ then $\psi^{s,s'}$ is the angle corresponding to a representation of

$$2\prod_j p_j^{\frac{|\gamma_j^s - \gamma_j^{s'}|}{2}}$$

The fact that the angles ϕ_j are linearly independent over the rationals allows us to conclude that $\psi^{s,s'}$ is the angle of a lattice point not on the coordinate axis, but on the circle of radius.

$$2^{\nu/2}\prod_j p_j^{\frac{|\gamma_j^s - \gamma_j^{s'}|}{4}} \quad \text{where } \nu = 0 \text{ or } 1\,.$$

Therefore, if $|||\ |||$ denotes distance to the integers we have:

$$\left|\left|\left|\psi^{s,s'}\right|\right|\right| \ge \frac{1}{2\pi\sqrt{2}\prod_j p_j^{|\gamma_j^s - \gamma_j^{s'}|/4}}$$

On the other hand our hypothesis about the location of the $m+1$ lattice points on an arc of length $\sqrt{2}R^{\alpha}$ yields:

$$\left|\left|\left|\psi^{s,s'}\right|\right|\right| \le \frac{1}{2\pi\sqrt{2}}R^{\alpha-1}$$

Multiplying these inequalities all together we get:

$$R^{(\alpha-1)\frac{m(m+1)}{2}} \geq \frac{1}{\prod_{s,s'}\prod_j p_j^{|\gamma_j^s-\gamma_j^{s'}|/4}} = \left[\frac{1}{\prod_j p_j^{\sum_{s,s'}|\gamma_j^s-\gamma_j^{s'}|}}\right]^{1/4}$$

Now we observe that because of the constrains $|\gamma_j^s| \leq \alpha_j$ we have:

$$\sum_{s,s'}\left|\gamma_j^s - \gamma_j^{s'}\right| \leq 2\alpha_j\frac{(m+1)^2-\delta(m)}{4}$$

$$\text{where} \qquad \delta(m) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ 1 & \text{if } m \text{ is even} \end{cases}$$

Therefore,

$$R^{(\alpha-1)\frac{m(m+1)}{2}} \geq \left[\frac{1}{\prod_j p_j^{\alpha_j/2}}\right]^{\frac{(m+1)^2-\delta(m)}{4}} = R^{-\frac{(m+1)^2-\delta(m)}{4}}$$

i.e.,

$$(1-\alpha)\frac{m(m+1)}{2} \leq \frac{(m+1)^2-\delta(m)}{4}$$

which yields,

$$\boxed{\alpha \geq \frac{1}{2} - \frac{1}{4[\frac{m}{2}]+2}}$$

The proof for other imaginary quadratic fields $Q(\sqrt{-d})$ follows a similar strategy. One has to assign angles to ideals, and the non-principal prime ideals complicate the program. But it can be carried out (see [2]). □

2. Allegro: Trigonometric Sums

Gaussian sums are an important object in Number Theory. Here we shall consider them in the particular form

$$S_N(x) = \sum_{N\leq k\leq 2N} e^{2\pi i k^2 x}$$

The following are well-known estimates:

i.

$$\left\|\sum_{N\leq k\leq 2N} e^{2\pi i k^2 x}\right\|_p \sim N^{1/2}, \quad 1 \leq p < 4$$

ii.

$$\mu\left\{x : \left|\sum_N^{2N} e^{2\pi i k^2 x}\right| \geq N^{1/2}\alpha\right\} \leq \frac{C}{\alpha^4}$$

for some universal constant $C < \infty$.

iii.

$$\left\|\sum_N^{2N} e^{2\pi i k^2 x}\right\|_4 \sim N^{1/2}(\log N)^{1/4}, N \to \infty .$$

An application of the method introduced in the proof of the previous theorem produces:

Theorem 2. ***(J. Cilleruelo, A. Córdoba)***

$$\int_0^1 \left| \sum_N^{N+N^\alpha} e^{2\pi i n^2 x} \right|^4 dx = 2N^{2\alpha} + 0\left(N^{3\alpha-1+\varepsilon}\right), \quad \textit{for every } \varepsilon > 0 \textit{ if } \frac{1}{2} < \alpha < 1 .$$

$$= 2N^{2\alpha} + 0(1), \quad \textit{if } \alpha \leq \frac{1}{2} .$$

It is an interesting open problem to extends Theorem 2 to the case of arbitrary coefficients, i.e., sums of the form

$$\sum_N^{N+N^\alpha} a_k e^{2\pi i k^2 x}$$

Trigonometric series whose frequencies are precisely the powers $\{n^k\}_{n=1,2,3,\ldots}$ are relevant in Number Theory. There is a conjecture about their L^p-behavior:

Conjecture 1.

If $\sum |a_n|^2 < \infty$, *then* $\sum a_n e^{2\pi i n^k x} \in L^p[0,1]$, *for* $p < 2k$.

In the following we shall consider the family of trigonometric series

$$S_{\alpha,k}(x) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} e^{2\pi i n^k x} .$$

They have an interesting history. According to Weierstrass [5], Riemann thought that $Im S_{2,2}(x)$ could be an example of a continuous but nowhere differentiable function. This was partially confirmed by Hardy [6], who proved that $Im S_{2,2}$ is not differentiable at any irrational value of x and at several types of rational values. However, years later Gerver [7] showed that there are infinitely many rational numbers in which the derivative exists.

With the help of a computer we have obtained the graphics in the following pages.

These graphics illustrate our next theorems about the fractal (box counting) dimension and differentiability properties of the functions

$$F_{\alpha,k}(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^\alpha} e^{2\pi i n^k x}$$

$0 < \liminf c_n \leq \limsup c_n < \infty$

Theorem 3. ***(F. Chamizo, A. Córdoba)***

If $1 < \alpha \leq k + 1/2$, *then we have*

$$\dim(F_{\alpha,k}) = 2 + \frac{1}{2k} - \frac{\alpha}{k} .$$

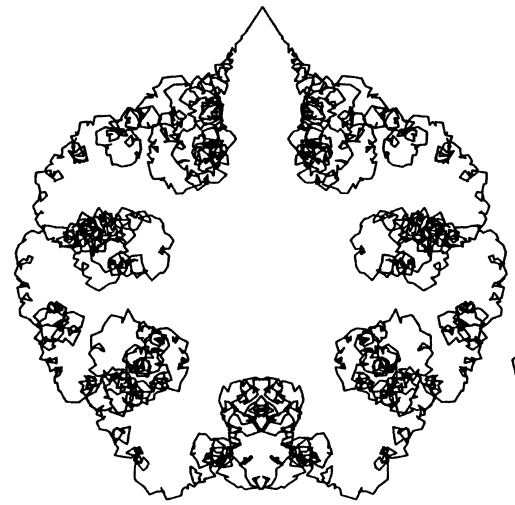
Theorem 4. ***(F. Chamizo, A. Córdoba)***

Let $a/q \in Q$ *be an irreducible fraction and* $S(a/q) = \sum_{n=1}^{q} e^{2\pi i \frac{a}{q} n^k}$.

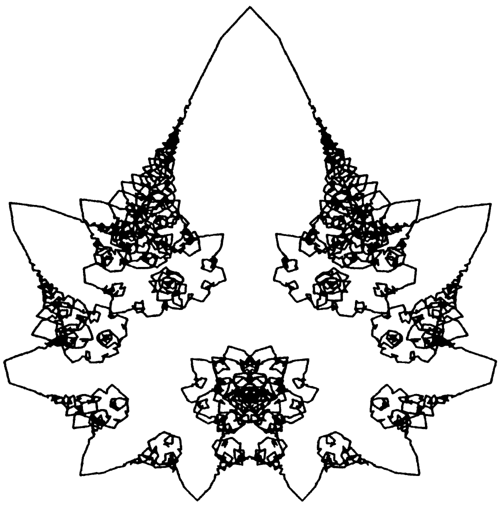
Then $S_{k,k}(x)$ *is differentiable at* a/q *if and only if* $S(a/q) = 0$. *Moreover, in this case,*

$$S'_{k,k}(x) = -\frac{2\pi i}{q} \sum_{n=1}^{q} n e^{2\pi i \frac{a}{q} n^k} .$$

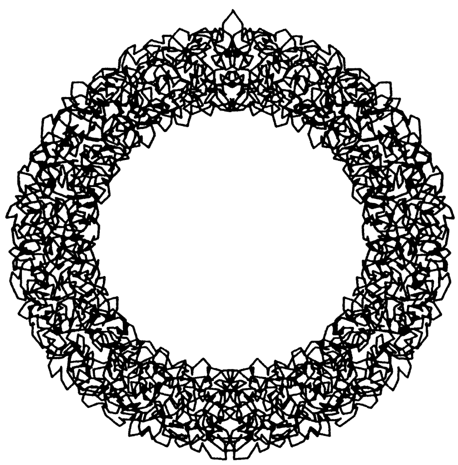
See [9, 10].



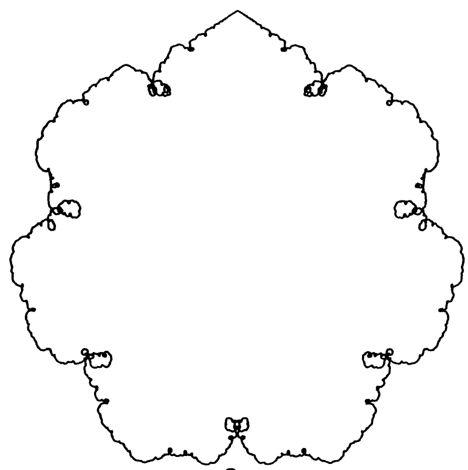
$S_{2,3}$



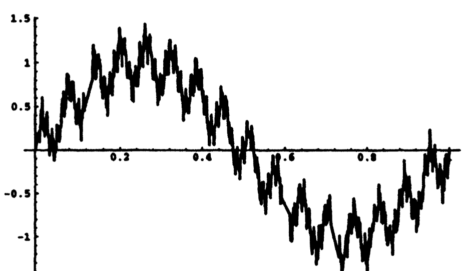
$S_{3/2,2}$



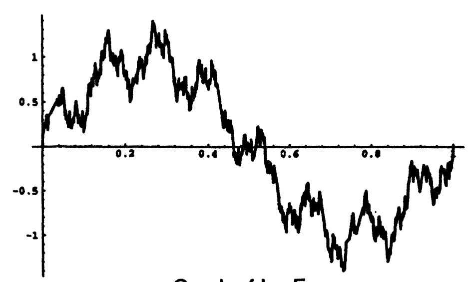
$S_{5/2,5}$



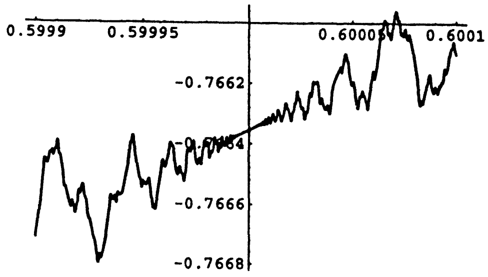
$S_{3,3}$



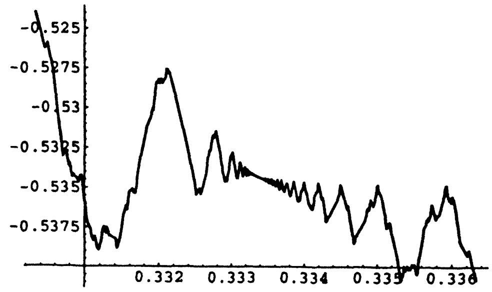
Graph of Im $F_{4,2}$



Graph of Im $F_{3,2}$



Re $F_{3,3}$: $0{,}5999 \leq \chi \leq 0{,}6001$



Re $F_{3,3}$: $0{,}331 \leq \chi \leq 0{,}336$

3. Molto Vivace: The Energy of a Large Atom

As we have seen before, an interesting problem in Number Theory is to estimate trigonometric sums of the following form:

$$S(N) = \sum_{k=1}^{N} f\left(\frac{k}{N}\right) \mu \left(N\phi\left(\frac{k}{N}\right)\right)$$

where:

(a) $|\phi''(x)| \geq C_0 > 0$.

(b) μ is a periodic function of average 0.

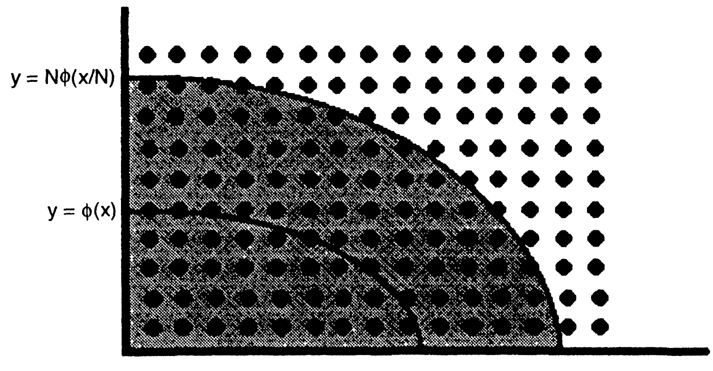
(c) The amplitude f is usually nice.

Example 1.

$f \equiv 1$, $\mu(x) = e^{2\pi i x}$, $\phi(x) = x^2$, then we obtain Gauss sums mod(N).

Example 2.

$f \equiv 1$, $\mu(x) = x - [x] - \frac{1}{2}$, then S represents the error term in the lattice point problem for a curve ϕ dilated by N.



In collaboration with Fefferman and Seco, the following example has been analyzed:

$$\begin{cases} \mu(x) = \operatorname{dist}\{x, \mathbf{Z}\}^2 - \dfrac{1}{12} \\ \phi(\Omega) = \dfrac{1}{\pi}\displaystyle\int \left(V_{TF}(r) - \frac{\Omega^2}{r^2}\right)_+^{1/2} dr \,. \\ f(\Omega) = \dfrac{\Omega}{P(\Omega)}\,,\ P(\Omega) = \displaystyle\int \left(V_{TF}(r) - \frac{\Omega^2}{r^2}\right)_+^{-1/2} dr \,. \\ N = Z^{1/3} \end{cases}$$

$V_{TF}^Z(r)$ is the Thomas–Fermi potential for an atom with charge Z which satisfies the perfect scaling:

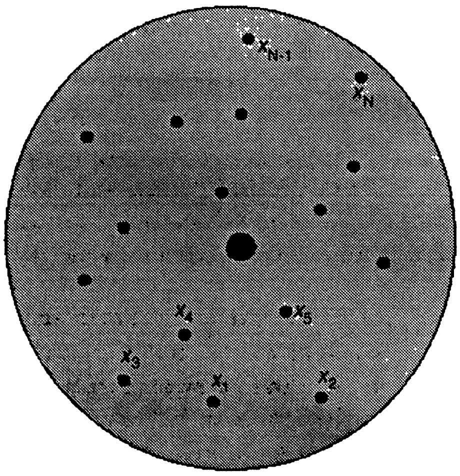
$$V_{TF}^Z(r) = Z^{4/3} V_{TF}\left(Z^{1/3} r\right) \,.$$

It happens that $-\phi''(\Omega) \geq C_0 > 0$, so we are in conditions to use Van der Corput's method.

For the function $\psi_Q(Z) = Z^{4/3} S(Z^{1/3})$, we have obtained the following estimates:

Theorem 5. (A. Córdoba, C. Fefferman, L. Seco)
a) $\exists C < \infty, |\psi_Q(z)| \leq C Z^{3/2}$.
b) $\limsup_{Z \to \infty} |Z^{-3/2} \psi_Q(Z)| \neq 0$.

The role of the function $\psi_Q(Z)$ in atomic physics is as follows:
Consider a non-relativistic atom, consisting of a nucleus of charge Z fixed at the origin and N quantized electrons at positions $x_i \in \mathbf{R}^3$.



The hamiltonian of such a system is given by $H_{Z,N} = \sum_{i=1}^N (-\Delta_{X_i} - \frac{Z}{|x_i|}) + \frac{1}{2}\sum_{j \neq k} \frac{1}{|x_j - x_k|}$ acting on functions $\Psi \in \mathcal{H} = \Lambda_{i=1}^N L^2(\mathbf{R}^3 \otimes Z_2)$

We define the energy

$$E(Z) = \inf_N E(Z; N), \quad E(Z; N) = \inf_{\substack{\psi \in \mathcal{H} \\ \|\psi\|=1}} < H_{Z,N}\psi, \psi >$$

Then we have the following asymptotic

$$E(Z) = C_{TF} Z^{7/3} + \frac{1}{8} Z^2 + C_{DS} Z^{5/3} + \psi_Q(Z) + 0\left(Z^{5/3-a}\right), \ a > 0 .$$

The rigorous proof of this formula is due to Lieb–Simon (1^o Term), Hughes (2^o Term) and Fefferman–Seco (Dirac–Schwinger term).

It has been known that the next correction is not just a power (fractionary) of Z. The work of Schwinger and Fefferman–Seco suggest that it is precisely $\psi_Q(Z)$ the right candidate for the next term of the expansion. Then its "oscillatory nature" became rather interesting as a first step in the project of understanding the periodic table from first principles.

The estimates for ψ_Q follow from Van der Corput's method; that is, stationary Phase and Poisson's summation formula, which allows us to make the connection with the last topic of this article.

4. Scherzo: X-Rays Crystallography

The spatial configuration of a crystal is usually obtained throughout X-ray diffraction data. The standard interpretation assigns diffraction peaks intensities to absolute values of the Fourier transform of the periodic electron density ρ.

The phase problem asks for the reconstruction of ρ from the knowledge of $|\hat{\rho}|$. In such general terms it is not well posed.[1] A rather interesting question is to analyze which kind of "chemical" or "geometrical" information about ρ is relevant to ensure the reconstruction.

A plausible model for the electronic density of one-dimensional crystals is given by sums of Dirac's deltas:

$$\rho = \sum_{j=1}^{N} b_j \sum_{-\infty}^{+\infty} \delta_{x_j+n}$$

where $b_j \in Z^*$ are positive integers and $0 \le x_j < 1$.

The phase problem asks to locate the positions $\{x_j\}$ (modulo translations or reflections $x_j' = 1 - x_j$) knowing the absolute values.

$$|F(m)| = \left| \sum_{j=1}^{N} b_j e^{2\pi i m \cdot x_j} \right|$$

$m \in Z$.

Example 3. (Gaussian molecule) Let p be an odd prime and consider the gaussian sum:

$$G_p(m) = \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) e^{2\pi i m \frac{n}{p}}$$

where

$$\left(\frac{n}{p}\right) = \text{Legendre symbol.} = \begin{cases} +1, & nRp \\ -1, & nNp . \end{cases}$$

Then

$$G_p(m) = \sum_{n=0}^{p-1} e^{2\pi i m \frac{n^2}{p}} = 1 + 2 \sum_{nRp} e^{2\pi i \frac{n}{p} m}$$

$$\left|G_p(m)\right| = \begin{cases} \sqrt{p} & \text{if } (m, p) = 1 \\ p & \text{if } p/m \end{cases}$$

[1]This was first observed by Pauling + Shappel (1930) in reference to the mineral bixbyite. Calderón and Pepinsky (1952) introduced a method to construct different homometric sets, i.e., $E \not\approx F$ and $|\hat{\chi}_E| = |\hat{\chi}_F|$.

Inverse problem: Are gaussian sums characterized by this property?

Theorem 6. (F. Chamizo, A. Córdoba)

Let $0 = x_1 < x_2 < \ldots < x_N < 1$ *be real numbers and* $\{b_j\}$ *positive integers such that*

$$|F(m)| = \left|\sum_{j=1}^{N} b_j e^{2\pi i x_j \cdot m}\right|$$

satisfies:

$$\begin{aligned} |F(m)| &= \Gamma \quad \textit{if} \quad (p,m) = 1 \\ |F(m)| &= \sum b_j \quad \textit{if} \quad p/m \end{aligned}$$

Then either

$$F(m) = AD_p\left(\frac{m}{p}\right) + Be^{2\pi i m \frac{k}{p}} G_p(m)$$

or

$$F(m) = AD_p\left(\frac{m}{p}\right) + Be^{2\pi i m \frac{k}{p}}$$

for suitables rational numbers A, B, *and integer* k.

We have used the notation:

$$\begin{cases} D_p(x) &= \sum_0^{p-1} e^{2\pi i n x} \quad \text{Dirichlet kernel} \\ G_p(m) &= \quad \text{gaussian sums} \end{cases}$$

Crucial lemma:
Let $\zeta_p = e^{2\pi i/p}$ and consider the field $Q(\zeta_p)$.

Lemma 1.

If all the algebraic conjugates of $w \in Q(\zeta_p)$ *have equal modulus, then:*

$$\begin{aligned} \textit{either} \quad w &= B\zeta_p^k \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \zeta_p^n \\ \textit{or} \quad w &= B\zeta_p^k \end{aligned}$$

for some rational number B *and integer* k.

Proof. Let σ be a generator of the Galois group of the extension $[Q(\zeta_p) : Q]$. The hypothesis about the algebraic conjugates yields

$$\frac{\sigma(w)}{w} = e^{2\pi i \frac{a}{b}} = (\zeta_b)^a, \, (a,b) = 1$$

Taking $a^*a \equiv 1$ mod (b) we obtain

$$\zeta_b = (\zeta_b^a)^{a^*} \in Q(\zeta_p)$$

Two cases:

1. If p/b then $[Q(\zeta_p) : Q(\zeta_b)] = \frac{\phi(p)}{\phi(b)}$, yields $b = p$ or $b = 2p$.
2. If $p \not| b$ then $Q(\zeta_p) = Q(\zeta_p \cdot \zeta_b) = Q(\zeta_{pb})$ and $\phi(p) = \phi(p \cdot b) \Longrightarrow b = 1, 2$.

i.e., we only have four possibilities

$$b = 1, 2, p, 2p$$

and it became easy to check that there exists e, $0 \leq e \leq p-1$ so that

$$\boxed{\frac{\sigma(w)}{w} = \pm\zeta_p^e}$$

Let us assume that $\sigma(\zeta_p) = \zeta_p^g$ and choose $k = e/(g-1)$ in Z_p^* ($g > 1$ because σ is a generator of the Galois group).

Then

$$\begin{aligned} \frac{\sigma(\zeta_p^{-k} w)}{\zeta_p^{-k}\cdot w} &= \frac{\sigma(\zeta_p)^{-k}}{\zeta_p^{-k}}\frac{\sigma(w)}{w} = \\ &= \zeta_p^{-kg+k}\frac{\sigma(w)}{w} = e^{2\pi i\frac{-k(g-1)}{p}}\frac{\sigma(w)}{w} = \\ &= \zeta_p^{-e}\frac{\sigma(w)}{w} = \pm 1\,. \end{aligned}$$

Similarly

$$\frac{\sigma^2(\zeta_p^{-k} w)}{\sigma(\zeta_p^{-k}\cdot w)} = \pm 1$$

Therefore, $\sigma^2(\zeta_p^{-k} w) = \zeta_p^{-k} w$, i.e.,

$$\begin{aligned} \zeta_p^{-k} w \in M &= \Big\{ a\left(\sigma(\zeta_p) + \sigma^3(\zeta_p) + \cdots + \sigma^{p-2}(\zeta_p)\right) \\ &\quad + b\left(\sigma^2(\zeta_p) + \sigma^4(\zeta_p) + \cdots + \sigma^{p-1}(\zeta_p)\right)\Big\}\,, \end{aligned}$$

where M is the subfield invariant under σ^2. Therefore,

$$w = \zeta_p^k \left\{ A\sum_{n\in R}\zeta_p^n + B\sum_{n\in N}\zeta_p^n \right\},\ A, B \in Q\,,$$

where R, N denotes, respectively, the set of quadratic and non-quadratic residues mod(p).

If $\sigma(\zeta_p^{-k} w) = \zeta_p^{-k} w$ then $\zeta^{-k} w \in Q$.

If $\sigma(\zeta_p^{-k} w) = -\zeta_p^{-k} w$, then we have $B = -A$ and $\zeta_p^{-k} w$ is a rational multiple of a gaussian sum. $\square$

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