AN EXPLICIT CONSTRUCTION OF THE UNIVERSAL DIVISION RING OF FRACTIONS OF $E\langle\langle x_1,\dots,x_d\rangle\rangle$

ANDREI JAIKIN-ZAPIRAIN

Abstract. We give a sufficient and necessary condition for a regular Sylvester matrix rank function on a ring R to be equal to its inner rank ρ_R. We apply it in two different contexts.

In our first application, we reprove a recent result of T. Mai, R. Speicher and S. Yin: if $X_1,\dots,X_d$ are operators in a finite von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ, then they generate the free division ring on $X_1,\dots,X_d$ in the algebra of unbounded operators affiliated to $\mathcal{M}$ if and only if the space of tuples $(T_1,\dots,T_d)$ of finite rank operators on $L^2(\mathcal{M},\tau)$ satisfying

$$\sum_{i=1}^{d}[T_k,X_k]=0,$$

is trivial.

In our second and main application we construct explicitly the universal division ring of fractions of $E\langle\langle x_1,\dots,x_n\rangle\rangle$, where E is a division ring, and we use it in order to show the following instance of pro-p Lück approximation.

Let F be a finitely generated free pro p-group, $F=F_1>F_2>\dots$ a chain of normal open subgroups of F with trivial intersection and A a matrix over $\mathbb{F}_p[[F]]$. Denote by A_i the matrix over $\mathbb{F}_p[F/F_i]$ obtained from the matrix A by applying the natural homomorphism $\mathbb{F}_p[[F]]\to\mathbb{F}_p[F/F_i]$. Then there exists the limit $\lim\limits_{i\to\infty}\dfrac{\mathrm{rk}_{\mathbb{F}_p}(A_i)}{|F:F_i|}$ and it is equal to the inner rank $\rho_{\mathbb{F}_p[[F]]}(A)$ of the matrix A.

1. Introduction

1.1. Sylvester matrix rank functions and Sylvester domains.

A **Sylvester matrix rank function** rk on a ring R is a function that assigns a non-negative real number to each matrix over R and satisfies the following conditions.

(SMat1) $\mathrm{rk}(A)=0$ if A is any zero matrix and $\mathrm{rk}(1)=1$;

(SMat2) $\mathrm{rk}(A_1A_2)\le\min\{\mathrm{rk}(A_1),\mathrm{rk}(A_2)\}$ for any matrices A_1 and A_2 which can be multiplied;

(SMat3) $\mathrm{rk}\begin{pmatrix}A_1&0\\0&A_2\end{pmatrix}=\mathrm{rk}(A_1)+\mathrm{rk}(A_2)$ for any matrices A_1 and A_2;

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(SMat4) $\operatorname{rk}\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \geq \operatorname{rk}(A_1) + \operatorname{rk}(A_2)$ for any matrices A_1, A_2 and A_3 of appropriate sizes.

We denote by $\mathbb{P}(R)$ the set of Sylvester matrix rank functions on R. We call $\operatorname{rk} \in \mathbb{P}(R)$ **faithfull** if $\operatorname{rk}(A) = 0$ only if A is a zero matrix.

A prototype of a Sylvester matrix rank function comes from **division R-rings**, that is the homomorphisms $\phi : R \to \mathcal{D}$, where $\mathcal{D}$ is a division ring. If we denote by $\operatorname{rk}_{\mathcal{D}}$ the $\mathcal{D}$-rank of matrices over $\mathcal{D}$, then the function rk, defined as

$$\operatorname{rk}(A) = \operatorname{rk}_{\mathcal{D}}(\phi(A)),$$

where A is a matrix over R, is an example of a Sylvester matrix rank function on R. The set of Sylvester matrix rank functions constructed in this way will be denoted by $\mathbb{P}_{div}(R)$.

A ring $\mathcal{U}$ is called **von Neumann regular** if for every $a \in \mathcal{U}$ there exists $b \in \mathcal{U}$ such that $aba = a$. The Sylvester matrix rank function on von Neumann regular rings are studied in detail in [6], where they are called pseudo-rank functions. A Sylvester matrix rank function rk on a ring R is called **regular** if there are a homomorphism $\phi : R \to \mathcal{U}$ from R to a von Neumann regular ring $\mathcal{U}$ and a faithful Sylvester matrix rank function rk' on $\mathcal{U}$ such that $\operatorname{rk} = \operatorname{rk}' \circ \phi$. In this case we will call the R-ring $\phi : R \to \mathcal{U}$ a **regular envelope** of rk. We denote by $\mathbb{P}_{reg}(R)$ the set of regular Sylvester matrix rank functions.

In general a regular Sylvester matrix rank function may have many different regular envelopes, but in some situations there is a canonical one. We say that two R-rings $\phi_1 : R \to S_1$ and $\phi_2 : R \to S_2$ are **isomorphic** if there exists an isomorphism $\alpha : S_1 \to S_2$ such that $\phi_2 = \alpha \circ \phi_1$. A division R-ring $\phi : R \to \mathcal{D}$ is called **epic** if $\mathcal{D}$ is generated by $\phi(R)$ as a division ring. If ϕ is an embedding, then we also say that $\mathcal{D}$ is a **division ring of fractions** of R. P. M. Cohn showed that two epic division R-rings are isomorphic if and only if the corresponding Sylvester matrix rank functions are equal (see, for example, [4]). By a result of P. Malcolmson [14], $\mathbb{P}_{div}(R)$ consists exactly of the Sylvester matrix rank function on R taking integer values. Thus, the Sylvester matrix rank functions taking integer values have a canonical regular envelope, which is a division ring.

If rk and $\operatorname{rk}' \in \mathbb{P}(R)$, we write $\operatorname{rk}' \leq \operatorname{rk}$ if $\operatorname{rk}'(A) \leq \operatorname{rk}(A)$ for any matrix A over R. Given a family $\mathcal{F}$ of Sylvester matrix rank functions, we say that $\operatorname{rk} \in \mathcal{F}$ is **universal** in $\mathcal{F}$ if for every Sylvester matrix rank function $\operatorname{rk}' \in \mathcal{F}$, $\operatorname{rk}' \leq \operatorname{rk}$. We say that a division R-ring $\phi : R \to \mathcal{D}$ is **universal** if the induced Sylvester matrix rank function is universal in $\mathbb{P}_{div}(R)$. If there exists a universal epic division R-ring, then it is unique up to R-isomorphism, and we denote it by $\mathcal{D}_R$. The corresponding Sylvester matrix rank function is denoted by rk_R.

The **inner rank** $\rho_R(A)$ of an $n \times m$ matrix A over a ring R is defined by

$$\rho_R(A) = \min\{k : \text{ there are } B \in \operatorname{Mat}_{n\times k}(R), C \in \operatorname{Mat}_{k\times m}(R) \text{ such that } A = BC\}.$$

The property (SMat2) implies that for any $\operatorname{rk} \in \mathbb{P}(R)$, $\operatorname{rk}(A) \leq \rho_R(A)$. However, observe that for most rings R, ρ_R is not a Sylvester matrix rank function. We say that R is a **Sylvester domain** if ρ_R is a Sylvester matrix rank function on R. In this case it is clear that ρ_R is universal in $\mathbb{P}(R)$ (and so in $\mathbb{P}_{div}(R)$) and $\rho_R = \operatorname{rk}_R$

(in other words, ρ_R is induced by the embedding of R into the universal division ring of fractions $\mathcal{D}_R$ of R).

The Sylvester domains appeared implicitly in a work of P. M. Cohn [3] and formally were defined in a paper of W. Dicks and E. D. Sontag [5]. A matrix $A \in \mathrm{Mat}_r(R)$ is called **full** if $\rho_R(A) = r$. P. M. Cohn [3] showed that if R is a Sylvester domain, then the ring $\mathcal{D}_R$ is isomorphic (as an R-ring) to the ring obtained by formally inverting all full matrices over R. It is not always easy to work with this construction of $\mathcal{D}_R$, and one would like to have more manageable presentations of $\mathcal{D}_R$ for concrete Sylvester domains R or, more generally, a way to know whether for a given embedding of R into a division ring (or, more generally, into a von Neumann regular algebra), the division closure of R is isomorphic to $\mathcal{D}_R$. In this paper we will give such a criterion.

A **fir** is a ring in which all submodules of a free module are free of unique rank. Among the most interesting examples of firs are free algebras over fields and the group algebras of free groups over fields (see [2]). P. M. Cohn [3] showed that a fir is a Sylvester domain. In the case of fir, we obtain the following characterization of universal embeddings.

Theorem 1.1. *Let R be a fir and* rk *a regular Sylvester matrix rank function on R with a regular envelope $\phi : R \to \mathcal{U}$. Then* $\mathrm{rk} = \rho_R$ *if and only if* $\mathrm{Tor}_1^R(\mathcal{U}, \mathcal{U}) = 0$.

In Theorem 2.4 we give a general criterion that can be applied not only to firs but to arbitrary rings R. Theorem 1.1 also suggests that the following stronger result might hold. Recall that if R is a subring of a ring S, the **division closure** of R in S is the smallest subring T of S, containing R, and such that for each $t \in T$, which is invertible in S, $t^{-1} \in T$ as well.

Conjecture 1. *Let R be a fir and $\phi : R \to \mathcal{U}$ a homomorphism of R into a von Neumann regular ring $\mathcal{U}$. If* $\mathrm{Tor}_1^R(\mathcal{U}, \mathcal{U}) = 0$*, then the division closure of $\phi(R)$ in $\mathcal{U}$ is R-isomorphic to $\mathcal{D}_R$.*

The difference between the statements of Theorem 1.1 and Conjecture 1 is that we do not assume in Conjecture 1 that $\mathcal{U}$ has a faithful Sylvester matrix rank function. Note that von Neumann regular rings without Sylvester matrix rank functions exist (see [6, Chapter 18]).

1.2. **Operators in von Neumann algebras.** Let $\mathcal{M}$ be a finite von Neumann algebra and $\tau : \mathcal{M} \to \mathbb{C}$ a faithful normal trace on $\mathcal{M}$. S. K. Berberian [1] showed that $\mathcal{M}$ satisfies the right and left Ore conditions and its classical ring of fractions $\mathcal{U}$ is a von Neumann regular ring. $\mathcal{U}$ can also be defined as the algebra of unbounded operators affiliated to $\mathcal{M}$. The trace τ induces a faithful Sylvester rank function rk_τ on $\mathcal{U}$, which can be defined in the following way. Given a matrix $A \in \mathrm{Mat}_{n\times m}(\mathcal{U})$, there is a unique $P \in \mathrm{Mat}_m(\mathcal{U})$ such that P is a projection ($PP^* = P$) and $\mathcal{U}^m \cdot P = \mathcal{U}^m \cdot (A^*A)$. Then we put

$$\mathrm{rk}_\tau(A) = \tau(P).$$

Thus, the study of representations of a ring R in finite von Neumann algebras lead to many interesting examples of regular Sylvester matrix rank functions on R. In

[15] T. Mai, R. Speicher and S. Yin studied representations of free $\mathbb{C}$-algebras. Theorem 1.1 implies the following result proved by them.

Theorem 1.2. [15, Theorem 1.1] *Let $\mathcal{M}$ be a finite von Neumann algebra and $\tau : \mathcal{M} \to \mathbb{C}$ a faithful normal trace on $\mathcal{M}$. Let $\mathcal{U}$ be the $*$-algebra of unbounded operators affiliated to $\mathcal{M}$. Let $R = \mathbb{C} < x_1, \dots, x_d >$ be the free $\mathbb{C}$-algebra on $x_1, \dots, x_d$. Consider a representation of R in $\mathcal{M}$ by sending the tuple $(x_1, \dots, x_d)$ to a tuple $(X_1, \dots, X_d)$ of operators in $\mathcal{M}$. Then the following are equivalent.*

(a) The division closure in $\mathcal{U}$ of the $\mathbb{C}$-subalgebra generated by $X_1, \dots, X_d$ is R-isomorphic to $\mathcal{D}_R$.
(b) The space of tuples $(T_1, \dots, T_d)$ of finite rank operators on $L^2(\mathcal{M}, \tau)$ satisfying $\sum_{k=1}^d [T_k, X_k] = 0$, is trivial.

1.3. **Universal division ring of fractions of $E\langle\langle X\rangle\rangle$.** A **semifir** is a ring in which all finitely generated submodules of a free module are free of unique rank. P. M. Cohn [3] showed that not only firs but also semifirs are Sylvester domains. This applies to the ring $R = E\langle\langle x_1, \dots, x_d\rangle\rangle$, where E is a division ring. In the following result we give the first explicit construction of universal division R-ring.

Theorem 1.3. *The embedding of $R = E\langle\langle x_1, \dots, x_d\rangle\rangle$ into $\mathcal{D}_{E\langle y_1,\dots,y_d\rangle}((t))$ by sending x_i to $y_i t$ is universal.*

1.4. **Universal Sylvester matrix rank function on $\mathbb{F}_p[[F]]$.** For a profinite group G we denote by $\mathbb{F}_p[[G]]$ the inverse limit of $\mathbb{F}_p[G/U]$, where U is a normal open subgroup of G. If F is a free pro-p group freely generated by $f_1, \dots, f_d$, then the continuous homomorphism $\mathbb{F}_p\langle\langle x_1, \dots, x_d\rangle\rangle \to \mathbb{F}_p[[F]]$ that sends x_i to $f_i - 1$, is an isomorphism. This allows us to apply Theorem 1.3 and give a new characterization of $\rho_{\mathbb{F}_p[[F]]}$.

Theorem 1.4. *Let F be a finitely generated free pro-p group, $F = F_1 > F_2 > \dots$ a chain of normal open subgroups of F with trivial intersection and A a matrix over $\mathbb{F}_p[[F]]$. Denote by A_i the matrix over $\mathbb{F}_p[F/F_i]$ obtained from the matrix A by applying the natural homomorphism $\mathbb{F}_p[[F]] \to \mathbb{F}_p[F/F_i]$. Then*

$$\rho_{\mathbb{F}_p[[F]]}(A) = \lim_{i\to\infty} \frac{\operatorname{rk}_{\mathbb{F}_p}(A_i)}{|F : F_i|}.$$

This theorem can be seen as a pro-p variation of Lück approximation (see [12, 8]). The limit $\lim_{i\to\infty} \frac{\operatorname{rk}_{\mathbb{F}_p}(A_i)}{|F : F_i|}$ has been already studied before. It was shown in [7, Theorem 2.3] that it does not depend on the chain $\{F_i\}$ and it always takes integer values. This property played a crucial role in the proof of the Hanna Neumann conjecture for free and Demushkin pro-p groups [7, 10]. Theorem 1.4 is used in the solution of an old conjecture of G. Baumslag in [9].

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2. A criterion for a Sylvester matrix rank function to be the inner rank

2.1. **Sylvester module rank function.** An alternative way to introduce Sylvester rank functions is via Sylvester module rank functions. A **Sylvester module rank function** dim on R is a function that assigns a non-negative real number to each finitely presented left R-module and satisfies the following conditions.

(SMod1) $\dim\{0\} = 0$, $\dim R = 1$;
(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;
(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \geq \dim M_2 \geq \dim M_3.$$

There exists a natural bijection between Sylvester matrix and module rank functions over a ring. Given a Sylvester matrix rank function rk on R and a finitely presented R-module $M \cong R^n/R^m A$ ($A \in \mathrm{Mat}_{m\times n}(R)$), the corresponding Sylvester module rank function dim is defined by means of $\dim(M) = n - \mathrm{rk}(A)$.

One advantage to consider Sylvester module rank functions and not only Sylvester matrix rank functions is the fact that they can be extended to functions on all R-modules (and not only finitely presented ones). This important result has been obtained recently by H. Li [11]. The construction of the extended Sylvester module rank function is done in the following way. If M is a finitely generated left R-module and dim a Sylvester module rank function, we put

$$\dim M = \inf\{\dim \widetilde{M} : \ \widetilde{M} \text{ is finitely presented and } M \text{ is a quotient of } \widetilde{M}\}.$$

For an arbitrary left R-module M we put

$$\dim M = \sup_{M_1} \inf_{M_2} (\dim M_2 - \dim(M_2/M_1)),$$

where $M_1 \leq M_2$ are finitely generated R-submodules of M. Observe that we allow $+\infty$ to be a value of $\dim M$. In the following proposition we collect some properties of this extension.

Proposition 2.1. *Let $f : R \to S$ be a ring homomorphism,* rk' *a Sylvester matrix rank function on S and* $\mathrm{rk} = \mathrm{rk}' \circ f$*. Let* dim *and* $\dim'$ *be the extended Sylvester module rank functions associated with* rk *and* rk' *respectively. Then*

(a) *For any R-module M,* $\dim M = \dim'(S \otimes_R M)$.
(b) *If S is von Neumann regular, then* $\dim'$ *is exact, i.e. for any exact sequence* $0 \to N_1 \to N_2 \to N_3 \to 0$ *of left S-modules,* $\dim' N_2 = \dim' N_1 + \dim' N_3$.

Proof. The first part follows from the discussion at the beginning of Section 7 of [11]. The second part was proved first by S. Virili [18] and follows also from [11, Corollary 4.3]. □

If R has the universal division R-ring $\mathcal{D}_R$, then the extended Sylvester module rank function that corresponds to rk_R is denoted by $\dim_R$. In this case we have

$$\dim_R M = \dim_{\mathcal{D}_R}(\mathcal{D}_R \otimes_R M).$$

If R is a Sylvester domain, then the universality of $\mathrm{rk}_R = \rho_R$ can be reformulated in terms of $\dim_R$ as $\dim_R(M) \le \dim(M)$ for every extended Sylvester module rank function dim on R and any finitely presented left R-module M. The inequality $\dim_R(M) \le \dim(M)$ still holds when M is finitely generated, but may not hold for all left R-modules.

2.2. **Properties of $\mathcal{D}_R$ as an R-module.** Let R be a Sylvester domain. The following proposition analyzes the structure of $\mathcal{D}_R$ as left and right R-module.

Proposition 2.2. *Let R be a Sylvester domain. Then*

(a) *For any R-submodule N of a right $\mathcal{D}_R$-module and any R-submodule M of a left $\mathcal{D}_R$-submodule,* $\mathrm{Tor}_1^R(N, M) = 0$ *and*

(b) *for any finitely generated left or right R-submodule L of a $\mathcal{D}_R$-module and any exact sequence $0 \to I \to R^n \to L \to 0$, I is a direct union of submodules isomorphic to R^k, where*

$$k = n - \dim_R L = \dim_R I.$$

Proof. Let L be a finitely generated left R-submodule of a $\mathcal{D}_R$-module and consider an exact sequence

$$0 \to I \to R^n \to L \to 0$$

of left R-modules. Let

$$k = n - \dim_R L = n - \dim_{\mathcal{D}_R} \mathcal{D}_R \otimes_R L.$$

We can find k elements $a_1, \dots, a_k \in I$ such that

$$\dim_R(R^n/(Ra_1 + \cdots + Ra_k)) = \dim_R L.$$

Put $J_0 = Ra_1 + \cdots + Ra_k$ and $\widetilde{L}_0 = R^n/J_0$. We want to show that for any finitely generated R-submodule J of I that contains J_0, there exists a free R-module J' of rank k such that $J \le J' \le I$. This would imply that I is a direct union of modules isomorphic to R^k.

Let $\widetilde{L} = R^n/J$. Observe that

$$\dim_R \widetilde{L}_0 \ge \dim_R \widetilde{L} \ge \dim_R L.$$

Hence, $\dim_R \widetilde{L} = \dim_R L$.

Represent J as $J = R^s A$ for some s by n matrix A over R. Since

$$\rho_R(A) = n - \dim_R \widetilde{L} = k,$$

we can write $A = BC$ where B is a s by k and C is a k by n matrices over R. In particular $J \le J' = R^k C$. Let $\widetilde{L}' = R^n/J'$.

Since J' is generated by k elements, $\dim_R \widetilde{L}' \ge n - k$. Therefore,

$$\dim_R L = \dim_R \widetilde{L}_0 \ge \dim_R \widetilde{L}' \ge n - k = \dim_R L.$$

Hence we obtain that $\dim_R \widetilde{L}' = \dim_R L$, and so,

$$\dim_R J' \ge n - \dim_R \widetilde{L}' = k.$$

Using again that J' is generated by k elements, we conclude that $J' \cong R^k$.

Assume that there exists $a \in J' \setminus I$. Since L is the maximal quotient of $\widetilde{L}_0$ that can be embedded in a left $\mathcal{D}_R$-module,

$$\dim_R(R^n/(J_0 + Ra)) = \dim_R(R^n/(I + Ra)),$$

and, since $a \notin I$,

$$\dim_R(R^n/(I + Ra)) < \dim_R L.$$

Thus, we obtain that

$$\dim_R L = \dim_R \widetilde{L}' \leq \dim_R(R^n/(J_0 + Ra)) < \dim_R L.$$

This implies that $J' \leq I$.

In particular, we have proved that L is of weak dimension at most 1. Thus, every left R-submodule of a $\mathcal{D}_R$-module is a direct union of left R-modules of weak dimension at most 1, and so, it is of weak dimension at most 1 as well. A symmetric argument shows that the weak dimension of any right R-submodule of a $\mathcal{D}_R$-module is also at most 1.

Now, let N be a R-submodule of a right $\mathcal{D}_R$-module $\mathcal{N}$ and M a R-submodule of a left $\mathcal{D}_R$-submodule $\mathcal{M}$. The exact sequence

$$0 \to N \to \mathcal{N} \to \mathcal{N}/N \to 0$$

induces the exact sequence

$$\mathrm{Tor}_2^R(\mathcal{N}/N, \mathcal{M}) \to \mathrm{Tor}_1^R(N, \mathcal{M}) \to \mathrm{Tor}_1^R(\mathcal{N}, \mathcal{M}).$$

Since $\mathcal{D}_R$ is the universal localization of R with respect to the set of all full matrices, by [17, Theorem 4.7 and Theorem 4.8], $\mathrm{Tor}_1^R(\mathcal{N}, \mathcal{M}) = 0$. On the other hand $\mathrm{Tor}_2^R(\mathcal{N}/N, \mathcal{M}) = 0$ because $\mathcal{M}$ is of weak dimension at most 1. Thus, $\mathrm{Tor}_1^R(N, \mathcal{M}) = 0$.

The exact sequence

$$0 \to M \to \mathcal{M} \to \mathcal{M}/M \to 0$$

induces the exact sequence

$$\mathrm{Tor}_2^R(N, \mathcal{M}/M) \to \mathrm{Tor}_1^R(N, M) \to \mathrm{Tor}_1^R(N, \mathcal{M}).$$

We have that $\mathrm{Tor}_2^R(N, \mathcal{M}/M) = 0$ because the weak dimension of N is smaller than 2. Thus, $\mathrm{Tor}_1^R(N, M) = 0$. This proves (a).

In particular (a) implies that $\mathrm{Tor}_1^R(\mathcal{D}_R, L) = 0$. Thus, applying $\mathcal{D}_R \otimes_R$ to

$$0 \to I \to R^n \to L \to 0,$$

we obtain the exact sequence of left $\mathcal{D}_R$-modules

$$0 \to \mathcal{D}_R \otimes_R I \to \mathcal{D}_R^n \to \mathcal{D}_R \otimes L \to 0.$$

Therefore,

$$k = n - \dim_R L = n - \dim_{\mathcal{D}_R} \mathcal{D}_R \otimes_R L = \dim_{\mathcal{D}_R} \mathcal{D}_R \otimes_R I = \dim_R I,$$

and so the claim (b) for left R-modules holds as well. The claim (b) for right R-modules is proved similarly.

□

2.3. **Universal embeddings.** In this subsection we show that the necessary conditions presented in Proposition 2.2 are also sufficient. But first we give a characterization of Sylvester domains, that generalizes the part (iii) of [5, Theorem 3].

Proposition 2.3. *Let R be a ring and let $\operatorname{rk} \in \mathbb{P}(R)$ satisfy $\operatorname{rk}(A) = \rho_R(A)$ for every full matrix A over R. Then R is a Sylvester domain and $\operatorname{rk} = \rho_R$.*

Proof. Let A and B be two n by n full matrices. Then $\operatorname{rk}(A) = \operatorname{rk}(B) = n$, and so, by [8, Proposition 5.1], $\operatorname{rk}(AB) = n$. Hence $\rho_R(AB) = n$ as well, and so, AB is full. Similarly, the diagonal sum of two full matrices is also full. Thus, the set of full matrices over R is closed under products (where defined) and diagonal sums. Hence, by [5, Proposition 2], the inner rank of a matrix over R is the maximum of the orders of its full submatrices. This implies that $\rho_R(A) \le \operatorname{rk}(A)$ for any matrix over R. Since rk is a Sylvester matrix rank function on R, $\rho_R \ge \operatorname{rk}$. Thus, $\rho_R = \operatorname{rk}$, and so R is a Sylvester domain. □

Now we are ready to prove the main result of this section.

Theorem 2.4. *Let R be a ring and $\operatorname{rk} \in \mathbb{P}_{reg}(R)$ with a regular envelope $\phi : R \to \mathcal{U}$. Let $\dim$ be the extended Sylvester module rank function on R associated with rk. Assume that*

(1) $\operatorname{Tor}_1^R(\mathcal{U},\mathcal{U}) = 0$ and
(2) for any finitely generated left or right R-submodule M of $\mathcal{U}$ and any exact sequence $0 \to I \to R^n \to M \to 0$, I is a direct union of submodules isomorphic to R^k, where $k = \dim I$.

Then $\operatorname{rk} = \rho_R$. In particular, R is a Sylvester domain and the division closure of $\phi(R)$ in $\mathcal{U}$ is R-isomorphic to $\mathcal{D}_R$.

Proof. Let rk' be a faithful Sylvester matrix rank function on $\mathcal{U}$ such that $\operatorname{rk} = \operatorname{rk}' \circ \phi$. We denote by $\dim'$ the extended Sylvester module rank function on $\mathcal{U}$ corresponding to rk'. In view of Proposition 2.3, we want to show that $\operatorname{rk}(A) = \rho_R(A)$ for an $n \times n$ full matrix A over R. We put $M = R^n/R^nA$. Observe that $\operatorname{rk}(A) = n - \dim M$. Thus, if $\dim(M) = 0$, then $\operatorname{rk}(A) = n$.

By way of contradiction, let us assume that $\dim M > 0$. Observe that $\mathcal{U} \otimes_R M$ is a finitely presented $\mathcal{U}$-module, and so, since $\mathcal{U}$ is von Neumann regular, it is also projective [6, Theorem 1.11]. Thus, by [6, Proposition 2.6], $\mathcal{U} \otimes_R M$ is a direct sum of cyclic projective $\mathcal{U}$-modules. Since $\mathcal{U} \otimes_R M$ is not trivial, $\mathcal{U} \otimes_R M$ maps onto a non-trivial projective cyclic $\mathcal{U}$-module $\mathcal{M}$. Let $\overline{M}$ denote the image of M in $\mathcal{M}$. Since $\mathcal{M}$ is a direct summand of $\mathcal{U}$, $\overline{M}$ is a R-submodule of $\mathcal{U}$, and since $\mathcal{M}$ is a quotient of $\mathcal{U} \otimes_R \overline{M}$, $\dim \overline{M} > 0$.

We can write $\overline{M} \cong R^n/I$, where I is an R-submodule of R^n and R^nA is an R-submodule of I. Let $J \cong R^k$ be a free R-submodule of rank $k = \dim I$ of I such that R^nA is contained in J. Since A is full, $k \ge n$.

Since $\mathcal{U}$ is a direct union of right R-modules of weak dimension at most 1, $\mathcal{U}$ is also of weak dimension at most 1. Therefore using that $\operatorname{Tor}_1^R(\mathcal{U},\mathcal{U}) = 0$, we conclude that $\operatorname{Tor}_1^R(\mathcal{U},\overline{M}) = 0$. Thus, applying $\mathcal{U}\otimes_R$ to $0 \to I \to R^n \to \overline{M} \to 0$, we obtain an exact sequence of left $\mathcal{U}$-modules

$$0 \to \mathcal{U} \otimes_R I \to \mathcal{U}^n \to \mathcal{U} \otimes_R \overline{M} \to 0.$$

Since $\mathcal{U}$ is von Neumann regular, Proposition 2.1 implies that

$$k = \dim I = \dim'(\mathcal{U} \otimes_R I) = n - \dim'(\mathcal{U} \otimes_R \overline{M}) = n - \dim \overline{M} < n \le k.$$

We have obtained a contradiction.

Thus, by Proposition 2.3, $\mathrm{rk} = \rho_R$ and R is a Sylvester domain. Since $\mathcal{U}$ is von Neumann regular and rk' is faithful, for a n by n matrix B over $\mathcal{U}$, $\mathrm{rk}'(B) = n$ if and only if B is invertible in $\mathcal{U}$. Therefore the images of all full over R matrices are invertible in $\mathcal{U}$. Hence, the division closure of $\phi(R)$ in $\mathcal{U}$ is R-isomorphic to $\mathcal{D}_R$.

□

We expect that the following weak version of the previous theorem can be applied in some concrete situations.

Corollary 2.5. *Let R be a ring and $\mathrm{rk} \in \mathbb{P}_{reg}(R)$ with a regular envelope $\phi : R \to \mathcal{U}$. Assume that*

(1) $\mathrm{Tor}_1^R(\mathcal{U},\mathcal{U}) = 0$ and
(2) for any finitely generated left or right R-submodule M of $\mathcal{U}$ and any exact sequence $0 \to I \to R^n \to M \to 0$, I is a free R-module.

Then $\mathrm{rk} = \rho_R$. In particular, R is a Sylvester domain and the division closure of $\phi(R)$ in $\mathcal{U}$ is R-isomorphic to $\mathcal{D}_R$.

Proof. If M is a left R-submodule of $\mathcal{U}$, then arguing as in the proof of Theorem 2.4, we conclude that $\mathrm{Tor}_1^R(\mathcal{U}, M) = 0$, and so I is of finite rank. Thus, Theorem 2.4 implies the corollary.

□

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. If R is a fir, then any submodule of R^n is free. Thus, the theorem is a consequence of Corollary 2.5 and Proposition 2.2. □

3. Operators in von Neumann algebras generating free division ring

Let R be a ring and let R_0 be a central subring of R. Assume that R is projective as an R_0-module (for example, R_0 is a field). We denote by R^{op} the opposite of R. Let M and N be a right and left R-module respectively. Then $M \otimes_{R_0} N$ has a right $R \otimes_{R_0} R^{op}$-module structure defined by means of

$$(m \otimes n) \cdot (a \otimes b) = ma \otimes bn, \ m \in M, n \in N, a, b \in R.$$

Also R becomes a left $R \otimes_{R_0} R^{op}$-module if we define

$$(a \otimes b) \cdot c = acb, \ a, b, c \in R.$$

Observe that $\mathrm{Tor}_1^R(M, N) \cong \mathrm{Tor}_1^{R \otimes_{R_0} R^{op}}(M \otimes_{R_0} N, R)$ (see, for example, [13, Exercise 7.3.3]).

In the case $R = R_0 < x_1, \ldots, x_d >$ is a free R_0-algebra or $R = R_0 < x_1^{\pm 1}, \ldots, x_d^{\pm 1} >$ is the group algebra over a finitely generated free group, we have the following free resolution of $R \otimes_{R_0} R^{op}$-module R.

$$0 \to (R \otimes_{R_0} R^{op})^d \xrightarrow{\psi_1} R \otimes_{R_0} R^{op} \xrightarrow{\psi_0} R \to 0,$$

where $\psi_0(a \otimes b) = ab$ and $\psi_1(v_1, \ldots, v_d) = \sum_{i=1}^d v_i(x_i \otimes 1 - 1 \otimes x_i)$. Thus, we obtain that

$$\text{(1)} \quad \mathrm{Tor}_1^R(M, N) \cong \mathrm{Tor}_1^{R \otimes_{R_0} R^{op}}(M \otimes_{R_0} N, R) \cong$$
$$\{(v_1, \ldots, v_d) \in (M \otimes_{R_0} N)^d : \sum_{i=1}^d v_i(x_i \otimes 1 - 1 \otimes x_i) = 0\}.$$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\phi : R \to \mathcal{U}$ be the map that sends x_i to X_i. Then the first condition of the theorem can be reformulated as $\mathrm{rk}_R = \mathrm{rk}_\tau \circ \phi$.

Denote by $\mathcal{F} = \mathcal{F}(L^2(\mathcal{M}, \tau))$ the space of finite rank operators on $L^2(\mathcal{M}, \tau)$. Then $\mathcal{F}$ is a right-$R \otimes_{\mathbb{C}} R^{op}$-module where the action is defined as

$$[f \cdot (a \otimes b)](x) = \phi(b) f(\phi(a)x), \ a, b \in R, \ x \in L^2(\mathcal{M}, \tau).$$

Hence, if $T \in \mathcal{F}$, then

$$\text{(2)} \qquad [T, X_i] = TX_i - X_iT = T \cdot (x_i \otimes 1 - 1 \otimes x_i).$$

The map $\mathcal{M} \times \mathcal{M} \to \mathbb{C}$ that sends (x, y) to $\tau(x^*y)$, extends continuously to a map $\langle \cdot, \cdot \rangle_\tau : L^2(\mathcal{M}, \tau) \times L^2(\mathcal{M}, \tau) \to \mathbb{C}$. Observe that the space $\mathcal{F}$ is isomorphic to $L^2(\mathcal{M}, \tau) \otimes_{\mathbb{C}} L^2(\mathcal{M}, \tau)$ with an explicit isomorphism $\alpha : L^2(\mathcal{M}, \tau) \otimes_{\mathbb{C}} L^2(\mathcal{M}, \tau) \to \mathcal{F}$ given by

$$[\alpha(u \otimes v)](x) = \langle u^*, x \rangle_\tau v, \ u, v, x \in L^2(\mathcal{M}, \tau).$$

Moreover, α is an $R \otimes_{\mathbb{C}} R^{op}$-module isomorphism if $L^2(\mathcal{M}, \tau) \otimes_{\mathbb{C}} L^2(\mathcal{M}, \tau)$ is considered with the following $R \otimes_{\mathbb{C}} R^{op}$-module structure:

$$(u \otimes v) \cdot (a \otimes b) = u\phi(a) \otimes \phi(b)v, \ a, b \in R, \ u, v \in L^2(\mathcal{M}, \tau).$$

Thus, taking into account (1) and (2), the second condition of the corollary can be reformulated as $\mathrm{Tor}_1^R(L^2(\mathcal{M}, \tau), L^2(\mathcal{M}, \tau)) = 0$.

Assume now that (a) holds. Thus, the division closure of $\phi(R)$ in $\mathcal{U}$ is isomorphic to $\mathcal{D}_R$ as a R-ring. Since $\mathrm{Tor}_1^R(\mathcal{D}_R, \mathcal{D}_R) = 0$ and $\mathcal{D}_R$ is a division algebra, $\mathrm{Tor}_1^R(M, N) = 0$ for any right $\mathcal{D}_R$-module M and any left $\mathcal{D}_R$-module N. In particular,

$$\text{(3)} \qquad \mathrm{Tor}_1^R(L^2(\mathcal{M}, \tau) \otimes_{\mathcal{M}} \mathcal{U}, \mathcal{U} \otimes_{\mathcal{M}} L^2(\mathcal{M}, \tau)) = 0.$$

Since $\mathcal{U}$ is the classical ring of fractions of $\mathcal{M}$, $L^2(\mathcal{M}, \tau)$ is embedded into $L^2(\mathcal{M}, \tau) \otimes_{\mathcal{M}} \mathcal{U}$ as a right R-module and analogously $L^2(\mathcal{M}, \tau)$ is embedded into $\mathcal{U} \otimes_{\mathcal{M}} L^2(\mathcal{M}, \tau)$ as a left R-module. Since R is of global dimension 1, (3) implies that

$$\mathrm{Tor}_1^R(L^2(\mathcal{M}, \tau), L^2(\mathcal{M}, \tau)) = 0$$

as well. Hence (b) holds.

Assume now, that (b) holds. Thus, $\mathrm{Tor}_1^R(L^2(\mathcal{M},\tau), L^2(\mathcal{M},\tau)) = 0$. Since R is of global dimension 1, $\mathrm{Tor}_1^R(\mathcal{M},\mathcal{M}) = 0$ as well. Since $\mathcal{U}$ is a flat left and right $\mathcal{M}$-module, we also obtain that $\mathrm{Tor}_1^R(\mathcal{U},\mathcal{U}) = 0$. Thus, by Theorem 1.1, $\rho_R = \mathrm{rk}_\tau \circ \phi$, and so (a) holds. □

4. The construction of a universal division $E\langle\langle x_1, \dots, x_d\rangle\rangle$-ring of fractions

4.1. **Filtered R-modules.** A **filtered** ring is a unital ring R with a descending chain of ideals of R,

$$R = \dots = R_{-1} = R_0 \geq R_1 \geq \cdots$$

such that $R_k R_l \leq R_{k+l}$. A **filtered** left R-modue is a left R-module M with a descending chain of submodules of M,

$$M = \dots = M_{l-1} = M_l > M_{l+1} \dots,$$

such that $R_n M_k \leq M_{k+n}$ for every $n \geq 0$ and $k \in \mathbb{Z}$. The family $\mathcal{F}(M) = \{M_k\}$ is called the **filtration** of M. An R-module M can have many filtrations, but it will be always clear from the context what filtration we use. A homomorphism of filtered left R-module is a homomorphism of R-modules $f : M \to N$ such that $f(M_k) \leq N_k$ for any $k \in \mathbb{Z}$.

The canonical filtration of a direct sum of filtered R-modules M and N will be $(M \oplus N)_k = M_k \oplus N_k$. In the same way we define the filtration for a direct sum of more than two or infinite number of filtered R-modules. If $v \in \mathbb{Z}$ we denote by $M(v)$ the filtered R-module that coincides with M as R-module but the filtration is given by $M(v)_k = M_{k-v}$.

We say that the filtration $\{M_k\}$ is **separated** if $\bigcap_k M_k = \{0\}$. In this case the filtration defines a distance on M: $d_M(a, b) = \inf\{2^{-k} : a - b \in M_k\}$. We say that M is **complete** if M is complete with respect to this distance. In general case we denote by $\widehat{M}$ the completion of $\overline{M} = M/\cap M_k$ with respect to the distance $d_{\overline{M}}$. Then $\widehat{M}$ is a complete filtered R-module with the filtration $\{\widehat{M_k}\}$, where $\widehat{M_k}$ is the closure of M_k in $\widehat{M}$. It is clear that $\widehat{M} \cong \varprojlim M/M_k$.

We say that a filtered left R-module M is **bounded** if the filtration of M is separated and all quotients M_k/M_{k+1} are left R-modules of finite length (observe that then they are also R/R_1-modules of finite length).

Proposition 4.1. *Let M be a bounded and complete filtered left R-module with respect to the filtration $\mathcal{F}_1 = \{M_i\}$. Let $\mathcal{F}_2 = \{N_i\}$ be another separated filtration on M such that for every i there exists j_i with $M_{j_i} \leq N_i$. Then for every i there exists k_i such that $N_{k_i} \leq M_i$. In particular, $\mathcal{F}_1$ and $\mathcal{F}_2$ induce the same topology on M.*

Proof. For each $i \geq 1$, let

$$L_i = \bigcap_k (N_k + M_i).$$

Since M/M_i is Artinian, there exists k_i such that $L_i = N_{k_i} + M_i$. In particular, for every l,

$$L_l \leq N_{k_{l+1}} + M_l \leq L_{l+1} + M_l \leq L_l.$$

Thus, $L_l = L_{l+1} + M_l$.

Assume that $L_i \neq M_i$. Let $a_i \in L_i \setminus M_i$ and for $l \geq i$ construct inductively $a_{l+1} \in L_{l+1}$ such that $a_{l+1} - a_l \in M_l$. Put

$$a = a_i + \sum_{l=i}^{\infty}(a_{l+1} - a_l) = a_s + \sum_{l=s}^{\infty}(a_{l+1} - a_l).$$

Then $0 \neq a$ because $a + M_i = a_i + M_i$ and also $a \in \bigcap_l L_l$. Observe that $L_s \leq N_l$ for every $s \geq j_l$, and so $\bigcap_l L_l = \{0\}$, because $\mathcal{F}_2$ is separated. This is a contradiction. Therefore, $L_i = M_i$, and so $N_{k_i} \leq M_i$.

□

If R is bounded as filtered left R-module, then any finitely generated separated filtered left R-module is bounded. Moreover, we have the following useful result.

Proposition 4.2. *Assume that R is bounded and complete as a left R-module. Let M be a filtered left R-module with a separated filtration. If M is finitely generated as a left R-module, then M is complete.*

Proof. We consider two filtrations on M. The original one we denote by $\mathcal{F}_2 = \{N_i\}$ and $\mathcal{F}_1 = \{M_i = R_i M\}$. If $M = N_k$ for some k, then $M_{i-k} \leq N_i$. Since $\mathcal{F}_2$ is separated, $\mathcal{F}_1$ is separated as well.

Let M be generated by $m_1, \ldots, m_n$ as a left R-module. Hence

$$M_i = R_i m_1 + \ldots + R_i m_n.$$

Thus, since R is complete as a left R-module, M is also complete with respect to $\mathcal{F}_1$. Now, the proposition follows from Proposition 4.1.

□

Corollary 4.3. *Assume that R is bounded and complete as a left R-module. Let M be a finitely generated filtered left R-module with the filtration $\mathcal{F} = \{M_i\}$. Then, $\widehat{M} \cong M/(\bigcap_i M_i)$. In particular, $\widehat{M}$ is finitely generated.*

Proof. Apply the previous proposition to the filtered module $M/(\bigcap_i M_i)$.

□

Let V be a multiset of integers. If M is a filtered R-module, we denote by $M(V)$ the filtered R-module $\oplus_{v \in V} M(v)$. We say that V is bounded if for every k there are only finitely many $v \in V$ for which $v \leq k$. Observe that $M(V)$ is bounded if M and V are bounded.

We denote by $\mathrm{Gr}(R)$ the graded ring $\oplus_{i \in \mathbb{Z}_{\geq 0}} R_i/R_{i+1}$. If M is a filtered left R-module, we put $\mathrm{Gr}(M) = \oplus_k M_k/M_{k+1}$. Then $\mathrm{Gr}(M)$ is a graded left $\mathrm{Gr}(R)$-module: for $r \in R_k$ and $m \in M_l$ we put

$$(r + R_{k+1})(m + M_{l+1}) = rm + M_{l+1+k}.$$

If $f : M \to N$ is a homomorphism of filtered R-modules, then $\mathrm{Gr}(f) : \mathrm{Gr}(M) \to \mathrm{Gr}(N)$ defined by means of

$$\mathrm{Gr}(f)(m + M_{k+1}) = f(m) + N_{k+1}$$

is a homomorphism of $\mathrm{Gr}(R)$-modules.

We say that a homomorphism $f : M \to N$ of filtered R-modules is **strict** if for every $k \in \mathbb{Z}$, $f(M_k) = \operatorname{Im} f \cap N_k$. That is, the filtrations on $\operatorname{Im} f$ induced by the map f and by the inclusion of $\operatorname{Im} f$ into N coincide. We say that a sequence of filtered R-modules $L_1 \xrightarrow{f} L_2 \xrightarrow{g} L_3$ is **strict exact** if the sequence is exact and f and g are strict. Observe that in this case the induced sequence

$$\operatorname{Gr}(L_1) \xrightarrow{\operatorname{Gr}(f)} \operatorname{Gr}(L_2) \xrightarrow{\operatorname{Gr}(g)} \operatorname{Gr}(L_3)$$

is exact as well. The following proposition (for example, see [16, Theorem D.III.3]) shows conditions under which the converse implication holds. We give its proof here for a completeness of exposition.

Proposition 4.4. *Consider the following sequence of filtered abelian groups*

$$N \xrightarrow{f} M \xrightarrow{g} L, \tag{*}$$

such that $g \circ f = 0$, and assume that

(1) the induced sequence

$$\operatorname{Gr}(N) \xrightarrow{\operatorname{Gr}(f)} \operatorname{Gr}(M) \xrightarrow{\operatorname{Gr}(g)} \operatorname{Gr}(L) \tag{Gr(*)}$$

is exact,

(2) N is complete and

(3) $\mathcal{F}(M)$ is separated.

Then (*) is strict exact.

Proof. Fist, let us see that g is strict. We want to show that $g(M_k) = L_k \cap \operatorname{Im} g$. Take $l = g(m) \in L_k$. We want to show that $l \in g(M_k)$. Without loss of generality we may assume that $m \neq 0$.

There exists s such that $m \in M_s$. If $s \geq k$ we are done. If not, then $\operatorname{Gr}(g)(m + M_{s+1}) = 0$. Since $(\operatorname{Gr}(*))$ is exact, there exist $n \in N_s$ such that $m + M_{s+1} = \operatorname{Gr}(f)(n + N_{s+1})$. Thus, $m - f(n) \in M_{s+1}$ and $g(m - f(n)) = g(m)$. Repeating this procedure we finally get that there exists a $m' \in M_k$ such that $l = g(m')$.

Now, we want to show that $f(N_k) = M_k \cap \ker g$. This clearly implies that (*) is exact and f is strict (because $\operatorname{Im} f \leq \ker g$). Take $m \in M_k \cap \ker g$. Since $\operatorname{Gr}(g)(m + M_{k+1}) = 0$ and $(\operatorname{Gr}(*))$ is exact, there exist $n_0 \in N_k$ such that $m + M_{k+1} = \operatorname{Gr}(f)(n_0 + N_{k+1})$. Thus we obtain that $m - f(n_0) \in M_{k+1}$. Repeating this procedure, we find $n_i \in N_{k+i}$ such that $m - f(\sum_{i=0}^{j} n_i) \in M_{k+j+1}$. Completeness of N allows to define $x = \sum_{i=0}^{\infty} n_i$. Since $\mathcal{F}(M)$ is separated, $m = f(x) \in f(N_k)$. $\square$

Proposition 4.5. *Let M be a complete and bounded filtered R-module. Then there are a bounded multiset V of $\mathbb{Z}$ and a strict exact sequence of filtered R-modules*

$$\widehat{R(V)} \to M \to 0.$$

Proof. Since M is bounded, we can found a set $S = \{m_i + M_{v_i+1} : i \in \mathbb{N}\}$ of homogeneous elements of $\operatorname{Gr}(M)$ such that S generates $\operatorname{Gr}(M)$ as a $\operatorname{Gr}(R)$-module and the multiset $V = \{\{v_i : i \in \mathbb{N}\}\}$ is bounded. One can easily check that the map $\widehat{R(V)} \to M$ that sends a generator of $R(v_i)$ to m_i is surjective and strict. $\square$

From now on we assume that R is *bounded and complete* as a left R-module. Since a closed R-submodule of a bounded complete filtered module is again bounded and complete, we obtain the following corollary of Proposition 4.5.

Corollary 4.6. *Let M be a complete and bounded filtered R-module. Then there are bounded multisets V_i of $\mathbb{Z}$ ($i \geq 0$) and a resolution of M*

$$(\star) \qquad \dots \widehat{R(V_i)} \to \dots \to \widehat{R(V_1)} \to \widehat{R(V_0)} \to M \to 0$$

such that $\mathrm{Gr}(\star)$ *is exact.*

Let now M be a filtered right R-module and N a filtered left R-module. Then $M \otimes_R N$ becomes a filtered abelian groups if we denote by $(M \otimes_R N)_s$ the abelian groups generated by $\{m \otimes n : m \in M_l, n \in N_k, s = k + l\}$. We denote by $M \widehat{\otimes}_R N$ the completion of $M \otimes_R N$ with respect to this filtration. Thus, we have that

$$M \widehat{\otimes}_R N \cong \varprojlim (M/M_i \otimes_R N/N_i).$$

In particular $M \widehat{\otimes}_R N \cong \widehat{M} \widehat{\otimes}_R \widehat{N}$.

There is a canonical surjective map

$$\phi_{M,N} : \mathrm{Gr}(M) \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(N) \to \mathrm{Gr}(M \otimes_R N)$$

defined by means of

$$\phi_{M,N}((m + M_{k+1}) \otimes (n + N_{l+1})) = (m \otimes n) + (M \otimes N)_{l+k+1}, \ m \in M_k, \ n \in N_l.$$

The following lemma is a direct consequence of the definitions.

Lemma 4.7. *Let M be a filtered right R-module and let $f : N \to L$ be a homomorphism of filtered left R-modules. Then the following diagram*

$$\begin{array}{ccc} \mathrm{Gr}(M) \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(N) & \xrightarrow{\mathrm{Id}_{\mathrm{Gr}(M)} \otimes \mathrm{Gr}(f)} & \mathrm{Gr}(M) \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(L) \\ \downarrow \phi_{M,N} & & \downarrow \phi_{M,L} \\ \mathrm{Gr}(M \otimes_R N) & \xrightarrow{\mathrm{Gr}(\mathrm{Id}_M \otimes f)} & \mathrm{Gr}(M \otimes_R L) \end{array}$$

is commutative.

Assume that $V = \{\{v_i : i \in \mathbb{N}\}\}$ (with $v_i \leq v_{i+1}$) is a bounded multiset of integers and let M be a complete filtered right R-module. Then any element m of $\widehat{M(V)}$ can be represented uniquely as $m = \sum_{i=0}^{\infty} m_i$, where $m_i \in M(v_i)$. Consider the map

$$\tau_{M,\widehat{R(V)}} : M \otimes_R \widehat{R(V)} \to \widehat{M(V)}$$

sending $m \otimes \sum_{i=0}^{\infty} f_i$ to $\sum_{i=0}^{\infty} m f_i$ (here $f_i \in R(v_i)$ and $m f_i \in M(v_i)$).

Lemma 4.8. *Let M be a complete filtered right R-module and let V be a bounded multiset of integers. Then the following holds*

(1) *The map $\mathrm{Gr}(\tau_{M,\widehat{R(V)}}) : \mathrm{Gr}(M \otimes_R \widehat{R(V)}) \to \mathrm{Gr}(\widehat{M(V)})$ is an isomorphism of abelian groups. In particular, $\tau_{M,\widehat{R(V)}}$ induces a strict isomorphism between $M \widehat{\otimes}_R \widehat{R(V)}$ and $\widehat{M(V)}$.*

(2) *The map $\phi_{M,\widehat{R(V)}}$ is an isomorphism of abelian groups.*

Proof. If V is finite, the lemma is clear. Now, assume that V is bounded. Let

$$V_n = \{\{v \in V : \ v \le n\}\}.$$

Observe that V_n is finite. There exists k such that $M_k = M$. Then

$$\widehat{M(V)}_n = M(V_{n-k})_n \oplus M(\widehat{V \setminus V_{n-k}})$$

and

$$(M \otimes_R \widehat{R(V)})_n = (M \otimes_R R(V_{n-k}))_n \oplus (M \otimes_R R(\widehat{V \setminus V_{n-k}})).$$

It is clear that

$$\tau_{M,\widehat{R(V)}}((M \otimes_R R(V_{n-k}))_n) = M(V_{n-k})_n$$

and

$$\tau_{M,\widehat{R(V)}}(M \otimes_R R(\widehat{V \setminus V_{n-k}})) \le M(\widehat{V \setminus V_{n-k}}).$$

Since $M \otimes_R R(\widehat{V \setminus V_{n-k}}) \le (M \otimes_R \widehat{R(V)})_{n+1}$ and $M(\widehat{V \setminus V_{n-k}}) \le (\widehat{M(V)})_{n+1}$, both statements for V follow from ones for $\{V_{n-k}\}$. □

We will show that under some additional conditions, $\tau_{M,\widehat{R(V)}}$ is a strict isomorphism.

Proposition 4.9. *Let M be a complete filtered right R-module. Let V be a bounded multiset of integers. Then the following holds.*

(1) If M is finitely presented, then $\tau_{M,\widehat{R(V)}}$ is a strict isomorphism of filtred abelian groups, and so $M \otimes_R \widehat{R(V)}$ is complete and coincides with $M \widehat{\otimes}_R \widehat{R(V)}$.

(2) Let $k \ge 2$. If M is of type FP_k, then $\operatorname{Tor}^R_{k-1}(M, \widehat{R(V)}) = 0$.

Proof. (1) First observe that the claim holds when M is a finitely generated free right R-module (with any filtration which makes it complete). Indeed, let $m_1, \dots, m_n$ be a free set of generators of M. Any element m of $\widehat{M(V)}$ can be written in a unique way as

$$m = \sum_{v \in V} m_1 r_{1,v} + \dots + m_n r_{n,v}, \text{ where } m_1 r_{1,v} + \dots + m_n r_{n,v} \in M(v).$$

Then we can define the inverse of $\tau_{M,\widehat{R(V)}}$ as

$$\tau^{-1}_{M,\widehat{R(V)}}(m) = m_1 \otimes \sum_{v \in V} r_{1,v} + \dots + m_n \otimes \sum_{v \in V} r_{n,v}.$$

Now, asume that M is an arbitrary finitely presented and complete filtered R-module. We can find an exact sequence of right R-modules

$$R^l \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0.$$

It is clear that we can define filtrations on R^k and R^n in such way that they become complete and the maps α and β are filtered.

After applying $\otimes_R \widehat{R(V)}$, we obtain the exact sequence of filtered K-spaces.

$$\widehat{R^l(V)} \xrightarrow{\widetilde{\alpha}} \widehat{R^n(V)} \xrightarrow{\widetilde{\beta}} M \otimes_R \widehat{R(V)} \to 0,$$

where $\widetilde{\alpha}$ is given by

$$\widetilde{\alpha}(\sum_{i=0}^{\infty} n_i) = \sum_{i=0}^{\infty} \alpha(n_i),\ n_i \in R^l(v_i).$$

Consider the map $\tau_{M,\widehat{R(V)}} \circ \widetilde{\beta} : \widehat{R^n(V)} \to \widehat{M(V)}$. Observe that

$$(\tau_{M,\widehat{R(V)}} \circ \widetilde{\beta})(\sum_{i=0}^{\infty} n_i) = \sum_{i=0}^{\infty} \beta(n_i),\ n_i \in R^n(v_i).$$

Thus, $\tau_{M,\widehat{R(V)}} \circ \widetilde{\beta}$ is surjective and $\ker(\tau_{M,\widehat{R(V)}} \circ \widetilde{\beta}) = \operatorname{Im} \widetilde{\alpha} = \ker \widetilde{\beta}$. Hence $\tau_{M,\widehat{R(V)}}$ is an isomorphism. In particular, the filtration of $M \otimes_R \widehat{R(V)}$ is separated. By Lemma 4.8(1), $\operatorname{Gr}(\tau_{M,\widehat{R(V)}})$ is isomorphism. Thus, applying Proposition 4.4, we obtain that $\tau_{M,\widehat{R(V)}}$ is strict.

(2) First assume that $k = 2$ (M is of type FP_2). Then there exists an exact sequence of right R-modules

$$0 \to L \to R^n \to M \to 0$$

with L finitely presented. Applying $\otimes_R \widehat{R(V)}$ to the previous sequence, and taking into account the part (1) we obtain an exact sequence of abelian groups

$$0 \to \operatorname{Tor}_1^R(M, \widehat{R(V)}) \to \widehat{L(V)} \xrightarrow{\alpha} \widehat{R^n(V)} \to \widehat{M(V)} \to 0.$$

Then it is clear that $\operatorname{Tor}_1^R(M, \widehat{R(V)}) \cong \ker \alpha = 0$.

Now, assume that $k \geq 3$. Then there exists an exact sequence of right R-modules

$$0 \to L \to R^{n_{k-3}} \to \ldots \to R^{n_0} \to M \to 0$$

with L of type FP_2. Then

$$\operatorname{Tor}_{k-1}^R(M, \widehat{R(V)}) \cong \operatorname{Tor}_1^R(L, \widehat{R(V)}) = 0.$$

□

Now we are ready to prove the main result of this subsection.

Theorem 4.10. *Let R be a filtered ring and $k \geq 1$. Assume that R is complete and bounded as a left and right R-module. Let M and N be complete filtered right and left respectively R-modules and assume that*

(1) M is of type FP_{k+1},
(2) N is bounded and
(3) $\operatorname{Tor}_k^{\operatorname{Gr}(R)}(\operatorname{Gr}(M), \operatorname{Gr}(N)) = 0$.

Then $\operatorname{Tor}_k^R(M, N) = 0$.

Proof. By Corollary 4.6, there are bounded multisets of integers $\{V_i\}$ and a strict exact sequence of filtered R-modules

$$(\star\star) \qquad \ldots \xrightarrow{\alpha_{i+1}} \widehat{R(V_i)} \xrightarrow{\alpha_i} \ldots \xrightarrow{\alpha_2} \widehat{R(V_1)} \xrightarrow{\alpha_1} \widehat{R(V_0)} \xrightarrow{\alpha_0} N \to 0.$$

By Proposition 4.9(2), $\operatorname{Tor}_j^R(M, \widehat{R(V_i)}) = 0$ for all i and $1 \leq j \leq k$. Thus,

$$\operatorname{Tor}_k^R(M, N) \cong \operatorname{Tor}_{k-1}^R(M, \operatorname{Im} \alpha_1) \cong \ldots \cong \operatorname{Tor}_1^R(M, \operatorname{Im} \alpha_{k-1}).$$

This implies that $\mathrm{Tor}_k^R(M,N)$ can be computed by applying $M\otimes_R$ to the sequence $(\star\star)$. Consider the part of the sequence $M\otimes_R(\star\star)$ that corresponds to the calculation of $\mathrm{Tor}_k^R(M,N)$:

$$(**)\qquad M\otimes_R \widehat{R(V_{k+1})}\to M\otimes_R \widehat{R(V_k)}\to M\otimes_R \widehat{R(V_{k-1})}.$$

We want to use Proposition 4.4 in order to show that $(**)$ is exact. By Proposition 4.9(1), $M\otimes_R \widehat{R(V_{k+1})}$ and $M\otimes_R \widehat{R(V_k)}$ are complete (and so, their filtrations are separated). By Lemma 4.7 and Lemma 4.8(2), the sequence $\mathrm{Gr}(**)$ is isomorphic to the sequence

$$(***)\quad \mathrm{Gr}(M)\otimes_{\mathrm{Gr}(R)}\mathrm{Gr}(\widehat{R(V_{k+1})})\to \mathrm{Gr}(M)\otimes_{\mathrm{Gr}(R)}\mathrm{Gr}(\widehat{R(V_k)})\to \mathrm{Gr}(M)\otimes_{\mathrm{Gr}(R)}\mathrm{Gr}(\widehat{R(V_{k-1})}).$$

Since, the sequence $(\star\star)$ is strict exact, the sequence $\mathrm{Gr}(\star\star)$ is a free resolution of $\mathrm{Gr}(N)$. Thus, the sequence $(***)$ is exact because $\mathrm{Tor}_k^{\mathrm{Gr}(R)}(\mathrm{Gr}(M),\mathrm{Gr}(N))=0$. Applying Proposition 4.4, we obtain that $(**)$ is exact. Hence $\mathrm{Tor}_k^R(M,N)=0$. $\square$

4.2. $E\langle\langle x_1,\dots,x_d\rangle\rangle$**-modules.** In this subsection we consider left and right R-modules where $R=E\langle\langle x_1,\dots,x_d\rangle\rangle$ and E is a division ring. All results that we will prove for left R-modules hold also for right R-modules and viceversa. We will consider R as a filtered ring with the filtration $R_k=(Rx_1+\dots+Rx_d)^k$ for $k\ge 0$. Then $\mathrm{Gr}(R)$ is isomorphic canonically to the free algebra $E\langle x_1,\dots,x_d\rangle$.

Proposition 4.11. *Let V be a bounded multiset of integers. Then any closed left R-submodule of $\widehat{R(V)}$ is isomorphic (as filtered R-module) to $\widehat{R(U)}$ for some bounded multiset of integers U.*

Proof. By [4, Proposition 2.9.8], R satisfies the inverse weak algorithm (it was shown in the case E is a field but the same proof works if E is a division ring). Now, the result follows from [4, Proposition 2.9.6]. $\square$

Corollary 4.12. *Let M be a complete bounded filtered left R-module. Then there are bounded multisets of integers V_0 and V_1 and a strict exact sequence.*

$$0\to\widehat{R(V_1)}\to\widehat{R(V_0)}\to M\to 0.$$

Let $\mathcal{D}=\mathcal{D}_{E\langle y_1,\dots,y_d\rangle}$ be the universal division $E\langle y_1,\dots,y_d\rangle$-ring. The ring $\mathcal{D}[[t]]$ is a filtered ring with the filtration $(\mathcal{D}[[t]])_i=t^i\mathcal{D}[[t]]$. We embed R into $\mathcal{D}[[t]]$ by sending x_i to y_it. This embedding respects the filtration, and so we obtain

$$E\langle x_1,\dots,x_d\rangle\cong \mathrm{Gr}(R)\hookrightarrow \mathrm{Gr}(\mathcal{D}[[t]])\cong\mathcal{D}[t]\hookrightarrow\mathcal{D}(t).$$

Observe that the resulting embedding $E\langle x_1,\dots,x_d\rangle\hookrightarrow\mathcal{D}(t)$, where x_i is sent to ty_i, is universal.

Proposition 4.13. *Let M be a finitely generated left R-submodule of $\mathcal{D}((t))$. Then $\dim_E \mathrm{Tor}_1^R(E,M)$ is finite.*

Proof. The embedding of M into $\mathcal{D}((t))$ induces a filtration on M: $M_k = t^k\mathcal{D}[[t]] \cap M$. Then $\mathrm{Gr}(M)$ can be seen as a $\mathrm{Gr}(R)$-submodule of $\mathcal{D}[t^{\pm 1}]$. By Proposition 4.2, M is complete with respect to this filtration.

Consider the exact sequence of strict morphisms from Corollary 4.12 for M.

$$0 \to \widehat{R(U)} \xrightarrow{\sigma} \widehat{R(V)} \xrightarrow{\delta} M \to 0. \tag{$\star$}$$

Since $(\star)$ is strict, the sequence $\mathrm{Gr}(\star)$

$$0 \to \mathrm{Gr}(R)(U) \to \mathrm{Gr}(R)(V) \to \mathrm{Gr}(M) \to 0$$

is exact. Since $\mathrm{Gr}(M)$ is a $\mathrm{Gr}(R)$-submodule of $\mathcal{D}(t)$, and the embedding $\mathrm{Gr}(R)$ into $\mathcal{D}(t)$ is universal, by Proposition 2.2, $\mathrm{Tor}_1^{\mathrm{Gr}(R)}(\mathcal{D}[t], \mathrm{Gr}(M)) = 0$. Thus the sequence $\mathcal{D}[t] \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(\star)$

$$0 \to \mathcal{D}[t] \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(R)(U) \xrightarrow{\overline{\alpha}} \mathcal{D}[t] \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(R)(V) \xrightarrow{\overline{\beta}} \mathcal{D}[t] \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(M) \to 0.$$

is also exact.

Claim 4.14. *The sequence* $\mathcal{D}[[t]]\widehat{\otimes}_R(\star)$

$$0 \to \mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(U)} \xrightarrow{\alpha} \mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(V)} \xrightarrow{\beta} \mathcal{D}[[t]]\widehat{\otimes}_R M \to 0$$

is exact.

Proof. First we show that α is injective. By Lemma 4.7, the following diagram is commutative.

$$\begin{array}{ccccc} 0 & \to & \mathrm{Gr}(\mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(U)}) & \xrightarrow{\mathrm{Gr}(\alpha)} & \mathrm{Gr}(\mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(V)}) \\ & & \uparrow \phi_{\mathcal{D}[[t]],\widehat{R(U)}} & & \uparrow \phi_{\mathcal{D}[[t]],\widehat{R(V)})} \\ 0 & \to & \mathcal{D}[t] \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(R)(U) & \xrightarrow{\overline{\alpha}} & \mathcal{D}[t] \otimes_{\mathrm{Gr}(R)} \mathrm{Gr}(R)(V) \end{array}.$$

By Lemma 4.8(2), the maps $\phi_{\mathcal{D}[[t]],\widehat{R(U)}}$ and $\phi_{\mathcal{D}[[t]],\widehat{R(V)})}$ are isomorphisms. Thus, the first row of the diagram is exact, and so, by Proposition 4.4, α is injective.

Let $x \in \ker\beta$. We want to show that $x \in \mathrm{Im}\,\alpha$. For each $k \geq 0$ we have the following exact sequence.

$$0 \to \widehat{R(U)}/\sigma^{-1}(\widehat{R(V)}_k) \xrightarrow{\sigma_k} \widehat{R(V)}/\widehat{R(V)}_k \xrightarrow{\delta_k} M/\delta(\widehat{R(V)}_k) \to 0$$

Since the tensor product is right exact we obtain the exact sequence

$$\begin{aligned}(\mathcal{D}[[t]]/\mathcal{D}[[t]]t^k) \otimes_R (\widehat{R(U)}/\sigma^{-1}(\widehat{R(V)}_k)) &\xrightarrow{\alpha_k} \\ (\mathcal{D}[[t]]/\mathcal{D}[[t]]t^k) \otimes_R (\widehat{R(V)}/\widehat{R(V)}_k) &\xrightarrow{\beta_k} (\mathcal{D}[[t]]/\mathcal{D}[[t]]t^k) \otimes_R (M/\delta(\widehat{R(V)}_k)).\end{aligned}$$

Let x_k be the image of x in $(\mathcal{D}[[t]]/\mathcal{D}[[t]]t^k)\otimes_R(\widehat{R(V)}/\widehat{R(V)}_k)$. Since $\beta_k(x_k) = 0$, we can find $y_k \in (\mathcal{D}[[t]]/\mathcal{D}[[t]]t^k) \otimes_R (\widehat{R(U)}/\sigma^{-1}(\widehat{R(V)}_k))$ such that $\alpha_k(y_k) = x_k$. Choose $z_k \in \mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(U)}$ such that y_k is the image of z_k in $(\mathcal{D}[[t]]/\mathcal{D}[[t]]t^k) \otimes_R (\widehat{R(U)}/\sigma^{-1}(\widehat{R(V)}_k))$. The construction of $\{z_k\}$ implies that $\lim_{k\to\infty} \alpha(z_k) = x$.

Observe that $\mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(U)}$ and $\mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(V)}$ are bounded and complete filtered left $\mathcal{D}[[t]]$-modules and α is a filtered $\mathcal{D}[[t]]$-homomorphism. Therefore, by

Proposition 4.1, the topology on $\mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(U)}$ coincides with the topology induced on it by α. Thus, since $\{\alpha(z_k)\}$ converges, $\{z_k\}$ converges in $\mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(U)}$. Let $z=\lim_{k\to\infty} z_k$. Then clearly $\alpha(z)=x$.

Finally, it is obvious that β is onto. Thus, the sequence $\mathcal{D}[[t]]\widehat{\otimes}_R(\star)$ is exact. $\square$

Since M is a finitely generated left R-module, $\mathcal{D}[[t]]\otimes_R M$ is a finitely generated $\mathcal{D}[[t]]$-module. Hence, by Corollary 4.3, $\mathcal{D}[[t]]\widehat{\otimes}_R M$ is a finitely generated $\mathcal{D}[[t]]$-module as well. In particular $\dim_{\mathcal{D}} \operatorname{Tor}_1^{\mathcal{D}[[t]]}(\mathcal{D},\mathcal{D}[[t]]\widehat{\otimes}_R M)$ is finite.

Since $\mathcal{D}[[t]]\widehat{\otimes}_R(\star)$ is exact, we can calculate $\operatorname{Tor}_1^{\mathcal{D}[[t]]}(\mathcal{D},\mathcal{D}[[t]]\widehat{\otimes}_R M)$ by applying $\mathcal{D}\otimes_{\mathcal{D}[[t]]}$ to it. Observe that

$$\mathcal{D}[[t]]\widehat{\otimes}_R\widehat{R(U)}\cong \widehat{\mathcal{D}[[t]](U)}$$

and, by Proposition 4.9(1),

$$\mathcal{D}\widehat{\otimes}_{\mathcal{D}[[t]]}\widehat{\mathcal{D}[[t]](U)}\cong \mathcal{D}\otimes_{\mathcal{D}[[t]]}\widehat{\mathcal{D}[[t]](U)}.$$

Therefore, we can calculate $\operatorname{Tor}_1^{\mathcal{D}[[t]]}(\mathcal{D},\mathcal{D}[[t]]\widehat{\otimes}_R M)$ by applying $\mathcal{D}\widehat{\otimes}_{\mathcal{D}[[t]]}$ to the sequence $\mathcal{D}[[t]]\widehat{\otimes}_R(\star)$. This can be done by applying directly $\mathcal{D}\widehat{\otimes}_R$ to the sequence $(\star)$, because $\mathcal{D}\widehat{\otimes}_R(\star)\cong\mathcal{D}\widehat{\otimes}_{\mathcal{D}[[t]]}(\mathcal{D}[[t]]\widehat{\otimes}_R(\star))$. Hence we obtain the exact sequence

$$0\to\operatorname{Tor}_1^{\mathcal{D}[[t]]}(\mathcal{D},\mathcal{D}[[t]]\widehat{\otimes}_R M)\to\mathcal{D}\widehat{\otimes}_R\widehat{R(U)}\to\mathcal{D}\widehat{\otimes}_R\widehat{R(V)}\to\mathcal{D}\widehat{\otimes}_R M\to 0.$$

Observe that a set of E-linear independent elements in $E\widehat{\otimes}_R\widehat{R(U)}$ remains $\mathcal{D}$-linear independent in $\mathcal{D}\widehat{\otimes}_R\widehat{R(U)}$. Therefore, since $\dim_{\mathcal{D}}\operatorname{Tor}_1^{\mathcal{D}[[t]]}(\mathcal{D},\mathcal{D}[[t]]\widehat{\otimes}_R M)$ is finite, $\dim_E\ker\alpha$ is finite in the sequence

$$E\widehat{\otimes}_R\widehat{R(U)}\xrightarrow{\alpha}E\widehat{\otimes}_R\widehat{R(V)}\to E\widehat{\otimes}_R M\to 0.$$

Note that E is of type FP_2 as a right R-module. Hence, by Proposition 4.9,

$$\ker\alpha\cong\operatorname{Tor}_1^R(E,M).$$

Therefore, $\dim_E\operatorname{Tor}_1^R(E,M)$ is finite.

$\square$

Corollary 4.15. *Let M be a finitely generated right R-submodule of $\mathcal{D}((t))$. Then for any exact sequence $0\to I\to R^d\to M\to 0$ of right R-modules, I is free of finite rank.*

Proof. Let $\alpha: R^d\to M$ be a surjective homomorphism of right R-submodules. By Proposition 4.11, $\ker\alpha\cong\widehat{R(V)}$ for some bounded multiset of integers. By Propsition 4.13, $\dim_E\operatorname{Tor}_1^R(E,M)$ is finite. Therefore, $\dim_E E\otimes_R\ker\alpha$ is finite, and so V is finite. $\square$

Proposition 4.16. *Let M and N be finitely generated filtered right and left respectively R-submodules of $\mathcal{D}((t))$. Then $\operatorname{Tor}_1^R(M,N)=0$.*

Proof. We want to use Theorem 4.10. The embeddings of M and N into $\mathcal{D}((t))$ induce filtrations $\{M_i=M\cap t^i\mathcal{D}[[t]]\}$ and $\{N_i=N\cap t^i\mathcal{D}[[t]]\}$ on M and N respectively. By Proposition 4.2, M and N are complete with respect to this filtrations. The right R-module M is of type FP_2 by Corollary 4.15. N is bounded because it is finitely generated. Since $\mathrm{Gr}(M)$ and $\mathrm{Gr}(N)$ are right and left, respectively, submodules of $\mathcal{D}(t)$ and the embedding $\mathrm{Gr}(R)$ into $\mathcal{D}(t)$ is universal, by Proposition 2.2, $\mathrm{Tor}_1^{\mathrm{Gr}(R)}(\mathrm{Gr}(M),\mathrm{Gr}(N))=0$. Therefore, we can apply Theorem 4.10 and obtain that $\mathrm{Tor}_1^R(M,N)=0$. □

Proof of Theorem 1.3. Observe that from Proposition 4.16 it follows that

$$\mathrm{Tor}_1^R(\mathcal{D}((t)),\mathcal{D}((t)))=0.$$

Corollary 4.15 shows that for any finitely generated left or right R-submodule M of $\mathcal{D}((t))$ and any exact sequence $0\to I\to R^n\to M\to 0$, I is free. Thus, the theorem is a consequence of Corollary 2.5. □

5. Another characterization of $\rho_{\mathbb{F}_p[[F]]}$

In this section we prove Theorem 1.4. Let F be a free pro-p group freely generated by $f_1,\dots,f_d$. Let us denote the limit

$$\lim_{i\to\infty}\frac{\mathrm{rk}_{\mathbb{F}_p}(A_i)}{|F:F_i|}$$

from Theorem 1.4 by $\mathrm{rk}(A)$. Then rk is a Sylvester matrix rank function on $\mathbb{F}_p[[F]]$. Let I_F be the augmentation ideal of $\mathbb{F}_p[[F]]$. Since $\mathrm{rk}(A)$ does not depend on the chain $\{F_i\}$, without loss of generality we may assume that

$$F_i=\{f\in F:\ f-1\in I_F^i\}.$$

Let $\mathcal{D}=\mathcal{D}_{\mathbb{F}_p\langle y_1,\dots,y_d\rangle}$ be the universal division ring of fractions of $\mathbb{F}_p\langle y_1,\dots,y_d\rangle$. By Theorem 1.3, the continuous map $\phi:\mathbb{F}_p[[F]]\to\mathcal{D}((t))$ which sends f_i to $1+ty_i$ is universal, and so the induced Sylvester matrix rank function coincides with the inner rank.

By Proposition 2.3, in order to show that $\mathrm{rk}=\rho_{\mathbb{F}_p[[F]]}$, it is enough to prove that $\mathrm{rk}(A)=\rho_{\mathbb{F}_p[[F]]}(A)$ for any full matrix A over $\mathbb{F}_p[[F]]$. Thus, let A be a full k by k matrix over $\mathbb{F}_p[[F]]$. Put $M=(\mathbb{F}_p[[F]])^k/(\mathbb{F}_p[[F]])^kA$. The statement $\mathrm{rk}(A)=k$ is equivalent to

$$\lim_{i\to\infty}\frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[F/F_i]\otimes_{\mathbb{F}_p[[F]]}M)}{|F:F_i|}=0.$$

We divide the proof of this equality in several claims. For any open subgroup H of F we define

$$l_H(M)=\dim_{\mathcal{D}}\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[H]]}M.$$

Claim 5.1. *We have that $l_F(M)$ is finite.*

Proof. Since $\phi:\mathbb{F}_p[[F]]\to\mathcal{D}((t))$ is universal,

$$\dim_{\mathcal{D}((t))}(\mathcal{D}((t))\otimes_{\mathbb{F}_p[[F]]}M)=k-\rho_{\mathbb{F}_p[[F]]}(A)=0.$$

Thus, $\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[F]]}M$ is a torsion $\mathcal{D}[[t]]$-module, and since it is finitely generated, it is of finite $\mathcal{D}$-dimension. □

Claim 5.2. *Let H_1 be an open subgroup of F and let H_2 an open subgroup of H_1 of index p. Then $l_{H_2}(M)\le p\cdot l_{H_1}(M)$. In particular, for any open subgroup H of F, $l_H(M)\le|F:H|l_F(M)$.*

Proof. Put $L=\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[H_2]]} M$. Let $g\in H_1\setminus H_2$. For $a\in\mathcal{D}[[t]]$ and $b\in M$ we put

$$\tau(a,b)=ag\otimes g^{-1}b-a\otimes b\in L.$$

Observe that for every $c\in\mathbb{F}_p[[H_2]]$ we have

$$\tau(ac,b)=acg\otimes g^{-1}b-ac\otimes b=ag(g^{-1}cg)\otimes g^{-1}b-a\otimes cb=$$
$$ag\otimes(g^{-1}cg)g^{-1}b-a\otimes cb=ag\otimes g^{-1}cb-a\otimes cb=\tau(a,cb).$$

Thus, there exists a unique map $\psi:L\to L$ such that

$$\psi(a\otimes b)=ag\otimes g^{-1}b-a\otimes b.$$

Observe that ψ is a homomorphism of left $\mathcal{D}[[t]]$-modules, $\psi^p=0$ and $\psi^i(L)/\psi^{i+1}(L)$ is a quotient of $L/\psi(L)\cong\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[H_1]]} M$. Thus, we obtain that

$$l_{H_2}(M)=\dim_{\mathcal{D}} L\le p\cdot\dim_{\mathcal{D}}\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[H_1]]} M=p\cdot l_{H_1}(M).$$

This proves the claim. $\square$

Claim 5.3. *For every $i\ge 1$, $l_{F_i}(M)\ge i\cdot\dim_{\mathcal{D}}\mathcal{D}\otimes_{\mathbb{F}_p[[F_i]]} M$.*

Proof. Recall that $M=(\mathbb{F}_p[[F]])^k/(\mathbb{F}_p[[F]])^kA$, where A is a full k by k matrix over $\mathbb{F}_p[[F]]$. Thus M viewed as a left $\mathbb{F}_p[[F_i]]$-module is isomorphic to

$$(\mathbb{F}_p[[F_i]])^{k|F:F_i|}/(\mathbb{F}_p[[F_i]])^{k|F:F_i|}B$$

for some $B\in\mathrm{Mat}_{k|F:F_i|}(\mathbb{F}_p[[F_i]])$. Clearly, the choice of B is not unique. In fact, we can change B by any matrix PBQ, where P and Q are invertible matrices over $\mathbb{F}_p[[F_i]]$. Thus, we can choose B such that

$$B=\begin{pmatrix} I_a+B_1 & B_2\\ B_3 & B_4\end{pmatrix},$$

where $I_a\in\mathrm{Mat}_a(\mathbb{F}_p)$ is the identity matrix and B_1, B_2, B_3 and B_4 are matrices with entries lying in the augmentation ideal of $\mathbb{F}_p[[F_i]]$. Observe that

$$\dim_{\mathcal{D}}(\mathcal{D}\otimes_{\mathbb{F}_p[[F_i]]} M)=k|F:F_i|-a. \tag{4}$$

The homomorphism ϕ embeds $\mathbb{F}_p[[F_i]]$ into $\mathcal{D}[[t]]$ and sends the augmentation ideal of $\mathbb{F}_p[[F_i]]$ into $t^i\mathcal{D}[[t]]$. Therefore, we can express the matrix $\phi(B)$ as a product of two matrices over $\mathcal{D}[[t]]$ as follows.

$$\phi(B)=\phi\begin{pmatrix} I_a+B_1 & B_2\\ B_3 & B_4\end{pmatrix}=\begin{pmatrix} I_a+\phi(B_1) & B_2'\\ \phi(B_3) & B_4'\end{pmatrix}\begin{pmatrix} I_a & 0\\ 0 & t^iI_{k|F:F_i|-a}\end{pmatrix},$$

where B_2' and B_4' are matrices over $\mathcal{D}[[t]]$. Since

$$\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[F_i]]} M\cong\mathcal{D}[[t]]^{k|F:F_i|}/\mathcal{D}[[t]]^{k|F:F_i|}\phi(B),$$

the $\mathcal{D}[[t]]$-module $(\mathcal{D}[[t]]/\mathcal{D}[[t]]t^i)^{k|F:F_i|-a}$ is a quotient of $\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[F_i]]} M$. Hence

$$l_{F_i}(M)=\dim_{\mathcal{D}}\mathcal{D}[[t]]\otimes_{\mathbb{F}_p[[F_i]]} M\ge\dim_{\mathcal{D}}(\mathcal{D}[[t]]/\mathcal{D}[[t]]t^i)^{k|F:F_i|-a}=$$
$$i(k|F:F_i|-a)\overset{\text{by (4)}}{=}i\dim_{\mathcal{D}}(\mathcal{D}\otimes_{\mathbb{F}_p[[F_i]]} M).$$

$\square$

Now we are ready to prove the final claim

Claim 5.4. *We have that* $\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[F/F_i]\otimes_{\mathbb{F}_p[[F]]} M)}{|F:F_i|} = 0.$

Proof. Observe that

$$\dim_{\mathbb{F}_p}(\mathbb{F}_p[F/F_i]\otimes_{\mathbb{F}_p[[F]]} M) = \dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{F}_p[[F_i]]} M) = \dim_{\mathcal{D}}(\mathcal{D}\otimes_{\mathbb{F}_p[[F_i]]} M).$$

Therefore,

$$\frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[F/F_i]\otimes_{\mathbb{F}_p[[F]]} M)}{|F:F_i|} = \frac{\dim_{\mathcal{D}}(\mathcal{D}\otimes_{\mathbb{F}_p[[F_i]]} M)}{|F:F_i|} \overset{\text{Claim 5.3}}{\leq} \frac{l_{F_i}(M)}{i|F:F_i|} \overset{\text{Claim 5.2}}{\leq} \frac{l_F(M)}{i}.$$

Since, by Claim 5.1, $l_F(M)$ is finite, $\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[F/F_i]\otimes_{\mathbb{F}_p[[F]]} M)}{|F:F_i|} = 0.$ □

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Departamento de Matemáticas, Universidad Autónoma de Madrid and Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM

E-mail address: `andrei.jaikin@uam.es`