Interface evolution: Water waves in 2-D

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Abstract

We study the free boundary evolution between two irrotational, incompressible and inviscid fluids in 2-D without surface tension. We prove local existence in Sobolev spaces when, initially, the difference of the gradients of the pressure in the normal direction has the proper sign, an assumption which is also known as the Rayleigh–Taylor condition. The well-posedness of the full water wave problem was first obtained by Wu (1997) [20]. The methods introduced in this paper allow us to consider multiple cases: with or without gravity, but also a closed boundary or a periodic boundary with the fluids placed above and below it. It is assumed that the initial interface does not touch itself, being a part of the evolution problem to check that such property prevails for a short time, as well as it does the Rayleigh–Taylor condition, depending conveniently upon the initial data. The addition of the pressure equality to the contour dynamic equations is obtained as a mathematical consequence, and not as a physical assumption, from the mere fact that we are dealing with weak solutions of Euler's equation in the whole space.
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1. Introduction

We consider the following evolution problem for the active scalar $\rho = \rho(x, t)$, $x \in \mathbb{R}^2$, and $t \geqslant 0$:

$$\rho_t + v \cdot \nabla \rho = 0, \tag{1.1}$$

with a velocity $v = (v_1, v_2)$ satisfying the Euler equation

$$\rho(v_t + v \nabla v) = -\nabla p - (0, g\rho), \tag{1.2}$$

and the incompressibility condition

$$\nabla \cdot v = 0. \tag{1.3}$$

The free boundary is given by the discontinuity on the densities of the fluids

$$\rho(x_1, x_2, t) = \begin{cases} \rho^1, & x \in \Omega^1(t), \\ \rho^2, & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t), \end{cases}$$

where $\rho^1 \neq \rho^2$ are constants.

We shall assume also that each fluid is irrotational, i.e. $\omega = \nabla \times u = 0$, in the interior of each domain Ω^j ($j = 1, 2$). The main purpose of this paper is to understand the evolution of the free boundary, but we shall also take the point of view of having weak solutions in the whole space presenting a discontinuity in the density along the interface. Under the hypothesis that at the initial time we have smooth velocity fields v^1, v^2 whose values at the interface differs only in the tangential direction it follows that, for a certain time $t > 0$, the vorticity ω will be supported on the free boundary curve $z(\alpha, t)$ and it has the form

$$\omega(x, t) = \varpi(\alpha, t)\delta(x - z(\alpha, t)).$$

Here we shall consider two types of geometries, namely periodicity in the horizontal space variable, says $z(\alpha + 2k\pi, t) = z(\alpha, t) + (2k\pi, 0)$, or the case of a closed contour $z(\alpha + 2k\pi, t) = z(\alpha, t)$. We shall assume also that we have infinite depth. In [15] fluids of finite depth were considered.

In Section 2 our first step will be to show the equality of pressure at each side of the free boundary, when we understand the system (1.1)–(1.3) in a weak sense (see Proposition 2.1).

The free boundary $z(\alpha, t)$ evolves with a velocity field coming from Biot–Savart law, which can be explicitly computed and it is given by the Birkhoff–Rott integral of the amplitude ϖ along the interface curve:

$$BR(z, \varpi)(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(z(\alpha, t) - z(\beta, t))^{\perp}}{|z(\alpha, t) - z(\beta, t)|^2} \varpi(\beta, t) \, d\beta, \tag{1.4}$$

where PV denotes principal value [19]. It gives us the velocity field at the interface to which we can subtract any term in the tangential direction without modifying the geometric evolution of the curve

$$z_t(\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t). \tag{1.5}$$

A wise choice of $c(\alpha, t)$, namely:

$$c(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int_{-\pi}^{\pi} \frac{\partial_\alpha z(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2} \cdot \partial_\alpha BR(z, \varpi)(\alpha, t) \, d\alpha$$

$$- \int_{-\pi}^{\alpha} \frac{\partial_\alpha z(\beta, t)}{|\partial_\alpha z(\beta, t)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) \, d\beta, \tag{1.6}$$

allows us to accomplish the fact that the length of the tangent vector to $z(\alpha, t)$ be just a function in the variable t only [14]:

$$A(t) = \left|\partial_\alpha z(\alpha, t)\right|^2.$$

Then we can close the system using Bernoulli's law with the equation:

$$\varpi_t(\alpha, t) = -2A_\rho \partial_t BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - A_\rho \partial_\alpha \left(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\right)(\alpha, t) + \partial_\alpha(c\varpi)(\alpha, t)$$

$$+ 2A_\rho c(\alpha, t) \partial_\alpha BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - 2A_\rho g \partial_\alpha z_2(\alpha, t), \tag{1.7}$$

where

$$A_\rho = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

is the Atwood number.

We shall use the notation T for the following operator (depending on the curve $z(\alpha, t)$) acting on $u(\alpha, t)$ by the formula

$$T(u)(\alpha, t) = 2BR(z, u)(\alpha, t) \cdot \partial_\alpha z(\alpha, t). \tag{1.8}$$

The inversibility of $(I + A_\rho T)$ (see [2]) allows us to write Eq. (1.7) in the following more convenient explicit manner:

$$\varpi_t(\alpha, t) = (I + A_\rho T)^{-1}\big(A_\rho R(z, \varpi) + \partial_\alpha(c\varpi)\big)(\alpha, t). \tag{1.9}$$

Next let us give the function which measures the arc-chord condition [13]

$$\mathcal{F}(z)(\alpha, \beta, t) = \frac{|\beta|}{|z(\alpha, t) - z(\alpha - \beta, t)|} \quad \forall \alpha, \beta \in (-\pi, \pi), \tag{1.10}$$

and

$$\mathcal{F}(z)(\alpha, 0, t) = \frac{1}{|\partial_\alpha z(\alpha, t)|}.$$

Finally following Refs. [3] and [1] we introduce the auxiliary function $\varphi(\alpha, t)$ which will allow us to integrate the evolution equation

$$\varphi(\alpha, t) = \frac{\varpi(\alpha, t)}{2|\partial_\alpha z(\alpha, t)|} - c(\alpha, t)|\partial_\alpha z(\alpha, t)|. \tag{1.11}$$

Our main result consists on local existence for the water wave problem: $\rho_1 = 0$. We prove that there is a positive time T (depending upon the initial condition) for which there exists a solution of Eqs. (1.4)–(1.7) with $\rho_1 = 0$ during the time interval $[0, T]$ so long as the initial data satisfy $z_0(\alpha) \in H^k$, $\varphi_0(\alpha) \in H^{k-\frac{1}{2}}$ and $\varpi_0(\alpha) \in H^{k-1}$ for $k \geqslant 4$, $\mathcal{F}(z_0)(\alpha, \beta) < \infty$, and

$$\sigma_0(\alpha) = -\big(\nabla p^2\big(z_0(\alpha), 0\big) - \nabla p^1\big(z_0(\alpha), 0\big)\big) \cdot \partial_\alpha^\perp z_0(\alpha) > 0,$$

where p^j denotes the pressure in Ω^j.

Theorem 1.1. *Let* $z_0(\alpha) \in H^k$, $\varphi_0(\alpha) \in H^{k-\frac{1}{2}}$ *and* $\varpi_0(\alpha) \in H^{k-1}$ *for* $k \geqslant 4$, $\mathcal{F}(z_0)(\alpha, \beta) < \infty$, *and*

$$\sigma_0(\alpha) = -\big(\nabla p^2\big(z_0(\alpha), 0\big) - \nabla p^1\big(z_0(\alpha), 0\big)\big) \cdot \partial_\alpha^\perp z_0(\alpha) > 0.$$

Then there exists a time T > 0 *so that we have a solution to* (1.4)–(1.7) *in the case* $\rho_1 = 0$, *where* $z(\alpha, t) \in C^1([0, T]; H^k)$ *and* $\varpi(\alpha, t) \in C^1([0, T]; H^{k-1})$ *with* $z(\alpha, 0) = z_0(\alpha)$ *and* $\varpi(\alpha, 0) = \varpi_0(\alpha)$.

The first results concerning the Cauchy problem for the linearized version in Sobolev spaces are due to [10,17,22]. In her important work [20] (see also [21]) S. Wu was able to prove that the presence of the gravitational field, together with the hypothesis about the asymptotic flatness of the fluid domains, implies that the Rayleigh–Taylor sign condition must hold so long as the interface is well-defined. In our treatment we can also get local solvability even in the absence of gravity, or for a closed contour, whenever the Rayleigh–Taylor and the arc-chord conditions are initially satisfied.

Besides the significant work of S. Wu that has been referred before, we can also quote the interesting paper [1] where they get energy estimates on the free boundary and the amplitude of the vorticity, under the time dependent assumption of the arc-chord property. These authors make also use of the fact obtained by Wu about the persistence of the Rayleigh–Taylor sign condition.

In our approach the explicit control upon the evolution of the arc-chord relation of the free boundary is especially emphasized, together with the inversion of the operator $(I + T)$, which gives us the equation for the time derivative of the vorticity amplitude in terms of the curve (see Eqs. (1.8)–(1.9) with $\rho_1 = 0$). The architecture of our proof relies upon different energy estimates for the quantities involved (Sobolev norms for z, ϖ, arc-chord and Rayleigh–Taylor condition). But in order to fix together its different parts it becomes crucial to get explicit upper bounds on the operator $(I + T)^{-1}$ on different Sobolev spaces. Here we continue the method introduced in [6,7], where conformal mappings, Hopf maximum principle and Dahlbert–Harnack inequality up to the boundary, for nonnegative harmonic functions, play a central role.

In the following interesting works by Christodoulou and Lindblad [5], Lindblad [16], Coutand and Shkoller [9], Shatah and Zeng [18] and Zhang and Zhang [23] the rotational case has been also considered. Let us point out that the evolution of the sign of Rayleigh–Taylor condition

is crucial in our proof [7], because it allows to get rid of the highest order derivatives in the evolution equation of the Sobolev norms of the curve (Section 8).

2. The evolution equation

We shall consider weak solutions of the system (1.1)–(1.3); that is for any smooth functions ζ, η and χ, compactly supported on $[0, T) \times \mathbb{R}^2$, i.e. lying in the space $C_c^\infty([0, T) \times \mathbb{R}^2)$, we have

$$\int_0^T \int_{\mathbb{R}^2} \rho(\zeta_t + v \cdot \nabla \zeta) \, dx \, dt + \int_{\mathbb{R}^2} \rho_0(x)\zeta(x, 0) \, dx = 0, \tag{2.1}$$

$$\int_0^T \int_{\mathbb{R}^2} \left(\rho v \cdot (\eta_t + v \cdot \nabla \eta) + p \nabla \cdot \eta - (0, g\rho) \cdot \eta \right) dx \, dt + \int_{\mathbb{R}^2} \rho_0(x)v_0(x) \cdot \eta(x, 0) \, dx = 0, \tag{2.2}$$

and

$$\int_0^T \int_{\mathbb{R}^2} v \cdot \nabla \chi \, dx \, dt = 0. \tag{2.3}$$

Here ρ is defined by

$$\rho(x_1, x_2, t) = \begin{cases} \rho^1, & x \in \Omega^1(t), \\ \rho^2, & x \in \Omega^2(t), \end{cases} \tag{2.4}$$

where $\rho^1 \neq \rho^2$. It is assumed that the vorticity is given by a delta function on the curve $\partial \Omega^j(t)$ multiplied by an amplitude and has the form

$$\omega(x, t) = \varpi(\alpha, t)\delta(x - z(\alpha, t)). \tag{2.5}$$

Then using the Biot–Savart law we get

$$v(x, t) = \frac{1}{2\pi} PV \int \frac{(x - z(\beta, t))^\perp}{|x - z(\beta, t)|^2} \varpi(\beta, t) \, d\beta \tag{2.6}$$

for x not lying on the curve $z(\alpha, t)$, and

$$v^2(z(\alpha, t), t) = BR(z, \varpi)(\alpha, t) + \frac{1}{2} \frac{\varpi(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2} \partial_\alpha z(\alpha, t),$$

$$v^1(z(\alpha, t), t) = BR(z, \varpi)(\alpha, t) - \frac{1}{2} \frac{\varpi(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2} \partial_\alpha z(\alpha, t), \tag{2.7}$$

where $v^j(z(\alpha, t), t)$ denotes the limit velocity field obtained approaching the boundary in the normal direction inside Ω^j and $BR(z, \varpi)(\alpha, t)$ is given by (1.4). It is easy to check that (2.3) is

satisfied by v given as in (2.6). Furthermore, we have that the identity of the weak formulation (2.1) is verified so long as the following equality holds (see [8]):

$$z_t(\alpha, t) \cdot \partial_\alpha^\perp z(\alpha, t) = BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha^\perp z(\alpha, t). \tag{2.8}$$

Proposition 2.1. *Let us consider a weak solution (ρ, v, p) satisfying (2.1)–(2.3) where ρ is given by (2.4) and* curl $v = \omega$ *by (2.5). Then we have the following identity*

$$p^1\big(z(\alpha, t), t\big) = p^2\big(z(\alpha, t), t\big),$$

where $p^j(z(\alpha, t), t)$ denotes the limit pressure obtained approaching the boundary in the normal direction inside Ω^j.

Proof. We shall show that the Laplacian of the pressure is as follows

$$\Delta p(x, t) = F(x, t) + f(\alpha, t)\delta\big(x - z(\alpha, t)\big),$$

where F is regular in $\Omega^j(t)$ although discontinuous on $z(\alpha, t)$, and the amplitude of the Dirac distribution f is regular. Then the inverse of the Laplacian by means of the Newtonian potential gives the continuity of the pressure on the free boundary (see [6]).

We also shall use an ad hoc integration by parts for the derivatives of the velocity. The expression for the conjugate of the velocity in complex variables

$$\bar{v}(z, t) = \frac{1}{2\pi i} PV \int \frac{1}{z - z(\alpha, t)} \varpi(\alpha, t)\, d\alpha,$$

for $z \neq z(\alpha, t)$ allows us to accomplish the fact that

$$\partial_z \bar{v}(z, t) = \frac{1}{2\pi i} PV \int \frac{-\varpi(\alpha, t)}{(z - z(\alpha, t))^2}\, d\alpha = \frac{1}{2\pi i} PV \int \frac{-\partial_\alpha z(\alpha, t)}{(z - z(\alpha, t))^2} \frac{\varpi(\alpha, t)}{\partial_\alpha z(\alpha, t)}\, d\alpha,$$

and therefore

$$\partial_z \bar{v}(z, t) = \frac{1}{2\pi i} PV \int \frac{1}{z - z(\alpha, t)} \partial_\alpha\left(\frac{\varpi}{\partial_\alpha z}\right)(\alpha, t)\, d\alpha, \tag{2.9}$$

for a regular parametrization with $\partial_\alpha z(\alpha, t) \neq 0$. In a similar way

$$\bar{v}_t(z, t) = \frac{1}{2\pi i} PV \int \frac{1}{z - z(\alpha, t)} \varpi_t(\alpha, t)\, d\alpha$$
$$- \frac{1}{2\pi i} PV \int \frac{1}{z - z(\alpha, t)} \partial_\alpha\left(\frac{z_t \varpi}{\partial_\alpha z}\right)(\alpha, t)\, d\alpha, \tag{2.10}$$

and

$$\partial_z^2 \bar{v}(z, t) = \frac{1}{2\pi i} PV \int \frac{1}{z - z(\alpha, t)} \partial_\alpha\left(\frac{1}{\partial_\alpha z} \partial_\alpha\left(\frac{\varpi}{\partial_\alpha z}\right)\right)(\alpha, t)\, d\alpha. \tag{2.11}$$

These identities help us to get the values of $\nabla v^j(z(\alpha, t), t)$, $v_t^j(z(\alpha, t), t)$ and $\nabla^2 v^j(z(\alpha, t), t)$ which are obtained as limits approaching the boundary in the normal direction inside $\Omega^j(t)$.

To get the stated formula for the pressure we start with identity (2.2) choosing $\eta(x, t) = \nabla\lambda(x, t)$. Then

$$
\int_0^T \int_{\mathbb{R}^2} p\Delta\lambda \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} (0, g\rho) \cdot \nabla\lambda \, dx \, dt - \int_0^T \int_{\mathbb{R}^2} \rho v \cdot \nabla\lambda_t \, dx \, dt
$$

$$
- \int_0^T \int_{\mathbb{R}^2} \rho v \cdot (v \cdot \nabla^2 \lambda) \, dx \, dt - \int_{\mathbb{R}^2} \rho_0(x) v_0(x) \cdot \nabla\lambda(x, 0) \, dx
$$

$$
= I_1 + I_2 + I_3 + I_4.
$$

Let us define $\Omega_\varepsilon^1(t) = \{x \in \Omega^1(t): \text{dist}(x, \partial\Omega^1(t)) \geqslant \varepsilon\}$ and $\Omega_\varepsilon^2(t) = \{x \in \Omega^2(t): \text{dist}(x, \partial\Omega^2(t)) \geqslant \varepsilon\}$, we have

$$
I_1 = \lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon^1(t)} g\rho^1 \partial_{x_2}\lambda \, dx \, dt + \int_0^T \int_{\Omega_\varepsilon^2(t)} g\rho^2 \partial_{x_2}\lambda \, dx \, dt
$$

$$
= \int_0^T \int_{-\pi}^{\pi} (\rho^2 - \rho^1) g\partial_\alpha z_1(\alpha, t)\lambda(z(\alpha, t), t) \, d\alpha \, dt,
$$

and we can consider the term $(\rho^2 - \rho^1) g\partial_\alpha z_1(\alpha, t)$ as being part of the function $f(\alpha, t)$.

Regarding the term I_2 we integrate by parts in the variable t to obtain

$$
I_2 = \lim_{\varepsilon\to 0} - \int_0^T \int_{\Omega_\varepsilon^1(t)} \rho^1 v^1 \cdot \nabla\lambda_t \, dx \, dt - \int_0^T \int_{\Omega_\varepsilon^2(t)} \rho^2 v^2 \cdot \nabla\lambda_t \, dx \, dt
$$

$$
= J_1 + J_2 - I_4
$$

where

$$
J_1 = \int_0^T \int_{\mathbb{R}^2} \rho v_t \cdot \nabla\lambda \, dx \, dt,
$$

and

$$
J_2 = \int_0^T \int_{-\pi}^{\pi} \left(\rho^2 v^2(z(\alpha, t), t) - \rho^1 v^1(z(\alpha, t), t)\right) \cdot \nabla\lambda(z(\alpha, t), t) z_t(\alpha, t) \cdot \partial_\alpha^\perp z(\alpha, t) \, d\alpha \, dt.
$$

In J_1 we use formula (2.10) to get the limit on the boundary of $v_t(x, t)$. Again we first integrate by parts in J_1 and then take the limit when $\epsilon \to 0$. Since in each $\Omega_\varepsilon^j(t)$ v_t is regular and div $v_t = 0$, it follows that

$$J_1 = \int_0^T \int_{-\pi}^\pi \left(\rho^2 v_t^2\big(z(\alpha, t), t\big) - \rho^1 v_t^1\big(z(\alpha, t), t\big)\right) \cdot \partial_\alpha^\perp z(\alpha, t) \lambda\big(z(\alpha, t), t\big) \, dx \, dt.$$

As before we may consider $(\rho^2 v_t^2(z(\alpha, t), t) - \rho^1 v_t^1(z(\alpha, t), t)) \cdot \partial_\alpha^\perp z(\alpha, t)$ as being a part of $f(\alpha, t)$.

Next (2.7) yields the splitting $J_2 = K_1 + K_2$ where

$$K_1 = \int_0^T \int_{-\pi}^\pi (\rho^2 - \rho^1) BR(z, \varpi)(\alpha, t) \cdot \nabla\lambda\big(z(\alpha, t), t\big) z_t(\alpha, t) \cdot \partial_\alpha^\perp z(\alpha, t) \, d\alpha \, dt,$$

$$K_2 = \int_0^T \int_{-\pi}^\pi (\rho^2 + \rho^1) \frac{\varpi(\alpha, t)}{2|\partial_\alpha z(\alpha, t)|^2} \partial_\alpha z(\alpha, t) \cdot \nabla\lambda\big(z(\alpha, t), t\big) z_t(\alpha, t) \cdot \partial_\alpha^\perp z(\alpha, t) \, d\alpha \, dt.$$

Integrating by parts in α we can write

$$K_2 = -\int_0^T \int_{-\pi}^\pi (\rho^2 + \rho^1) \lambda\big(z(\alpha, t), t\big) \partial_\alpha \left(\frac{\varpi}{2|\partial_\alpha z|^2} z_t \cdot \partial_\alpha^\perp z\right)(\alpha, t) \, d\alpha \, dt,$$

giving us another term of $f(\alpha, t)$.

Let us introduce now the decomposition $I_3 = J_3 + J_4 + J_5 + J_6$ where

$$J_3 = -\int_0^T \int_{\mathbb{R}^2} \rho(v_1)^2 \partial_{x_1}^2 \lambda \, dx \, dt, \qquad J_4 = -\int_0^T \int_{\mathbb{R}^2} \rho v_1 v_2 \partial_{x_2} \partial_{x_1} \lambda \, dx \, dt,$$

$$J_5 = -\int_0^T \int_{\mathbb{R}^2} \rho v_1 v_2 \partial_{x_1} \partial_{x_2} \lambda \, dx \, dt, \qquad J_6 = -\int_0^T \int_{\mathbb{R}^2} \rho(v_2)^2 \partial_{x_2}^2 \lambda \, dx \, dt.$$

Using the sets $\Omega_\varepsilon^j(t)$ and the identity (2.9) we get

$$J_3 = \int_0^T \int_{\mathbb{R}^2} 2\rho v_1 \partial_{x_1} v_1 \partial_{x_1} \lambda \, dx \, dt$$

$$+ \int_0^T \int_{-\pi}^\pi \left(\rho^2 \big(v_1^2(z(\alpha, t), t)\big)^2 - \rho^1 \big(v_1^1(z(\alpha, t), t)\big)^2\right) \partial_{x_1} \lambda\big(z(\alpha, t), t\big) \partial_\alpha z_2(\alpha, t) \, d\alpha \, dt$$

$$= K_3 + K_4.$$

The term K_3 trivializes because the ad hoc integration by parts formula together with the identity (2.11) gives

$$K_3 = - \int\limits_0^T \int\limits_{\mathbb{R}^2} 2\rho \big(v_1 \partial_{x_1}^2 v_1 + (\partial_{x_1} v_1)^2\big) \lambda \, dx \, dt - \int\limits_0^T \int\limits_{-\pi}^{\pi} \widetilde{f}(\alpha, t) \lambda\big(z(\alpha, t), t\big) \, d\alpha \, dt,$$

where $\widetilde{f}(\alpha, t) = 2(\rho^2 v_1^2(z(\alpha, t), t) \partial_{x_1} v_1^2(z(\alpha, t), t) - \rho^1 v_1^1(z(\alpha, t), t) \partial_{x_1} v_1^1(z(\alpha, t), t)) \partial_\alpha z_2(\alpha, t)$, and the first term in K_3 is part of $F(x, t)$ while the second lies in $f(\alpha, t)$.

We can rewrite K_4 as follows

$$K_4 = \big(\rho^2 - \rho^1\big) \int\limits_0^T \int\limits_{-\pi}^{\pi} \left[(BR_1)^2 + \frac{\varpi^2}{4} \frac{(\partial_\alpha z_1)^2}{|\partial_\alpha z|^4} \right] \partial_{x_1} \lambda(z) \partial_\alpha z_2 \, d\alpha \, dt$$

$$+ \big(\rho^2 + \rho^1\big) \int\limits_0^T \int\limits_{-\pi}^{\pi} \varpi BR_1 \frac{\partial_\alpha z_1}{|\partial_\alpha z|^2} \partial_{x_1} \lambda(z) \partial_\alpha z_2 \, d\alpha \, dt. \tag{2.12}$$

Next we continue analogously with J_4

$$J_4 = \int\limits_0^T \int\limits_{\mathbb{R}^2} \rho (v_2 \partial_{x_2} v_1 + v_1 \partial_{x_2} v_2) \partial_{x_1} \lambda \, dx \, dt$$

$$- \int\limits_0^T \int\limits_{-\pi}^{\pi} \big(\rho^2 \big(v_1^2 v_2^2\big) (z(\alpha, t), t) - \rho^1 \big(v_1^1 v_2^1\big) (z(\alpha, t), t)\big) \partial_{x_1} \lambda\big(z(\alpha, t), t\big) \partial_\alpha z_1(\alpha, t) \, d\alpha \, dt$$

$$= K_5 + K_6,$$

and K_5 is treated as K_3 (a term in K_5 is part of $F(x, t)$ and another of $f(\alpha, t)$). K_6 can be written in the following manner

$$K_6 = -\big(\rho^2 - \rho^1\big) \int\limits_0^T \int\limits_{-\pi}^{\pi} \left[BR_1 BR_2 + \frac{\varpi^2}{4} \frac{\partial_\alpha z_1 \partial_\alpha z_2}{|\partial_\alpha z|^4} \right] \partial_{x_1} \lambda(z) \partial_\alpha z_1 \, d\alpha \, dt$$

$$- \big(\rho^2 + \rho^1\big) \int\limits_0^T \int\limits_{-\pi}^{\pi} \left[\frac{\varpi}{2} BR_1 \frac{\partial_\alpha z_2}{|\partial_\alpha z|^2} + \frac{\varpi}{2} BR_2 \frac{\partial_\alpha z_1}{|\partial_\alpha z|^2} \right] \partial_{x_1} \lambda(z) \partial_\alpha z_1 \, d\alpha \, dt. \tag{2.13}$$

Regarding J_5 we have the splitting

$$J_5 = \int\limits_0^T \int\limits_{\mathbb{R}^2} \rho (v_2 \partial_{x_1} v_1 + v_1 \partial_{x_1} v_2) \partial_{x_2} \lambda \, dx \, dt$$

$$+ \int_0^T \int_{-\pi}^{\pi} \left(\rho^2 (v_1^2 v_2^2)(z(\alpha,t),t) - \rho^1 (v_1^1 v_2^1)(z(\alpha,t),t) \right) \partial_{x_2} \lambda(z(\alpha,t),t) \partial_\alpha z_2(\alpha,t) \, d\alpha \, dt$$

$$= K_7 + K_8.$$

K_7 again can be treated like K_3, and we obtain for K_8 the following expression

$$K_8 = (\rho^2 - \rho^1) \int_0^T \int_{-\pi}^{\pi} \left[BR_1 BR_2 + \frac{\varpi^2}{4} \frac{\partial_\alpha z_1 \partial_\alpha z_2}{|\partial_\alpha z|^4} \right] \partial_{x_2} \lambda(z) \partial_\alpha z_2 \, d\alpha \, dt$$

$$+ (\rho^2 + \rho^1) \int_0^T \int_{-\pi}^{\pi} \left[\frac{\varpi}{2} BR_1 \frac{\partial_\alpha z_2}{|\partial_\alpha z|^2} + \frac{\varpi}{2} BR_2 \frac{\partial_\alpha z_1}{|\partial_\alpha z|^2} \right] \partial_{x_2} \lambda(z) \partial_\alpha z_2 \, d\alpha \, dt. \quad (2.14)$$

Next for J_6

$$J_6 = \int_0^T \int_{\mathbb{R}^2} 2\rho v_2 \partial_{x_2} v_2 \partial_{x_2} \lambda \, dx \, dt$$

$$- \int_0^T \int_{-\pi}^{\pi} \left(\rho^2 (v_2^2(z(\alpha,t),t))^2 - \rho^1 (v_2^1(z(\alpha,t),t))^2 \right) \partial_{x_2} \lambda(z(\alpha,t),t) \partial_\alpha z_2(\alpha,t) \, d\alpha \, dt$$

$$= K_9 + K_{10},$$

and for K_9 we proceed as before. Finally we have

$$K_{10} = -(\rho^2 - \rho^1) \int_0^T \int_{-\pi}^{\pi} \left[(BR_2)^2 + \frac{\varpi^2}{4} \frac{(\partial_\alpha z_2)^2}{|\partial_\alpha z|^4} \right] \partial_{x_2} \lambda(z) \partial_\alpha z_1 \, d\alpha \, dt$$

$$- (\rho^2 + \rho^1) \int_0^T \int_{-\pi}^{\pi} \varpi BR_2 \frac{\partial_\alpha z_2}{|\partial_\alpha z|^2} \partial_{x_2} \lambda(z) \partial_\alpha z_1 \, d\alpha \, dt. \quad (2.15)$$

Using Eqs. (2.12)–(2.15) we get the following sum $K_4 + K_6 + K_8 + K_{10} = (\rho^2 - \rho^1) L_1 + (\rho^2 + \rho^1) L_2$ where

$$L_1 = - \int_0^T \int_{-\pi}^{\pi} BR(z,\varpi)(\alpha,t) \cdot \nabla \lambda(z(\alpha,t),t) BR(z,\varpi)(\alpha,t) \cdot \partial_\alpha^\perp z(\alpha,t) \, d\alpha \, dt,$$

and

$$L_2 = - \int_0^T \int_{-\pi}^{\pi} \frac{\varpi(\alpha,t)}{2|\partial_\alpha z(\alpha,t)|^2} BR(z,\varpi)(\alpha,t) \cdot \partial_\alpha^\perp z(\alpha,t) \partial_\alpha z(\alpha,t) \cdot \nabla \lambda(z(\alpha,t),t) \, d\alpha \, dt.$$

Integration by parts in the variable α in L_2 gives the last term of $f(\alpha, t)$. Then identity (2.8) gives $K_1 + (\rho^2 - \rho^1)L_1 = 0$ and the stated formula for the Laplacian of p is proved. $\square$

Identity (2.8) allows us to choose the velocity of the curve as follows:

$$z_t(\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t), \tag{2.16}$$

where the scalar $c(\alpha, t)$ is given by

$$c(\alpha) = \frac{\alpha + \pi}{2\pi} \int\limits_{-\pi}^{\pi} \frac{\partial_\beta z(\beta)}{|\partial_\beta z(\beta)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta)\, d\beta - \int\limits_{-\pi}^{\alpha} \frac{\partial_\beta z(\beta)}{|\partial_\beta z(\beta)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta)\, d\beta, \tag{2.17}$$

and has been taken in such a way that the length of the tangent vector only depends on the variable t:

$$\left|\partial_\alpha z(\alpha, t)\right|^2 = A(t). \tag{2.18}$$

Since $c(\alpha, t)$ has to be periodic, we obtain

$$A'(t) = 2\partial_\alpha z_t(\alpha, t) \cdot \partial_\alpha z(\alpha, t) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \partial_\alpha z(\alpha, t) \cdot \partial_\alpha BR(z, \varpi)(\alpha, t)\, d\alpha. \tag{2.19}$$

Next we close the system giving the evolution equation for the amplitude of the vorticity $\varpi(\alpha, t)$ by means of Bernoulli's law. This fact allows us to satisfy (2.2) showing that we have a weak solution. Using (2.6) for $x \neq z(\alpha, t)$ we get $v(x, t) = \nabla\phi(x, t)$ where

$$\phi(x, t) = \frac{1}{2\pi} PV \int \arctan\left(\frac{x_2 - z_2(\beta, t)}{x_1 - z_1(\beta, t)}\right) \varpi(\beta, t)\, d\beta.$$

Let us define

$$\Pi(\alpha, t) = \phi^2(z(\alpha, t), t) - \phi^1(z(\alpha, t), t),$$

where again $\phi^j(z(\alpha, t), t)$ denotes the limit obtained approaching the boundary in the normal direction inside Ω^j. It is clear that

$$\partial_\alpha \Pi(\alpha, t) = \left(\nabla\phi^2(z(\alpha, t), t) - \nabla\phi^1(z(\alpha, t), t)\right) \cdot \partial_\alpha z(\alpha, t)$$
$$= \left(v^2(z(\alpha, t), t) - v^1(z(\alpha, t), t)\right) \cdot \partial_\alpha z(\alpha, t) = \varpi(\alpha, t),$$

and therefore

$$\int\limits_{-\pi}^{\pi} \varpi(\alpha, t)\, d\alpha = 0.$$

Now we observe that

$$\phi^2\big(z(\alpha, t), t\big) = IT(z, \varpi)(\alpha, t) + \frac{1}{2}\Pi(\alpha, t),$$

$$\phi^1\big(z(\alpha, t), t\big) = IT(z, \varpi)(\alpha, t) - \frac{1}{2}\Pi(\alpha, t), \qquad (2.20)$$

where

$$IT(z, \varpi)(\alpha, t) = \frac{1}{2\pi} PV \int \arctan\left(\frac{z_2(\alpha, t) - z_2(\beta, t)}{z_1(\alpha, t) - z_1(\beta, t)}\right) \varpi(\beta, t)\, d\beta.$$

Using the Bernoulli's law in (1.2), inside each domain, we have

$$\rho\left(\phi_t(x, t) + \frac{1}{2}\big|v(x, t)\big|^2 + gx_2\right) + p(x, t) = 0.$$

Next we take limits to get

$$\rho^j\left(\phi_t^j\big(z(\alpha, t), t\big) + \frac{1}{2}\big|v^j\big(z(\alpha, t), t\big)\big|^2 + gz_2(\alpha, t)\right) + p^j\big(z(\alpha, t), t\big) = 0,$$

and since $p^1(z(\alpha, t), t) = p^2(z(\alpha, t), t)$, we obtain

$$[\rho\phi_t](\alpha, t) + \frac{\rho^2}{2}\big|v^2\big(z(\alpha, t), t\big)\big|^2 - \frac{\rho^1}{2}\big|v^1\big(z(\alpha, t), t\big)\big|^2 + \big(\rho^2 - \rho^1\big)gz_2(\alpha, t) = 0, \qquad (2.21)$$

where we have introduced the following notation:

$$[\rho\phi_t](\alpha, t) = \rho^2\phi_t^2\big(z(\alpha, t), t\big) - \rho^1\phi_t^1\big(z(\alpha, t), t\big).$$

Then it is clear that $\phi_t^j(z(\alpha, t), t) = \partial_t(\phi^j(z(\alpha, t), t)) - z_t(\alpha, t) \cdot \nabla\phi^j(z(\alpha, t), t)$, and using (2.20) we find that

$$[\rho\phi_t] = \frac{\rho^2 + \rho^1}{2}\Pi_t + \big(\rho^2 - \rho^1\big)\partial_t\big(IT(z, \varpi)\big) - z_t \cdot \big(\rho^2 v^2(z, t) - \rho^1 v^1(z, t)\big).$$

Introducing Eqs. (2.7) and (2.16) into (2.21) we get

$$\Pi_t(\alpha, t) = -2A_\rho \partial_t\big(IT(z, \varpi)\big)(\alpha, t) + c(\alpha, t)\varpi(\alpha, t) + A_\rho\big|BR(z, \varpi)(\alpha, t)\big|^2$$

$$+ 2A_\rho c(\alpha, t)BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - A_\rho \frac{|\varpi(\alpha, t)|^2}{4|\partial_\alpha z(\alpha, t)|^2}$$

$$- 2A_\rho gz_2(\alpha, t). \qquad (2.22)$$

Since the equality

$$\partial_\alpha \partial_t\big(IT(z, \varpi)\big) = \partial_t\big(BR(z, \varpi) \cdot \partial_\alpha z\big) = \partial_t BR(z, \varpi) \cdot \partial_\alpha z + BR(z, \varpi) \cdot \partial_\alpha BR(z, \varpi)$$

$$+ cBR(z, \varpi) \cdot \partial_\alpha^2 z + \partial_\alpha cBR(z, \varpi) \cdot \partial_\alpha z$$

can be proved easily, we can take then a derivative in (2.22) and use the above identity to find the desired formula for ϖ:

$$\varpi_t(\alpha, t) = -2A_\rho \partial_t BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - A_\rho \partial_\alpha \left(\frac{|\varpi|^2}{4|\partial_\alpha z|^2} \right)(\alpha, t) + \partial_\alpha(c\varpi)(\alpha, t)$$

$$+ 2A_\rho c(\alpha, t)\partial_\alpha BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - 2A_\rho g\partial_\alpha z_2(\alpha, t). \tag{2.23}$$

Our next step will be to get the formula for the difference of the gradients of the pressure in the normal direction:

$$\sigma(\alpha, t) = -\left(\nabla p^2(z(\alpha, t), t) - \nabla p^1(z(\alpha, t), t) \right) \cdot \partial_\alpha^\perp z(\alpha, t), \tag{2.24}$$

which we shall find in the singular terms of the evolution equation.

We will consider the case $\rho_1 = 0$, which gives $-\nabla p(x, t) = 0$ inside $\Omega^1(t)$ and therefore $\nabla p^1(z(\alpha, t), t) = 0$. Let us define the Lagrangian coordinates for the free boundary with the velocity v^2

$$Z_t(\gamma, t) = v^2(Z(\gamma, t), t),$$

$$Z(\gamma, 0) = z_0(\gamma).$$

We have two different parameterizations for the same curve $Z(\gamma, t) = z(\alpha(\gamma, t), t)$ and also two equations for its velocity, namely

$$Z_t(\gamma, t) = z_t(\alpha, t) + \alpha_t(\gamma, t)\partial_\alpha z(\alpha, t)$$

$$= BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t) + \alpha_t(\gamma, t)\partial_\alpha z(\alpha, t)$$

and another one given by the limit

$$Z_t(\gamma, t) = BR(z, \varpi)(\alpha, t) + \frac{1}{2} \frac{\varpi(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2} \partial_\alpha z(\alpha, t). \tag{2.25}$$

The dot product with the tangential vector gives

$$\alpha_t(\gamma, t) = \frac{1}{2} \frac{\varpi(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2} - c(\alpha) = \frac{\varphi(\alpha, t)}{|\partial_\alpha z(\alpha, t)|}.$$

And taking a time derivative in (2.25) yields

$$Z_{tt}(\gamma, t) \cdot \partial_\alpha^\perp z(\alpha, t) = \left(\partial_t BR(z, \varpi)(\alpha, t) + \alpha_t(\gamma, t)\partial_\alpha BR(z, \varpi)(\alpha, t) \right) \cdot \partial_\alpha^\perp z(\alpha, t)$$

$$+ \frac{1}{2} \frac{\varpi(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2} \left(\partial_\alpha z_t(\alpha, t) + \alpha_t(\gamma, t)\partial_\alpha^2 z(\alpha, t) \right) \cdot \partial_\alpha^\perp z(\alpha, t).$$

Therefore

$$\frac{\sigma(\alpha,t)}{\rho^2} = \left(\partial_t BR(z,\varpi)(\alpha,t) + \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|} \partial_\alpha BR(z,\varpi)(\alpha,t) \right) \cdot \partial_\alpha^\perp z(\alpha,t)$$

$$+ \frac{1}{2} \frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} \left(\partial_\alpha z_t(\alpha,t) + \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|} \partial_\alpha^2 z(\alpha,t) \right) \cdot \partial_\alpha^\perp z(\alpha,t)$$

$$+ g\partial_\alpha z_1(\alpha,t). \tag{2.26}$$

Remark 2.2. Let us consider ρ_2 and ρ_1 to be now arbitrary densities, then using the Lagrangian coordinates for the free boundary of the fluid in $\Omega^1(t)$

$$Z'_t(\gamma,t) = v^1\big(Z'(\gamma,t),t\big),$$

$$Z'(\gamma,0) = z_0(\gamma),$$

it is easy to check that

$$\frac{\sigma(\alpha,t)}{\rho^2 + \rho^1} = A_\rho \left(\partial_t BR(z,\varpi)(\alpha,t) + \frac{|\varpi(\alpha,t)|^2}{4|\partial_\alpha z(\alpha,t)|^4} \partial_\alpha^2 z(\alpha,t) \right) \cdot \partial_\alpha^\perp z(\alpha,t)$$

$$+ \left(\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} - A_\rho c(\alpha,t) \right) \partial_\alpha BR(z,\varpi)(\alpha,t) \cdot \partial_\alpha^\perp z(\alpha,t) + gA_\rho \partial_\alpha z_1(\alpha,t).$$

3. The evolution equation in terms of $\varphi(\alpha,t)$

We will consider $\rho_1 = 0$ and therefore $A_\rho = 1$. Using (2.23) we can write

$$\varpi_t(\alpha,t) = -2\partial_t BR(z,\varpi)(\alpha,t) \cdot \partial_\alpha z(\alpha,t) - \partial_\alpha \left(\frac{|\varpi|^2}{4|\partial_\alpha z|^2} \right)(\alpha,t) + \partial_\alpha(c\varpi)(\alpha,t)$$

$$+ 2c(\alpha,t)\partial_\alpha BR(z,\varpi)(\alpha,t) \cdot \partial_\alpha z(\alpha,t) - 2g\partial_\alpha z_2(\alpha,t). \tag{3.1}$$

In the case $A_\rho = 0$ the expression (2.23) yields

$$\varpi_t(\alpha,t) = \partial_\alpha(c\varpi)(\alpha,t),$$

that is, we obtain the vortex sheet problem for which the Kelvin–Helmholtz instability arises [4,12]. For $A_\rho = 1$ this term again appears in the evolution equation, and in order to absorb it we shall make use of the parameter $\varphi(\alpha,t)$ [1,3]. The fact that $|\partial_\alpha z(\alpha,t)|^2 = A(t)$ yields

$$2A(t)\partial_\alpha c = \frac{1}{\pi} \int_{-\pi}^{\pi} \partial_\alpha z(\alpha,t) \cdot \partial_\alpha BR(z,\varpi)(\alpha,t)\, d\alpha - 2\partial_\alpha z \cdot \partial_\alpha BR(z,\varpi)$$

and therefore

$$2c\partial_\alpha z \cdot \partial_\alpha BR(z,\varpi) = -\partial_\alpha\big(Ac^2\big) + \frac{c}{\pi} \int_{-\pi}^{\pi} \partial_\alpha z(\alpha,t) \cdot \partial_\alpha BR(z,\varpi)(\alpha,t)\, d\alpha.$$

Substituting the formula above in (3.1) we find

$$\varpi_t = -2\partial_t BR(z,\varpi) \cdot \partial_\alpha z - \partial_\alpha(\varphi^2) + \frac{c}{\pi} \int_{-\pi}^{\pi} \partial_\alpha z(\alpha,t) \cdot \partial_\alpha BR(z,\varpi)(\alpha,t)\, d\alpha - 2g\partial_\alpha z_2, \quad (3.2)$$

for φ given by (1.11). From that identity we have

$$\varphi_t(\alpha,t) = \frac{\varpi_t(\alpha,t)}{2|\partial_\alpha z(\alpha,t)|} - \frac{\varpi(\alpha,t)}{2|\partial_\alpha z(\alpha,t)|^3}\partial_\alpha z(\alpha,t) \cdot \partial_\alpha z_t(\alpha,t) - \partial_t\big(c|\partial_\alpha z|\big)(\alpha,t) \quad (3.3)$$

which together with (3.2) and (2.19) yields

$$\varphi_t = -\partial_t BR(z,\varpi) \cdot \frac{\partial_\alpha z}{|\partial_\alpha z|} - \frac{\partial_\alpha(\varphi^2)}{2|\partial_\alpha z|} + c\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|} \cdot \partial_\alpha BR(z,\varpi)(\alpha,t)\, d\alpha - g\frac{\partial_\alpha z_2}{|\partial_\alpha z|}$$

$$- \frac{\varpi}{2|\partial_\alpha z|^2}\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|} \cdot \partial_\alpha BR(z,\varpi)(\alpha,t)\, d\alpha - \partial_t\big(c|\partial_\alpha z|\big),$$

that is

$$\varphi_t = -\frac{\partial_\alpha(\varphi^2)}{2|\partial_\alpha z|} - B(t)\varphi - \partial_t BR(z,\varpi) \cdot \frac{\partial_\alpha z}{|\partial_\alpha z|} - g\frac{\partial_\alpha z_2}{|\partial_\alpha z|} - \partial_t\big(c|\partial_\alpha z|\big), \quad (3.4)$$

where

$$B(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} \cdot \partial_\alpha BR(z,\varpi)(\alpha,t)\, d\alpha.$$

It is easy to check in the equation above that the singular term $\partial_\alpha(c\varpi)$ takes part of the transport term $\partial_\alpha(\varphi^2)$.

Now let us remember that the evolution equation for the quantity $\Pi(\alpha,t)$ was discovered using the continuity of the pressure on $z(\alpha,t)$ (Proposition 2.1). Analogously the evolution equation for $\partial_\alpha \Pi(\alpha,t) = \varpi(\alpha,t)$ can be obtained throughout the following identity:

$$-\big(\nabla p^2\big(z(\alpha,t),t\big) - \nabla p^1\big(z(\alpha,t),t\big)\big) \cdot \partial_\alpha z(\alpha,t) = 0.$$

(Observe nevertheless that the Rayleigh–Taylor condition refers the jump of the pressure in the normal direction (2.24).)

With the help of property (2.18) we find that

$$\partial_\alpha^2 z(\alpha,t) \cdot \partial_\alpha z(\alpha,t) = 0,$$

and therefore

$$\partial_\alpha^2 z(\alpha,t) = \frac{\partial_\alpha^2 z(\alpha,t) \cdot \partial_\alpha^\perp z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha^\perp z(\alpha,t).$$

In the above formula we get the normal direction in the second derivative of z. Using this fact in (3.4) we obtain

$$
\partial_\alpha \varphi_t = -\frac{\partial_\alpha^2(\varphi^2)}{2|\partial_\alpha z|} - B(t)\partial_\alpha \varphi - \left(\partial_t BR(z,\varpi) \cdot \partial_\alpha^\perp z + g\partial_\alpha z_1\right)\frac{\partial_\alpha^2 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}
$$
$$
- \partial_t \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}\cdot \partial_\alpha BR(z,\varpi)(\alpha,t)\,d\alpha\right) + \partial_t\left(\frac{\partial_\alpha z}{|\partial_\alpha z|}\right)\cdot \partial_\alpha BR(z,\varpi).
$$

In $\partial_t\left(\frac{\partial_\alpha z}{|\partial_\alpha z|}\right)$ the perpendicular direction also appears, so that completing the formula for σ (2.26) we get

$$
\partial_\alpha \varphi_t = -\frac{\partial_\alpha^2(\varphi^2)}{2|\partial_\alpha z|} - B(t)\partial_\alpha \varphi - \frac{\sigma}{\rho^2}\frac{\partial_\alpha^2 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3} - \partial_t\left(|\partial_\alpha z|B\right)(t) + \partial_\alpha BR(z,\varpi)\cdot \partial_\alpha^\perp z \frac{\partial_\alpha z_t \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}
$$
$$
+ \left(\frac{\varphi}{|\partial_\alpha z|}\partial_\alpha BR(z,\varpi)\cdot \partial_\alpha^\perp z + \frac{1}{2}\frac{\varpi}{|\partial_\alpha z|^2}\left(\partial_\alpha z_t \cdot \partial_\alpha^\perp z + \frac{\varphi}{|\partial_\alpha z|}\partial_\alpha^2 z \cdot \partial_\alpha^\perp z\right)\right)\frac{\partial_\alpha^2 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}
$$

and therefore

$$
\partial_\alpha \varphi_t = -\frac{\partial_\alpha^2(\varphi^2)}{2|\partial_\alpha z|} - B(t)\partial_\alpha \varphi - \frac{\sigma}{\rho^2}\frac{\partial_\alpha^2 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3} - \partial_t\left(|\partial_\alpha z|B\right)(t)
$$
$$
+ \frac{1}{|\partial_\alpha z|^3}\left(\partial_\alpha BR(z,\varpi)\cdot \partial_\alpha^\perp z + \frac{\varpi}{2|\partial_\alpha z|^2}\partial_\alpha^2 z \cdot \partial_\alpha^\perp z\right)\left(\partial_\alpha z_t \cdot \partial_\alpha^\perp z + \frac{\varphi}{|\partial_\alpha z|}\partial_\alpha^2 z \cdot \partial_\alpha^\perp z\right).
$$

Finally after a straightforward calculation we obtain the following:

$$
\partial_\alpha \varphi_t = -\frac{\partial_\alpha^2(\varphi^2)}{2|\partial_\alpha z|} - B\partial_\alpha \varphi - \frac{\sigma}{\rho^2}\frac{\partial_\alpha^2 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3} - \partial_t\left(|\partial_\alpha z|B\right)
$$
$$
+ \frac{1}{|\partial_\alpha z|^3}\left(\partial_\alpha BR(z,\varpi)\cdot \partial_\alpha^\perp z + \frac{\varpi}{2|\partial_\alpha z|^2}\partial_\alpha^2 z \cdot \partial_\alpha^\perp z\right)^2. \tag{3.5}
$$

4. The basic operator

Let the operator T be defined by the formula

$$
T(u)(\alpha) = 2BR(z,u)(\alpha)\cdot \partial_\alpha z(\alpha). \tag{4.1}
$$

Lemma 4.1. *Suppose that $\|\mathcal{F}(z)\|_{L^\infty} < \infty$ (1.10) and $z \in C^{2,\delta}$ with $0 < \delta < 1/2$. Then $T : L^2 \to H^1$ and*

$$
\|T\|_{L^2 \to H^1} \leqslant \left\|\mathcal{F}(z)\right\|_{L^\infty}^4 \|z\|_{C^{2,\delta}}^4. \tag{4.2}
$$

Proof. Here we shall show the argument in the case of a closed curve. The other case was treated in [6].

Since the formula (1.4) yields

$$T(u)(\alpha) = \frac{1}{\pi} \partial_\alpha \int\limits_{-\pi}^{\pi} u(\beta) \arctan\left(\frac{z_2(\alpha) - z_2(\beta)}{z_1(\alpha) - z_1(\beta)} \right) d\beta,$$

we have

$$\int\limits_{-\pi}^{\pi} T(u)(\alpha) \, d\alpha = 0,$$

which implies $\|T(u)\|_{L^2} \leqslant \|\partial_\alpha T(u)\|_{L^2}$.

Let us write first:

$$\partial_\alpha T(u) = 2BR(z, u)(\alpha) \cdot \partial_\alpha^2 z(\alpha) + 2\partial_\alpha z(\alpha) \cdot \partial_\alpha BR(z, u)(\alpha) = I_1 + I_2.$$

For I_1 we have the expression

$$I_1 = 2\left(BR(z, u)(\alpha) - \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2} H(u)(\alpha) \right) \cdot \partial_\alpha^2 z(\alpha) + 2H(u)(\alpha) \frac{\partial_\alpha^\perp z(\alpha) \cdot \partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2}$$
$$= J_1 + J_2,$$

where $H(u)$ is the (periodic) Hilbert transform of the function u.

Then

$$J_1 = \frac{1}{\pi} \partial_\alpha^2 z(\alpha) \cdot \int\limits_{-\pi}^{\pi} u(\alpha - \beta) \left[\frac{(z(\alpha) - z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{2|\partial_\alpha z(\alpha)|^2 \tan(\beta/2)} \right] d\beta.$$

Let us define

$$C_1(\alpha, \beta) = \frac{(z(\alpha) - z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{2|\partial_\alpha z(\alpha)|^2 \tan(\beta/2)}, \tag{4.3}$$

then we shall show that $\|C_1\|_{L^\infty} \leqslant C\|\mathcal{F}(z)\|_{L^\infty}^2 \|z\|_{C^2}^2$ and therefore $J_1 \leqslant C\|\mathcal{F}(z)\|_{L^\infty}^2 \times \|z\|_{C^2}^3 \|u\|_{L^2}$. Since the estimate $J_2 \leqslant C\|\mathcal{F}(z)\|_{L^\infty} \|z\|_{C^2} |H(u)(\alpha)|$ is immediate, we finally get

$$|I_1| \leqslant C\|\mathcal{F}(z)\|_{L^\infty}^2 \|z\|_{C^2}^3 \left(\|u\|_{L^2} + \left| H(u)(\alpha) \right| \right). \tag{4.4}$$

Next we split $C_1 = D_1 + D_2 + D_3$ where

$$D_1 = \frac{(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta)^\perp}{|z(\alpha) - z(\alpha - \beta)|^2}, \qquad D_2 = \partial_\alpha^\perp z(\alpha) \left[\frac{\beta}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2 \beta} \right],$$

and

$$D_3 = \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \left[\frac{1}{\beta} - \frac{1}{2\tan(\beta/2)} \right].$$

The inequality

$$\left| z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta \right| \leqslant \|z\|_{C^2} |\beta|^2 \tag{4.5}$$

yields easily $|D_1| \leqslant \|z\|_{C^2} \|\mathcal{F}(z)\|_{L^\infty}^2$.

Then we can rewrite D_2 as follows:

$$D_2 = \partial_\alpha^\perp z(\alpha) \left[\frac{(\partial_\alpha z(\alpha)\beta - (z(\alpha) - z(\alpha - \beta))) \cdot (\partial_\alpha z(\alpha)\beta + (z(\alpha) - z(\alpha - \beta)))}{|z(\alpha) - z(\alpha - \beta)|^2 |\partial_\alpha z(\alpha)|^2 \beta} \right],$$

and, in particular, we have

$$|D_2| \leqslant \frac{|\partial_\alpha z(\alpha)\beta - (z(\alpha) - z(\alpha - \beta))|(|\partial_\alpha z(\alpha)\beta| + |z(\alpha) - z(\alpha - \beta)|)}{|z(\alpha) - z(\alpha - \beta)|^2 |\partial_\alpha z(\alpha)||\beta|}.$$

Using (4.5) we find that $|D_2| \leqslant 2\|z\|_{C^2} \|\mathcal{F}(z)\|_{L^\infty}^2$.

Next let us observe that $[-\pi, \pi]$ gives $|D_3| \leqslant C\|\mathcal{F}(z)\|_{L^\infty}$.

The identity $\partial_\alpha z(\alpha) \cdot \partial_\alpha^\perp z(\alpha) = 0$ allows us to write I_2 as follows:

$$I_2 = -\frac{2}{\pi} \int_{-\pi}^{\pi} u(\beta) \frac{(z(\alpha) - z(\beta))^\perp \cdot \partial_\alpha z(\alpha)(z(\alpha) - z(\beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\beta)|^4} \, d\beta$$

and therefore

$$I_2 = -\frac{2}{\pi} \int_{-\pi}^{\pi} u(\alpha - \beta) \frac{(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta)^\perp \cdot \partial_\alpha z(\alpha)(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4} \, d\beta.$$

Next we take $I_2 = J_3 + J_4 + J_5 + J_6 + J_7$ where

$$J_3 = -\frac{2}{\pi} \int_{-\pi}^{\pi} u(\alpha - \beta) \frac{(E(\alpha, \beta))^\perp \cdot \partial_\alpha z(\alpha)(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4} \, d\beta,$$

$$J_4 = -\left(\partial_\alpha^2 z(\alpha)\right)^\perp \cdot \partial_\alpha z(\alpha) \frac{1}{\pi} \int_{-\pi}^{\pi} u(\alpha - \beta) \frac{\beta^2 (z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4} \, d\beta,$$

$$J_5 = -\left(\partial_\alpha^2 z(\alpha)\right)^\perp \cdot \partial_\alpha z(\alpha) |\partial_\alpha z(\alpha)|^2 \frac{1}{\pi} \int_{-\pi}^{\pi} u(\alpha - \beta) \left[\frac{\beta^3}{|z(\alpha) - z(\alpha - \beta)|^4} - \frac{1}{|\partial_\alpha z(\alpha)|^4 \beta} \right] d\beta,$$

$$J_6 + J_7 = -\frac{(\partial_\alpha^2 z(\alpha))^\perp \cdot \partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} u(\alpha - \beta) \left[\frac{1}{\beta} - \frac{1}{2\tan(\beta/2)} \right] d\beta + H(u)(\alpha) \right),$$

and $E(\alpha, \beta) = z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta - \frac{1}{2}\partial_\alpha^2 z(\alpha)\beta^2$. Using the bound

$$\big|E(\alpha, \beta)\big| \leqslant \frac{1}{2}\|z\|_{C^{2,\delta}}|\beta|^{2+\delta}, \tag{4.6}$$

one gets easily that

$$|J_3| \leqslant C\|z\|_{C^{2,\delta}}^3 \big\|\mathcal{F}(z)\big\|_{L^\infty}^3 \int\limits_{-\pi}^{\pi} |u(\beta)||\beta|^{\delta-1}.$$

Then reasoning as before the inequality (4.5) gives $|J_4| \leqslant C\|z\|_{C^2}^4\|\mathcal{F}(z)\|_{L^\infty}^4\|u\|_{L^2}$. Regarding D_2, we have $|J_5| \leqslant C\|z\|_{C^2}^4\|\mathcal{F}(z)\|_{L^\infty}^3\|u\|_{L^2}$, and it is easy to get $|J_6| \leqslant C\|z\|_{C^2}\|\mathcal{F}(z)\|_{L^\infty}\|u\|_{L^2}$. Finally we have

$$|I_2| \leqslant C\big\|\mathcal{F}(z)\big\|_{L^\infty}^4\|z\|_{C^{2,\delta}}^4\left(\|u\|_{L^2} + \big|H(u)(\alpha)\big| + \int\limits_{-\pi}^{\pi} |\beta|^{\delta-1}\big|u(\alpha-\beta)\big|\,d\beta\right).$$

This last inequality together with (4.4) gives us

$$\big|\partial_\alpha T(u)(\alpha)\big| \leqslant C\big\|\mathcal{F}(z)\big\|_{L^\infty}^4\|z\|_{C^{2,\delta}}^4\left(\|u\|_{L^2} + \big|H(u)(\alpha)\big| + \int\limits_{-\pi}^{\pi} |\beta|^{\delta-1}\big|u(\alpha-\beta)\big|\,d\beta\right).$$

To finish we use the L^2 boundedness of H and Minkowski's inequality to obtain the estimate

$$\big\|\partial_\alpha T(u)\big\|_{L^2} \leqslant C\big\|\mathcal{F}(z)\big\|_{L^\infty}^4\|z\|_{C^{2,\delta}}^4\|u\|_{L^2}. \qquad \Box$$

5. Estimates on the inverse operator $(I + T)^{-1}$

As it was shown in Ref. [6], under our hypothesis about the curve z, $T(u) = 2BR(z, u)(\alpha) \cdot \partial_\alpha z(\alpha)$ defines a compact operator in Sobolev spaces. Its adjoint T^* is given as the real part of the Cauchy integral and it does not has real eigenvalues λ such that $|\lambda| \geqslant 1$ [2]. Therefore the existence of the bounded operator $(I + T)^{-1}$ follows from the standard theory.

Let $F(z)$ be given by

$$F(z) = \frac{1}{2\pi i}\int \frac{u(\beta)\partial_\alpha z(\beta)}{z - z(\beta)}\,d\beta,$$

and $f(z) = \text{Re}(F(z))$, which can be considered either in the periodic setting, where we have two periodic domains Ω^1, Ω^2 (see Ref. [6]), or in the bounded domain case (Ω^2 bounded). In both situations $F(z)$ can be evaluated in the interior of both domains, and T^* appears when we take limits approaching the boundary from the interior of each Ω^j: $z = z(\alpha) + \varepsilon\partial_\alpha^\perp z(\alpha)$, $\epsilon \to 0$ ($\epsilon > 0$, Ω^1; $\epsilon < 0$, Ω^2):

$$f\big(z(\alpha)\big) = T^*(u) - \text{sign}(\varepsilon)u(\alpha).$$

The periodic case was treated in Ref. [6] (Proposition 4.2). Therefore we shall consider here the bounded domain case.

Let $\mathcal{H}^j$ denote the Hilbert transform associated to Ω^j, we have:

$$\left(\mathcal{H}^j\right)^2 = -I,$$
$$F^1 = F/\Omega^1 = f^1 + ig^1,$$
$$F^2 = F/\Omega^2 = f^2 + ig^2,$$
$$f^1/\partial\Omega = T^*u - u,$$
$$f^2/\partial\Omega = T^*u + u,$$
$$g^1/\partial\Omega = g^2/\partial\Omega = \mathcal{G}(u),$$
$$u - T^*u = \mathcal{H}^1\left(\mathcal{G}(u)\right),$$
$$u + T^*u = \mathcal{H}^2\left(\mathcal{G}(u)\right).$$

Theorem 5.1. *The norm of the operator $(I + T)^{-1}$ from L^2 to L^2 is bounded from above by* $\exp(C\|z\|^p)$ *with* $\|z\| = \|z\|_{H^3} + \|\mathcal{F}(z)\|_{L^\infty}$, *for some universal constants C and p.*

Proof. As in Proposition 4.2 (Ref. [6]) the proof follows from the estimate

$$\left\|\mathcal{H}^j\right\|_{L^2(\partial\Omega^j)} \leqslant \exp\left(C\|z\|^p\right).$$

Let ϕ be a conformal mapping of Ω^2 into the unit disc D such that $\phi(z_0) = 0$ where z_0 satisfies $\text{dist}(z_0, \partial\Omega^1) \gg \frac{1}{\|z\|}$, then

$$\mathcal{H}^2 f = H\left(f \circ \phi^{-1}\right) \circ \phi$$

where H is the Hilbert transform in the unit disc D. Since $\partial\Omega^2$ is smooth enough ($C^{2,\alpha}$) we know from general theory that ϕ and ϕ' have continuous extensions to $\partial\Omega^2$ and our problem is reduced to obtain a weighted estimate for the Hilbert transform H in ∂D with respect to the weight $w(\tau) = |(\phi^{-1})'(\tau)|$, $|\tau| = 1$. But that is a consequence of the inequality

$$e^{-C\|z\|^p} \leqslant \frac{w(\tau_1)}{w(\tau_2)} \leqslant e^{C\|z\|^p}$$

for arbitrary τ_j, $|\tau_j| = 1$.

Following Riemann let us write $\phi(z) = (z - z_0)e^{R(z)+iS(z)}$ where the real harmonic function $R(z)$ is the solution of the following Dirichlet's problem:

$$\Delta R = 0 \quad \text{in } \Omega^2,$$
$$R(z) = -\log|z - z_0|, \quad z \in \partial\Omega^2.$$

Since Ω^2 is a regular domain whose boundary has tangent balls of radius $\frac{1}{\|z\|}$ contained in Ω^2, it follows from the standard theory that $|\nabla R|_{L^\infty} \ll \|z\| \log(\|z\|)$. This estimate also holds for the conjugate harmonic functions $S(z)$ implying $|\phi'(\tau)| \ll \|z\| \log(\|z\|)$, $\tau \in \partial\Omega^2$.

Given $\tau_0 \in \partial \Omega^2$ the arc $\gamma = \{\tau \in \partial \Omega^2 \colon \mathrm{dist}(\tau, \tau_0) < \frac{1}{\||z\|| \log(\||z\||)}\}$ is then mapped by ϕ into the semicircle $\phi(\gamma) = \{z \in \partial D \colon \mathrm{dist}(z, \phi(\tau_0)) \leqslant \sqrt{2}\}$.

Let us consider the Cayley transform $C_{\phi(\tau_0)} \colon D \to \mathbb{R}_+^2$

$$C_{\phi(\tau_0)}(z) = -\frac{1 - \overline{\phi(\tau_0)} \cdot z}{1 + \overline{\phi(\tau_0)} \cdot z}$$

verifying that

$$V = \mathrm{Im}(C_{\phi(\tau_0)} \circ \phi) \geqslant 0 \quad \text{in } \Omega^2,$$
$$V/\partial \Omega^2 = 0,$$
$$w(\gamma) = \mathrm{Re}(C_{\phi(\tau_0)} \circ \phi)(\gamma) \subset [-1, +1],$$
$$w(\tau_0) = 0.$$

Applying Hopf's maximum principle to the nonnegative harmonic function V in a disc of radius $1/\||z\||$ tangent to $\partial \Omega^2$ in τ, we get an estimate for the normal derivative of V at τ, i.e. for $\|\nabla V(\tau)\|$ (since $\partial \Omega^2$ is a level set of V), namely:

$$\left| \frac{\partial V}{\partial \nu}(\tau) \right| \gg \frac{1}{\||z\||} V(\tau^*)$$

where τ^* is the center of the disc.

To get an upper bound we may use the Poisson's kernel representation of V in a $C^{2,\alpha}$-domain $\widetilde{\Omega}$ contained in Ω^2 whose boundary consists of γ and its parallel arc γ^* at distance $1/\||z\||$, together with two "vertical" connecting arcs chosen in such a way that the $C^{2,\alpha}$-norm of $\partial \widetilde{\Omega}$ is controlled by $\||z\||$. Since $V/\partial \Omega^2 \equiv 0$ we obtain the estimate:

$$\left| \frac{\partial V}{\partial \nu}(\tau) \right| \ll \||z\|| \log(\||z\||) \sup_{\tau \in \widetilde{\Omega}} V(\tau)$$

for

$$\tau \in \frac{1}{2}\gamma = \left\{ \tau \in \partial \Omega^2, \ \mathrm{dist}(\tau, \tau_0) \leqslant \frac{1}{2C\||z\|| \log(\||z\||)} \right\}.$$

We are then in condition to invoke Dahlberg's Harnack inequality up to the boundary [11] to conclude that

$$\left| \frac{\partial V}{\partial \nu}(\tau) \right| \ll \||z\|| \log(\||z\||) V(\tau^*), \quad \tau \in \frac{1}{2}\gamma.$$

Next we use the standard Harnack's inequality in the parallel curve γ^* to conclude that

$$\frac{\|\nabla V(\tau_1)\|}{\|\nabla V(\tau_2)\|} \ll \||z\||^2 \log(\||z\||)$$

for any two points $\tau_1, \tau_2 \in \frac{1}{2}\gamma$.

But since $\frac{1}{2} \leqslant |C'(\phi(\tau))| \leqslant 2$, $\tau \in \gamma$, we get the bound

$$\left| \frac{\phi'(\tau_1)}{\phi'(\tau_2)} \right| \ll \|z\|^2 \log(\|z\|), \quad \tau_1, \tau_2 \in \frac{1}{2}\gamma.$$

Let us observe now that the length of $\partial\Omega^2$ is controlled by $\|z\|$ giving us a number of, at most, $C\|z\|^2 \log(\|z\|)$ different arcs $\frac{1}{2}\gamma$ needed to cover $\partial\Omega^2$. Then an iteration of the inequality above yields

$$e^{-C\|z\|^2 \log(\|z\|)} \leqslant \left| \frac{\phi'(\tau_1)}{\phi'(\tau_2)} \right| \leqslant e^{C\|z\|^2 \log(\|z\|)}$$

for any two arbitrary points $\tau_1, \tau_2 \in \partial\Omega^2$, allowing us to finish the proof in the case $\mathcal{H}^2$. The transformation $z \to 1/(z - z_0)$ where, as before, $z_0 \in \Omega^2$, $\mathrm{dist}(z_0, \partial\Omega^2) \gg 1/\|z\|$, allows us to reduce the estimate for $\mathcal{H}^1$ to the previous case. $\quad\square$

6. Preliminary estimates

The following subsection are devoted to show the regularity of the different elements involved in the problem: the Birkhoff–Rott integral, $z_t(\alpha, t)$, $\varpi_t(\alpha, t)$, $\varpi(\alpha, t)$; the difference of the gradient of the pressure in the normal direction $\sigma(\alpha, t)$ and its time derivative $\sigma_t(\alpha, t)$. We shall concentrate our attention in the case of a closed contour, because for a periodic domain in the horizontal space variable the treatment is completely analogous (see [6]).

6.1. Estimates for $BR(z, \varpi)$

In this section we show that the Birkhoff–Rott integral is as regular as $\partial_\alpha z$.

Lemma 6.1. *The following estimate holds*

$$\left\| BR(z, \varpi) \right\|_{H^k} \leqslant C \left(\left\| \mathcal{F}(z) \right\|_{L^\infty}^2 + \|z\|_{H^{k+1}}^2 + \|\varpi\|_{H^k}^2 \right)^j, \tag{6.1}$$

for $k \geqslant 2$, where C and j are constants independent of z and ϖ.

Remark 6.2. Using this estimate for $k = 2$ we find easily that

$$\left\| \partial_\alpha BR(z, \varpi) \right\|_{L^\infty} \leqslant C \left(\left\| \mathcal{F}(z) \right\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2 \right)^j, \tag{6.2}$$

which shall be used through out the paper.

Proof. We shall present the proof for $k = 2$. Let us write

$$BR(z, \varpi)(\alpha, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C_1(\alpha, \beta) \varpi(\alpha - \beta) \, d\beta + \frac{\partial_\alpha^\perp z(\alpha)}{2|\partial_\alpha z(\alpha)|^2} H(\varpi)(\alpha)$$

where C_1 is given by (4.3). The boundedness of the term C_1 in L^∞ gives us easily

$$\left\| BR(z,\varpi) \right\|_{L^2} \leqslant C \left\| \mathcal{F}(z) \right\|_{L^\infty}^2 \|z\|_{C^2}^2 \|\varpi\|_{L^2}. \tag{6.3}$$

In $\partial_\alpha^2 BR(z,\varpi)$, the most singular terms are given by

$$P_1(\alpha) = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \partial_\alpha^2 \varpi(\alpha-\beta) \frac{(z(\alpha)-z(\alpha-\beta))^\perp}{|z(\alpha)-z(\alpha-\beta)|^2} \, d\beta,$$

$$P_2(\alpha) = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \varpi(\alpha-\beta) \frac{\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta)}{|z(\alpha)-z(\alpha-\beta)|^2} \, d\beta,$$

$$P_3(\alpha) = -\frac{1}{\pi} PV \int_{-\pi}^{\pi} \varpi(\alpha-\beta) \frac{(z(\alpha)-z(\alpha-\beta))^\perp}{|z(\alpha)-z(\alpha-\beta)|^4}$$
$$\times \left((z(\alpha)-z(\alpha-\beta)) \cdot (\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)) \right) d\beta.$$

Again we have the expression

$$P_1(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C_1(\alpha,\beta) \partial_\alpha^2 \varpi(\alpha-\beta) \, d\beta + \frac{\partial_\alpha^\perp z(\alpha)}{2|\partial_\alpha z(\alpha)|^2} H(\partial_\alpha^2 \varpi)(\alpha) \, d\alpha,$$

giving us

$$\left| P_1(\alpha) \right| \leqslant C \left\| \mathcal{F}(z) \right\|_{L^\infty}^j \|z\|_{C^2}^j \left(\left\| \partial_\alpha^2 \varpi \right\|_{L^2} + \left| H(\partial_\alpha^2 \varpi)(\alpha) \right| \right). \tag{6.4}$$

Next let us write $P_2 = Q_1 + Q_2 + Q_3$ where

$$Q_1(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\varpi(\alpha-\beta) - \varpi(\alpha) \right) \frac{\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta)}{|z(\alpha)-z(\alpha-\beta)|^2} \, d\beta,$$

$$Q_2(\alpha) = \frac{\varpi(\alpha)}{2\pi} \int_{-\pi}^{\pi} \left(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta) \right) \left(\frac{1}{|z(\alpha)-z(\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2|\beta|^2} \right) d\beta,$$

$$Q_3(\alpha) = \frac{1}{2\pi} \frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2} \int_{-\pi}^{\pi} \left(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta) \right) \left(\frac{1}{|\beta|^2} - \frac{1}{4\sin^2(\beta/2)} \right) d\beta$$
$$+ \frac{1}{2} \frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2} \Lambda(\partial_\alpha^2 z)(\alpha),$$

where $\Lambda = \partial_\alpha H$.

Using that

$$\left|\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha - \beta)\right| \leqslant |\beta|^\delta \|z\|_{C^{2,\delta}},$$

we get $|Q_1(\alpha)| + |Q_2(\alpha)| \leqslant \|\varpi\|_{C^1} \|\mathcal{F}(z)\|^j \|z\|_{C^{2,\delta}}^j$, while for Q_3 we have

$$\left|Q_3(\alpha)\right| \leqslant C\|\varpi\|_{L^\infty} \|\mathcal{F}(z)\|_{L^\infty} \left(\|z\|_{C^2} + \left|\Lambda\big(\partial_\alpha^2 z\big)(\alpha)\right|\right),$$

that is

$$\left|P_2(\alpha)\right| \leqslant \left(1 + \left|\Lambda\big(\partial_\alpha^2 z\big)(\alpha)\right|\right) \|\varpi\|_{C^1} \|\mathcal{F}(z)\|^j \|z\|_{C^{2,\delta}}^j. \tag{6.5}$$

Let us now consider $P_3 = Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9$, where

$$\begin{aligned}
Q_4 = \frac{-1}{\pi} \int_{-\pi}^{\pi} &\big(\varpi(\alpha - \beta) - \varpi(\alpha)\big) \frac{(z(\alpha) - z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^4} \\
&\times \left((z(\alpha) - z(\alpha - \beta)) \cdot \big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha - \beta)\big)\right) d\beta,
\end{aligned}$$

$$\begin{aligned}
Q_5 = -\frac{\varpi(\alpha)}{\pi} \int_{-\pi}^{\pi} &\frac{(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta)^\perp}{|z(\alpha) - z(\alpha - \beta)|^4} \\
&\times \left((z(\alpha) - z(\alpha - \beta)) \cdot \big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha - \beta)\big)\right) d\beta,
\end{aligned}$$

$$Q_6 = -\frac{\varpi(\alpha)\partial_\alpha^\perp z(\alpha)}{\pi} \int_{-\pi}^{\pi} \frac{\beta(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta) \cdot (\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha - \beta))}{|z(\alpha) - z(\alpha - \beta)|^4} \, d\beta,$$

$$\begin{aligned}
Q_7 = -\frac{\varpi(\alpha)\partial_\alpha^\perp z(\alpha)}{\pi} &\partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \beta^2 \big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha - \beta)\big) \\
&\times \left(\frac{1}{|z(\alpha) - z(\alpha - \beta)|^4} - \frac{1}{|\partial_\alpha z(\alpha)|^4 |\beta|^4}\right) d\beta,
\end{aligned}$$

$$Q_8 = -\frac{\varpi(\alpha)\partial_\alpha^\perp z(\alpha)}{\pi |\partial_\alpha z(\alpha)|^4} \partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha - \beta)\big) \left(\frac{1}{|\beta|^2} - \frac{1}{4\sin^2(\beta/2)}\right) d\beta,$$

and

$$Q_9 = -\frac{\varpi(\alpha)\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4} \partial_\alpha z(\alpha) \cdot \Lambda\big(\partial_\alpha^2 z(\alpha)\big).$$

Proceeding as before we get

$$\left|P_3(\alpha)\right| \leqslant C\left(1 + \left|\Lambda\big(\partial_\alpha^2 z\big)(\alpha)\right|\right) \|\varpi\|_{C^1} \|\mathcal{F}(z)\|_{L^\infty}^j \|z\|_{C^{2,\delta}}^j,$$

which together with (6.4) and (6.5) gives us the estimate

$$\left|(P_1 + P_2 + P_3)(\alpha)\right| \leqslant C\left(1 + \left|\Lambda\left(\partial_\alpha^2 z\right)(\alpha)\right| + \left|H\left(\partial_\alpha^2 \varpi\right)(\alpha)\right|\right)\|\varpi\|_{C^1}\left(\left\|\mathcal{F}(z)\right\|_{L^\infty}^j + \|z\|_{H^3}^j\right).$$

For the rest of the terms in $\partial_\alpha^2 BR(z, \varpi)$ we obtain analogous estimates allowing us to conclude the inequality

$$\left\|\partial_\alpha^2 BR(z, \varpi)\right\|_{L^2} \leqslant C\left(1 + \left\|\partial_\alpha^3 z\right\|_{L^2} + \left\|\partial_\alpha^2 \varpi\right\|_{L^2}\right)\|\varpi\|_{C^1}\left\|\mathcal{F}(z)\right\|_{L^\infty}^j \|z\|_{C^{2,\delta}}^j.$$

Finally the Sobolev inequalities yield (6.1) for $k = 2$. □

6.2. Estimates for $z_t(\alpha, t)$

This section is devoted to show that z_t is as regular as $\partial_\alpha z$.

Lemma 6.3. *The following estimate holds*

$$\|z_t\|_{H^k} \leqslant C\left(\left\|\mathcal{F}(z)\right\|_{L^\infty}^2 + \|z\|_{H^{k+1}}^2 + \|\varpi\|_{H^k}^2\right)^j, \tag{6.6}$$

for $k \geqslant 2$.

Proof. It follows easily from formulas (2.16), (2.17) together with the estimates obtained in the last section. □

6.3. Estimates for ϖ_t

This section is devoted to show that ϖ_t is as regular as $\partial_\alpha \varpi$.

Lemma 6.4. *The following estimate holds*

$$\|\varpi_t\|_{H^k} \leqslant C \exp\left(C\|z\|^p\right)\left(\left\|\mathcal{F}(z)\right\|_{L^\infty}^2 + \|z\|_{H^{k+2}}^2 + \|\varpi\|_{H^{k+1}}^2 + \|\varphi\|_{H^{k+1}}^2\right)^j, \tag{6.7}$$

for $k \geqslant 1$.

Proof. In the following we shall work the details of the proof only when $k = 1$, since the cases $k \geqslant 2$ can be treated analogously. Formula (3.2) yields

$$\varpi_t(\alpha, t) + T(\varpi_t)(\alpha, t) = I_1(\alpha, t) + I_2(\alpha, t) - 2\varphi(\alpha, t)\partial_\alpha\varphi(\alpha, t) + R(\alpha, t), \tag{6.8}$$

where

$$I_1 = \frac{-1}{\pi} \int_{-\pi}^{\pi} \frac{(z_t(\alpha) - z_t(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta)\, d\beta,$$

$$I_2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}\left(z(\alpha) - z(\alpha - \beta)\right) \cdot \left(z_t(\alpha) - z_t(\alpha - \beta)\right)\varpi(\alpha - \beta)\, d\beta,$$

and

$$R = \frac{c(\alpha, t)}{\pi} \int_{-\pi}^{\pi} \partial_\alpha z(\alpha, t) \cdot \partial_\alpha BR(z, \varpi)(\alpha, t) \, d\alpha + 2g \partial_\alpha z_2(\alpha, t).$$

From Theorem 5.1 we get

$$\|\varpi_t\|_{L^2} \leqslant \|(I + T)^{-1}\|_{L^2 \to L^2} (\|I_1\|_{L^2} + \|I_1\|_{L^2} + 2\|\varphi \partial_\alpha \varphi\|_{L^2} + \|R\|_{L^2}),$$

and proceeding as before, using the estimates above, we obtain

$$\|w_t\|_{L^2} \leqslant \exp(C\|z\|^p)(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2 + \|\varphi\|_{H^2}^2)^j. \tag{6.9}$$

Next we shall show that in the singular case we have:

$$\|\partial_\alpha w_t\|_{L^2} \leqslant \exp(C\|z\|^p)(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2 + \|\varphi\|_{H^2}^2)^j. \tag{6.10}$$

To see it let us take a derivative in (6.8) to obtain the identity

$$\partial_\alpha \varpi_t(\alpha, t) + T(\partial_\alpha \varpi_t)(\alpha, t) = J_1(\alpha, t) + J_2(\alpha, t) + J_3(\alpha, t) + \partial_\alpha I_1(\alpha, t) + \partial_\alpha I_2(\alpha, t)$$
$$- \partial_\alpha^2(\varphi^2)(\alpha, t) + \partial_\alpha R(\alpha, t), \tag{6.11}$$

where

$$J_1 = \frac{-1}{\pi} \int_{-\pi}^{\pi} \frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi_t(\alpha - \beta) \, d\beta,$$

$$J_2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$
$$\times (z(\alpha) - z(\alpha - \beta)) \cdot (\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta)) \varpi_t(\alpha - \beta) \, d\beta,$$

and

$$J_3 = \frac{-1}{\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta))^\perp \cdot \partial_\alpha^2 z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi_t(\alpha - \beta) \, d\beta.$$

Using Theorem 5.1 in (6.11) we get

$$\|\partial_\alpha \varpi_t\|_{L^2} \leqslant \|(I + T)^{-1}\|_{L^2 \to L^2}$$
$$\times \left(\sum_{l=1}^{3} \|J_l\|_{L^2} + \|\partial_\alpha I_1\|_{L^2} + \|\partial_\alpha I_2\|_{L^2} + \|\partial_\alpha^2(\varphi^2)\|_{L^2} + \|\partial_\alpha R\|_{L^2} \right).$$

A straightforward calculation yields

$$\left\| \partial_\alpha^2(\varphi^2) \right\|_{L^2} + \|\partial_\alpha R\|_{L^2} \leqslant C \left(\left\| \mathcal{F}(z) \right\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2 + \|\varphi\|_{H^2}^2 \right)^j.$$

To estimate the other terms we write:

$$J_1 = \frac{-1}{\pi} \int\limits_{-\pi}^{\pi} C_2(\alpha, \beta) \varpi_t(\alpha - \beta) \, d\beta - \frac{(\partial_\alpha^2 z(\alpha))^\perp \cdot \partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} H(\varpi_t)(\alpha),$$

where

$$C_2(\alpha, \beta) = \left[\frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{(\partial_\alpha^2 z(\alpha))^\perp \cdot \partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2 2 \tan(\beta/2)} \right].$$

Then

$$\left| J_1(\alpha) \right| \leqslant C \left\| \mathcal{F}(z) \right\|_{L^\infty}^k \|z\|_{C^{2,\delta}}^k \left(\int\limits_{-\pi}^{\pi} |\beta|^{\delta - 1} \left| \varpi_t(\alpha - \beta) \right| d\beta + \left| H(\varpi_t)(\alpha) \right| \right),$$

and using (6.9) we have

$$\|J_1\|_{L^2} \leqslant \exp\left(C \|z\|^p \right) \left(\left\| \mathcal{F}(z) \right\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2 + \|\varphi\|_{H^2}^2 \right)^j.$$

Next we rewrite J_2 as follows

$$\frac{2}{\pi} \int\limits_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta)^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \left(z(\alpha) - z(\alpha - \beta) \right) \cdot \left(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta) \right) \varpi_t(\alpha - \beta) \, d\beta,$$

which is a more regular term than J_1. Since J_3 is also more regular than J_1 we finally get

$$\|J_2\|_{L^2} + \|J_3\|_{L^2} \leqslant \exp\left(C \|z\|^p \right) \left(\left\| \mathcal{F}(z) \right\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2 + \|\varphi\|_{H^2}^2 \right)^j.$$

The most singular term in $\partial_\alpha I_1$ is given by

$$K_1 = \frac{-1}{\pi} \int\limits_{-\pi}^{\pi} \frac{(\partial_\alpha z_t(\alpha) - \partial_\alpha z_t(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta) \, d\beta,$$

and will be estimated using the following splitting $K_1 = L_1 + L_2 + L_3 + L_4$ where

$$L_1 = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \frac{(\partial_\alpha z_t(\alpha) - \partial_\alpha z_t(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \big(\varpi(\alpha) - \varpi(\alpha - \beta)\big) d\beta,$$

$$L_2 = \frac{-\varpi(\alpha)}{\pi} \int\limits_{-\pi}^{\pi} \big(\partial_\alpha z_t(\alpha) - \partial_\alpha z_t(\alpha - \beta)\big)^\perp \cdot \partial_\alpha z(\alpha) \left[\frac{1}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2 |\beta|^2} \right] d\beta,$$

$$L_3 = \frac{-1}{\pi} \frac{\varpi(\alpha)\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \int\limits_{-\pi}^{\pi} \big(\partial_\alpha z_t(\alpha) - \partial_\alpha z_t(\alpha - \beta)\big)^\perp \left[\frac{1}{\beta^2} - \frac{1}{4\sin^2(\beta/2)} \right] d\beta,$$

and

$$L_4 = \frac{-1}{\pi} \frac{\varpi(\alpha)\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \Lambda(\partial_\alpha z_t).$$

Since $|\partial_\alpha z_t(\alpha) - \partial_\alpha z_t(\alpha - \beta)| \leqslant |\beta| \int_0^1 |\partial_\alpha^2 z_t(\alpha + (s-1)\beta)| \, ds$ we have

$$|K_1| \leqslant C \|\mathcal{F}(z)\|_{L^\infty}^k \|z\|_{H^3}^k \|\varpi\|_{C^1} \left(\int\limits_0^1 \big|\partial_\alpha^2 z_t\big(\alpha + (s-1)\beta\big)\big| \, ds + \big|\Lambda(\partial_\alpha z_t)(\alpha)\big| \right).$$

From (6.6) we obtain the estimates

$$\|K_1\|_{L^2} \leqslant C \big(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2\big)^j$$

and

$$\|\partial_\alpha I_1\|_{L^2} \leqslant C \big(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2\big)^j.$$

Next we rewrite I_2 in the form

$$\frac{2}{\pi} \int\limits_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta)^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \big(z(\alpha) - z(\alpha - \beta)\big) \cdot \big(z_t(\alpha) - z_t(\alpha - \beta)\big) \varpi(\alpha - \beta) \, d\beta,$$

which shows that I_2 is more regular than I_1 and, therefore, the estimate for $\partial_\alpha I_2$ follows easily with the same methods that we used with $\partial_\alpha I_1$, allowing us to finish the proof. $\quad\square$

6.4. Estimates for ϖ

In this section we show that the amplitude of the vorticity ϖ lies at the same level than $\partial_\alpha z$. We shall consider $z \in H^k(\mathbb{T})$, $\varphi \in H^{k-\frac{1}{2}}(\mathbb{T})$ and $\varpi \in H^{k-2}(\mathbb{T})$ as part of the energy estimates. The inequality below yields $\varpi \in H^{k-1}(\mathbb{T})$.

Lemma 6.5. *The following estimate holds*

$$\|\varpi\|_{H^k} \leqslant C\big(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^{k+1}}^2 + \|\varpi\|_{H^{k-1}}^2 + \|\varphi\|_{H^k}^2\big)^j, \tag{6.12}$$

for $k \geqslant 2$.

Proof. We shall present the proof for $k = 2$, being the rest of the cases completely analogous. Since $\varpi = 2|\partial_\alpha z|\varphi + 2|\partial_\alpha z|^2 c$ the identity $|\partial_\alpha z|^2 = A(t)$ gives us the equality

$$\partial_\alpha^2 \varpi(\alpha) = 2\big|\partial_\alpha z(\alpha)\big|\partial_\alpha^2 \varphi(\alpha) - \partial_\alpha\big(2\partial_\alpha z \cdot \partial_\alpha BR(z,\varpi)\big)(\alpha),$$

from which we easily get

$$\big\|\partial_\alpha^2 \varpi\big\|_{L^2} \leqslant 2\|z\|_{C^1}\big\|\partial_\alpha^2 \varphi\big\|_{L^2} + \big\|\partial_\alpha\big(2\partial_\alpha z \cdot \partial_\alpha BR(z,\varpi)\big)\big\|_{L^2}.$$

Therefore in order to get the estimate (6.12) for $k = 2$ we need to show that the following inequality holds

$$\big\|\partial_\alpha\big(2\partial_\alpha z \cdot \partial_\alpha BR(z,\varpi)\big)\big\|_{L^2} \leqslant C\|\mathcal{F}(z)\|_{L^\infty}^j \|z\|_{H^3}^j \|\varpi\|_{H^1}. \tag{6.13}$$

To see that we can write

$$2\partial_\alpha z(\alpha) \cdot \partial_\alpha BR(z,\varpi)(\alpha) = T(\partial_\alpha \varpi)(\alpha) + R_1(\alpha) + R_2(\alpha),$$

where

$$R_1(\alpha) = \frac{1}{\pi}\partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2}\varpi(\alpha - \beta)\,d\beta,$$

and

$$R_2(\alpha) = -\frac{2}{\pi}\partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^4}$$
$$\times \big(z(\alpha) - z(\alpha - \beta)\big) \cdot \big(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta)\big)\varpi(\alpha - \beta)\,d\beta.$$

Then we have $\|T(\partial_\alpha \varpi)\|_{H^1} \leqslant C\|\mathcal{F}(z)\|_{L^\infty}^4 \|z\|_{C^{2,\delta}}^4 \|\partial_\alpha \varpi\|_{L^2}$ from (4.2), so that we only need to estimate $\partial_\alpha R_1$ and $\partial_\alpha R_2$ in L^2 to get (6.13).

Next we consider the most singular terms in $\partial_\alpha R_1$, namely:

$$S_1(\alpha) = \frac{1}{\pi}\partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \frac{(\partial_\alpha^2 z(\alpha) - \partial_\alpha z^2(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2}\varpi(\alpha - \beta)\,d\beta,$$

$$S_2(\alpha) = \frac{1}{\pi}\partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2}\partial_\alpha \varpi(\alpha - \beta)\,d\beta,$$

and we use the decomposition

$$S_2(\alpha) = \frac{1}{\pi} \partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \left[\frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{(\partial_\alpha^2 z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^2 2 \tan(\beta/2)} \right] \partial_\alpha \varpi(\alpha - \beta) \, d\beta$$

$$- \frac{\partial_\alpha^\perp z(\alpha) \cdot \partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2} H(\partial_\alpha \varpi)(\alpha)$$

to obtain

$$\left| S_2(\alpha) \right| \leqslant C \| \mathcal{F}(z) \|_{L^\infty}^j \| z \|_{C^{2,\delta}}^j \left(\| \partial_\alpha \varpi \|_{L^2} + \left| H(\partial_\alpha \varpi)(\alpha) \right| + \int_{-\pi}^{\pi} |\beta|^{\delta-1} \left| \partial_\alpha \varpi(\alpha - \beta) \right| d\beta \right),$$

that is $\| S_2 \|_{L^2} \leqslant C \| \mathcal{F}(z) \|_{L^\infty}^j \| z \|_{C^{2,\delta}}^j \| \partial_\alpha \varpi \|_{L^2}$.

In S_1 we have the splitting $U_1 + U_2 + U_3 + U_4$ where

$$U_1(\alpha) = \frac{1}{\pi} \partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \frac{(\partial_\alpha^2 z(\alpha) - \partial_\alpha z^2(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2} \left(\varpi(\alpha - \beta) - \varpi(\alpha) \right) d\beta,$$

$$U_2(\alpha) = \frac{1}{\pi} \varpi(\alpha) \partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \left(\partial_\alpha^2 z(\alpha) - \partial_\alpha z^2(\alpha - \beta) \right)^\perp$$

$$\times \left[\frac{1}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2 \beta^2} \right] d\beta,$$

$$U_3(\alpha) = \frac{1}{\pi} \varpi(\alpha) \frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \int_{-\pi}^{\pi} \left(\partial_\alpha^2 z(\alpha) - \partial_\alpha z^2(\alpha - \beta) \right)^\perp \left[\frac{1}{\beta^2} - \frac{1}{4 \sin^2(\beta/2)} \right] d\beta,$$

and

$$U_4(\alpha) = \varpi(\alpha) \frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \Lambda \left(\partial_\alpha^2 z \right)^\perp(\alpha).$$

Then in U_1 we use the identity

$$\partial_\alpha^2 z(\alpha) - \partial_\alpha z^2(\alpha - \beta) = \beta \int_0^1 \partial_\alpha^3 z(\alpha + (s-1)\beta) \, ds \tag{6.14}$$

to get

$$\left| U_1(\alpha) \right| \leqslant C \| \mathcal{F}(z) \|_{L^\infty}^2 \| z \|_{C^{2,\delta}} \| \varpi \|_{C^\delta} \int_0^1 \int_{-\pi}^{\pi} |\beta|^{\delta-1} \left| \partial_\alpha^3 z(\alpha + (s-1)\beta) \right| d\beta \, ds,$$

and therefore $\| U_1 \|_{L^2} \leqslant C \| \mathcal{F}(z) \|_{L^\infty}^2 \| z \|_{H^3}^2 \| \varpi \|_{H^1}$.

To estimate U_2 and U_3 we can use again (6.14). For U_4 the control is easier. To finish the argument we rewrite R_2 as follows:

$$-\frac{2}{\pi}\partial_\alpha z(\alpha) \cdot \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta)^\perp}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \big(z(\alpha) - z(\alpha - \beta)\big) \cdot \big(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta)\big)\varpi(\alpha - \beta)\,d\beta,$$

expressing the fact that with the same method, $\partial_\alpha R_2$ is easier to estimate than $\partial_\alpha R_1$.　□

6.5. Estimates for σ

Here we prove that σ, the difference of the gradient of the pressure in the normal direction, is at the same level than $\partial_\alpha^2 z$.

Lemma 6.6. *The following estimate holds*

$$\|\sigma\|_{H^k} \leqslant C \exp\big(C\|z\|^p\big)\big(\big\|\mathcal{F}(z)\big\|_{L^\infty}^2 + \|z\|_{H^{k+2}}^2 + \|\varpi\|_{H^{k+1}}^2 + \|\varphi\|_{H^{k+1}}^2\big)^j, \qquad (6.15)$$

for $k \geqslant 2$.

Proof. We shall give the details of the case $k = 2$. Let us recall the formula for $\sigma(\alpha)$:

$$\frac{\sigma}{\rho_2} = \left(\partial_t BR(z, \varpi) + \frac{\varphi}{|\partial_\alpha z|}\partial_\alpha BR(z, \varpi)\right) \cdot \partial_\alpha^\perp z$$

$$+ \frac{1}{2}\frac{\varpi}{|\partial_\alpha z|^2}\left(\partial_\alpha z_t + \frac{\varphi}{|\partial_\alpha z|}\partial_\alpha^2 z\right) \cdot \partial_\alpha^\perp z + g\partial_\alpha z_1, \qquad (6.16)$$

then from previous sections we have:

$$\|\sigma\|_{L^2} \leqslant C \exp\big(C\|z\|^p\big)\big(\big\|\mathcal{F}(z)\big\|_{L^\infty}^2 + \|z\|_{H^4}^2 + \|\varpi\|_{H^3}^2 + \|\varphi\|_{H^3}^2\big)^j.$$

To control $\|\partial_\alpha^2 \sigma\|_{L^2}$ we only have to deal with $\partial_\alpha^2(\partial_t BR(z, \varpi) \cdot \partial_\alpha^\perp z)$, because the remainder terms have been already estimated. Again we shall consider the most singular parts:

$$I_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2}\partial_\alpha^2 \varpi_t(\alpha - \beta)\,d\beta,$$

$$I_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\partial_\alpha^2 z_t(\alpha) - \partial_\alpha^2 z_t(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2}\varpi(\alpha - \beta)\,d\beta,$$

$$I_3 = \frac{-1}{\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \big(z(\alpha) - z(\alpha - \beta)\big) \cdot \big(\partial_\alpha^2 z_t(\alpha) - \partial_\alpha^2 z_t(\alpha - \beta)\big)\varpi(\alpha - \beta)\,d\beta.$$

We have

$$I_1 = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} E(\alpha, \beta) \partial_\alpha^2 \varpi_t(\alpha - \beta) \, d\beta + \frac{1}{2} H\big(\partial_\alpha^2 \varpi_t\big)(\alpha),$$

where

$$E(\alpha, \beta) = \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{1}{2 \tan(\beta/2)}.$$

Since $\|E\|_{L^\infty} \leqslant C \|\mathcal{F}(z)\|_{L^\infty}^2 \|z\|_{C^2}^2$ we can estimate I_1 throughout inequality (6.7).

The equality

$$\partial_\alpha^2 z_t(\alpha) - \partial_\alpha^2 z_t(\alpha - \beta) = \beta \int\limits_0^1 \partial_\alpha^3 z_t\big(\alpha + (s - 1)\beta\big) \, ds$$

allows us to get

$$|I_2| + |I_3| \leqslant C \big\|\mathcal{F}(z)\big\|_{L^\infty}^2 \|z\|_{C^2}^2 \|\varpi\|_{C^1} \left(\int\limits_0^1 \int\limits_{-\pi}^{\pi} \big|\partial_\alpha^3 z_t\big(\alpha + (s - 1)\beta\big)\big| \, ds + \big|\Lambda\big(\partial_\alpha^2 z_t\big)(\alpha)\big| \right)$$

and (6.6) takes care of the rest. $\square$

6.6. Estimate for σ_t

In this section we obtain an upper bound for the L^∞ norm of σ_t that will be used in the energy inequalities and in the treatment of the Rayleigh–Taylor condition.

Lemma 6.7. *The following estimate holds*

$$\|\sigma_t\|_{L^\infty} \leqslant C \exp\big(C\|z\|^p\big)\big(\big\|\mathcal{F}(z)\big\|_{L^\infty}^2 + \|z\|_{H^4}^2 + \|\varpi\|_{H^3}^2 + \|\varphi\|_{H^3}^2\big)^j. \qquad (6.17)$$

Proof. Let us consider (6.16) the splitting $\sigma/\rho_2 = P_1 + P_2 + P_3 + P_4 + P_5$ where

$$P_1 = \partial_t BR(z, \varpi) \cdot \partial_\alpha^\perp z, \qquad P_2 = \frac{\varphi}{|\partial_\alpha z|} \partial_\alpha BR(z, \varpi) \cdot \partial_\alpha^\perp z,$$

$$P_3 = \frac{1}{2} \frac{\varpi}{|\partial_\alpha z|^2} \partial_\alpha z_t \cdot \partial_\alpha^\perp z, \qquad P_4 = \frac{\varphi}{|\partial_\alpha z|} \partial_\alpha^2 z \cdot \partial_\alpha^\perp z, \qquad P_5 = g \partial_\alpha z_1.$$

Estimate (6.6) yields $\|\partial_t P_5\|_{L^\infty} \leqslant \|g \partial_t \partial_\alpha z_1\|_{H^1} \leqslant C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 + \|\varpi\|_{H^2}^2)^j$. For P_3 we write

$$P_3 = \frac{1}{2} \frac{\varpi}{|\partial_\alpha z|^2} \big(\partial_\alpha BR(z, \varpi) \cdot \partial_\alpha^\perp z + \partial_\alpha^2 z \cdot \partial_\alpha^\perp z\big),$$

and we get

$$|\partial_t P_3| \leqslant C\left(\left\|\mathcal{F}(z)\right\|_{L^\infty}^2 + \|z\|_{H^4}^2 + \|\varpi\|_{H^3}^2\right)^m$$
$$\times \left(|\varpi_t| + |\partial_\alpha z_t| + |\partial_\alpha \varpi_t| + |\partial_\alpha^2 z_t| + |H(\partial_\alpha \varpi_t)| + |\Lambda(\partial_\alpha z_t)|\right).$$

It yields

$$\|\partial_t P_3\|_{L^\infty} \leqslant C\left(\left\|\mathcal{F}(z)\right\|_{L^\infty}^2 + \|z\|_{H^4}^2 + \|\varpi\|_{H^3}^2\right)^j \left(\|\varpi_t\|_{H^2} + \|z_t\|_{H^3}\right)$$

by the Sobolev embedding. The inequalities (6.7) and (6.6) take care of the rest.

In $\partial_t P_4$ we have the term

$$\partial_t \varphi = \frac{\varpi_t}{2|\partial_\alpha z|^2} - \varpi \frac{\partial_\alpha z \cdot \partial_\alpha z_t}{2|\partial_\alpha z|^3} - \partial_t\left(|\partial_\alpha z|c\right)(\alpha, t),$$

but estimates (6.7) and (6.6) yield easily the appropriate bounds for $\|\varphi_t\|_{L^\infty}$ and $\|\partial_t P_4\|_{L^\infty}$.

In a similar way we control $\|\partial_t P_2\|_{L^\infty}$. Regarding $\partial_t P_1$ the most singular terms are given by

$$Q_1 = \frac{1}{2} H(\varpi_{tt}), \qquad Q_2 = -\frac{1}{2|\partial_\alpha z|^2} \Lambda(z_{tt} \cdot \partial_\alpha z).$$

For Q_2 we decompose further $Q_2 = R_1 + R_2$ where

$$R_1 = -\frac{1}{2|\partial_\alpha z|^2} H\left(z_{tt} \cdot \partial_\alpha^2 z\right), \qquad R_2 = -\frac{1}{2|\partial_\alpha z|^2} H(\partial_\alpha z_{tt} \cdot \partial_\alpha z).$$

Then we take a time derivative in (2.16) to estimate R_1 in L^∞, and for R_2 we use the fact that $\partial_\alpha z_t \cdot \partial_\alpha z$ only depends on t (see (2.19)). Next the identity $\partial_\alpha z_{tt} \cdot \partial_\alpha z = \partial_t(\partial_\alpha z_t \cdot \partial_\alpha z) - |\partial_\alpha z_t|^2$ allows us to write

$$R_2 = \frac{1}{2|\partial_\alpha z|^2} H\left(|\partial_\alpha z_t|^2\right).$$

From estimates (6.6) we get control of R_2 in L^∞.

For Q_1 we have

$$\|Q_1\|_{L^\infty} \leqslant C\|\varpi_{tt}\|_{C^\delta}.$$

To continue we will need estimates on $\|\varpi_{tt}\|_{C^\delta}$ for which we may use the identity (6.8), and the inequality $\|f\|_{C^\delta} \leqslant C(\|f\|_{L^2} + \|f\|_{\overline{C}^\delta})$ where

$$\|f\|_{\overline{C}^\delta} = \sup_{\alpha \neq \beta} \frac{|f(\alpha) - f(\beta)|}{|\alpha - \beta|^\delta}.$$

Then formula (6.8) gives

$$\varpi_{tt} + T(\varpi_{tt}) = \partial_t I_1 + \partial_t I_2 - 2\varphi_t \partial_\alpha \varphi - 2\varphi \partial_\alpha \varphi_t + \partial_t R + J_1 + J_2, \qquad (6.18)$$

where

$$J_1 = -\frac{1}{\pi} \int\limits_{-\pi}^{\pi} \frac{(z_t(\alpha) - z_t(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi_t(\alpha - \beta) \, d\beta,$$

and

$$J_2 = \frac{2}{\pi} \int\limits_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \left(z(\alpha) - z(\alpha - \beta)\right) \cdot \left(z_t(\alpha) - z_t(\alpha - \beta)\right) \varpi_t(\alpha - \beta) \, d\beta.$$

As before we use the invertibility of $(I + T)$ to get appropriate estimates on $\|\varpi_{tt}\|_{L^2}$:

$$\|\varpi_{tt}\|_{L^2} \leqslant C \exp\left(C\|z\|^p\right)\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^4}^2 + \|\varpi\|_{H^3}^2 + \|\varphi\|_{H^3}^2\right)^j. \qquad (6.19)$$

We shall show with some details how to get the most singular case $\|\varpi_{tt}\|_{\overline{C}^\delta}$.

Formula (6.18) yields

$$\|\varpi_{tt}\|_{\overline{C}^\delta} \leqslant \left\|T(\varpi_{tt})\right\|_{\overline{C}^\delta} + \|\partial_t I_1 + \partial_t I_2 - 2\varphi_t \partial_\alpha \varphi - 2\varphi \partial_\alpha \varphi_t + \partial_t R + J_1 + J_2\|_{\overline{C}^\delta},$$

and therefore

$$\|\varpi_{tt}\|_{\overline{C}^\delta} \leqslant \left\|T(\varpi_{tt})\right\|_{H^1} + \|\partial_t I_1 + \partial_t I_2 - 2\varphi_t \partial_\alpha \varphi - 2\varphi \partial_\alpha \varphi_t + \partial_t R + J_1 + J_2\|_{H^1}.$$

Then the inequality $\|T(\varpi_{tt})\|_{H^1} \leqslant \|T\|_{L^2 \to H^1} \|w_{tt}\|_{L^2}$, together with (4.2) and (6.19) yields the desired estimate. In $\partial_t I_1$ we find the term $\Lambda(z_{tt})$ therefore we need to control $\|\Lambda(z_{tt})\|_{H^1} = \|\partial_\alpha^2 z_{tt}\|_{L^2}$, but formula (2.16) let us obtain that bound. In $\partial_t I_2$ we have again the extra cancellation given by

$$\left(z(\alpha) - z(\alpha - \beta)\right)^\perp \cdot \partial_\alpha z(\alpha) = \left(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta\right)^\perp \cdot \partial_\alpha z(\alpha),$$

which yields the appropriate estimate. We have also to control $\|\partial_\alpha^2 \varphi_t\|_{L^2}$, but formula (3.3) gives

$$\partial_\alpha^2 \varphi_t(\alpha, t) = \frac{\partial_\alpha^2 \varpi_t(\alpha, t)}{2|\partial_\alpha z(\alpha, t)|} - \frac{\partial_\alpha^2 \varpi(\alpha, t)}{2|\partial_\alpha z(\alpha, t)|^3} \partial_\alpha z(\alpha, t) \cdot \partial_\alpha z_t(\alpha, t) - \partial_t\left(\partial_\alpha \left(\partial_\alpha z \cdot \partial_\alpha BR(z, \varpi)\right)\right),$$

showing that it can be estimated as before. Finally, the remainder terms are less singular in derivatives, allowing us to finish the proof. $\quad \square$

7. A priori energy estimates

Let us consider for $k \geqslant 4$ the following definition of energy $E(t)$:

$$E^2(t) = \|z\|^2_{H^{k-1}}(t) + \int\limits_{-\pi}^{\pi} \frac{\sigma(\alpha, t)}{\rho^2 |\partial_\alpha z(\alpha, t)|^2} \left| \partial_\alpha^k z(\alpha, t) \right|^2 d\alpha$$

$$+ \left\| \mathcal{F}(z) \right\|^2_{L^\infty}(t) + \|\varpi\|^2_{H^{k-2}}(t) + \|\varphi\|^2_{H^{k-\frac{1}{2}}}(t), \tag{7.1}$$

so long as $\sigma(\alpha, t) > 0$. In the next section we shall show a proof of the following lemma.

Lemma 7.1. *Let $z(\alpha, t)$ and $\varpi(\alpha, t)$ be a solution of* (1.4)–(1.7) *in the case $\rho_1 = 0$. Then, the following a priori estimate holds:*

$$\frac{d}{dt} E^p(t) \leqslant \frac{C}{m^q(t)} \exp\left(CE^p(t)\right), \tag{7.2}$$

for $m(t) = \min_{\alpha \in [-\pi, \pi]} \sigma(\alpha, t) = \sigma(\alpha_t, t) > 0$, $k \geqslant 4$ and C, q and p some universal constants.

We shall present the details when $k = 4$. Regarding $\|\partial_\alpha^4 z\|^2_{L^2}$ let us remark that we have

$$\|\partial_\alpha^4 z\|^2_{L^2}(t) = \int_{\mathbb{T}} \frac{\sigma(\alpha, t)}{\sigma(\alpha, t)} \left| \partial_\alpha^4 z(\alpha, t) \right|^2 d\alpha \leqslant \frac{1}{m(t)} \int_{\mathbb{T}} \sigma(\alpha, t) \left| \partial_\alpha^4 z(\alpha, t) \right|^2 d\alpha.$$

7.1. Energy estimates on the curve

In this section we give the proof of the following lemma when, again, $k = 4$. The case $k > 4$ is left to the reader.

Lemma 7.2. *Let $z(\alpha, t)$ and $\varpi(\alpha, t)$ be a solution of* (1.4)–(1.7) *in the case $\rho_1 = 0$. Then, the following a priori estimate holds:*

$$\frac{d}{dt} \left(\|z\|^2_{H^{k-1}} + \int\limits_{-\pi}^{\pi} \frac{\sigma(\alpha)}{\rho^2 |\partial_\alpha z(\alpha)|^2} \left| \partial_\alpha^k z(\alpha) \right|^2 d\alpha \right)(t) \leqslant S(t) + \frac{C}{m^q(t)} \exp\left(CE^p(t)\right), \tag{7.3}$$

for

$$S(t) = \int\limits_{-\pi}^{\pi} \frac{2\sigma(\alpha)}{\rho^2} \frac{\partial_\alpha^k z(\alpha) \cdot \partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^3} \Lambda\left(\partial_\alpha^{k-1} \varphi\right)(\alpha) \, d\alpha, \tag{7.4}$$

and $k \geqslant 4$.

(We have denoted with S a nonintegrable term which shall appear in the equation of the evolution of φ but with the opposite sign.)

Proof. Using (6.6) one gets easily

$$\frac{d}{dt}\|z\|_{H^3}^2 \leqslant C\int_{-\pi}^{\pi}\left(|z(\alpha)||z_t(\alpha)| + |\partial_\alpha^3 z(\alpha)||\partial_\alpha^3 z_t(\alpha)|\right)d\alpha \leqslant \frac{C}{m^q(t)}\exp\left(CE^p(t)\right).$$

Then we have

$$\frac{d}{dt}\int_{-\pi}^{\pi}\frac{\sigma(\alpha)}{\rho^2|\partial_\alpha z(\alpha)|^2}|\partial_\alpha^4 z(\alpha)|^2\,d\alpha = \int_{-\pi}^{\pi}\frac{1}{\rho^2}\left(\frac{\sigma_t(\alpha)}{|\partial_\alpha z(\alpha)|^2} - \frac{\sigma(\alpha)2\partial_\alpha z(\alpha)\cdot\partial_\alpha z_t(\alpha)}{|\partial_\alpha z(\alpha)|^4}\right)|\partial_\alpha^4 z(\alpha)|^2\,d\alpha$$

$$+ \int_{-\pi}^{\pi}\frac{2\sigma(\alpha)}{\rho^2|\partial_\alpha z(\alpha)|^2}\partial_\alpha^4 z(\alpha)\cdot\partial_\alpha^4 z_t(\alpha)\,d\alpha$$

$$= I_1 + I_2.$$

The bound (6.17) gives us

$$I_1 \leqslant \frac{C}{m^q(t)}C\exp\left(CE^p(t)\right).$$

Next for I_2 we write

$$I_2 = \int_{-\pi}^{\pi}\frac{2\sigma(\alpha)}{\rho^2|\partial_\alpha z|^2}\partial_\alpha^4 z(\alpha)\cdot\partial_\alpha^4 BR(z,\varpi)(\alpha)\,d\alpha + \int_{-\pi}^{\pi}\frac{2\sigma(\alpha)}{\rho^2|\partial_\alpha z|^2}\partial_\alpha^4 z(\alpha)\cdot\partial_\alpha^4(c\partial_\alpha z)(\alpha)\,d\alpha$$

$$= J_1 + J_2.$$

The most singular terms in J_1 are given by K_1, K_2 and K_3:

$$K_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sigma(\alpha)}{\rho^2|\partial_\alpha z|^2}\partial_\alpha^4 z(\alpha)\cdot\frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha-\beta))^\perp}{|z(\alpha) - z(\alpha-\beta)|^2}\varpi(\alpha-\beta)\,d\beta\,d\alpha,$$

$$K_2 = -\frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sigma(\alpha)}{\rho^2|\partial_\alpha z|^2}\partial_\alpha^4 z(\alpha)\cdot\frac{(z(\alpha) - z(\alpha-\beta))^\perp}{|z(\alpha) - z(\alpha-\beta)|^4}C(\alpha,\beta)\varpi(\alpha-\beta)\,d\beta\,d\alpha,$$

and

$$K_3 = \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sigma(\alpha)}{\rho^2|\partial_\alpha z|^2}\partial_\alpha^4 z(\alpha)\cdot\frac{(z(\alpha) - z(\alpha-\beta))^\perp}{|z(\alpha) - z(\alpha-\beta)|^2}\partial_\alpha^4\varpi(\alpha-\beta)\,d\beta,$$

where $C(\alpha,\beta) = (z(\alpha) - z(\alpha-\beta))\cdot(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha-\beta))$.

Then we write:

$$
K_1 = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \frac{\sigma(\alpha)}{\rho^2 |\partial_\alpha z|^2} \partial_\alpha^4 z(\alpha) \cdot \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\beta))^\perp}{|z(\alpha) - z(\beta)|^2} \varpi(\beta) \, d\beta \, d\alpha
$$

$$
= \frac{1}{\pi \rho^2 |\partial_\alpha z|^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \partial_\alpha^4 z(\alpha) \cdot \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\beta))^\perp}{|z(\alpha) - z(\beta)|^2} \frac{\sigma(\alpha)\varpi(\beta) + \sigma(\beta)\varpi(\alpha)}{2} \, d\beta \, d\alpha
$$

$$
+ \frac{1}{\pi \rho^2 |\partial_\alpha z|^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \partial_\alpha^4 z(\alpha) \cdot \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\beta))^\perp}{|z(\alpha) - z(\beta)|^2} \frac{\sigma(\alpha)\varpi(\beta) - \sigma(\beta)\varpi(\alpha)}{2} \, d\beta \, d\alpha
$$

$$
= L_1 + L_2.
$$

That is we have performed a kind of integration by parts in K_1, allowing us to show that L_1, its most singular term, vanishes:

$$
L_1 = -\frac{1}{\pi \rho^2 |\partial_\alpha z|^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \partial_\alpha^4 z(\beta) \cdot \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\beta))^\perp}{|z(\alpha) - z(\beta)|^2} \frac{\sigma(\alpha)\varpi(\beta) + \sigma(\beta)\varpi(\alpha)}{2} \, d\beta \, d\alpha
$$

$$
= \frac{1}{2\pi \rho^2 |\partial_\alpha z|^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} (\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\beta))
$$

$$
\cdot \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\beta))^\perp}{|z(\alpha) - z(\beta)|^2} \frac{\sigma(\alpha)\varpi(\beta) + \sigma(\beta)\varpi(\alpha)}{2} \, d\beta \, d\alpha
$$

$$
= 0,
$$

whether for L_2 we have

$$
L_2 = -\frac{1}{\pi \rho^2 |\partial_\alpha z|^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \partial_\alpha^4 z(\alpha) \cdot \frac{(\partial_\alpha^4 z(\beta))^\perp}{|z(\alpha) - z(\beta)|^2} \frac{(\sigma(\alpha) - \sigma(\beta))\varpi(\beta)}{2} \, d\beta \, d\alpha
$$

$$
- \frac{1}{\pi \rho^2 |\partial_\alpha z|^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \partial_\alpha^4 z(\alpha) \cdot \frac{(\partial_\alpha^4 z(\beta))^\perp}{|z(\alpha) - z(\beta)|^2} \frac{\sigma(\beta)(\varpi(\beta) - \varpi(\alpha))}{2} \, d\beta \, d\alpha.
$$

In L_2 the kernels have degree -1 so long as the arc-chord condition is satisfied, so they can be estimated by

$$
L_2 \leqslant C \|\mathcal{F}(z)\|_{L^\infty}^k \|z\|_{H^3}^k \|\varpi\|_{C^{1,\delta}} \|\sigma\|_{C^{1,\delta}} \|\partial_\alpha^4 z\|_{L^2}^2 \leqslant \frac{C}{m^q(t)} \exp(CE^p(t)).
$$

The term $C(\alpha, \beta)$ in K_2 can be written as follows:

$$C(\alpha, \beta) = \left(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta\right) \cdot \left(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha - \beta)\right)$$
$$- \beta\left(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta)\right) \cdot \partial_\alpha^4 z(\alpha - \beta)$$
$$+ \beta\left(\partial_\alpha z(\alpha) \cdot \partial_\alpha^4 z(\alpha) - \partial_\alpha z(\alpha - \beta) \cdot \partial_\alpha^4 z(\alpha - \beta)\right),$$

then using that

$$\partial_\alpha z(\alpha) \cdot \partial_\alpha^4 z(\alpha) = -3\partial_\alpha^2 z(\alpha) \cdot \partial_\alpha^3 z(\alpha),$$

we can split K_2 as a sum of kernels of degree -1 operating on $\partial_\alpha^4 z(\alpha)$, plus a kernel of degree -2 acting in three derivatives $\partial_\alpha^3 z(\alpha)$, allowing us to obtain again the estimate

$$K_2 \leqslant \frac{C}{m^q(t)} \exp\left(CE^p(t)\right).$$

The term K_3 is a sum of a kernel of degree zero acting on four derivatives of ϖ

$$L_3 = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \frac{\sigma(\alpha)}{\rho^2 |\partial_\alpha z|^2} \partial_\alpha^4 z(\alpha)$$

$$\cdot \int\limits_{-\pi}^{\pi} \left[\frac{(z(\alpha) - z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2 2\tan(\beta/2)}\right] \partial_\alpha^4 \varpi(\alpha - \beta)\, d\beta\, d\alpha,$$

plus the following term:

$$L_4 = \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \frac{\sigma(\alpha)}{\rho^2} \frac{\partial_\alpha^4 z(\alpha) \cdot \partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4} H\left(\partial_\alpha^4 \varpi\right)(\alpha)\, d\beta.$$

We can integrate by parts on L_3 with respect to β writing $\partial_\alpha^4 \varpi(\alpha - \beta) = -\partial_\beta(\partial_\alpha^3 \varpi(\alpha - \beta))$ and then pass the derivative to the kernel of degree zero. This calculation gives three derivatives in ϖ and kernels of degree -1 which can be estimated as before.

Next for L_4 we write

$$L_4 = \int\limits_{-\pi}^{\pi} \frac{2\sigma(\alpha)}{\rho^2} \frac{\partial_\alpha^4 z(\alpha) \cdot \partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^3} \Lambda\left(\partial_\alpha^3\left(\frac{\varpi}{2|\partial_\alpha z|}\right)\right)(\alpha)\, d\alpha$$

$$= \int\limits_{-\pi}^{\pi} \frac{2\sigma(\alpha)}{\rho^2} \frac{\partial_\alpha^4 z(\alpha) \cdot \partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^3} \left[\Lambda(\partial_\alpha^3 \varphi)(\alpha) - \Lambda\left(\partial_\alpha^2\left(\frac{\partial_\alpha z}{|\partial_\alpha z|} \cdot \partial_\alpha BR(z, \varpi)\right)\right)(\alpha)\right] d\alpha$$

$$= S + M_0,$$

for $S(t)$ given by (7.4). For M_0 we have

$$\frac{\rho^2}{2} M_0 = \int\limits_{-\pi}^{\pi} H\left(\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \partial_\alpha^3\left(\frac{\partial_\alpha z}{|\partial_\alpha z|} \cdot \partial_\alpha BR(z, \varpi)\right)(\alpha)\, d\alpha = N_1 + N_2 + N_3,$$

where

$$N_1 = \int\limits_{-\pi}^{\pi} H\left(\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \frac{\partial_\alpha^4 z}{|\partial_\alpha z|} \cdot \partial_\alpha BR(z, \varpi)(\alpha)\, d\alpha,$$

$$N_2 = \int\limits_{-\pi}^{\pi} H\left(\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \frac{\partial_\alpha z}{|\partial_\alpha z|} \cdot \partial_\alpha^4 BR(z, \varpi)(\alpha)\, d\alpha,$$

and N_3 is given by the rest of the terms which can be controlled easily with the estimate that we already have for the Birkhoff–Rott integral.

Regarding N_1 a straightforward calculation gives

$$N_1 \leqslant C \left\|\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right\|_{L^2} \left\|\frac{\partial_\alpha^4 z}{|\partial_\alpha z|} \cdot \partial_\alpha BR(z, \varpi)\right\|_{L^2}$$

$$\leqslant C \|\sigma\|_{L^\infty} \|\mathcal{F}(z)\|_{L^\infty}^k \|\partial_\alpha BR(z, \varpi)\|_{L^\infty} \|\partial_\alpha^4 z\|_{L^2}^2.$$

Again, in N_2 we consider the most singular terms given by

$$O_1 = \int\limits_{-\pi}^{\pi} H\left(\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|} \cdot \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta)\, d\alpha,$$

$$O_2 = -\int\limits_{-\pi}^{\pi} H\left(\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|} \cdot \frac{1}{4\pi} \int\limits_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^4} C(\alpha, \beta) \varpi(\alpha - \beta)\, d\alpha,$$

$$O_3 = \int\limits_{-\pi}^{\pi} H\left(\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \frac{\partial_\alpha z}{|\partial_\alpha z|} \cdot \partial_\alpha BR(z, \partial_\alpha^3 \varpi)(\alpha)\, d\alpha.$$

Using the above decomposition for $C(\alpha, \beta)$ we can easily estimate O_2. In O_3 we may write

$$\partial_\alpha z(\alpha) \cdot \partial_\alpha BR(z, \partial_\alpha^3 \varpi)(\alpha) = \frac{1}{2} \partial_\alpha T(\partial_\alpha^3 \varpi) - \partial_\alpha^2 z(\alpha) \cdot BR(z, \partial_\alpha^3 \varpi)(\alpha)$$

to obtain

$$\left\|\partial_\alpha z \cdot \partial_\alpha BR(z, \partial_\alpha^3 \varpi)\right\|_{L^2} \leqslant \left\|T(\partial_\alpha^3 \varpi)\right\|_{H^1} + \left\|\partial_\alpha^2 z\right\|_{L^\infty} \left\|BR(z, \partial_\alpha^3 \varpi)\right\|_{L^2}$$

allowing us to control O_3.

Next we split O_1 into several kernels of degree one acting on $(\partial_\alpha^4 z(\alpha))^\perp$, which can be estimated as before, plus the term

$$P_1 = \frac{1}{2} \int_{-\pi}^{\pi} H\left(\sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \frac{\varpi(\alpha) \partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^3} \cdot \Lambda\left((\partial_\alpha^4 z)^\perp\right)(\alpha) \, d\alpha.$$

Then the following estimate for the commutator,

$$\left\| \frac{\varpi \partial_\alpha z}{|\partial_\alpha z|^3} \cdot \Lambda\left((\partial_\alpha^4 z)^\perp\right) - \Lambda\left(\frac{\varpi \partial_\alpha z}{|\partial_\alpha z|^3} \cdot (\partial_\alpha^4 z)^\perp\right) \right\|_{L^2} \leqslant \|\mathcal{F}(z)\|_{L^\infty}^3 \|w\|_{H^2} \|z\|_{H^3} \|\partial_\alpha^4 z\|_{L^2},$$

yields

$$P_1 \leqslant \|\mathcal{F}(z)\|_{L^\infty}^3 \|w\|_{H^2} \|z\|_{H^3} \|\partial_\alpha^4 z\|_{L^2} - \frac{1}{2} \int_{-\pi}^{\pi} \sigma \frac{\partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3} \partial_\alpha \left(\frac{\varpi \partial_\alpha^4 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha) \, d\alpha$$

using that

$$\int_{-\pi}^{\pi} Hf(\alpha)\Lambda g(\alpha) \, d\alpha = - \int_{-\pi}^{\pi} f(\alpha) \partial_\alpha g(\alpha) \, d\alpha,$$

and a straightforward integration by parts let us control P_1.

So finally we have controlled J_1 in the following manner:

$$J_1 \leqslant \frac{C}{m^q(t)} \exp\left(CE^p(t)\right) + S.$$

To finish the proof let us observe that the term J_2 can be estimated integrating by parts, using the identity $\partial_\alpha^4 z(\alpha, t) \cdot \partial_\alpha z(\alpha, t) = -3\partial_\alpha^3 z(\alpha, t) \cdot \partial_\alpha^2 z(\alpha, t)$ to treat its most singular component. We have obtained

$$\int_{\mathbb{T}} \frac{\sigma(\alpha)}{\rho^2 |\partial_\alpha z|^2} \partial_\alpha^4 z(\alpha) \cdot \partial_\alpha z(\alpha) \partial_\alpha^4 c(\alpha) \, d\alpha = 3 \int_{\mathbb{T}} \frac{1}{\rho^2 |\partial_\alpha z|^2} \partial_\alpha \left(\sigma \partial_\alpha^3 z \cdot \partial_\alpha^2 z\right)(\alpha) \partial_\alpha^3 c(\alpha) \, d\alpha$$

and this yields the desired control. □

7.2. Energy estimates for the arc-chord condition

In this section we analyze the evolution of the quantity $\|\mathcal{F}(z)\|_{L^\infty}(t)$, which gives the local control of the arc-chord condition.

Lemma 7.3. *The following estimate holds*

$$\frac{d}{dt} \|\mathcal{F}(z)\|_{L^\infty}^2(t) \leqslant C\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t) + \|\varpi\|_{H^2}^2(t)\right)^j. \tag{7.5}$$

Proof. First we compute the time derivative of the function $\mathcal{F}(z)$ as follows

$$\frac{d}{dt}\mathcal{F}(z)(\alpha,\beta)(t) = -\frac{|\beta|(z(\alpha,t) - z(\alpha - \beta, t)) \cdot (z_t(\alpha, t) - z_t(\alpha - \beta, t))}{|z(\alpha, t) - z(\alpha - \beta, t)|^3},$$

obtaining

$$\frac{d}{dt}\mathcal{F}(z)(\alpha,\beta)(t) \leqslant \frac{|\beta||z_t(\alpha,t) - z_t(\alpha - \beta, t)|}{|z(\alpha,t) - z(\alpha - \beta, t)|^2} \leqslant \left(\mathcal{F}(z)(\alpha,\beta)(t)\right)^2 \|\partial_\alpha z_t\|_{L^\infty}(t).$$

Sobolev estimates and (6.6) yield

$$\frac{d}{dt}\mathcal{F}(z)(\alpha,\beta)(t) \leqslant C\left(\mathcal{F}(z)(\alpha,\beta)(t)\right)^2 \left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t) + \|\varpi\|_{H^2}^2(t)\right)^j,$$

and therefore

$$\frac{d}{dt}\mathcal{F}(z)(\alpha,\beta)(t) \leqslant C\mathcal{F}(z)(\alpha,\beta)(t)\|\mathcal{F}(z)\|_{L^\infty}(t)\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t) + \|\varpi\|_{H^2}^2(t)\right)^j.$$

We shall denote $G(t) = C\|\mathcal{F}(z)\|_{L^\infty}(t)(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t) + \|\varpi\|_{H^2}^2(t))^j$, so that after integration in the time variable t we get

$$\mathcal{F}(z)(t+h) \leqslant \mathcal{F}(z)(t)\exp\left(\int_t^{t+h} G(s)\,ds\right),$$

and therefore

$$\|\mathcal{F}(z)\|_{L^\infty}(t+h) \leqslant \|\mathcal{F}(z)\|_{L^\infty}(t)\exp\left(\int_t^{t+h} G(s)\,ds\right),$$

which yields

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty}(t) = \lim_{h\to 0^+}\left(\|\mathcal{F}(z)\|_{L^\infty}(t+h) - \|\mathcal{F}(z)\|_{L^\infty}(t)\right)h^{-1}$$

$$\leqslant \|\mathcal{F}(z)\|_{L^\infty}(t)\lim_{h\to 0^+}\left(\exp\left(\int_t^{t+h} G(s)\,ds\right) - 1\right)h^{-1} \leqslant \|\mathcal{F}(z)\|_{L^\infty}(t)G(t),$$

allowing us to finish the proof of Lemma 7.3. $\quad\square$

7.3. Energy estimates for ϖ and φ

In this section we complete the estimate (7.2) with the following result.

Lemma 7.4. *Let $z(\alpha, t)$ and $\varpi(\alpha, t)$ be a solution of (1.4)–(1.7) in the case $\rho_1 = 0$. Then, the following a priori estimate holds:*

$$\frac{d}{dt}\left(\|\varpi\|_{H^{k-2}}^2 + \|\varphi\|_{H^{k-\frac{1}{2}}}^2\right)(t) \leqslant -S(t) + \frac{C}{m^q(t)}\exp\left(CE^p(t)\right), \tag{7.6}$$

for $k \geqslant 4$.

Proof. We shall present the details in the case $k = 4$, leaving the other cases to the reader.
Formula (6.7) shows easily that

$$\frac{d}{dt}\|\varpi\|_{H^2}^2(t) \leqslant \exp\left(C\|z\|^p(t)\right)\left(\left\|\mathcal{F}(z)\right\|_{L^\infty}^2(t) + \|z\|_{H^4}^2(t) + \|\varpi\|_{H^3}^2(t) + \|\varphi\|_{H^3}^2(t)\right)^j$$

which together with (6.12) yields

$$\frac{d}{dt}\|\varpi\|_{H^2}^2(t) \leqslant \frac{1}{m^q(t)}C\exp\left(CE^p(t)\right).$$

Using the estimates obtained before one has

$$\frac{d}{dt}\|\varphi\|_{L^2}^2(t) \leqslant \frac{1}{m^q(t)}C\exp\left(CE^p(t)\right).$$

Next (3.5) yields

$$\frac{d}{dt}\left\|\Lambda^{1/2}\left(\partial_\alpha^3\varphi\right)\right\|_{L^2}^2(t) = \int_{\mathbb{T}}\partial_\alpha^3\varphi(\alpha)\Lambda\left(\partial_\alpha^3\varphi_t\right)(\alpha)\,d\alpha = I_1 + I_2 + I_3 + I_4, \tag{7.7}$$

where

$$I_1 = -\int_{\mathbb{T}}\frac{1}{2|\partial_\alpha z|}\partial_\alpha^3\varphi(\alpha)\Lambda\left(\partial_\alpha^4(\varphi^2)\right)(\alpha)\,d\alpha, \qquad I_2 = -\int_{\mathbb{T}}B(t)\partial_\alpha^3\varphi(\alpha)\Lambda\left(\partial_\alpha^3\varphi\right)d\alpha,$$

$$I_3 = -\int_{\mathbb{T}}\frac{1}{\rho^2|\partial_\alpha z|^3}\partial_\alpha^3\varphi(\alpha)\Lambda\left(\partial_\alpha^2\left(\sigma\partial_\alpha^2 z\cdot\partial_\alpha^\perp z\right)\right)(\alpha)\,d\alpha,$$

and

$$I_4 = -\int_{\mathbb{T}}\frac{1}{|\partial_\alpha z|^3}\partial_\alpha^3\varphi(\alpha)\Lambda\left(\partial_\alpha^2\left(\partial_\alpha BR(z,\varpi)\cdot\partial_\alpha^\perp z + \frac{\varpi}{2|\partial_\alpha z|^2}\partial_\alpha^2 z\cdot\partial_\alpha^\perp z\right)^2\right)(\alpha)\,d\alpha.$$

The most singular term in I_1 is given by

$$J_1 = -\int_{\mathbb{T}} \frac{1}{|\partial_\alpha z|} \partial_\alpha^3 \varphi(\alpha) \Lambda \left(\varphi \partial_\alpha^4 \varphi\right)(\alpha)\, d\alpha,$$

and we have

$$J_1 = \int_{\mathbb{T}} \frac{1}{|\partial_\alpha z|} \Lambda^{\frac{1}{2}} \left(\partial_\alpha^3 \varphi\right)(\alpha) \left[\varphi(\alpha) \Lambda^{\frac{1}{2}} \left(\partial_\alpha^4 \varphi\right)(\alpha) - \Lambda^{\frac{1}{2}} \left(\varphi \partial_\alpha^4 \varphi\right)(\alpha)\right] d\alpha$$

$$+ \int_{\mathbb{T}} \frac{\partial_\alpha \varphi(\alpha)}{2|\partial_\alpha z|} \left|\Lambda^{\frac{1}{2}} \left(\partial_\alpha^3 \varphi\right)(\alpha)\right|^2 d\alpha.$$

The following estimate for the commutator $\|g \Lambda^{\frac{1}{2}}(\partial_\alpha f) - \Lambda^{\frac{1}{2}}(g \partial_\alpha f)\|_{L^2} \leqslant \|g\|_{C^2} \|f\|_{H^{\frac{1}{2}}}$ yields

$$J_1 \leqslant \left\|\mathcal{F}(z)\right\|_{L^\infty} \|\varphi\|_{H^{4-\frac{1}{2}}}^3,$$

allowing us to get the estimate $I_1 \leqslant \frac{1}{m^q(t)} C \exp(C E^p(t))$.

The boundedness of the term $B(t)$ gives us a similar control of I_2

$$I_2 \leqslant \frac{1}{m^p(t)} C \exp\left(C E^p(t)\right).$$

Next we write the term I_4 as follows:

$$I_4 = \int_{\mathbb{T}} \frac{1}{|\partial_\alpha z|^3} H\left(\partial_\alpha^3 \varphi\right)(\alpha) \partial_\alpha^3 \left(\partial_\alpha BR(z,\varpi) \cdot \partial_\alpha^\perp z + \frac{\varpi}{2|\partial_\alpha z|^2} \partial_\alpha^2 z \cdot \partial_\alpha^\perp z\right)^2 (\alpha)\, d\alpha,$$

where the most singular part is given by

$$J_2 = \int_{\mathbb{T}} \frac{2}{|\partial_\alpha z|^3} H\left(\partial_\alpha^3 \varphi\right)(\alpha) D(\alpha) \partial_\alpha^3 D(\alpha)\, d\alpha,$$

where

$$D(\alpha) = \partial_\alpha BR(z,\varpi) \cdot \partial_\alpha^\perp z + \frac{\varpi}{2|\partial_\alpha z|^2} \partial_\alpha^2 z \cdot \partial_\alpha^\perp z. \tag{7.8}$$

To analyze $\partial_\alpha^3(D)$, let us observe that the most singular terms are given by

$$E_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta)\, d\alpha,$$

$$E_2 = \frac{-1}{\pi} \int\limits_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \big(z(\alpha) - z(\alpha - \beta)\big) \cdot \big(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha - \beta)\big) \varpi(\alpha - \beta) \, d\alpha,$$

$$E_3 = BR\big(z, \partial_\alpha^4 \varpi\big) \cdot \partial_\alpha^\perp z + \partial_\alpha^3 \left(\frac{\varpi}{2|\partial_\alpha z|^2} \partial_\alpha^2 z \cdot \partial_\alpha^\perp z \right).$$

Since the terms E_1 and E_2 are singular only in the tangential directions, we can again use the following identity

$$\partial_\alpha z(\alpha) \cdot \partial_\alpha^4 z(\alpha) = -3 \partial_\alpha^2 z(\alpha) \cdot \partial_\alpha^3 z(\alpha), \tag{7.9}$$

to obtain the desired control.

In E_3 the term $BR(z, \partial_\alpha^4 \varpi) \cdot \partial_\alpha^\perp z$ can be written as the sum of $\frac{1}{2} H(\partial_\alpha^4 \varpi)$ plus kernels of degree zero in $\partial_\alpha^4 \varpi$, which are bounded in L^2. Therefore we can write it as follows

$$BR\big(z, \partial_\alpha^4 \varpi\big) \cdot \partial_\alpha^\perp z = \frac{1}{2} H\big(\partial_\alpha^4 \varpi\big) + \text{``bounded terms in } L^2\text{''}.$$

The identity

$$\frac{1}{2} \partial_\alpha^4 \varpi = |\partial_\alpha z| \partial_\alpha^4 \varphi - \partial_\alpha^3 \big(\partial_\alpha BR(z, \varpi) \cdot \partial_\alpha z\big)$$

yields

$$\partial_\alpha^3 \big(\partial_\alpha BR(z, \varpi) \cdot \partial_\alpha z\big) = H\left(\partial_\alpha^3 \left(\frac{\varpi}{2|\partial_\alpha z|^2} \big(\partial_\alpha^2 z\big)^\perp \cdot \partial_\alpha z \right) \right) + \text{``bounded terms in } L^2\text{''}.$$

That is

$$\frac{1}{2} H\big(\partial_\alpha^4 \varpi\big) = H\big(|\partial_\alpha z| \partial_\alpha^4 \varphi\big) - H^2 \left(\partial_\alpha^3 \left(\frac{\varpi}{2|\partial_\alpha z|^2} \big(\partial_\alpha^2 z\big)^\perp \cdot \partial_\alpha z \right) \right) + \text{``bounded terms in } L^2\text{''}$$

and therefore

$$\frac{1}{2} H\big(\partial_\alpha^4 \varpi\big) = H\big(|\partial_\alpha z| \partial_\alpha^4 \varphi\big) + \left(\partial_\alpha^3 \left(\frac{\varpi}{2|\partial_\alpha z|^2} \big(\partial_\alpha^2 z\big)^\perp \cdot \partial_\alpha z \right) \right) + \text{``bounded terms in } L^2\text{''}.$$

The above equality gives $E_3 = |\partial_\alpha z| H(\partial_\alpha^4 \varphi) + \text{``bounded terms in } L^2\text{''}$.

Finally for J_2 we have

$$J_2 = \int\limits_{\mathbb{T}} \frac{2}{|\partial_\alpha z|^2} H\big(\partial_\alpha^3 \varphi\big)(\alpha) H\big(\partial_\alpha^4 \varphi\big)(\alpha) D(\alpha) \, d\alpha + \text{``bounded terms''},$$

and integration by parts gives us the desired estimate.

For I_3 it is important to arrange conveniently the derivatives

$$I_3 = -S + J_3 + \text{"bounded terms"},$$

where

$$J_3 = \int_{\mathbb{T}} \frac{\partial_\alpha^2 z \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^3} H(\partial_\alpha^3 \varphi)(\alpha) \partial_\alpha^3 \sigma(\alpha) \, d\alpha. \tag{7.10}$$

Then, because of its sign, the term involving the highest derivative can be eliminated and we are left with the task of estimating J_3. In order to do that we shall study the singular term $\partial_\alpha^3 \sigma(\alpha)$ using the splitting

$$\partial_\alpha^3 \sigma = \partial_\alpha^3 \left(\left(\partial_t BR(z, \varpi) + \frac{\varphi}{|\partial_\alpha z|} \partial_\alpha BR(z, \varpi) \right) \cdot \partial_\alpha^\perp z \right)$$

$$+ \partial_\alpha^3 \left(\frac{1}{2} \frac{\varpi}{|\partial_\alpha z|^2} \left(\partial_\alpha z_t + \frac{\varphi}{|\partial_\alpha z|} \partial_\alpha^2 z \right) \cdot \partial_\alpha^\perp z \right) + g \partial_\alpha^4 z_1$$

$$= F_1 + F_2 + F_3.$$

The term F_3 trivializes, whether for F_2 we have

$$\partial_\alpha^3 \left(\frac{1}{2} \frac{\varpi}{|\partial_\alpha z|^2} \left(\partial_\alpha BR(z, \varpi) \cdot \partial_\alpha^\perp z + \frac{\varpi}{2|\partial_\alpha z|^2} \partial_\alpha^2 z \cdot \partial_\alpha^\perp z \right) \right) = \partial_\alpha^3 \left(\frac{1}{2} \frac{\varpi}{|\partial_\alpha z|^2} D \right)$$

where D is given by (7.8) and the integral can be estimated like I_4 or J_2. Finally we are left with F_1, and we shall show that

$$F_1 = |\partial_\alpha z| H(\partial_\alpha^3 \varphi_t) - c|\partial_\alpha z| H(\partial_\alpha^4 \varphi) + \text{"bounded terms in } L^2\text{"}. \tag{7.11}$$

Plugging the above decomposition in J_3 (7.10) we can control this term as before using the formula for $\partial_\alpha^3 \varphi_t$ (3.5).

Next we split $F_1 = G_1 + G_2$ where

$$G_1 = \partial_\alpha^3 \left(\partial_t BR(z, \varpi) \cdot \partial_\alpha^\perp z \right), \qquad G_2 = \partial_\alpha^3 \left(\frac{\varphi}{|\partial_\alpha z|} \partial_\alpha BR(z, \varpi) \cdot \partial_\alpha^\perp z \right),$$

and again we will consider the more singular terms. In G_1 we have

$$O_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\partial_\alpha^3 z_t(\alpha) - \partial_\alpha^3 z_t(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta) \, d\beta,$$

$$O_2 = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \left(z(\alpha) - z(\alpha - \beta) \right) \cdot \left(\partial_\alpha^3 z_t(\alpha) - \partial_\alpha^3 z_t(\alpha - \beta) \right) \varpi(\alpha - \beta) \, d\beta,$$

and

$$O_3 = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \partial_\alpha^3 \varpi_t(\alpha - \beta) \, d\beta.$$

Let us write $O_1 = P_1 + P_2$ where

$$P_1 = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{\partial_\alpha^3 z_t(\alpha) \cdot \partial_\alpha z(\alpha) - \partial_\alpha^3 z_t(\alpha - \beta) \cdot \partial_\alpha z(\alpha - \beta)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta) \, d\beta,$$

and

$$P_2 = -\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{\partial_\alpha^3 z_t(\alpha - \beta) \cdot (\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta) \, d\beta.$$

The term P_2 has a kernel of degree -1 in $\partial_\alpha^3 z_t$, giving us a Hilbert integral of $\partial_\alpha^3 z_t$ which can be estimated using (6.6). From its expression it follows that P_1 can be written as the sum of terms involving kernels of degree -1 and the operator Λ, that is:

$$P_1 = \frac{\varpi}{2|\partial_\alpha z|^2} \Lambda(\partial_\alpha^3 z_t \cdot \partial_\alpha z) + \text{``bounded terms in } L^2\text{''}.$$

Since $A'(t) = 2\partial_\alpha z_t(\alpha, t) \cdot \partial_\alpha z(\alpha, t)$ we have

$$\partial_\alpha^3 z_t \cdot \partial_\alpha z = -2\partial_\alpha^2 z_t \cdot \partial_\alpha^2 z - \partial_\alpha z_t \cdot \partial_\alpha^3 z,$$

which yields

$$P_1 = \frac{\varpi}{2|\partial_\alpha z|^2} \left(-2\Lambda(\partial_\alpha^2 z_t \cdot \partial_\alpha^2 z) - \Lambda(\partial_\alpha z_t \cdot \partial_\alpha^3 z) \right) + \text{``bounded terms in } L^2\text{''}.$$

Then, as it was shown before, the estimates for z and z_t give us the control of the term P_1 in the L^2 norm.

Regarding O_2 we introduce into its integral expression the following identity

$$\begin{aligned}
\left(z(\alpha) - z(\alpha - \beta) \right) \cdot \left(\partial_\alpha^3 z_t(\alpha) - \partial_\alpha^3 z_t(\alpha - \beta) \right) \\
= \beta \partial_\alpha z(\alpha) \cdot \left(\partial_\alpha^3 z_t(\alpha) - \partial_\alpha^3 z_t(\alpha - \beta) \right) \\
+ \left(z(\alpha) - z(\alpha - \beta) - \partial_\alpha z(\alpha)\beta \right) \cdot \left(\partial_\alpha^3 z_t(\alpha) - \partial_\alpha^3 z_t(\alpha - \beta) \right)
\end{aligned}$$

and then we just take the same steps that we followed with O_1.

Using the estimates (6.7) for ϖ_t we get

$$O_3 = \frac{1}{2} H(\partial_\alpha^3 w_t) + \text{``bounded terms in } L^2\text{''},$$

and therefore

$$G_1 = \frac{1}{2} H\left(\partial_\alpha^3 w_t\right) + \text{"bounded terms in } L^2\text{"}.$$

The formula for G_2 gives us more singular terms, namely the following ones

$$O_4 = \frac{\varphi}{2\pi |\partial_\alpha z|} \int_{-\pi}^{\pi} \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \varpi(\alpha - \beta)\, d\beta,$$

$$O_5 = -\frac{\varphi}{\pi |\partial_\alpha z|} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^4}$$

$$\times \left(z(\alpha) - z(\alpha - \beta)\right) \cdot \left(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha - \beta)\right) \varpi(\alpha - \beta)\, d\beta,$$

and

$$O_6 = \frac{\varphi}{2\pi |\partial_\alpha z|} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha - \beta)) \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha - \beta)|^2} \partial_\alpha^4 \varpi(\alpha - \beta)\, d\beta.$$

Using the identity (7.9) we can estimate O_4 and O_5 as before. Furthermore we have that

$$O_6 = \frac{\varphi}{2|\partial_\alpha z|} H\left(\partial_\alpha^4 \varpi\right) + \text{"bounded terms in } L^2\text{"},$$

and

$$G_2 = \frac{\varphi}{2|\partial_\alpha z|} H\left(\partial_\alpha^4 \varpi\right) + \text{"bounded terms in } L^2\text{"}.$$

Then we get

$$F_1 = \frac{1}{2} H\left(\partial_\alpha^3 w_t\right) + \frac{\varphi}{2|\partial_\alpha z|} H\left(\partial_\alpha^4 \varpi\right) + \text{"bounded terms in } L^2\text{"}. \tag{7.12}$$

We shall continue deducing (7.11) from (7.12); in order to do that let us write

$$\frac{1}{2} w_t = \partial_t\left(|\partial_\alpha z|\right) \frac{w}{2|\partial_\alpha z|} + |\partial_\alpha z|\left(\varphi_t + \partial_t\left(|\partial_\alpha z| c\right)\right),$$

$$\frac{1}{2} \partial_\alpha^3 w_t = \partial_t\left(|\partial_\alpha z|\right) \frac{\partial_\alpha^3 w}{2|\partial_\alpha z|} + |\partial_\alpha z| \partial_\alpha^3 \varphi_t - |\partial_\alpha z| \partial_\alpha^2 \partial_t\left(\frac{\partial_\alpha z}{|\partial_\alpha z|} \cdot \partial_\alpha BR(z, \varpi)\right).$$

We have

$$|\partial_\alpha z| \partial_\alpha^2 \partial_t\left(\frac{\partial_\alpha z}{|\partial_\alpha z|} \cdot \partial_\alpha BR(z, \varpi)\right) = \partial_\alpha^2\left(\partial_\alpha z \cdot \partial_\alpha \partial_t BR(z, \varpi)\right) + \partial_\alpha^2\left(\frac{\partial_\alpha z_t \cdot \partial_\alpha^\perp z}{|\partial_\alpha z|^2} \partial_\alpha^\perp z \cdot \partial_\alpha BR(z, \varpi)\right)$$

$$= \partial_\alpha^2\left(\partial_\alpha z \cdot \partial_\alpha \partial_t BR(z, \varpi)\right) + \text{"bounded terms in } L^2\text{"}.$$

The last two identities allow us to consider

$$\frac{1}{2}\partial_\alpha^3 w_t = |\partial_\alpha z|\partial_\alpha^3\varphi_t - \partial_\alpha^2\big(\partial_\alpha z \cdot \partial_\alpha \partial_t BR(z,\varpi)\big) + \text{"bounded terms in } L^2\text{"}$$

and therefore

$$\frac{1}{2}H\big(\partial_\alpha^3 w_t\big) = |\partial_\alpha z|H\big(\partial_\alpha^3\varphi_t\big) - H\big(\partial_\alpha^2\big(\partial_\alpha z \cdot \partial_\alpha \partial_t BR(z,\varpi)\big)\big) + \text{"bounded terms in } L^2\text{"}.$$

This formula indicates that to prove (7.11) it is enough to obtain

$$c|\partial_\alpha z|H\big(\partial_\alpha^4\varphi\big) = \frac{\varphi}{2|\partial_\alpha z|}H\big(\partial_\alpha^4\varpi\big) - G_3 + \text{"bounded terms in } L^2\text{"},\qquad(7.13)$$

where

$$G_3 = H\big(\partial_\alpha^2\big(\partial_\alpha z \cdot \partial_\alpha \partial_t BR(z,\varpi)\big)\big).\qquad(7.14)$$

Again let us consider the most singular terms in $\partial_\alpha^2(\partial_\alpha z \cdot \partial_\alpha \partial_t BR(z,\varpi))$:

$$O_7 = \partial_\alpha\big(\partial_\alpha z \cdot BR\big(z,\partial_\alpha^2\varpi_t\big)\big),$$

$$O_8 = \frac{-1}{\pi}\int_{-\pi}^{\pi}\frac{(z(\alpha) - z(\alpha-\beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^4}$$
$$\times\big(z(\alpha) - z(\alpha-\beta)\big)\cdot\big(\partial_\alpha^3 z_t(\alpha) - \partial_\alpha^3 z_t(\alpha-\beta)\big)\varpi(\alpha-\beta)\,d\beta,$$

and

$$O_9 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha^3 z_t(\alpha) - \partial_\alpha^3 z_t(\alpha-\beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - z(\alpha-\beta)|^2}\varpi(\alpha-\beta)\,d\beta.$$

The term $O_7 = \frac{1}{2}\partial_\alpha T(\partial_\alpha^2\varpi_t)$ is estimated in L^2 by using the operator T. In O_8 we substitute $(z(\alpha) - z(\alpha-\beta))^\perp \cdot \partial_\alpha z(\alpha)$ by $(z(\alpha) - z(\alpha-\beta) - \partial_\alpha z(\alpha)\beta)^\perp \cdot \partial_\alpha z(\alpha)$ inside the integral and then we split the integral in two terms (one is multiplied by $\partial_\alpha^3 z_t(\alpha)$ and the other is an operator $R(\partial_\alpha^3 z_t)$ with kernel of degree -1) allowing us to integrate O_8.

Regarding O_9 we have that

$$O_9 = \frac{-1}{2|\partial_\alpha z|^2}\Lambda\big(\partial_\alpha^3 z_t \cdot \partial_\alpha^\perp z\varpi\big) + \text{"bounded terms in } L^2\text{"},$$

and therefore the identity $H(\Lambda(f)) = -\partial_\alpha f$ yields for G_3 in (7.14) the following configuration:

$$G_3 = \frac{1}{2|\partial_\alpha z|^2}\partial_\alpha\big(\partial_\alpha^3 z_t \cdot \partial_\alpha^\perp z\varpi\big) + \text{"bounded terms in } L^2\text{"}$$

$$= \frac{1}{2|\partial_\alpha z|^2}\big(\partial_\alpha\big(\partial_\alpha^3 BR(z,\varpi) \cdot \partial_\alpha^\perp z\varpi\big) + \partial_\alpha\big(c\partial_\alpha^4 z \cdot \partial_\alpha^\perp z\varpi\big)\big) + \text{"bounded terms in } L^2\text{"}$$

$$= \frac{1}{2|\partial_\alpha z|^2}\left(\frac{1}{2}H\big(\partial_\alpha^4\varpi\big)\varpi + c\varpi\,\partial_\alpha\big(\partial_\alpha^4 z \cdot \partial_\alpha^\perp z\big)\right) + \text{"bounded terms in } L^2\text{"}.$$

With this identity in (7.13) we obtain

$$\frac{\varphi}{2|\partial_\alpha z|} H\big(\partial_\alpha^4 \varpi\big) - G_3 = -\frac{c}{2} H\big(\partial_\alpha^4 \varpi\big) - \frac{c\varpi}{2|\partial_\alpha z|^2} \partial_\alpha\big(\partial_\alpha^4 z \cdot \partial_\alpha^\perp z\big) + \text{"bounded terms in } L^2\text{"}$$

$$= -cH\big(|\partial_\alpha z|\partial_\alpha^4 \varphi\big) - G_4 + \text{"bounded terms in } L^2\text{",}$$

for

$$G_4 = cH\big(|\partial_\alpha z|^2 \partial_\alpha^4 c\big) + \frac{c\varpi}{2|\partial_\alpha z|^2} \partial_\alpha\big(\partial_\alpha^4 z \cdot \partial_\alpha^\perp z\big).$$

Finally we only have to show that G_4 is a bounded term in L^2. But this follows because we have

$$|\partial_\alpha z|^2 \partial_\alpha^4 c = -\partial_\alpha^3\big(\partial_\alpha z \cdot \partial_\alpha BR(z, \varpi)\big)$$

$$= \frac{1}{2|\partial_\alpha z|^2} \Lambda\big(\partial_\alpha^4 z \cdot \partial_\alpha^\perp z \varpi\big) + \text{"bounded terms in } L^2\text{".} \qquad \square$$

8. The addition of the Rayleigh–Taylor condition to the energy

Our final step is to use the a priori estimates to prove local existence (Theorem 1.1). For that purpose we introduce a regularized evolution equation which is well-posed independently of the sign condition on $\sigma(\alpha, t)$ at $t = 0$. But for $\sigma(\alpha, 0) > 0$, we shall find a time of existence uniformly in the regularization, allowing us to take the limit.

Let $z^\varepsilon(\alpha, t)$ be a solution of the following system:

$$z_t^\varepsilon(\alpha, t) = BR\big(z^\varepsilon, \varpi^\varepsilon\big)(\alpha, t) + c^\varepsilon(\alpha, t)\partial_\alpha z^\varepsilon(\alpha, t),$$

$$\varpi_t^\varepsilon = -2\partial_t BR\big(z^\varepsilon, \varpi^\varepsilon\big) \cdot \partial_\alpha z^\varepsilon - \partial_\alpha\big((\varphi^\varepsilon)^2\big) + 2|\partial_\alpha z^\varepsilon|B^\varepsilon c^\varepsilon - 2g\partial_\alpha z_2^\varepsilon + \varepsilon 2|\partial_\alpha z^\varepsilon|\Delta\varphi^\varepsilon,$$

$z^\varepsilon(\alpha, 0) = z_0(\alpha)$ and $\varpi^\varepsilon(\alpha, 0) = \varpi_0(\alpha)$ for $\varepsilon > 0$, where

$$c^\varepsilon(\alpha) = \frac{\alpha + \pi}{2\pi} \int_{-\pi}^{\pi} \frac{\partial_\alpha z^\varepsilon(\alpha)}{|\partial_\alpha z^\varepsilon(\alpha)|^2} \cdot \partial_\alpha BR\big(z^\varepsilon, \varpi^\varepsilon\big)(\alpha)\, d\alpha - \int_{-\pi}^{\alpha} \frac{\partial_\alpha z^\varepsilon(\beta)}{|\partial_\alpha z^\varepsilon(\beta)|^2} \cdot \partial_\beta BR\big(z^\varepsilon, \varpi^\varepsilon\big)(\beta)\, d\beta,$$

$$\varphi^\varepsilon = \frac{\varpi^\varepsilon}{2|\partial_\alpha z^\varepsilon|} - |\partial_\alpha z^\varepsilon|c^\varepsilon, \qquad B^\varepsilon(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial_\alpha z^\varepsilon(\alpha, t)}{|\partial_\alpha z^\varepsilon(\alpha, t)|^2} \cdot \partial_\alpha BR\big(z^\varepsilon, \varpi^\varepsilon\big)(\alpha, t)\, d\alpha.$$

Proceeding as in Section 3 we find

$$\partial_\alpha\varphi_t^\varepsilon = -\frac{\partial_\alpha^2((\varphi^\varepsilon)^2)}{2|\partial_\alpha z^\varepsilon|} - B^\varepsilon(t)\partial_\alpha\varphi^\varepsilon - \frac{\sigma^\varepsilon}{\rho^2}\frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^3} - \partial_t\big(|\partial_\alpha z^\varepsilon|B^\varepsilon\big)$$

$$+ \frac{1}{|\partial_\alpha z^\varepsilon|^3}\bigg(\partial_\alpha BR\big(z^\varepsilon, \varpi^\varepsilon\big) \cdot \partial_\alpha^\perp z^\varepsilon + \frac{\varpi^\varepsilon}{2|\partial_\alpha z^\varepsilon|^2}\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon\bigg)^2 + \varepsilon\Delta\partial_\alpha\varphi^\varepsilon, \quad (8.1)$$

where

$$\frac{\sigma^\varepsilon}{\rho^2} = \left(\partial_t BR(z^\varepsilon, \varpi^\varepsilon) + \frac{\varphi^\varepsilon}{|\partial_\alpha z^\varepsilon|} \partial_\alpha BR(z^\varepsilon, \varpi^\varepsilon) \right) \cdot \partial_\alpha^\perp z^\varepsilon$$

$$+ \frac{1}{2} \frac{\varpi^\varepsilon}{|\partial_\alpha z^\varepsilon|^2} \left(\partial_\alpha z_t^\varepsilon + \frac{\varphi^\varepsilon}{|\partial_\alpha z^\varepsilon|} \partial_\alpha^2 z^\varepsilon \right) \cdot \partial_\alpha^\perp z^\varepsilon + g \partial_\alpha z_1^\varepsilon.$$

For this system there is local existence for initial data satisfying $\mathcal{F}(z_0)(\alpha, \beta) < \infty$ even if $\sigma^\varepsilon(\alpha, 0)$ does not have the proper sign. In the following we shall show briefly how to obtain a solution of the regularized system with $z^\varepsilon, \varphi^\varepsilon \in C^1([0, T^\varepsilon], H^k)$ for $k \geqslant 4$. We shall prove the same a priori estimates given in Sections 6.1, 6.2 and 6.4, but the estimates corresponding to Sections 6.3 and 6.5 are respectively

$$\left\| \varpi_t^\varepsilon \right\|_{H^k} \leqslant C \exp\left(C \|z^\varepsilon\|^p\right) \left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^{k+2}}^2 + \|\varpi^\varepsilon\|_{H^{k+1}}^2 + \|\varphi^\varepsilon\|_{H^{k+1}}^2 \right)^j$$

$$+ \varepsilon C \exp\left(C \|z^\varepsilon\|^p\right) \|\Delta \partial_\alpha^k \varphi^\varepsilon\|_{L^2}, \tag{8.2}$$

$$\left\| \sigma^\varepsilon \right\|_{H^k} \leqslant C \exp\left(C \|z^\varepsilon\|^p\right) \left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^{k+2}}^2 + \|\varpi^\varepsilon\|_{H^{k+1}}^2 + \|\varphi^\varepsilon\|_{H^{k+1}}^2 \right)^j$$

$$+ \varepsilon C \exp\left(C \|z^\varepsilon\|^p\right) \|\Delta \partial_\alpha^k \varphi^\varepsilon\|_{L^2}, \tag{8.3}$$

for $k \geqslant 2$.

Then following the same steps of Section 6 we have

$$\frac{d}{dt} \left(\|z^\varepsilon\|_{H^k}^2 + \|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|\varpi^\varepsilon\|_{H^{k-2}}^2 + \|\varphi^\varepsilon\|_{H^k}^2 \right)^2 (t)$$

$$\leqslant C(\varepsilon) \exp\left(\left(\|z^\varepsilon\|_{H^k}^2 + \|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|\varpi^\varepsilon\|_{H^{k-2}}^2 + \|\varphi^\varepsilon\|_{H^k}^2 \right)^p (t) \right)$$

where the only difference appears in the following new term

$$I = -\int_{-\pi}^{\pi} \partial_\alpha^{k-1} \left(\frac{\sigma^\varepsilon}{\rho^2} \frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^3} \right) \partial_\alpha^k \varphi \, d\alpha \leqslant C(\varepsilon) \left\| \frac{\sigma^\varepsilon}{\rho^2} \frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^3} \right\|_{H^{k-2}}^2 + \varepsilon \left\| \partial_\alpha^{k+1} \varphi^\varepsilon \right\|_{L^2}^2$$

which is controlled by the Laplacian dissipation term introduced in the regularization.

The next step is to integrate the system during a time T independent of ε. We will show that for this system we have

$$\frac{d}{dt} E^p(t) \leqslant \frac{C}{(m^\varepsilon)^q (t)} \left(\|\sigma_t^\varepsilon\|_{L^\infty} + 1 \right) \exp\left(C E^p(t) \right), \tag{8.4}$$

where $E(t)$ is given by the analogous formula (7.1) for the ε-system,

$$m^\varepsilon(t) = \min_{\alpha \in [-\pi, \pi]} \sigma^\varepsilon(\alpha, t) = \sigma^\varepsilon(\alpha_t, t) > 0$$

and C, p and q are universal constants independent of ε.

In the following we shall select only the most singular terms, showing for them the corresponding uniform estimates for $k = 4$ and leaving to the reader the remainder easier cases. Let us consider the one corresponding to I_3 in Section 7.3, we have

$$I_3^\varepsilon = -\int_\pi^\pi \frac{1}{|\partial_\alpha z^\varepsilon|^3} \partial_\alpha^3 \varphi^\varepsilon(\alpha) \Lambda\big(\partial_\alpha^2(\sigma^\varepsilon \partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon)\big) d\alpha.$$

We split this term as $I_3^\varepsilon = -S^\varepsilon + J_2^\varepsilon + J_3^\varepsilon +$ "bounded terms" where S^ε corresponds to S in (7.4) and

$$J_2^\varepsilon = \int_\pi^\pi \frac{C}{|\partial_\alpha z^\varepsilon|^3} H(\partial_\alpha^3 \varphi^\varepsilon)(\alpha) \partial_\alpha^2(\sigma^\varepsilon) \partial_\alpha(\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon) d\alpha,$$

$$J_3^\varepsilon = \int_\pi^\pi \frac{1}{|\partial_\alpha z^\varepsilon|^3} H(\partial_\alpha^3 \varphi^\varepsilon)(\alpha) \partial_\alpha^3(\sigma^\varepsilon) \partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon \, d\alpha.$$

In J_2^ε we use (8.3) to get

$$J_2^\varepsilon \leqslant \frac{C}{(m^\varepsilon)^q(t)} \exp\big(CE^p(t)\big) + \varepsilon^2 \|\partial_\alpha^4 \varphi^\varepsilon\|_{L^2}^2.$$

The similarity with (7.10) together with the use of the corresponding version of (7.11) allows us to get

$$J_3^\varepsilon = \int_\pi^\pi \frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^2} H(\partial_\alpha^3 \varphi^\varepsilon) H(\partial_\alpha^3 \varphi_t^\varepsilon) d\alpha + M\varepsilon^2 \|\partial_\alpha^4 \varphi^\varepsilon\|_{L^2}^2 + \text{"bounded terms"}$$

that by formula (8.1) becomes

$$J_3^\varepsilon = -\varepsilon \int_\pi^\pi \frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^2} H(\partial_\alpha^3 \varphi^\varepsilon) H(\partial_\alpha^5 \varphi^\varepsilon) d\alpha + M\varepsilon^2 \|\partial_\alpha^4 \varphi^\varepsilon\|_{L^2}^2 + \text{"bounded terms"}.$$

Then we can write it as follows

$$J_3^\varepsilon = -\varepsilon \int_\pi^\pi \Lambda^{\frac{1}{2}}\bigg(\frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^2} H(\partial_\alpha^3 \varphi^\varepsilon)\bigg) \Lambda^{\frac{1}{2}}(\partial_\alpha^4 \varphi^\varepsilon) d\alpha + M\varepsilon^2 \|\partial_\alpha^4 \varphi^\varepsilon\|_{L^2}^2 + \text{"bounded terms"},$$

and therefore

$$J_3^\varepsilon \leqslant M\varepsilon^2 \|\Lambda^{\frac{1}{2}} \partial_\alpha^4 \varphi^\varepsilon\|_{L^2}^2 + \text{"bounded terms"}.$$

Now the use of the Laplacian dissipative term introduced in the evolution equation yields

$$\frac{d}{dt} E^2(t) \leqslant \frac{C}{(m^\varepsilon)^q(t)} \left(\|\sigma_t^\varepsilon\|_{L^\infty} + 1 \right) \exp\!\left(C E^p(t) \right) + \left(M\varepsilon^2 - \varepsilon \right) \big\| \Lambda^{\frac{1}{2}} \partial_\alpha^4 \varphi^\varepsilon \big\|_{L^2}^2,$$

where the constant M is fixed. This finally shows (8.4) for ε small enough.

Our regularization damages the estimates for the term $\|\sigma_t^\varepsilon\|_{L^\infty}$ in (6.17). But this control is necessary only once in the argument and therefore enough derivatives in the definition of energy gives the desired control. Since we wish to keep the result for four derivatives, we can go around the problem just by regularizing the initial data. At the end of the argument, when the local-existence theorem holds for $\varepsilon = 0$, then the a priori energy estimate for $k = 4$ allows us to take the limit in the regularization of the initial data. With this strategy and taking enough derivatives in the definition of the energy, we find in (8.4) the following inequality

$$\frac{d}{dt} E^p(t) \leqslant \frac{C}{(m^\varepsilon)^q(t)} \exp\!\left(C E^p(t) \right). \tag{8.5}$$

Now let us observe that if $z_0(\alpha) \in H^k$, $\varpi_0(\alpha) \in H^{k-1}$ and $\varphi_0(\alpha) \in H^{k-\frac{1}{2}}$, then we have the solution in $[0, T^\varepsilon]$ of the regularized system. And if initially $\sigma(\alpha, 0) > 0$, there is a time depending on ε, denoted by T^ε again, in which $\sigma^\varepsilon(\alpha, t) > 0$. Now, for $t \leqslant T^\varepsilon$ we have (8.5). Let us mention that at this point of the proof we cannot assume local existence, because we have the above estimate for $t \leqslant T^\varepsilon$, and if we let $\varepsilon \to 0$, it could be possible that $T^\varepsilon \to 0$, i.e. we cannot assume that if the initial data satisfy $\sigma(\alpha, 0) > 0$, there must be a time T, independent of ε, in which the following important quantity

$$m^\varepsilon(t) = \min_{\alpha \in [-\pi, \pi]} \sigma^\varepsilon(\alpha, t) = \sigma^\varepsilon(\alpha_t, t)$$

is strictly greater than zero. In fact, everything in the evolution problem depends upon the sign of $\sigma^\varepsilon(\alpha, t)$ (the higher order derivatives), since otherwise the problem is ill-posed [12]. In other words, at this stage of the proof we do not have local existence when $\varepsilon \to 0$, but the following argument will allow us to continue: First let us introduce the Rayleigh–Taylor condition in a new definition of energy as follows:

$$E_{RT}(t) = E^p(t) + \frac{1}{m^\varepsilon(t)}.$$

Sobolev inequalities shows that $\sigma^\varepsilon(\alpha, t) \in C^1([0, T^\varepsilon] \times [-\pi, \pi])$ and therefore $m^\varepsilon(t)$ is a Lipschitz function differentiable almost everywhere by Rademacher's theorem. With an analogous argument to the one used in [6] and [7], we can calculate the derivative of $m^\varepsilon(t)$, to obtain

$$\left(m^\varepsilon \right)'(t) = \sigma_t^\varepsilon(\alpha_t, t)$$

for almost every t. Then it follows that

$$\frac{d}{dt}\left(\frac{1}{m^\varepsilon} \right)(t) = -\frac{\sigma_t^\varepsilon(\alpha_t, t)}{(m^\varepsilon)^2(t)}$$

almost everywhere. The control of the quantity $\|\sigma_t^\varepsilon\|_{L^\infty}$, independently of ε, by its formula together with inequality (8.5) yields

$$\frac{d}{dt} E_{RT}(t) \leqslant C \exp\big(C E_{RT}(t)\big),$$

and therefore

$$E_{RT}(t) \leqslant -\frac{1}{C} \ln(\exp\big(-C E_{RT}(0) - C^2 t\big).$$

Now we are in position to extend the time of existence T^ε so long as the above estimate works and obtain a time T depending only on the initial data (arc-chord and Rayleigh–Taylor). Finally we can let ε tend to 0 to conclude the existence result.

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