

# A geometrical constraint for capillary jet breakup

A. Córdoba,<sup>a</sup> D. Córdoba,<sup>b</sup> C.L. Fefferman,<sup>c,\*</sup> and  
M.A. Fontelos<sup>d</sup>

<sup>a</sup> *Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain*

<sup>b</sup> *Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas,  
Serrano 123, 28006 Madrid, Spain*

<sup>c</sup> *Department of Mathematics, Princeton University, Fine Hall, Princeton, NJ 08540, USA*

<sup>d</sup> *Departamento de Ciencia e Ingeniería, Universidad Rey Juan Carlos, Tulipán s/n, 28933 Móstoles,  
Madrid, Spain*

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## Abstract

The formation of thinning filaments is commonly observed previously to the break-up of a very viscous jet. This paper shows that a fluid under capillary forces cannot break-up through the uniform collapse of a filament.

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## 1. Introduction

A mass of fluid bounded by a free surface and occupying a simply connected domain may evolve in such a way that, after some time, the domain becomes disconnected. The simplest example in which this transition is observed consists of a fluid jet emerging from a faucet. At a certain distance of the faucet, the jet breaks into drops.

An obvious question is whether one can deduce from the equations for the fluids (Euler and Navier–Stokes under the action of surface tension) this kind of transition

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\*Corresponding author. Fax: +1-609-258-1367.

E-mail address: [cf@math.princeton.edu](mailto:cf@math.princeton.edu) (C.L. Fefferman).

or not. If the answer is positive, then this fact supports the self-consistency of the theory. If not, one would have to modify the equations in a physically reasonable manner in order to accommodate these phenomena.

The evolution and break-up of fluid jets has attracted the attention of scientists since the early 19th century. In 1833 Savart [9] performed experiments in order to measure the size of drops resulting out of the break-up of a jet. In 1879, Rayleigh [8] presented the first analytical study of the problem. He showed that a stationary jet, which is a solution for both Euler or Navier–Stokes systems, is unstable and computed the dispersion relation for small perturbations. This dispersion relation is such that it attains a maximum at a wavelength coherent with the size of the drops measured by Savart. The celebrated linear theory of Rayleigh, nevertheless, fails to show that breakup follows from the equations. The process is inherently nonlinear, which is the main analytical obstacle that one has to face. At the end of the last century, the problem was attacked again using the theoretical, computational and experimental tools available at the time. A close experimental observation of the evolution and breakup processes revealed their high degree of complexity, spanning several time and length scales and being strongly dependent on physical parameters. On the theoretical side, the main result is a universal self-similar breakup mechanism postulated by Eggers [4]. The breakup happens at a point and in its neighborhood the jet thins, close to the breakup time  $T$ , at a rate  $\alpha(T - t)$ . Experimental and computational evidence shows that this mechanism is consistent with a large number of observations but not with many others. In particular, in the limits of very low and very high viscosity fluids, events unfold very differently. In low viscosity fluids, there is an overturning phenomenon by which breakup happens at a point “inside” a drop (see [12]). In high viscosity fluids, the breakup is preceded by the formation of long and thin filaments (cf. [6,10]). In experiments, these filaments thin uniformly up to a diameter of the order of a micron. Sometimes they generate new and smaller filaments (see [10]), sometimes they become unstable and break. In [5] it is proved the formation of filaments for very viscous fluids under the slender jet approximation. The criterion obtained in [2] shows that in order to have a filament collapse at time  $T$  is necessary for the quantity  $\int_0^T |u|_{L^\infty} ds$  to diverge.

The question that it is then natural to ask is whether breakup is possible in a jet through the uniform collapse of a fluid filament or not. In the following, we prove that the answer to this question is negative.

In our case we have coordinates  $(r, z)$  where  $z$  is the vertical coordinate, and  $r$  denote the distance to the axis of symmetry. Let us denote by  $h(z, t)$  the distance of a point of the boundary of the tube to its axis.

We understand by collapse of a filament at time  $T$  the following:

$$\lim_{t \rightarrow T} h(z, t) = 0 \quad \text{for every } z \in I, \quad (1.1)$$

where  $I$  is an interval that we take  $[-L, L]$ . The collapse will be uniform if

$$\frac{1}{C} \bar{h}(t) \leq h(z, t) \leq C \bar{h}(t) \quad \text{for every } z \in I, \quad (1.2)$$

where  $C$  is a constant and  $\bar{h}(t)$  is the average of  $h(z, t)$  over  $I$ . We will also impose

$$|h_z(z, t)| \leq C \quad \text{for every } z \in I \text{ and every } t. \quad (1.3)$$

Our main result is the following theorem:

**Theorem 1.** *Under conditions (1.2) and (1.3) in a given interval  $I$ , the uniform collapse of a filament (in the sense of (1.1)) is impossible. Moreover, the volume of fluid enclosed by the filament satisfies*

$$V(t) \geq Ce^{-Ct^2}$$

for some positive constant  $C$ .

These results were announced in [1]. Similar approach was used in [3] for the formation of fronts.

The article is organized as follows: In Section 2 we present the equations that describe mathematically the evolution of a fluid tube in absence of external forces such as gravity and show a well-known energy inequality. In Section 3 we deduce an inequality satisfied by the volume enclosed by a filament. In Section 4 an inequality satisfied over the cross sections of the tube is deduced. Finally, in Section 5 we finish the proof of Theorem 1. Section 6 is devoted to the analysis of the problem when external forces are present.

## 2. The equations and an energy identity

The equations describing the evolution of a Newtonian fluid in a bounded domain  $\Omega(t)$  limited by a free surface  $\partial\Omega(t)$  are the Navier–Stokes system:

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p + \mu \Delta \vec{v} + \vec{F}, \quad (2.1)$$

$$\nabla \cdot \vec{v} = 0, \quad (2.2)$$

together with the boundary condition

$$\left[ -p\delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right] n_j = \sigma H n_i \quad \text{in } \partial\Omega(t), \quad (2.3)$$

where  $\vec{n}$  is the field of outer normal vectors to  $\Omega(t)$  and  $H$  is the mean curvature of  $\partial\Omega(t)$ , and the following kinematic condition for the evolution of  $\partial\Omega(t)$ :

$$V_N = \vec{v} \cdot \vec{n}, \quad (2.4)$$

expressing the fact that the particles of the boundary move with a velocity whose normal component  $V_N$  equals the normal component of the velocity field defined in it. The parameters  $\rho$  and  $\mu$  are the density and viscosity of the fluid, respectively, while  $\sigma$  denotes the surface tension coefficient of the interface which depends upon the fluid itself and the surrounding media.  $\vec{F}$  denotes an external force that we will take, for the sake of simplicity, as zero. In Section 6 we will retake the problem with  $\vec{F} \neq \vec{0}$ .

From Eqs. (2.1) and (2.2) it is very simple to deduce the following equation:

$$\frac{d}{dt} \left[ \int_{\Omega(t)} \frac{1}{2} \rho |\vec{v}|^2 dV + \sigma |\partial\Omega(t)| \right] = -\mu \int_{\Omega(t)} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV, \quad (2.5)$$

where  $|\partial\Omega(t)|$  denotes the area of  $\partial\Omega(t)$ . It follows then

$$\int_{\Omega(t)} \frac{1}{2} \rho |\vec{v}|^2 dV + \sigma |\partial\Omega(t)| + \mu \int_0^t \int_{\Omega(t)} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV dt = C. \quad (2.6)$$

In order to obtain (2.5) one takes the dot product of (2.1) with  $\vec{v}$ , integrate over the volume of  $\Omega(t)$ , integrate by parts, and use (2.2). Then

$$\begin{aligned} & \rho \int_{\Omega(t)} \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} |\vec{v}|^2 \right) + \vec{v} \cdot (\vec{v} \cdot \nabla) \vec{v} \right] dV \\ &= -\mu \int_{\Omega(t)} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV + \sigma \int_{\partial\Omega(t)} H \vec{v} \cdot \vec{n} dS. \end{aligned} \quad (2.7)$$

Since  $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = \frac{d\vec{v}}{dt}$  (the material derivative of  $\vec{v}$ ), we have that the left-hand side of (2.7) is

$$\frac{1}{2} \int_{\Omega(t)} \frac{d|\vec{v}|^2}{dt} dV = \frac{1}{2} \int_{\Omega(0)} \frac{d|\vec{v}|^2}{dt} dV^* = \frac{1}{2} \frac{d}{dt} \int_{\Omega(0)} |\vec{v}|^2 dV^* = \frac{1}{2} \frac{d}{dt} \int_{\Omega(t)} |\vec{v}|^2 dV,$$

where we have performed a change to Lagrangian coordinates in which the domain  $\Omega$  remains fixed, extracted the time derivative outside the integral, and returned to Eulerian coordinates.

As for the second term at the right-hand side of (2.7), we take into account the following: let  $\vec{x}(u, v)$  be a parametrization of a surface  $\partial\Omega$  and generate another surface  $\partial\Omega'$  parametrized by  $\vec{x}(u, v) + \delta(u, v) \vec{n}$  (with  $\vec{n}$  being the field of unitary vectors normal to  $S$ ). The variation of the area is (cf. [7]):

$$|\partial\Omega'| - |\partial\Omega| = - \int_S \delta H dS + O(\delta^2).$$

If  $\delta = \vec{v} \cdot \vec{n} dt$ , then we conclude

$$\frac{d|\partial\Omega(t)|}{dt} = - \int_{S(t)} H \vec{v} \cdot \vec{n} dS.$$

In the case of a fluid (with viscosity  $\mu_1$  and density  $\rho_1$ ) surrounded by another fluid (with viscosity  $\mu_2$  and density  $\rho_2$ ), each fluid satisfies Navier–Stokes equations and the kinematic condition at the interface is

$$[T_{ij}^{(1)} - T_{ij}^{(2)}]n_j = \sigma H n_i \quad \text{in } \partial\Omega(t) \quad (2.8)$$

with

$$T_{ij}^{(k)} = \left[ -p\delta_{ij} + \mu_k \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right]$$

and  $\sigma$  being the surface tension coefficient for the interface between both fluids. Also, continuity of the velocity field across the interface has to be imposed (cf. [11]).

One can deduce the following energy identity:

$$\begin{aligned} & \int_{\Omega(t)} \frac{1}{2} \rho_1 |\vec{v}|^2 dV + \mu_1 \int_0^t \int_{\Omega(t)} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV dt + \sigma |\partial\Omega(t)| \\ & + \int_{\mathbb{R}^3 \setminus \Omega(t)} \frac{1}{2} \rho_2 |\vec{v}|^2 dV + \mu_2 \int_0^t \int_{\mathbb{R}^3 \setminus \Omega(t)} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV dt = C. \end{aligned}$$

An immediate consequence of this is the following inequality:

$$\min_i(\rho_i) \int_{\mathbb{R}^3} \frac{1}{2} |\vec{v}|^2 dV + \min_i(\mu_i) \int_0^t \int_{\mathbb{R}^3} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV dt + \sigma |\partial\Omega(t)| \leq C. \quad (2.9)$$

### 3. A differential inequality for the volume enclosed by a filament

Given a mass of fluid in the interval  $[-z_0, z_0]$ , the variation of its volume with time is given by the equation:

$$\begin{aligned} \frac{dV(t; z_0)}{dt} &= \int_0^{2\pi} \int_0^{h(z_0, \theta, t)} v_z(z_0, \rho, \theta, t) \rho d\rho d\theta \\ &\quad - \int_0^{2\pi} \int_0^{h(-z_0, \theta, t)} v_z(-z_0, \rho, \theta, t) \rho d\rho d\theta. \end{aligned}$$

Next, we integrate the previous equation in  $z_0$  between 0 and  $L$  to obtain

$$\begin{aligned} \frac{d}{dt} \int_0^L V(t; z) dz &= \int_0^L \int_0^{2\pi} \int_0^{h(z, \theta, t)} v_z(z, \rho, \theta, t) \rho \, d\rho \, d\theta \, dz \\ &\quad - \int_0^L \int_0^{2\pi} \int_{z=0}^{h(-z, \theta, t)} v_z(-z, \rho, \theta, t) \rho \, d\rho \, d\theta \, dz. \end{aligned} \quad (3.1)$$

The triple integrals at the right-hand side of (3.1) can be viewed as integrals over the domain  $0 \leq z \leq L$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \rho \leq h(z, \theta, t)$ . Let us introduce now the following change of variables:

$$z' = z,$$

$$\rho' = \frac{\bar{h}}{h(z, \theta, t)} \rho,$$

$$\theta' = \theta,$$

where  $\bar{h}$  denotes the average of  $h$  over the interval  $z \in [-L, L]$ . The Jacobian of this transformation is

$$\left| \frac{\partial(z', \rho', \theta')}{\partial(z, \rho, \theta)} \right| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{h_z(z, \theta, t)\bar{h}}{h^2(z, \theta, t)}\rho & \frac{\bar{h}}{h(z, \theta, t)} & -\frac{h_\theta(z, \theta, t)\bar{h}}{h^2(z, \theta, t)}\rho \\ 0 & 0 & 1 \end{vmatrix} = \frac{\bar{h}}{h(z, \theta, t)} = \frac{\rho'}{\rho}.$$

Then, the integrals at the right-hand side of (3.1) are

$$\begin{aligned} &\int_0^L \int_0^{2\pi} \int_0^{\bar{h}} v_z(z', \theta', \rho', t) \left( \frac{h(z', \theta', t)}{\bar{h}} \right)^2 \rho' d\rho' d\theta' dz' \\ &\quad - \int_0^L \int_0^{2\pi} \int_0^{\bar{h}} v_z(-z', \theta', \rho', t) \left( \frac{h(z', \theta', t)}{\bar{h}} \right)^2 \rho' d\rho' d\theta' dz' \\ &= \int_{-L}^L \int_0^{2\pi} \int_0^{\bar{h}} \text{sign}(z') v_z(z', \theta', \rho', t) \left( \frac{h(z', \theta', t)}{\bar{h}} \right)^2 \rho' d\rho' d\theta' dz' \\ &= \int_0^{\bar{h}} \left[ \int_{-L}^L \int_0^{2\pi} \text{sign}(z') v_z(z', \theta', \rho', t) \left( \frac{h(z', \theta', t)}{\bar{h}} \right)^2 d\theta' dz' \right] \rho' d\rho'. \end{aligned}$$

We can estimate

$$\left| \int_0^{\bar{h}} \left[ \int_{-L}^L \int_0^{2\pi} \text{sign}(z') v_z(z, \theta', \rho', t) \left( \frac{h(z', \theta', t)}{\bar{h}} \right)^2 d\theta' dz' \right] \rho' d\rho' \right| \\ \leq C \bar{h}^2 |\ln \bar{h}|^{\frac{1}{2}} \int_{-L}^L \int_0^{2\pi} \sup_{\rho'} \frac{|v_z(z', \theta', \rho', t)|}{|\ln \rho'|^{\frac{1}{2}}} d\theta' dz'.$$

Under the hypothesis of uniform collapse (1.2) and (1.3), we can conclude, for a given  $L$  and  $\bar{h}$  small enough

$$\pi \bar{h}^2 |\ln \bar{h}|^{\frac{1}{2}} \leq C \bar{V}(t) |\ln \bar{V}(t)|^{\frac{1}{2}},$$

where

$$\bar{V}(t) \equiv \frac{1}{L} \int_0^L V(t; z) dz$$

and we arrive to the inequality

$$\frac{d\bar{V}(t)}{dt} \geq -C \left( \int_{-L}^L \int_0^{2\pi} \sup_{\rho'} \frac{|v_z(z', \theta', \rho', t)|}{|\ln \rho'|^{\frac{1}{2}}} d\theta' dz' \right) \bar{V}(t) |\ln \bar{V}(t)|^{\frac{1}{2}}. \quad (3.2)$$

#### 4. An inequality in a disc

Consider a radial function  $v(r)$  defined in the disc  $0 \leq r \leq R$  (we assume, without loss of generality,  $R < \frac{1}{2}$ ). Given a point  $a > 0$  we have

$$v(a) = v(r) + \int_r^a \frac{dv(\rho)}{d\rho} d\rho.$$

Hence,

$$v^2(a) \leq v^2(r) + 2 \left( \int_r^a \frac{dv(\rho)}{d\rho} d\rho \right)^2 \\ \leq v^2(r) + 2 \left| \int_r^a \frac{1}{\rho} d\rho \right| \left| \int_r^a \left( \frac{dv(\rho)}{d\rho} \right)^2 \rho d\rho \right| \\ \leq v^2(r) + 2 \left| \ln \frac{r}{a} \right| \int_0^R \left( \frac{dv(\rho)}{d\rho} \right)^2 \rho d\rho.$$

We multiply the previous inequality by  $r$  and integrate to obtain

$$\frac{R^2}{2} v^2(a) \leq 2 \int_0^R v^2(r) r \, dr + 2 \int_0^R r \left| \ln \frac{r}{a} \right| dr \left| \int_0^R \left( \frac{dv(\rho)}{d\rho} \right)^2 \rho \, d\rho \right|.$$

Now we can take

$$\int_0^R r \left| \ln \frac{r}{a} \right| dr = a^2 \int_0^{\frac{R}{a}} u |\ln u| du \simeq |\ln a| \quad \text{as } a \rightarrow 0$$

and globally,

$$\int_0^R r \left| \ln \frac{r}{a} \right| dr \leq C |\ln a|,$$

so that

$$\begin{aligned} \frac{v^2(a)}{|\ln a|} &\leq \frac{C}{|\ln a|} \int_0^R v^2(r) r \, dr + C \int_0^R \left( \frac{dv(r)}{dr} \right)^2 r \, dr \\ &\leq C \left( \int_0^R v^2(r) r \, dr + \int_0^R \left( \frac{dv(r)}{dr} \right)^2 r \, dr \right), \end{aligned} \quad (4.1)$$

where  $C$  depends only on  $R$ .

Then, for the function  $v_z(z, \theta, \rho, t)$  one has

$$\begin{aligned} \sup_r \frac{|v_z(z, \theta, r, t)|}{|\ln r|^{\frac{1}{2}}} &\leq C \left( \int_0^R v_z^2(z, \theta, r, t) r \, dr + \int_0^R \left| \frac{\partial v_z}{\partial r}(z, \theta, r, t) \right|^2 r \, dr \right)^{\frac{1}{2}} \\ &\equiv C \Gamma^{\frac{1}{2}}(z, \theta, t). \end{aligned} \quad (4.2)$$

Hence,

$$\begin{aligned} \int_0^{2\pi} \sup_r \frac{|v_z(z, \theta, r, t)|}{|\ln r|^{\frac{1}{2}}} d\theta &\leq C \int_0^{2\pi} \Gamma^{\frac{1}{2}}(z, \theta, t) d\theta \\ &\leq C \sqrt{2\pi} \left( \int_0^{2\pi} \Gamma(z, \theta, t) d\theta \right)^{\frac{1}{2}} \equiv C \sqrt{2\pi} \Phi^{\frac{1}{2}}(z, t). \end{aligned} \quad (4.3)$$

**Remark 1.** The  $\ln a$  at the left-hand side of (4.1) cannot be eliminated from the estimate. Notice at this respect that the functions  $v(r) = |\ln r|^\alpha$  with  $0 < \alpha < 1$  are not bounded at  $r = 0$  but the integrals at the right-hand side of (4.1) are bounded.

## 5. Proof of Theorem 1

In this section we put together the results of the previous sections and prove Theorem 1.

First let us observe the inequality (see the appendix):

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial v_z}{\partial r} \right|^2 dV dt \leq \frac{3}{2} \int_0^t \int_{\mathbb{R}^3} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV dt, \quad (5.1)$$

which implies, by (2.9),

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial v_z}{\partial r} \right|^2 dV dt \leq C. \quad (5.2)$$

From (2.9) we also obtain, integrating once in  $t$ ,

$$\int_0^t \int_{\Omega(t)} \frac{1}{2} |\vec{v}|^2 dV \leq Ct. \quad (5.3)$$

Second, we can estimate by (4.3),

$$\begin{aligned} \int_{-L}^L \int_0^{2\pi} \sup_{\rho'} \frac{|v_z(z', \theta', \rho', t)|}{|\ln \rho'|^{\frac{1}{2}}} d\theta' dz' &\leq C\sqrt{2\pi} \left( \int_{-L}^L \Phi^{\frac{1}{2}}(z', t) dz' \right) \\ &\leq C\sqrt{2\pi}\sqrt{2L} \left( \int_{-L}^L \Phi(z', t) dz' \right)^{\frac{1}{2}}, \end{aligned}$$

so that

$$\begin{aligned} \int_0^t \int_{-L}^L \sup_{\rho'} \frac{|v_z(z', \rho', t)|}{|\ln \rho'|^{\frac{1}{2}}} dz' dt &\leq C\sqrt{2\pi}\sqrt{2L} \int_0^t \left( \int_{-L}^L \Phi(z', t) dz' \right)^{\frac{1}{2}} dt \\ &\leq C\sqrt{2\pi}\sqrt{2Lt} \left( \int_0^t \int_{-L}^L \Phi(z', t) dz' dt \right)^{\frac{1}{2}} \\ &\leq C(1+t), \end{aligned} \quad (5.4)$$

where we have used inequality (4.2), the definition of  $\Phi$  in (4.3), and inequalities (5.2) and (5.3).

Finally, we can take inequality (3.2) and obtain

$$\overline{V}(t) \geq Ce^{-Ct^2}.$$

Under the assumptions of uniform collapse, analogous inequality follows for  $V(t; L)$ .

**Remark 2.** We have assumed  $\Omega(t)$  bounded, but this assumption is not essential. The results easily extend to infinite periodic domains (jets) provided that the integrals in (2.9) are taken in a period or even in domains with part of its boundary being a solid wall since the boundary integrals are zero there.

## 6. The case $\vec{F} \neq \vec{0}$

We assume in this section the existence of an external force  $\vec{F}$  such that

$$\|\vec{F}\|_{L^2} \leq C.$$

Then, multiplying (2.1) by  $\vec{v}$  and integrating over the volume, we obtain the inequality

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega(t)} \frac{1}{2} \rho_1 |\vec{v}|^2 dV + \mu_1 \int_{\Omega(t)} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV + \sigma |\partial \Omega(t)| \\ & + \frac{d}{dt} \int_{\mathbb{R}^3 \setminus \Omega(t)} \frac{1}{2} \rho_2 |\vec{v}|^2 dV + \mu_2 \int_{\mathbb{R}^3 \setminus \Omega(t)} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV = \int_{\mathbb{R}^3 \setminus \Omega} \vec{F} \cdot \vec{v} dV \\ & \leq \|\vec{F}\|_{L^2} \|\vec{v}\|_{L^2} \leq C \|\vec{v}\|_{L^2}. \end{aligned}$$

Hence,

$$\frac{d}{dt} \|\vec{v}\|_{L^2}^2 \leq C \|\vec{v}\|_{L^2},$$

which implies  $\|\vec{v}(t)\|_{L^2} \leq \|\vec{v}(0)\|_{L^2} + \frac{1}{2} C t$ . Then, the equivalent to inequality (2.9) is:

$$\min_i(\rho_i) \int_{\mathbb{R}^3} \frac{1}{2} |\vec{v}|^2 dV + \min_i(\mu_i) \int_0^t \int_{\mathbb{R}^3} |\partial_{x_i} v_j + \partial_{x_j} v_i|^2 dV dt + \sigma |\partial \Omega(t)| \leq C(1 + t^2).$$

We can use this last inequality and operate as in (5.4) to conclude

$$\int_0^t \int_{-L}^L \sup_{\rho'} \frac{|v_z(z', \rho', t)|}{|\ln \rho'|^{\frac{1}{2}}} dz' dt \leq C(1 + t^{\frac{3}{2}}).$$

Finally, using inequality (3.2) we find

$$\overline{V}(t) \geq C e^{-C t^3}.$$

The filament cannot collapse in finite time.

## 7. A proof of inequality (5.1)

We present here a simple proof of inequality (5.1). Notice that

$$\begin{aligned} \sum_{i,j} \left\| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 &= 4 \sum_i \left\| \frac{\partial v_i}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 + 2 \sum_{i < j} \left\| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 \\ &= 4 \sum_i \left\| \frac{\partial v_i}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 + 2 \sum_{i < j} \left( \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \frac{\partial v_j}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 \right. \\ &\quad \left. + 2 \int_{\mathbb{R}^3} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dV \right). \end{aligned} \quad (7.1)$$

Integration by parts in the last integral yields

$$\int_{\mathbb{R}^3} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dV = \int_{\mathbb{R}^3} \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} dV.$$

Hence

$$\begin{aligned} \sum_{i,j} \left\| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 &= 4 \sum_i \left\| \frac{\partial v_i}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 + 2 \sum_{i \neq j} \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^2(\mathbb{R}^3)}^2 + 4 \sum_{i < j} \int_{\mathbb{R}^3} \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} dV \\ &\geq 4 \sum_i \left\| \frac{\partial v_i}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 + 2 \sum_{i \neq j} \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^2(\mathbb{R}^3)}^2 \\ &\quad - 2 \sum_{i < j} \left( \left\| \frac{\partial v_i}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \frac{\partial v_j}{\partial x_j} \right\|_{L^2(\mathbb{R}^3)}^2 \right) \\ &= 2 \sum_{i \neq j} \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \quad (7.2)$$

By the first line in (7.1) and (7.2) we have

$$\sum_i \left\| \frac{\partial v_i}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2 + \sum_{i \neq j} \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^2(\mathbb{R}^3)}^2 \leq \left( \frac{1}{2} + \frac{1}{4} \right) \sum_{i,j} \left\| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\|_{L^2(\mathbb{R}^3)}^2. \quad (7.3)$$

Finally, observe that given the definition of the radial coordinate  $r$ ,

$$\int_{\mathbb{R}^3} \left| \frac{\partial v_z}{\partial r} \right|^2 dV \leq 2 \int_{\mathbb{R}^3} \left( \left| \frac{\partial v_z}{\partial x} \right|^2 + \left| \frac{\partial v_z}{\partial y} \right|^2 \right) dV,$$

so that, using (7.3), inequality (5.1) follows.

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