

A path integral approach to lattice point problems

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Abstract

The main problem in (planar) lattice point theory consists in counting lattice points under the graph of positive functions supported on $[0, M]$ and with radius of curvature comparable to M . We prove that, in some sense motivated by Feynman path integral formulation of Quantum Mechanics, for “most” functions the lattice error term in the area approximation is $O(M^{1/2+\varepsilon})$. This complements Jarník construction of curves with an optimal $O(M^{2/3})$ error term.

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1. Introduction

Standard (two dimensional) lattice point problems reduce to counting lattice points under positive graphs $y = g(x)$ with

$$g \in \mathcal{C}_N^2 = \left\{ f \in C^2([0, N+1]): 0 < \frac{C_1}{N} < -f'' < \frac{C_2}{N}, f(0) = f(N+1) = 0 \right\},$$

where C_1, C_2 are constants and $N \in \mathbb{Z}^+$. Well known arguments allow to obtain asymptotic formulas in which the lattice error term is given by the difference between the area under g and the number of lattice points (lattice points on the real axis count one half). Variations on an old argument due to Jarník [7] prove that the bound $O(N^{2/3})$ is best possible (see [6, Section 1]). After a century of advances in the method of exponential sums this bound has been improved by Huxley [6] to $O(N^{46/73+\varepsilon})$ under stronger regularity conditions over g , namely three continuous derivatives. But this is still far from Hardy’s conjecture

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$O(N^{1/2+\varepsilon})$ for the case of a circle [5]. There is some evidence of an extended conjecture for closed convex curves supported by average results over centers and radii (see [1,8]). However, this latter kind of results are not completely satisfactory because they can be understood as an average over the lattice rather than a real arbitrary variation of the curve. With this idea in mind and taking Feynman path integral formulation of Quantum Mechanics [3] as our main motivation, we want to prove in this paper that the integral of the squared lattice error term over all arcs $g \in \mathcal{C}_N^2$ agrees with Hardy's conjecture and hence that for "most of the arcs" belonging to \mathcal{C}_N^2 the optimal $O(N^{1/2+\varepsilon})$ error bound holds without further regularity conditions.

It is not the first time that the number theoretical problem of counting lattice points in convex region has been related to Quantum Mechanics. At the contrary, that connection appeared already in the pioneering work of H. Weyl: considering a potential V in Euclidean space \mathbb{R}^n , then the number of bounded states of $-\Delta + V$ (sometimes associated to lattice points) can be estimated in a semiclassical approximation, by the volume $|\{(\xi, x): |\xi|^2 + V(x) < \infty\}|$ in phase space. More references can be found in [2]. On the other hand, Ya.G. Sinai, has also considered random lattice point problems in connection with Physics.

Summarizing the ideas exposed below, we want to give a sense to the integral $\int |\Delta(f)|^2 d\mu(f)$ where $d\mu$ is a normalized measure on \mathcal{C}_N^2 and $\Delta(f)$ is the lattice error function which is given by:

$$\Delta(f) = \int_0^{N+1} f(t) dt - \sum_{i=0}^{N+1} \left([f(i)] + \frac{1}{2} \right),$$

where $[\cdot]$ denotes the integral part.

Euler–Mac Laurin summation formula or truncation error formula for trapezoidal rule yields

$$\int_0^{N+1} f(t) dt = \sum_{i=0}^{N+1} f(i) + O(1).$$

Hence we infer that any reasonable definition of $\int |\Delta(f)|^2 d\mu(f)$ should coincide, up to a negligible $O(1)$ term, with

$$\int |\Delta(x_1, x_2, \dots, x_N)|^2 d\mu_N(f),$$

where $x_i = f(i)$,

$$\Delta(x_1, x_2, \dots, x_N) = \sum_{i=1}^N \psi(x_i) \quad \text{with } \psi(x) = x - [x] - 1/2$$

and $d\mu_N$ factors into N measures supported on the vertical slices $x = x_1, x = x_2, \dots, x = x_N$.

In this discrete setting we can say that \mathcal{C}_N^2 transforms into

$$\left\{ (x_0, x_1, \dots, x_{N+1}) \in \mathbb{R}^{N+2}; 0 < \frac{C_1}{N} < -f_i^{(2)} < \frac{C_2}{N}, x_0 = x_{N+1} = 0 \right\},$$

where $f_i^{(2)}$ is the discrete second derivative at i :

$$f_i^{(2)} = \frac{(x_{i+1} - x_i) - (x_i - x_{i-1})}{2}.$$

Then, the most natural way to define the measure $d\mu_N(f)$ is penalizing large or small values of the (discrete) second derivative. Hence we consider:

$$d\mu_N(f) = d\mu_N(x_1, \dots, x_N) = K_N \prod_{i=1}^N \phi \left(\frac{(x_i - x_{i-1}) - (x_{i+1} - x_i)}{2} N \right) dx_i, \quad (1)$$

where $\phi \in C_0^\infty((C_1, C_2))$, $\phi > 0$ and K_N is a normalizing constant.

Definition. We shall call *all arcs variance* to the value of the integral:

$$\mathcal{V} = \int_{\mathbb{R}^N} |\Delta(x_1, x_2, \dots, x_N)|^2 d\mu_N(x_1, x_2, \dots, x_N),$$

where $d\mu_N$ is as in (1) with $\int \phi = 1$ and $K_N = (N+1)(N/2)^N$.

Remark. We shall see later that with this value of K_N the measure is actually normalized, i.e., $\int d\mu_N = 1$.

Now we state our main result.

Theorem 1. *Let \mathcal{V} be the all arcs variance, then for every $\varepsilon > 0$ it holds*

$$\mathcal{V} = O(N^{1+\varepsilon}).$$

In particular, Hardy's conjecture $|\Delta| = O(N^{1/2+\varepsilon})$ holds except in a set of μ_N -vanishing measure.

The scheme of the proof is as follows: We shall firstly smooth out Δ to get an analytic approximation in terms of some oscillatory sums. Secondly we shall perform the integration over \mathbb{R}^N to express all arcs variance as an exponential sum with coefficients. Finally we shall estimate these coefficients and the corresponding exponential sum.

Apart from the special notation already introduced, henceforth with ε we shall mean a small enough positive constant (non-necessarily always the same) and the O -constants usually will depend on ε and collapse when $\varepsilon \rightarrow 0^+$. We shall also use extensively the

abbreviation $e(t)$ for $e^{2\pi i t}$. Finally, we shall employ convolution and Fourier transform with the standard normalization:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt, \quad \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e(-t\xi) dt.$$

2. Smoothing process

Firstly we shall smooth the function $\psi(x)$. This can be done in several ways but it will be convenient here to choose one leading to a finite Fourier series.

Lemma 1. *There exists $\sigma \in C_0^\infty((-1, 1))$ such that the functions*

$$S_\pm(y) = - \sum_{h=1}^{\infty} \frac{\sin(2\pi h(y \pm \delta))}{\pi h} \sigma\left(\frac{h}{N^{1/2}}\right) \quad \text{with } \delta = N^{-1/2+\varepsilon},$$

satisfy

$$S_+(y) + O(\delta) \leq \psi(y) \leq S_-(y) + O(\delta).$$

Proof. By the periodicity we can assume $y > 2\delta$. Let $s \in C_0^\infty((-1/2, 1/2))$ be any real valued non-negative even function such that $\|s\|_2 = 1$ then we take $\sigma = s * s$. It is easy to prove that σ is even, compactly supported inside $(-1, 1)$ and satisfies:

$$\sigma, \widehat{\sigma} \geq 0, \quad \widehat{\sigma}(x) = O((1 + |x|)^{-1/(2\varepsilon)}), \quad \sigma(0) = \int \widehat{\sigma}(x) dx = 1.$$

Consider the convolutions, say p_y^\pm , of $N^{1/2}\widehat{\sigma}(N^{1/2}x)$ and the characteristic function of the interval $[-y \mp \delta, y \pm \delta]$, then we have:

$$p_y^\pm(x) = \int_{I_\pm} \widehat{\sigma}(t) dt \quad \text{with } I_\pm = [(x-y)N^{1/2} \mp N^\varepsilon, (x+y)N^{1/2} \pm N^\varepsilon].$$

The properties of $\widehat{\sigma}$ imply that $0 \leq p_y^\pm(x) \leq 1$ and

$$\begin{aligned} p_y^-(x) &= O(\delta(1 + |x| - y)^{-2}) \quad \text{for } |x| \geq y, \\ p_y^+(x) &= 1 + O(\delta(1 + y - |x|)^{-2}) \quad \text{for } |x| \leq y. \end{aligned}$$

Hence

$$\sum_{h=-\infty}^{\infty} p_y^-(h) + O(\delta) \leq 2[y] + 1 \leq \sum_{h=-\infty}^{\infty} p_y^+(h) + O(\delta).$$

We apply Poisson's summation formula to obtain:

$$y - \frac{1}{2} \sum_{h=-\infty}^{\infty} \widehat{p_y^+}(h) + O(\delta) \leq \psi(y) \leq y - \frac{1}{2} \sum_{h=-\infty}^{\infty} \widehat{p_y^-}(h) + O(\delta). \quad (2)$$

Then the definition of p_y^\pm and the properties of the convolution and Fourier transform allow us to deduce

$$\widehat{p_y^\pm}(\xi) = \frac{\sin(2\pi(y \pm \delta)\xi)}{\pi\xi} \sigma\left(\frac{\xi}{N^{1/2}}\right) \quad \text{for } \xi \neq 0$$

and $\widehat{p_y^\pm}(0) = 2y \pm 2\delta$. Substituting in (2) the results follows. \square

A consequence of the previous result is the following:

Lemma 2. Let $\Delta = \Delta(x_1, \dots, x_N)$ be as in the introduction, then

$$|\Delta|^2 \leq |\Delta_+|^2 + |\Delta_-|^2 + O(N^{1+\varepsilon}),$$

where

$$\Delta_\pm = \sum_{h=1}^{\infty} \sum_{m=1}^N \frac{e(hx_m \pm h\delta)}{h} \sigma\left(\frac{h}{N^{1/2}}\right) \quad \text{with } \delta = N^{-1/2+\varepsilon}.$$

Proof. From Lemma 1 we obtain the estimate:

$$\sum_{m=1}^N S_+(x_m) + O(N\delta) \leq \Delta \leq \sum_{m=1}^N S_-(x_m) + O(N\delta).$$

Therefore, taking absolute values and since $\sin(2\pi t) = \text{Im } e(t)$ we obtain:

$$|\Delta| \leq \frac{1}{\pi} |\Delta_+| + \frac{1}{\pi} |\Delta_-| + O(N^{1/2+\varepsilon})$$

and the proof follows by convexity. \square

3. The exponential sum

Our next step in the proof of Theorem 1 is to estimate all arcs variance by an exponential sum. In this process we shall need the following technical result.

Lemma 3. Let J be the $N \times N$ Jacobi matrix:

$$J_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1/2 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and let (b_{ij}) its inverse matrix J^{-1} , then

$$(a) \quad \det J = \frac{N+1}{2^N}.$$

$$(b) \quad b_{lk} - b_{mk} = \begin{cases} 2k(m-l)/(N+1) & \text{if } k \leq l \leq m, \\ 2l - 2k(1 - (m-l)/(N+1)) & \text{if } l \leq k \leq m, \\ 2(m-l)(1 - k/(N+1)) & \text{if } l \leq m \leq k. \end{cases}$$

Proof. An elementary argument proves that the value of the determinant as a function of N , say d_N , satisfies the recursive formula:

$$d_{N+2} = d_{N+1} - \frac{1}{4}d_N \quad \text{with } d_1 = 1, \quad d_2 = \frac{3}{4},$$

and (a) follows by induction.

On the other hand, (b) is a direct consequence of

$$b_{ij} = \frac{2 \min(i, j)(N+1 - \max(i, j))}{N+1}.$$

The proof of this formula reduces to check $\sum_j J_{ij} b_{jk} = \delta_{ik}$. We can write this as

$$-\frac{1}{2}b_{i-1k} + b_{ik} - \frac{1}{2}b_{i+1k} = \delta_{ik}, \quad (3)$$

for $1 \leq i, k \leq N$ (according with the formula for b_{ij} we define $b_{0k} = b_{N+1k} = 0$). For $k = i$, we have:

$$-\frac{(i-1)(N+1-i)}{N+1} + \frac{2i(N+1-i)}{N+1} - \frac{i(N+1-i-1)}{N+1} = 1.$$

And for $k < i$,

$$-\frac{k(N+1-i+1)}{N+1} + \frac{2k(N+1-i)}{N+1} - \frac{k(N+1-i-1)}{N+1} = 0.$$

The case $k > i$ follows by the symmetry. Hence (3) holds. \square

Remark. Note that (a) implies that $\int_{\mathbb{R}^N} d\mu_N = 1$ (use the change of variables $\vec{y} = J\vec{x}$ and $\int \phi = 1$), i.e., the measure $d\mu_N$ is normalized.

Remark. From the point of view of Numerical Analysis the matrix J appears in approximating the second derivative in finite differences schemes, and its Jordan canonical form is a known object (see [4]). The formula for its eigenvalues (which are related to the stability of some algorithms) can be used to prove (a) and the whole canonical form to prove (b), but in both cases some auxiliary trigonometric identities are needed. We are indebted to Blanca Ayuso for supplying us with some references in this connection.

The function ϕ appearing in $d\mu_N$ is “off-centered” which causes an oscillation on its Fourier transform and it will be convenient to separate the non-oscillatory part. In order to do that, let us define:

$$\rho(\xi) = e(\alpha\xi)\widehat{\phi}(\xi) \quad \text{with } \alpha = \int x\phi(x) dx.$$

Then $\phi \geq 0$ and $\int \phi = 1$ can be easily used to show that $\rho(0) = 1$, $\rho'(0) = 0$, $\rho''(0) < 0$ and $|\rho(\xi)| \leq 1$. Hence there exists a positive constant $0 < C_0 < 1$ such that

$$|\rho(x)| \leq \max(1 - C_0x^2, 1 - C_0).$$

It turns out that all arcs variance is controlled by an exponential sum whose coefficients are related to ρ . They are precisely

$$a_{lm}(h) = \prod_{k=1}^N \rho\left(\frac{h}{N}(b_{lk} - b_{mk})\right).$$

The explicit form of the exponential sum and its relation with our problem is the content of the following result.

Proposition 1. *Let $S(M, h)$ be the exponential sum,*

$$S(M, h) = \sum_{M \leq l < m < 2M} a_{lm}(h) e\left(\frac{\alpha h}{N}(l - m)(m + l - N - 1)\right)$$

then

$$\mathcal{V} = O\left(N^{1+\varepsilon} + N^\varepsilon \sup_{h < N^{1/2}} \sup_{2M < N+1} |S(M, h)|\right).$$

Proof. By Lemma 2, we have

$$\mathcal{V} \leq \int_{\mathbb{R}^N} |\Delta_+|^2 d\mu_N + \int_{\mathbb{R}^N} |\Delta_-|^2 d\mu_N + O(N^{1+\varepsilon}). \quad (4)$$

Dyadic subdivision on h and m (note that there are $O(N^\varepsilon)$ dyadic blocks) and Cauchy's inequality give:

$$\int_{\mathbb{R}^N} |\Delta_+|^2 d\mu_N = O\left(N^\varepsilon \sup_{h < N^{1/2}} \sup_{2M < N+1} \int_{\mathbb{R}^N} \left| \sum_{M \leq m < 2M} e(hx_m) \right|^2 d\mu_N\right).$$

The same bound applies to $\int |\Delta_-|^2 d\mu_N$ and substituting in (4) we obtain:

$$\mathcal{V} = O\left(N^{1+\varepsilon} + N^\varepsilon \sup_{h < N^{1/2}} \sup_{2M < N+1} \left| \sum_{M \leq l < m < 2M} \int_{\mathbb{R}^N} e(h(x_l - x_m)) d\mu_N \right| \right). \quad (5)$$

Using the definition of $d\mu_N$, the change of variables $\vec{y} = J\vec{x}$ and Lemma 3(a), we obtain:

$$\begin{aligned} \int_{\mathbb{R}^N} e(h(x_l - x_m)) d\mu_N &= K_N \int_{\mathbb{R}^N} e(h(x_l - x_m)) \prod_{k=1}^N \phi((J\vec{x})_k N) dx_k \\ &= \prod_{k=1}^N \widehat{\phi}\left(\frac{h}{N}(b_{lk} - b_{mk})\right). \end{aligned} \quad (6)$$

By Lemma 3(b),

$$\sum_{k=1}^N (b_{lk} - b_{mk}) = (m - l)(m + l - N - 1)$$

and the result follows from (5) and (6). \square

4. End of the proof

According to the last proposition the bound $S(M, h) = O(N^{1+\varepsilon})$ suffices to complete the proof of our main result.

Proposition 2. For $h \leq N^{1/2}$ and $2M \leq N + 1$, it holds that

$$S(M, h) = O(N^{1+\varepsilon}).$$

Proof. Since $|\rho| \leq 1$ and $l < m$ we have:

$$\begin{aligned} |a_{lm}(h)| &= \prod_{k=1}^N \left| \rho\left(\frac{h}{N}(b_{lk} - b_{mk})\right) \right| \\ &\leq \prod_{1 \leq k \leq l} \left| \rho\left(\frac{h}{N}(b_{lk} - b_{mk})\right) \right| \cdot \prod_{m \leq k \leq N} \left| \rho\left(\frac{h}{N}(b_{lk} - b_{mk})\right) \right|. \end{aligned}$$

Note that the right-hand side of the inequality

$$|\rho(x)| \leq \max(1 - C_0 x^2, 1 - C_0)$$

is non-increasing for $x > 0$. Hence if $m - l > CN^{1/2+\varepsilon}h^{-1}$, with a small enough constant C , the first product is bounded by:

$$\prod_{1 \leq k \leq l} \max\left(1 - C_0 \frac{4k^2 h^2 (m-l)^2}{N^2 (N+1)^2}, 1 - C_0\right) \leq \prod_{1 \leq k \leq l} \left(1 - C'_0 \frac{k^2}{(N+1)^{3-\varepsilon}}\right)$$

and the second product is bounded by

$$\begin{aligned} & \prod_{m \leq k \leq N} \max\left(1 - 4C_0 \frac{h^2 (m-l)^2}{N^2} \left(1 - \frac{k}{N+1}\right)^2, 1 - C_0\right) \\ & \leq \prod_{m \leq k \leq N} \left(1 - C'_0 \frac{(N+1-k)^2}{(N+1)^{3-\varepsilon}}\right). \end{aligned}$$

Noting that l and m belong to the same dyadic interval we deduce from both bounds that $|a_{lm}(h)|$ has an exponential decay with N when $m - l > CN^{1/2+\varepsilon}h^{-1}$. On the other hand, the terms with $0 < m - l < N^\varepsilon$ contribute trivially $O(N^{1+\varepsilon})$. Therefore we have:

$$S(M, h) = S^*(M, h) + O(N^{1+\varepsilon}), \quad (7)$$

where $S^*(M, h)$ is the sum $S(M, h)$ but restricting the range of summation to $N^\varepsilon \leq m - l \leq CN^{1/2+\varepsilon}h^{-1}$.

Assume that $a_{lm}(h) \neq 0$ and that $\frac{h}{N}(b_{lk} - b_{mk})$ is small enough, say $O(N^{-\varepsilon/4})$, for $l < k \leq m$. The definition of $a_{lm}(h)$ implies that

$$1 + \frac{a_{l+1m+1}(h) - a_{lm}(h)}{a_{lm}(h)} = \frac{a_{l+1m+1}(h)}{a_{lm}(h)} = \prod_{k=l+1}^m \frac{\rho(\frac{h}{N}(b_{l+1k} - b_{m+1k}))}{\rho(\frac{h}{N}(b_{lk} - b_{mk}))}.$$

As $\rho'(0) = 0$ and ρ'' is bounded, for $x = b_{lk} - b_{mk}$, $l < k \leq m$, Taylor expansion gives:

$$\frac{\rho(\frac{h}{N}(x+2))}{\rho(\frac{h}{N}x)} - 1 = \frac{\rho(\frac{h}{N}(x+2)) - \rho(\frac{h}{N}x) - \frac{2h}{N}\rho'(0)}{\rho(\frac{h}{N}x)} = O\left(\frac{h^2}{N^2}x\right).$$

Note that $x + 2 = b_{l+1k} - b_{m+1k}$ and $|x| \leq 2(m-l)$. Hence

$$\log\left(1 + \frac{a_{l+1m+1}(h) - a_{lm}(h)}{a_{lm}(h)}\right) = O\left(\sum_{k=l+1}^m \frac{h^2}{N^2}(m-l)\right) = O(N^{-1+\varepsilon}),$$

and $|a_{lm}(h)| \leq 1$ implies

$$|a_{l+1m+1}(h) - a_{lm}(h)| = O(N^{-1+\varepsilon}).$$

This formula is valid in general, because if $a_{lm}(h) = 0$ then $a_{l+1m+1}(h) = 0$ except when $\rho(\frac{h}{N}(b_{lk} - b_{mk}))$ vanishes for some $l < k \leq m$. But in this case Taylor expansion proves that $\frac{h}{N}(b_{lk} - b_{mk})$ is not $O(N^{-\varepsilon/4})$ and then $a_{lm}(h)$ and $a_{l+1m+1}(h)$ are exponentially small (in their definitions are at least N^ε terms greater than $1 - CN^{-\varepsilon/2}$).

Therefore partial summation in $S^*(M, h)$ with the new variables $r = m - l, s = m$ gives:

$$S^*(M, h) = O\left(N^\varepsilon \sum_{N^\varepsilon \leq r \leq CN^{1/2+\varepsilon}h^{-1}} \max_{M \leq M' < 2M} \left| \sum_{M' \leq s < 2M} e\left(\frac{2\alpha h}{N}rs\right) \right| \right).$$

The innermost sum is bounded by $2 \min(M, \|2\alpha hrN^{-1}\|^{-1})$ where $\|\cdot\|$ is the distance to the nearest integer. A final substitution in (7) allows us to conclude:

$$S(M, h) = O(N^{1+\varepsilon})$$

which is the desired result. \square

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