Drops: The collapse of capillary jets

Antonio Cordoba*, Diego Cordoba†, Charles Fefferman‡§, and Marco A. Fontelos¶

*Departamento de Matemáticas, Universidad Autonoma de Madrid, Carretera de Colmenar Viejo, Kilometro 15, 28049 Madrid, Spain; †Consejo Superior de Investigaciones Cientificas, c/Serrano 123, 28006 Madrid, Spain; ‡Department of Mathematics, Princeton University, Washington Road, Princeton, NJ 08540-1000; and ¶Departamento de Matemática Aplicada, Universidad Rey Juan Carlos I, c/Tulipan s/n, 28933 Mostoles-Madrid, Spain

Contributed by Charles Fefferman, May 28, 2002

The appearance of fluid filaments during the evolution of a viscous fluid jet is a commonly observed phenomenon. It is shown here that the break-up of such a jet subject to capillary forces is impossible through the collapse of a uniform filament.

Model

The formation of thinning filaments is commonly observed previously to the break-up of a viscous jet.

In "Traité du movement des eaux et des autres corps fluides," in 1686, E. Mariotte already attacked the problem of drop formation. Later on, in 1833, F. Savart performed experiments to estimate the size of drops resulting from the breaking-up of a jet (see ref. 1). In 1879 Lord Rayleigh presented the first analytical study of that problem (see ref. 2). He showed the instability of the stationary-jet solutions to the Navier–Stokes equations, explaining, at least partially, Savart's observations.

With the more sophisticated experimental and numerical tools now available, the subject has recently gained considerable momentum. This is in part due to the technological importance of controlling the drops generation mechanism, which is relevant, for example, to the modern ink-jet printing systems. Let us refer to ref. 3 for an updated presentation of the state of this art.

In high viscosity fluids, the break-up is preceded by the formation of long filaments, which, in experiments, are thin uniformly up to a diameter of the order of a micron (cf. ref. 4 where experimental data were collected by using a high-resolution charge-coupled device sensor). Sometimes they generate new and smaller drops, but often they become unstable and break (cf. ref. 5). At this small scale it is possible that molecular forces, which are not considered in a continuum description, come into play, but it is important to know whether the continuum equations predict break-up or not. In particular, it is important to rule out the possibility of collapse of a whole filament, so that the pinching has to occur at "isolated" points. In the process of understanding this phenomenon, M. A. Fontelos has proved the formation of filaments for very viscous fluids under the slender jet approximation (unpublished work). In connection to this, let us mention the existence of a self-similar mechanism for break-up at isolated points suggested by Eggers (cf. ref. 6).

In ref. 7 it was shown that under the mild assumption

$$\int_0^T \sup_x |u(x,t)|dt < \infty,$$

the volume of a "regular tube" moving with the flow cannot reach zero thickness at time T. In particular, the thickness of the neck of a drop cannot reach zero thickness in finite time, unless it bends and twists so violently that no part of it forms a regular tube.

Estimates

In this article we shall consider the case of a fluid of viscosity μ_1, density ρ_1, and velocity field v_1, occupying at time t a region $\Omega(t)$, and being surrounded by another fluid whose respective parameters are μ_2 and ρ_2 and whose velocity is v_2. We shall assume a smooth external force (such as gravity), which together with the fundamental laws (Navier–Stokes equations) and an appropriate interphase condition, involving surface tension forces, will provide us the mathematical model.

In Eulerian coordinates we shall consider our domain $\Omega(t)$ to be a tube whose lateral wall is the interphase. The fluid will be "entering" throughout the two "cups" $D_1(t)$, $D_2(t)$, while $\Gamma_t = \partial\Omega(t) - (D_1(t) \cup D_2(t))$ is moving with the fluid.

The tube $\Omega(t)$ is uniformly collapsing to a smooth curve γ at time T, if there exists a function $h(t)$ such that:

$$i)\ \lim_{t\to T} h(t) = 0$$

$$ii)\ \frac{1}{C}h(t) \le dist(x,\gamma) \le Ch(t) \text{ for each } x \in \Gamma_t.$$

We may state our main result:

Theorem 1. *Under the conditions described above the uniform collapse of a fluid tube (filament) is impossible. Moreover, the volume V(t) of fluid enclosed by the tube satisfies*:

$$V(t) \ge Ce^{-Ct^2}$$

for some positive constant C.

The proof starts with the Navier–Stokes equations for each one of our two incompressible fluids. Then one has to use the boundary conditions, implying that there is a continuity of the velocity field at $\partial\Omega(t)$ together with the following pressure equilibrium:

$$[T_{ij}^{(1)} - T_{ij}^{(2)}]n_j = \sigma H n_i \text{ in } \partial\Omega(t),$$

with

$$T_{ij}^{(k)} = \left[-p\delta_{ij} + \mu_k\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]$$

$k = 1, 2$, on $\partial\Omega(t)$. Here n is the field of normal vectors to $\Omega(t)$, H is the mean curvature of $\partial\Omega(t)$, and $\sigma > 0$ is the surface tension coefficient.

Then one obtains the fundamental estimate:

$$\min_i(\rho_i)\int_{\mathbb{R}^3}\frac{1}{2}|\vec{v}|^2 dV + \min_i(\mu_i)\int_0^t\int_{\mathbb{R}^3}|\partial_{x_i}v_j + \partial_{x_j}v_i|^2 dV\,dt$$
$$+ \sigma|\partial\Omega(t)| \le C.$$

where

$$v = \begin{cases} v_1 & \text{in } \Omega(t) \\ v_2 & \text{in } R^3 - \Omega(t). \end{cases}$$

§To whom reprint requests should be addressed. E-mail: cf@math.princeton.edu.

www.pnas.org/cgi/doi/10.1073/pnas.152318399

From that formula it follows the differential inequality:

$$\frac{dV(t)}{dt} \geq -CV(t)|\ln V(t)|^{1/2},$$

which allows us to conclude the estimate:

$$V(t) \geq Ce^{-Ct^2}$$

for some positive constant C.

A.C. was partially supported by Ministerio de Ciencia y Tecnologa Grant PB 93-0281. C.F. was partially supported by National Science Foundation Grant DMS 0070692.

1. Savart, F. (1833) *Ann. Chim. Phys.* **53,** 337–386.
2. Rayleigh, W. (Strutt, J. W.) (1879) *Proc. London Math. Soc.* **10,** 4–13.
3. Eggers, J. (1997) *Rev. Mod. Phys.* **60,** 865–929.
4. Kowalewski, T. A. (1996) *Fluid Dyn. Res.* **17,** 121–145.
5. Brenner, M. P., Nagel, S. R. & Shi, X. D. (1994) *Science* **265,** 219–222.
6. Eggers, J. (1993) *Phys. Rev. Lett.* **71,** 3458.
7. Cordoba, D. & Fefferman, C. (2001) *Comm. Math. Phys.* **222,** 293–298.