Encounters at the interface between Number Theory and Harmonic Analysis

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In his Habilitationschrift (*Uber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*) B. Riemann states the following: "...the usefullness of Fourier series is not limited to research in Physics; they have been succesfully applied also to a field in pure mathematics, namely Number Theory, and here it seems to be of importance precisely to consider those functions whose representability by trigonometric series has not been yet investigated...".

Therefore the subject of this talk has a long and interesting history and even the title, or a rather similar one, has been already used by several authors. For example by H. Montgomery: *Ten lectures on the interface between analytic number theory and harmonic analysis* (CBMS **84**, Amer. Math. Soc., 1994).

Along this paper we shall write $f \gg g$ to denote that $f(x) \geq C\, g(x)$ for some positive constant C independent of other parameters relevant to the problem. Similarly $f \asymp g$ will express the existence of universal positive constants C_j such that: $C_1 f(x) \leq g(x) \leq C_2 f(x)$ for every x.

I. Euler's evaluation of $\zeta(2)$

To begin let us consider the famous identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

whose original proof given by L. Euler uses the infinite product

$$\frac{\sin(\pi z)}{\pi z} = \prod \left(1 - \frac{z^2}{n^2}\right)$$

and it is a beautiful expression of the power of eighteen century calculus. Euler obtained several proofs but, years after him, we can find another one based on Bessel's identity applied to the Fourier series

$$\{x\} = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin(2\pi n x)$$

where

$$\{x\} = \begin{cases} x - m, & \text{if } |x - m| < \frac{1}{2}, \\ 0, & \text{if } x = m + \frac{1}{2}, \end{cases} \qquad m \in \mathbb{Z}$$

which is, certainly, the more transparent example to be covered under the title of this talk.

In the following I shall present a much more recent proof [6].

Since $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$, we have:

$$\zeta(2) = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{4}{3} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 s^{2n} t^{2n} ds \, dt$$

$$= \frac{4}{3} \int_0^1 \int_0^1 \frac{ds \, dt}{1 - s^2 t^2} = \frac{1}{3} \int_{-1}^1 \int_{-1}^1 \frac{ds \, dt}{1 - s^2 t^2} \, .$$

The change of variables

$$s = \tanh(u) = \frac{\frac{1}{2}\left(e^u - e^{-u}\right)}{\frac{1}{2}\left(e^u + e^{-u}\right)} = \frac{\sinh(u)}{\cosh(u)} \, ,$$

$$t = \tanh(v) = \frac{\frac{1}{2}\left(e^v - e^{-v}\right)}{\frac{1}{2}\left(e^v + e^{-v}\right)} = \frac{\sinh(v)}{\cosh(v)}$$

yields

$$\zeta(2) = \frac{1}{3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{du \, dv}{\cosh^2(u) \cosh^2(v) - \sinh^2(u) \sinh^2(v)}$$

$$= \frac{1}{3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{ds \, dt}{\cosh(u + v) \cdot \cosh(u - v)} \, .$$

Then we introduce the new variables $s = u - v$, $t = u + v$, to obtain

$$\zeta(2) = \frac{1}{6} \left(\int_{-\infty}^{+\infty} \frac{ds}{\cosh(s)} \right) \left(\int_{-\infty}^{+\infty} \frac{dt}{\cosh(t)} \right) = \frac{1}{6} \left(\int_{-\infty}^{+\infty} \frac{ds}{\cosh(s)} \right)^2 = \frac{\pi^2}{6} \, .$$

Clearly it is a proof which could have been given by the eighteen century mathematicians, and whose main new ingredient is the change of variables $s = \tanh(u)$. As a matter of curiosity let me add that the functions $y = \tanh \frac{x}{\varepsilon}$, $\varepsilon \to 0$, appear in the Ginzburg-Landau phase transition model, about which I was writing a paper when J. Cilleruelo asked me to contribute with an expository article (about the irrationality of $\zeta(2)$ and $\zeta(3)$) for his *Devil of Numbers* section in the *Gaceta Matemática, 6.*

As far as I know it was Riemann, in the same thesis quoted before, who gave one of the first application of Euler's identity:

Riemann introduces the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2}$$

and shows that f is discontinuous precisely at the rational points of the form $\frac{a}{2b}$, $\mathrm{mcd}(a, 2b) = 1$, which constitute a dense subset of the real line.

Furthermore using the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \left(\text{or } \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \right)$$

he proves that at those points of discontinuity there is a jump

$$f^-\left(\tfrac{a}{2b}\right) - f^+\left(\tfrac{a}{2b}\right) = \frac{\pi^2}{8b^2}\,.$$

Since $|f(x)| \leq \frac{\pi^2}{6}$ and the set of discontinuity is countable, it follows that f is Riemann's integrable and its indefinite integral

$$F(x) = \int_0^x f(t)\, dt$$

turns out to be a continuous but non-differentiable function at those rational points

$$\lim_{h \to 0^+} \frac{F\left(\tfrac{a}{2b} + h\right) - F\left(\tfrac{a}{2b}\right)}{h} = f^+\left(\tfrac{a}{2b} + h\right)$$

$$\lim_{h \to 0^-} \frac{F\left(\tfrac{a}{2b} + h\right) - F\left(\tfrac{a}{2b}\right)}{h} = f^-\left(\tfrac{a}{2b} + h\right)\,.$$

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II. Gaussian trigonometric series

In part IV of Riemann's Habilitationschrift one find the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n} \, .$$

We know that f is Lebesgue-integrable but not Riemann-integrable, because its oscillation is unbounded on any interval. It can be represented by the trigonometric series

$$f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{d_{\mathrm{o}}(n) - d_{\mathrm{e}}(n)}{n} \, \sin(\pi n x)$$

where $d_{\mathrm{o}}(n)$ (respectively $d_{\mathrm{e}}(n)$) is the number of odd (respectively even) divisors of n.

Riemann writes: "One can obtain similar examples with series of the form

$$\sum_{n=0}^{\infty} C_n \, e^{2\pi i n^2 x}$$

when the positive quantities C_n are decreasing to 0 but for which $\sum C_n = \infty$".

We find here a very interesting problem whose solution remains open.

Question: *If f is a Lebesgue-integrable function whose Fourier spectrum is contained in the set of square numbers, does it follow that $\|f\|_p \ll \|f\|_1$, for every $p < 4$?*

An equivalent formulation is the following: let S be the Fourier multiplier operator given by the characteristic function of the set of square numbers, i.e., if f has the Fourier series $\sum \widehat{f}(k) e^{2\pi i k x}$ then the Fourier series of Sf is given by

$$\sum_{n=0}^{\infty} \widehat{f}(n^2) \, e^{2\pi i n^2 x} \, .$$

Question: *Is S a bounded operator from $L^2[0,1]$ to $L^p[0,1]$, $2 \le p < 4$?*

By duality it is equivalent to the boundedness of S from $L^q[0,1]$ to $L^2[0,1]$, for every $q > 4/3$. A positive answer would have very interesting arithmetical consequences. It would imply, for example, that any arithmetic progression of length N, $\{a + br \,:\, 0 \le r \le N-1\}$, may contains, at most, $O\big(N^{\frac{1}{2}+\varepsilon}\big)$ square numbers for every $\varepsilon > 0$ (see reference [4] for more details about this problem).

If $G(x) = \sum_{r=0}^{N-1} e^{2\pi i(a+br)x}$ then we have $\|G\|_p \sim N^{\frac{p-1}{p}}$, $p > 1$. On the other hand if $n_1^2 < n_2^2 < \cdots < n_k^2$ are the square numbers contained in the arithmetic progression $\{a+br\}_{r=0,\ldots,N-1}$, and assuming a positive answer to our question about S we obtain

$$k^{1/2} = \|S(G)\|_2 \ll \|S(G)\|_p \ll N^{\frac{p-1}{p}}$$

for every $p > \frac{4}{3}$, implying that $k = O\big(N^{\frac{1}{2}+\varepsilon}\big)$ where $p = \frac{1}{3/4-\varepsilon}$.

Proposition 1. *Suppose that $\{a_n\}$ is a monotonically decreasing sequence of non-negative real numbers such that $\sum_{n=0}^{\infty} a_n e^{2\pi i n^2 x}$ is the Fourier series of an integrable function f. Then $f \in L^p[0,1]$ for every p, $1 \leq p < 4$, and satisfies the estimate*

$$\|f\|_p \ll \|f\|_1.$$

The proof of the above proposition will be based on the following fact.

Lemma 2. *Let f be continuous in $[0,1]$ and such that:*

i) $\|f\|_\infty \leq N$, $\|f\|_2 \leq N^{\frac{1}{2}}$

ii) *If $\left|x - \frac{p}{q}\right| \leq \frac{1}{q^2}$ then*

$$|f(x)| \leq C_0 \Big\{ q^{1/k} + \frac{N}{q^{1/k}} \Big\}, \quad \text{for some constant } C_0.$$

Then f is in the space weak $\big(L^{2k}\big)$ and satisfies the estimate:

$$\mu\big\{x \ : \ |f(x)| > \alpha > 0\big\} \leq C\,\frac{N^k}{\alpha^{2k}}$$

for every $\alpha > 0$, where the constant $C = C(C_0)$, is independent of α and N, and μ denotes Lebesgue measure in $[0,1]$.

Proof. Without loss of generality we may assume that $C_0 = 1$, because if $C_0 > 1$ we will just consider then the function f/C_0. It will be equivalent to show that for every $\alpha > 0$ we have

$$\mu(E_\alpha) = \mu\big\{|f(x)| \geq 2N^{\frac{1}{2}}\alpha\big\} \leq \widetilde{C}\,\frac{1}{\alpha^{2k}}$$

with $\widetilde{C}$ independent of N and α.

Observe that it is enough to prove it in the case $\frac{1}{2}N^{\frac{1}{2}} \geq \alpha \geq 1$, because if $\alpha < 1$ then the inequality trivializes, and if $\alpha \geq \frac{1}{2}N^{\frac{1}{2}}$ we have that $E_\alpha = \emptyset$.

If

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cdots}}}$$

is the continuous fraction expansion of the irrational number $x \in E_\alpha$ with convergents $\left\{ \frac{P_\nu}{Q_\nu} \right\}$ then the following holds

$$Q_\nu^{1/k} + \frac{N}{Q_\nu^{1/k}} \geq 2N^{\frac{1}{2}}\alpha \, .$$

That is, for every ν

$$\text{either} \quad Q_\nu^{1/k} \geq N^{\frac{1}{2}}\alpha$$

$$\text{or} \quad \frac{N}{Q_\nu^{1/k}} \geq N^{\frac{1}{2}}\alpha \, .$$

We have two possible cases:

1st Case: $Q_1^{1/k} \geq N^{\frac{1}{k}}\alpha \implies Q_1 \geq N^{\frac{k}{2}}\alpha^k$.

Since $x \leq \frac{1}{Q_1}$ we have that $x \in I_0 = \left[0 \, , \frac{1}{N^{k/2}\alpha^k} \right]$ and $\mu(I_0) = \frac{1}{N^{k/2}\alpha^k} \leq \frac{2^k}{\alpha^{2k}}$.

2nd Case: There exists $\nu \geq 1$ such that

- $\frac{N}{Q_\nu^{1/k}} \geq N^{\frac{1}{2}}\alpha \implies Q_\nu \leq \frac{N^{k/2}}{\alpha^k}$

- $Q_{\nu+1}^{1/k} \geq N^{\frac{1}{2}}\alpha \implies Q_{\nu+1} \geq N^{\frac{k}{2}}\alpha^k$.

We have

$$\left| x - \frac{P_\nu}{Q_\nu} \right| \leq \frac{1}{Q_\nu \, Q_{\nu+1}} \leq \frac{1}{Q_\nu} \frac{1}{N^{k/2}\alpha^k}$$

that is

$$x \in \left(\frac{P_\nu}{Q_\nu} - \frac{1}{Q_\nu} \frac{1}{N^{k/2}\alpha^k} \, , \, \frac{P_\nu}{Q_\nu} + \frac{1}{Q_\nu} \frac{1}{N^{k/2}\alpha^k} \right) .$$

Given integers r, s such that $r < s$, $\mathrm{mcd}(r,s) = 1$, $s \leq \frac{N^{k/2}}{\alpha^k}$, let us consider the interval

$$I_{r,s} = \left(\frac{r}{s} - \frac{1}{s} \frac{1}{N^{k/2}\alpha^k} \, , \, \frac{r}{s} + \frac{1}{s} \frac{1}{N^{k/2}\alpha^k} \right)$$

Then the previous observations give us the inclusion

$$E_\alpha - \{\text{rationals}\} \subset I_0 \cup \left(\bigcup_{r,s} I_{r,s} \right).$$

Therefore

$$\mu(E_\alpha) \le \mu(I_0) + \sum_{\substack{r=1 \\ (r,s)=1}}^{s-1} \sum_{s=1}^{\frac{N^{k/2}}{\alpha^k}} \mu(I_{r,s})$$

$$\le \frac{2^k}{\alpha^{2k}} + \sum_{s=1}^{\frac{N^{k/2}}{\alpha^k}} \sum_{r=1}^{s-1} \frac{2}{s} \frac{1}{N^{k/2}\alpha^k} \le \frac{2^k}{\alpha^{2k}} + \sum_{s=1}^{\frac{N^{k/2}}{\alpha^k}} \frac{2}{N^{k/2}\alpha^k} \le \frac{2 + 2^k}{\alpha^{2k}}. \blacksquare$$

Next let us consider the trigonometric polynomials

$$S_N(x\,;\,y) = \sum_{n=1}^{N} e^{2\pi i (n^2 x + ny)}$$

and its Carleson maximal operator

$$C_N^*(x\,;\,y) = \sup_{1 \le M \le N} \left| S_M(x\,;\,y) \right|.$$

We will use now Carleson's maximal theorem to continue the proof of proposition 1 showing that, for every $y \in [0,1]$, $S_N(x\,;\,y)$ verifies the hypothesis of lemma 2 in the case $k = 2$.

Lemma 3. *Let x have a rational approximation of the form $|x - \frac{p}{q}| \le \frac{1}{q^2}$, $1 \le q \le N^2$, $\mathrm{mcd}(p,q) = 1$, then*

$$C_N^*(x\,;\,y) \ll \left\{ q^{1/2} + \frac{N}{q^{1/2}} \right\}.$$

This result was known to Hardy and Littlewood who proved it using the approximate functional equation for θ-functions. The following proof of E. Bombieri [1] emphasizes the relationship between number theory and harmonic analysis throughout the use of Carleson's theorem [2].

Proof. First let us observe that if $|x - x_0| \le \frac{1}{4N^2}$ and $|y - y_0| \le \frac{1}{4N}$ then we have the inequality

$$\mathcal{C}_N^*(x \,;\, y) \le 100\, \mathcal{C}_N^*(x_0 \,;\, y_0)\,.$$

This is because

$$S_M(x \,;\, y) = \sum_{n=1}^{M} e^{2\pi i(n^2 x + ny)}$$

$$= \sum_{n=1}^{M} e^{2\pi i(n^2 x_0 + ny_0)} \cdot e^{2\pi i(n^2(x-x_0) + n(y-y_0))}$$

$$= \sum_{n=1}^{M} \big(S_n(x_0 \,;\, y_0) - S_{n-1}(x_0 \,;\, y_0)\big) e^{2\pi i(n^2(x-x_0) + n(y-y_0))}$$

$$= \sum_{n=1}^{M-1} S_n(x_0 \,;\, y_0)\Big(e^{2\pi i(n^2(x-x_0) + n(y-y_0))} - e^{2\pi i((n+1)^2(x-x_0) + (n+1)(y-y_0))}\Big)$$

$$+ S_M(x_0 \,;\, y_0) e^{2\pi i(M^2(x-x_0) + M(y-y_0))}$$

Taking the supremum in M, $1 \le M \le N$, we obtain

$$\mathcal{C}_N^*(x \,;\, y) \le 2\, \mathcal{C}_N^*(x_0 \,;\, y_0)\Big[1 + \sum_{n=1}^{N} \big(|x - x_0|4\pi n + 4\pi|y - y_0|\big)\Big]$$

$$\le 100\, \mathcal{C}_N^*(x_0 \,;\, y_0)\,.$$

Next we have $\forall M$, $1 \le M \le N$, the following inequality

$$\mathcal{C}_N^*(x \,;\, y) \le 2\, \mathcal{C}_{2N}^*(x \,;\, y - 2Mx)$$

This is because

$$\big|S_K(x \,;\, y)\big| = \Big|\sum_{n=1}^{K} e^{2\pi i(n^2 x + ny)}\Big| = \Big|\sum_{n=M+1}^{K+M} e^{2\pi i((n-M)^2 x + (n-M)y)}\Big|$$

$$= \Big|\sum_{n=M+1}^{K+M} e^{2\pi i(n^2 x + n(y-2Mx))}\Big| \le 2\, \mathcal{C}_{2N}^*(x \,;\, y - 2Mx)\,.$$

Taking the supremum over $K \le N$, rising the inequality to the square power and averaging over M, we get

$$\big|\mathcal{C}_N^*(x \,;\, y)\big|^2 \le \frac{4}{N} \sum_{M=1}^{N} \big|\mathcal{C}_{2N}^*(x \,;\, y - 2Mx)\big|^2$$

$$\le \frac{4}{N} \sum_{n=1}^{M} (100)^2 8N \int_{I(2Mx-y)} \big|\mathcal{C}_{2N}^*(x \,;\, y_0)\big|^2 dy_0$$

where $I(z) = \left\{ w \, : \, |z - w| \leq \frac{1}{8N} \right\}$.

An elementary calculation gives us an upper bound for the overlapping of the family of intervals $\left\{ I_{(2Mx-y)} \right\}$

$$\Big\| \sum_{M=1}^{N} \chi_{I_{(2Mx-y)}} \Big\|_{\infty} \ll \left(\frac{N}{q} + \frac{q}{N} \right),$$

and applying Carleson's maximal theorem we obtain

$$\mathcal{C}_N^*(x \,;\, y)^2 \ll \left\{ q + \frac{N^2}{q} \right\}. \qquad \blacksquare$$

Proof of proposition 1. Let us consider the maximal function

$$\mathcal{C}_N^*(x) = \sup_{1 \leq M \leq N} \Big| \sum_{n=1}^{N} e^{2\pi i n^2 x} \Big|,$$

then lemmas 2 and 3 give us the estimate

$$\mu\left(\left\{ \mathcal{C}_N^*(x) \geq 2 N^{\frac{1}{2}} \alpha \right\} \right) \ll \frac{1}{\alpha^4}$$

and, in particular, we obtain

$$\left\| \mathcal{C}_N^* \right\|_p \leq C_p \, N^{\frac{1}{2}}, \qquad \text{for } 2 \leq p < 4.$$

Given the Fourier series $f = \sum a_n e^{2\pi i n^2 x}$ we have the Littlewood-Paley equivalence of norms

$$\|f\|_p \sim \Big\| \Big(\sum_k |\Delta_k(x)|^2 \Big)^{\frac{1}{2}} \Big\|_p, \quad 1 < p < \infty$$

where

$$\Delta_k(x) = \sum_{n=2^k}^{2^{k+1}-1} a_n e^{2\pi i n^2 x}.$$

In particular, for $2 \leq p < 4$ we have

$$\|f\|_p^p \ll \Big[\int_0^1 \Big(\sum_k |\Delta_k(x)|^2 \Big)^{\frac{p}{2}} \Big]^{\frac{2}{p} \cdot \frac{1}{2}}$$

$$\leq \Big[\sum_k \Big(\int_0^1 |\Delta_k|^p \Big)^{\frac{2}{p}} \Big]^{\frac{1}{2}} = \Big(\sum_k \|\Delta_k\|_p^2 \Big)^{\frac{1}{2}}.$$

But

$$\Delta_k(x) = \sum_{n=2^k}^{2^{k+1}-1} a_n e^{2\pi i n^2 x} = \sum_{n=2^k}^{2^{k+1}-1} a_n \big[S_n(x) - S_{n-1}(x)\big]$$

$$= \sum_{n=2^k}^{2^{k+1}-1} (a_n - a_{n+1})S_n(x) + a_{2^{k+1}-1}S_{2^{k+1}-1}(x) \,,$$

and since the coefficients are monotonically decreasing we get

$$\big|\Delta_k(x)\big| \le 2\mathcal{C}^*_{2^{k+1}}(x) \cdot a_{2^k} \,.$$

Therefore

$$\|f\|_p \ll \Big(\sum_k a_{2^k}^2 \cdot 2^k \Big)^{\frac{1}{2}} = \|f\|_2 \,. \qquad \blacksquare$$

III. The fractal dimension of Riemann's graphs

In his thesis Riemann introduces also several examples of continuous functions lacking derivatives in a dense set of points, a project culminated by Weierstrass who produced the first known continuous and nowhere differentiable function using lacunary trigonometric series. It seems that Weierstrass was inspired by the erroneous claim, attributed to Riemann, that the following two continuous functions are nowhere differentiable

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^2} \,, \qquad G(x) = \sum_{n=1}^{\infty} \frac{\cos(2\pi n^2 x)}{n^2} \,.$$

These functions have been studied by several authors and nowadays we have a rather complete knowledge of its points of differentiability. In the following we shall be concerned about the Minkowski or box counting dimension of their graphs.

Given $0 < \delta \le 1$ let us consider the family of continuous functions

$$F_\delta(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^{1+\delta}} \,, \qquad G_\delta(x) = \sum_{n=1}^{\infty} \frac{\cos(2\pi n^2 x)}{n^{1+\delta}} \,.$$

Theorem 4. $\dim_{\mathrm{B}}\big(\mathrm{graph}(F_\delta)\big) = \dim_{\mathrm{B}}\big(\mathrm{graph}(G_\delta)\big) = \frac{7}{4} - \frac{\delta}{2}$.

Proof. With a fixed (big) integer N let us denote $J_{a/q}$ be the arc containing a/q in the Farey disection of $(0,1]$ of order $N^{\frac{1}{2}}$, that is

$$(0,1] = \bigcup J_{a/q} \,, \qquad J_{a/q} = \big(\tfrac{a'+a}{q'+q}, \ \tfrac{a+a''}{q+q''}\big]$$

where $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ are three consecutive fractions in the Farey sequence $F_{N^{1/2}}$.

It happens that

$$J_{a/q} \subset \left\{ x \in (0,1] \ : \ \left| x - \tfrac{a}{q} \right| < \tfrac{1}{q\, N^{1/2}} \right\}.$$

Next we shall consider incomplete gaussians sums

$$S_{n,m}(x) = \sum_{k=m+1}^{n} e^{2\pi i k^2 x}, \qquad S_n(x) = S_{n,0}(x).$$

From the previous section we know the estimate

$$\left| S_{n,m}(x) \right| \ll \frac{n}{\sqrt{q}} + \sqrt{q}$$

so long as $\left| x - \tfrac{a}{q} \right| \le \tfrac{1}{q^2}$.

For the complete sums we have the explicit evaluation

$$\sum_{k=1}^{q} e^{2\pi i \frac{a}{q} k^2} = \varepsilon_q \sqrt{q} \left(\frac{a}{q} \right), \qquad \text{if } \mathrm{mcd}(a,q) = 1$$

where

$$\varepsilon_q = \begin{cases} 1 & \text{if } q \equiv 1 \mod (4) \\ i & \text{if } q \equiv 3 \mod (4) \\ 0 & \text{if } q \equiv 2 \mod (4) \\ 1 + i & \text{if } q \equiv 0 \mod (4). \end{cases}$$

Given $n' = m + q\left[\tfrac{n-m}{q} \right]$ we have

$$S_{n,m}\left(\frac{a}{q} \right) = S_{n'-1,m}\left(\frac{a}{q} \right) + S_{n,n'}\left(\frac{a}{q} \right) = \varepsilon_q\left(\frac{a}{q} \right) \frac{n-m}{\sqrt{q}} + O\left(\sqrt{q} \right).$$

Let $\alpha\left(\tfrac{a}{q} \, ; \, N \right)$ be the number of boxes of the mesh

$$\mathcal{M}_N = \left\{ x = \frac{\ell}{N} \, , \, y = \frac{m}{n} \, ; \, 0 \le \ell \le N \right\}$$

which are needed to cover the intersection $\mathrm{graph}(F_\delta) \cap (J_{a/q} \times \mathbb{R})$, and denote by $\alpha\left(\tfrac{a}{q} \, ; \, N \, , \, k \right)$ the number of them contained inside the strip $I_k \times \mathbb{R}$, where $I_k = \left[\tfrac{k}{N} \, , \, \tfrac{k+1}{N} \right]$.

i) We shall prove first the upper bound

$$\overline{\dim}_{\mathrm{B}}\left(\mathrm{graph}(F_\delta) \right) \le \frac{7}{4} - \frac{\delta}{2}.$$

Let M be a positive integer to be chosen later. From the definition of $\alpha\left(\frac{a}{q}\,;\,N\,,\,k\right)$ and after applying the mean value theorem we get

$$\alpha\left(\tfrac{a}{q}\,;\,N\,,\,k\right) \le 2 + N \sup_{x,y\in I_k\cap J_{a/q}} |F_\delta(x) - F_\delta(y)|$$

$$\le 2 + 2\pi\left| \sum_{n\le M} n^{1-\delta} e^{2\pi in^2\xi}\right| + N\left| \sum_{n\ge M} \frac{1}{n^{1+\delta}}\left(e^{2\pi in^2 x_0} - e^{2\pi in^2 y_0}\right)\right|$$

where ξ, x_0, $y_0 \in I_k\cap J_{a/q}$.

Partial summation yields

$$\alpha\left(\tfrac{a}{q}\,;\,N\,,\,k\right) \ll 2 + 2\pi\left|\sum_{n\le M}\left[n^{1-\delta} - (n+1)^{1-\delta}\right]S_n(\xi)\right|$$

$$+ \left|S_M(\xi)\right| M^{1-\delta} + (1+\delta)N\sum_{n\ge M}\frac{|S_{n,M}(x_0)| + |S_{n,M}(y_0)|}{n^{2+\delta}}\,.$$

Now we choose $M = \left[\sqrt{N}\right]$ and from the known estimates for gaussian sums we get

$$\alpha\left(\tfrac{a}{q}\,;\,N\,,\,k\right) \ll \left(\frac{M}{\sqrt{q}} + \sqrt{q}\right) + \left(\frac{M}{\sqrt{q}} + \sqrt{q}\right)M^{1-\delta}$$

$$+ N\left[\frac{1}{q^{1/2}}\cdot\frac{1}{N^{\frac{1}{2}(\frac{1}{2}+\delta)}}\right]$$

$$\ll N^{1-\frac{\delta}{2}}\frac{1}{\sqrt{q}} + N^{\frac{1}{2}-\frac{\delta}{2}}\sqrt{q}\,.$$

Next we observe that

$$\alpha\left(\tfrac{a}{q}\,;\,N\right) \ll \frac{|J_{a/q}|}{1/N}\left(N^{1-\frac{\delta}{2}}\frac{1}{\sqrt{q}} + N^{1-\frac{\delta}{2}}\sqrt{q}\right)$$

$$\ll \frac{N^{\frac{3}{2}-\frac{\delta}{2}}}{q^{3/2}} + N^{1-\frac{\delta}{2}}\frac{1}{q^{1/2}}\,.$$

Therefore the number of boxes of the mesh $\mathcal{M}_N$ needed to cover the graph is bounded by

$$\sum_{q\le N^{1/2}}\sum_{(a,q)=1}\left(\frac{N^{\frac{3}{2}-\frac{\delta}{2}}}{q^{3/2}} + N^{1-\frac{\delta}{2}}\frac{1}{q^{1/2}}\right)$$

$$\ll \sum_{q\le N^{1/2}}\left(\frac{N^{\frac{3}{2}-\frac{\delta}{2}}}{q^{1/2}} + N^{1-\frac{\delta}{2}}q^{1/2}\right) \ll N^{\frac{7}{4}-\frac{\delta}{2}}$$

implying the upper bound

$$\overline{\dim}_{\mathrm{B}}\big(\mathrm{graph}(F_\delta)\big) \leq \limsup_{N\to\infty} \frac{\log\left(N^{\frac{7}{4}-\frac{\delta}{2}}\right)}{\log N} = \frac{7}{4} - \frac{\delta}{2}\,.$$

Obviously the same proof also shows that

$$\overline{\dim}_{\mathrm{B}}\big(\mathrm{graph}(G_\delta)\big) \leq \frac{7}{4} - \frac{\delta}{2}\,.$$

ii) We shall prove now the lower bounds

$$\underline{\dim}_{\,\mathrm{B}}\big(\mathrm{graph}(F_\delta)\big) \geq \frac{7}{4} - \frac{\delta}{2}$$

$$\underline{\dim}_{\,\mathrm{B}}\big(\mathrm{graph}(G_\delta)\big) \geq \frac{7}{4} - \frac{\delta}{2}\,.$$

Here we need to separate both cases. For F_δ we choose P to be the set of prime numbers such that $p \equiv 3 \mod (4)$ and $p \asymp \sqrt{N}$ (i.e. $C_1\sqrt{N} \leq p \leq C_2\sqrt{N}$, where $0 < C_1 < 1$, $1 < C_2 < 2$ are fixed).

For G_δ we will proceed similarly but changing to the primes $p \equiv 1 \mod (4)$ for obvious reasons.

From the definition of Minkowski dimension we have

$$\underline{\dim}_{\,\mathrm{B}}\big(\mathrm{graph}(F_\delta)\big) \geq \liminf_{N\to\infty} \frac{\log\left(N\sum_{p\in P}\sum_{0\leq a\leq p}\left|F_\delta\big(\tfrac{a}{p}\big) - F_\delta\big(\tfrac{a}{p}+\tfrac{1}{p^2}\big)\right|\right)}{\log N}$$

Next we shall consider only the set $\mathcal{R}(p)$ of values of a which are quadratic residues $\mathrm{mod}(p)$. The corresponding sum over those a is greater than the absolute value of

$$\sum_{a\in\mathcal{R}(p)}\left(F_\delta\big(\tfrac{a}{p}\big) - F_\delta\big(\tfrac{a}{p}+\tfrac{1}{p^2}\big)\right) = \mathrm{Im}\left(\sum_n \sum_{r=1}^{\frac{p-1}{2}} \frac{e^{2\pi i r^2 \frac{n^2}{p^2}}}{n^{1+\delta}}\left(1 - e^{2\pi i \frac{n^2}{p^2}}\right)\right)$$

$$= \mathrm{Im}\left(\sum_n \frac{\left(-1+i\sqrt{p}\right)\left(1 - e^{2\pi i \frac{n^2}{p^2}}\right)}{2n^{1+\delta}}\right)$$

$$= \sqrt{p}\sum_n \frac{\sin^2\left(\pi\frac{n^2}{q^2}\right)}{n^{1+\delta}} + \sum_n \frac{\sin\left(\pi\frac{n^2}{q^2}\right)}{2n^{1+\delta}}$$

$$\gg p^{\frac{1}{2}-\delta}\,.$$

The prime number theorem in arithmetic progressions yields

$$\underline{\dim}_{\,\mathrm{B}}\big(\mathrm{graph}(F_\delta)\big) \geq \lim_{N\to\infty} \frac{\log\left(N N^{\frac{1}{4}-\frac{\delta}{2}} N^{\frac{1}{2}}/\log N^{1/2}\right)}{\log N} = \frac{7}{4} - \frac{\delta}{2}\,.$$

∎

IV. Convergence and divergence of Fourier series

For any function $f \in L^1[0,1]$ we can write its Fourier series

$$f(x) \sim \sum_{-\infty}^{+\infty} \widehat{f}(k)e^{2\pi ikx}, \qquad \widehat{f}(k) = \int_0^1 f(x)e^{-2\pi ikx}\, dx\,.$$

Then the traditional way of reconstructing f from its Fourier coefficients is to consider the partial sums

$$S_N f(x) = \sum_{|k| \le N} \widehat{f}(k)e^{2\pi ikx}\,.$$

It is well known (Marcel Riesz's theorem) that we have the norm convergence

$$\lim_{N \to \infty} S_N f = f \quad \text{in} \ \ L^p[0,1]\,,\ 1 < p < \infty\,.$$

Furthermore Lennart Carleson [2] proved that there is also convergence at almost every point

$$\lim_{N \to \infty} S_N f = f \quad \text{a.e. } x\,, \text{ for } f \in L^p[0,1]\,,\ p > 1\,.$$

However, in many applications it seems natural to pay more attention to the set of bigger Fourier coefficients, and to reconstruct the function ordering them in decreasing magnitude. The mathematical expression of this fact leads us to consider, for each $\lambda > 0$, the partial sum

$$\widetilde{S}_\lambda f(x) = \sum_{|\widehat{f}(k)| \ge \lambda} \widehat{f}(k)e^{2\pi ikx}$$

and their limit when $\lambda \to 0^+$.

In reference [9] T. Körner answered in the negative a question asked by L. Carleson and R. Coifman, proving the existence of a function $f \in L^2[0,1]$ such that

$$\limsup_{\lambda \to 0^+} \Big| \sum_{|\widehat{f}(k)| \ge \lambda} \widehat{f}(k)e^{2\pi ikx} \Big| = \infty\,, \text{ a.e. } x \in [0,1]\,.$$

Körner's proof is based on an ingenious modification of a construction due to Olevskii for the Haar system, and it also uses a probabilistic lemma of Salem and Zygmund. In the following we shall present several number theoretical arguments to analyze this type of convergence (see reference [5]).

Theorem 5. a) *Define the maximal operator*

$$\widetilde{S}^* f(x) = \sup_{\lambda>0} \left| \widetilde{S}_\lambda f(x) \right|.$$

Then for all $1 \le p < 2$, there is a function $f \in L^p[0,1]$ (explicitly constructed) such that

$$\left\| \widetilde{S}^* f \right\|_p = \infty.$$

b) *For each $p < 2$ there exists $f \in L^p[0,1]$ such that*

$$\limsup_{\lambda \to 0^+} \left\| \widetilde{S}_\lambda f \right\|_p = \infty.$$

The details of the proof are given in [5] but they rely on several number theoretical estimates. A typical one is the following.

Lemma 6. *Let*

$$P_N^*(x) = \max_{1 \le j \le N} \left| \sum_{\substack{p \, \text{prime} \\ N \le p \le N+j}} e^{2\pi i p x} \right|,$$

then

$$\left\| P_N^* \right\|_r \gg \frac{N^{\frac{3}{4} - \frac{1}{2r}}}{\left(\log N \right)^{e(r)}}, \qquad 1 < r \le 2, \quad e(r) = 1 + \frac{1}{r}.$$

Assuming the lemma and with a given $\alpha > 0$ let us introduce the functions

$$f_0(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} B_k^0(x), \qquad f(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} B_k(x)$$

where

$$B_k^0 = \sum_{n=2^k}^{2^{k+1}-1} e^{2\pi i n x}$$

$$B_k = \sum_{n=2^k}^{2^{k+1}-1} c_n^k e^{2\pi i n x}, \qquad c_n^k = \begin{cases} 1 + \frac{1}{n}, & \text{if } n \text{ is prime} \\ 1 - \frac{1}{2^k}, & \text{if } n \text{ is composite.} \end{cases}$$

Then we have

$$\|f\|_p \ll 1 + \|f_0\|_p \le 1 + 2^{-k\alpha} \sum_{k=0}^{\infty} \|B_k^0\|_p \ll 1 + \sum_{k=0}^{\infty} 2^{k\left(1 - \frac{1}{p} - \alpha\right)} < \infty$$

if $\alpha > 1 - \frac{1}{p}$.

Next let us define $a_{k,j} = \dfrac{1}{2^{k\alpha}}\left(1 + \dfrac{1}{j}\right)$ for every $k = 0, 1, 2, \ldots$, and j such that $2^k \le j < 2^{k+1}$. Then

$$\widetilde{S}_{a_{k,j}} f(x) = \sum_{|\nu| \le 2^k} \widehat{f}(\nu)e^{2\pi i \nu x} + \frac{1}{2^{k\alpha}} \sum_{\substack{p \text{ prime} \\ 2^k \le p \le j}} e^{2\pi i p x} + \frac{1}{2^{k\alpha}} \sum_{\substack{p \text{ prime} \\ 2^k \le p \le j}} \frac{1}{p} e^{2\pi i p x}.$$

Therefore

$$\sup_{\lambda > 0} \left|\widetilde{S}_\lambda f(x)\right| \ge \frac{1}{2^{k\alpha}} P^*_{2^k}(x) - \mathcal{C}^* f(x) - O(1)$$

where $\mathcal{C}^* f(x)$ designs the Carleson's maximal operator. In particular we obtain

$$\left\| \sup_\lambda \left|\widetilde{S}_\lambda f\right| \right\|_p \gg \frac{1}{2^{k\alpha}} \frac{2^{k\left(\frac{3}{4} - \frac{1}{2p} - \alpha\right)}}{k^{e(p)}} \longrightarrow \infty$$

if $1 - \frac{1}{p} < \alpha < \frac{3}{4} - \frac{1}{2p} \iff p < 2$.

There are variations of this construction which prove part b) of theorem 5. To finish let us sketch the proof of lemma 6.

First we consider the primes q in the interval $\sqrt{N} \le q < \sqrt{2N}$ and for each $1 \le a \le q - 1$ we have the arc

$$J_{a/q} = \left(\frac{a}{q} - \frac{1}{8q^2}, \frac{a}{q} + \frac{1}{8q^2}\right).$$

It is easy to see that the $J_{a/q}$ are disjoints. Let us introduce the set

$$E_N = \bigcup_{\substack{q \text{ prime} \\ \sqrt{N} \le q < \sqrt{2N}}} \bigcup_{a=1}^{q-1} J_{a/q}.$$

Then

$$\left\| P^*_N \right\|^r_r = \int_0^1 \left(P^*_N(x)\right)^r dx \ge \int_{E_N} \left(P^*_N(x)\right)^r dx$$

$$= \sum_{\substack{q \text{ prime} \\ \sqrt{N} \le q < \sqrt{2N}}} \sum_{a=1}^{q-1} J_{a/q} \cdot \int_{J_{a/q}} \left(P^*_N(x)\right)^r dx.$$

But we have seen in (II) that $|x - y| < \dfrac{1}{8q^2}$, $q \ge \sqrt{n}$ implies that

$P_N^*(x) \geq C P_N^*(y)$, for some universal positive constant C. Therefore

$$\|P_N^*\|_r^r \gg \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \sum_{a=1}^{q-1} \int_{J_{a/q}} \left| P_N^*\left(\tfrac{a}{q}\right) \right|^r dx$$

$$\gg \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \frac{1}{q^2} \sum_{a=1}^{q-1} \left| \sum_{\substack{p \text{ prime} \\ N \leq p < 2N}} e^{2\pi i p \frac{a}{q}} \right|^r$$

$$= \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \frac{1}{q^2} \sum_{a=1}^{q-1} \left| \sum_{s=1}^{q-1} \sum_{\substack{p \text{ prime} \\ N \leq p < 2N \\ p \equiv s(q)}} e^{2\pi i \frac{as}{q}} \right|^r$$

$$= \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \frac{1}{q^2} \sum_{a=1}^{q-1} \left| \sum_{s=1}^{q-1} e^{2\pi i \frac{as}{q}} \big(\pi(2N\,;\,q\,,\,s) - \pi(N\,;\,q\,,\,s) \big) \right|^r$$

where $\pi(x\,;\,\alpha\,,\,\beta)$, $\mathrm{mcd}(\alpha, \beta) = 1$, counts the number of primes less than or equal to x in the arithmetical progression

$$\beta\,,\,\beta + a\,,\,\beta + 2\alpha\,,\,\beta + 3\alpha\,,\,\ldots$$

Then writing $b_{N,q,s} = \pi(2N\,;\,q\,,\,s) - \pi(N\,;\,q\,,\,s)$ and since $1 \leq r \leq 2$, we get

$$\|P_N^*\|_r^r \gg \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \frac{1}{q^2} \left(\sum_{a=1}^{q-1} \left| \sum_{s=1}^{q-1} e^{2\pi i s \frac{a}{q}} b_{N,q,s} \right|^2 \right)^{r/2}$$

$$= \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \frac{1}{q^2} \left\{ (q-1) \sum_{s=1}^{q-1} b_{N,q,s}^2 + \sum_{s \neq s'} b_{N,q,s} b_{N,q,s'} \sum_{a=1}^{q-1} e^{2\pi i a \frac{s-s'}{q}} \right\}^{r/2}$$

$$= \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \frac{1}{q^2} \left\{ (q-1) \sum_{s=1}^{q-1} b_{N,q,s}^2 - \sum_{s \neq s'} b_{N,q,s} b_{N,q,s'} \right\}^{r/2}$$

$$\geq \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \frac{1}{q^2} \left\{ \sum_{s=1}^{q-1} b_{N,q,s}^2 \right\}^{r/2} \gg \frac{1}{N} \sum_{\substack{q \text{ prime} \\ \sqrt{N} \leq q\sqrt{2N}}} \left\{ \sum_{s=1}^{q-1} b_{N,q,s}^2 \right\}^{r/2}$$

where we have used the inequality

$$M \sum_{j=1}^{M} a_j^2 - \sum_{j \neq k} a_j a_k \geq \sum_{j=1}^{M} a_j^2 \,.$$

To finish let us observe that Cauchy's inequality together with the prime number theorem yields

$$(q-1)\sum_{s=1}^{q-1} b_{N,q,s}^2 \geq \Big(\sum_{s=1}^{q-1} b_{N,q,s}\Big)^2 \gg \frac{N^2}{\log^2 N}$$

and, therefore

$$\big\|P_N^*\big\|_r^r \gg \frac{1}{N}\sum_{\substack{q \text{ prime}\\ \sqrt{N}\leq q\sqrt{2N}}} \Big(\frac{N^2}{(q-1)\log^2 N}\Big)^{r/2}$$

$$\gg \frac{N^r}{N(\log N)^r N^{r/4}}\frac{N^{1/2}}{\log N} \gg \frac{N^{\frac{3r}{4}-\frac{1}{2}}}{(\log N)^{1+r}}.$$

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