Free $\mathbb{Q}$-groups are residually torsion-free nilpotent

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http://matematicas.uam.es/~andrei.jaikin/preprints/baumslag.pdf

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Free $\mathbb{Q}$-groups

A group G is a $\mathbb{Q}$-**group** if for every $x \in G$, $n \in \mathbb{N}$ there exists a unique $y \in G$ such that $y^n = x$.

G. Baumslag (1960): introduced them as $\mathcal{D}$-groups

Examples: $(\mathbb{Q}, +)$; nilpotent $\mathbb{Q}$-groups are in one-to-one correspondence with nilpotent Lie $\mathbb{Q}$-algebras.

$\mathbb{Q}$-groups form a variety of algebras with one binary operation (the multiplication) and a collection of unary operations $g \mapsto g^q$ for each $q \in \mathbb{Q}$ (that satisfy the obvious defining relations).

$F^{\mathbb{Q}}(X)$ is the **free** $\mathbb{Q}$-**group** on a set X (it is a free object in this variety).
$F^{\mathbb{Q}}(X)$ contains X and it is uniquely determined by the universal property: if G is a $\mathbb{Q}$-group and $\phi : X \to G$ a map, then there exists a unique homomorphism of $\mathbb{Q}$-groups $F^{\mathbb{Q}}(X) \to G$ which extends ϕ.
$F^{\mathbb{Q}}(X) = \{\text{all the possible formal expressions}\}/\sim$

The main difficulty to work with $F^{\mathbb{Q}}(X)$: there is no obvious canonical expression for an element of $F^{\mathbb{Q}}(X)$.

Theorem (G. Baumslag, 1960)

If X is countable, then there exists a sequence $G_0 < G_1 < G_2 < \dots$ of subgroups of $F^{\mathbb{Q}}(X)$ such that

1. $G_0 = F(X)$ and $\displaystyle\bigcup_{i=0}^{\infty} G_i = F^{\mathbb{Q}}(X)$.
2. For each $i \geq 0$, $G_{i+1} = G_i *_{g_i = h_i^{n_i}} \langle h_i \rangle$, where g_i generates a maximal abelian subgroup in G_i.

Baumslag's conjecture I

$F^{\mathbb{Q}} = F^{\mathbb{Q}}(X)$: the end result of repeatedly freely adjoining nth roots to the free group $F = F(X)$.

Conjecture (G. Baumslag)

The group $F^{\mathbb{Q}}(X)$ is residually torsion-free nilpotent.

A group G is called **residually torsion-free nilpotent** if for any $g \in G$ there exists a map $\phi : G \to \overline{G}$, where $\overline{G}$ is torsion-free nilpotent and $\phi(g) \neq 1$.

Let I be a set and $X = \{x_i : i \in I\}$ and $Y = \{y_i : i \in I\}$.

$\mathbb{Q}\langle\langle Y \rangle\rangle$ the ring of non-commutative power series over Y

W. Magnus: $\Phi_F : F(X) \to 1 + \sum_{i\in I} y_i\mathbb{Q}\langle\langle Y \rangle\rangle$, $x_i \mapsto 1 + y_i$, is injective.

Since, $1 + \sum_{i\in I} y_i\mathbb{Q}\langle\langle Y \rangle\rangle$ is a $\mathbb{Q}$-group, Φ_F extends uniquely to the map

$$\Phi_{F^{\mathbb{Q}}} : F^{\mathbb{Q}}(X) \to 1 + \sum_{i\in I} y_i\mathbb{Q}\langle\langle Y \rangle\rangle.$$

An observation of Baumslag

$F^{\mathbb{Q}}(X)$ is residually torsion-free nilpotent if and only if $\Phi_{F^{\mathbb{Q}}}$ injective.

$F^{\mathbb{Q}}(X)$ is residually torsion-free nilpotent if and only if it is locally residually torsion-free nilpotent.

Baumslag's conjecture II

Baumslag's Conjecture (reformulation)

Let $G_0, G_1, \ldots, G_k$ be a sequence of groups such that G_0 is a finitely generated free group and for $0 \le i \le k-1$, $G_{i+1} = G_i *_{g_i = h_i^{n_i}} \langle h_i \rangle$, where $C_{G_i}(g_i) =< g_i >$. Then G_k is residually torsion-free nilpotent.

G. Baumslag (1968) proved that it holds when $k = 1$ and he wrote:

"... it seems likely that free $\mathcal{D}$-groups are residually torsion-free nilpotent. However the complicated nature of free $\mathcal{D}$-groups makes it difficult to substantiate such a remark."

Theorem A (A. Jaikin, 2019)

Let $G_0, G_1, \ldots, G_k$ be a sequence of groups such that $G_0 = F$ is a finitely generated free group and for $0 \le i \le k-1$, $G_{i+1} = G_i *_{A_i} B_i$, where B_i is a torsion-free finitely generated abelian group and A_i is maximal abelian in G_i. Let p be a prime such that there is no p-torsion in each B_i/A_i. Then G_k can be embedded into the pro-p completion $\mathbf{F}$ of F.

The group $\mathbf{F}$ is a free pro-p group and it is residually torsion-free nilpotent: the closure of $\langle 1+y_1, \ldots, 1+y_d \rangle$ in $\mathbb{Z}_p\langle\langle y_1, \ldots, y_d \rangle\rangle$ is isomorphic to the free pro-p group on d generators.

L^2-Betti numbers

M. Atiyah (1974) introduced L^2-Betti numbers in the context of Riemann manifolds

J. Dodziuk (1977) extended the notion of L^2-Betti numbers to the more general context of free cocompact actions of discrete groups G on CW-complexes X and defined $\beta_k^{(2)}(X, G)$.

J. Cheeger, M. Gromov (1986) extended the definition to the case of actions without the cocompatc condition.

W. Lück (1998) introduced a definition that uses dimensions of $\mathcal{N}(G)$-modules.

P. Linnell (1993): if G is torsion free and satisfy the strong Atiyah conjecture, then

$$\beta_k^{(2)}(X, G) = \dim_{\mathcal{D}(G)} H_k(X; \mathcal{D}(G)).$$

The strong Atiyah conjecture for torsion-free groups

Let G be a torsion-free group. All L^2-Betti numbers $\beta_k^{(2)}(X, G)$ are integers.

P. Linnell (1993): the strong Atiyah conjecture holds for free groups

T. Schick (2000): the strong Atiyah conjecture holds for residually torsion-free nilpotent groups.

A group G is locally indicable if any non-trivial finitely generated subgroup of G maps onto $\mathbb{Z}$.
A. Jaikin, D. López-Álvarez (2019): a locally indicable group satisfies the strong Atiyah conjecture.

Can we describe the $\mathbb{Q}G$-ring $\mathcal{D}(G)$ algebraically?

Our motivation: We want to define mod-p L^2-Betti numbers.

Epic division R-rings

Let R be a unital ring. An epic division R-ring is a homomorphism $\phi : R \to \mathcal{D}$, where $\mathcal{D}$ is a division ring and the division closure of $\phi(R)$ is equal to $\mathcal{D}$.

$\phi_1 : R \to \mathcal{D}_1$ and $\phi_2 : R \to \mathcal{D}_2$ are isomorphic if there exists an isomorphism $\alpha : \mathcal{D}_1 \to \mathcal{D}_2$ such that $\phi_2 = \alpha \circ \phi_1$.

The theory of epic division R-rings has been developed by P. Cohn, P. Malcolmson, A. Schofield, G.Bergman, W. Dicks, ...

Examples:

- $\mathbb{Q}G \hookrightarrow \mathcal{D}(G)$ if G satisfies the Atiyah conjecture,
- the epic division $\mathbb{Z}$-rings are $\mathbb{F}_p$ (p a prime) and $\mathbb{Q}$.
- if R is a commutative ring, then there exists one to one correspondence between $\mathrm{Spec}(R)$ and the isomorphism classes of epic R-fields.
- D. Herbera, J. Sánchez (2015): the free K-algebra $K < x, y >$ has infinitely many non-isomorphic faithful epic division $K < x, y >$-rings

Universal division ring of fractions (P. Cohn)

For each division R-ring $\phi : R \to \mathcal{D}$ and any R-module we define M
$$\dim_{\mathcal{D}} M = \dim_{\mathcal{D}}(\mathcal{D} \otimes_R M).$$
The universal division ring of fraction of R is an epic division R-ring $\phi : R \to \mathcal{D}_R$ such that

1. ϕ is an embedding and
2. for every epic division R-ring $\psi : R \to \mathcal{E}$ and every finitely presented R-module M, $\dim_{\mathcal{D}_R} M \leq \dim_{\mathcal{E}} M$.

Example: $\mathbb{Q}$ is the universal division $\mathbb{Z}$-ring of fractions.

A. Jaikin (2019): If $\mathbb{Q}G$ has the universal division ring of fractions, then G is locally indicable.

Conjecture

Let G be a locally indicable group and E a division ring. Then there exists a universal division EG-ring of fractions.

A. Jaikin (2019): the conjecture holds for residually torsion-free nilpotent groups. Moreover, $\mathcal{D}(G)$ is isomorphic to $\mathcal{D}_{\mathbb{Q}G}$ as $\mathbb{Q}G$-rings, and so

$$\beta_k^{(2)}(X, G) = \dim_{\mathcal{D}(G)} H_k(X; \mathcal{D}(G)) = \dim_{\mathcal{D}_{\mathbb{Q}G}} H_k(X; \mathcal{D}_{\mathbb{Q}G}).$$

Definition of mod-p L^2-Betti numbers for residually torsion-free nilpotent groups

If G is residually torsion-free nilpotent then, $\beta_k^{\mathrm{mod}-p}(X, G) = \dim_{\mathcal{D}_{\mathbb{F}_pG}} H_k(X; \mathcal{D}_{\mathbb{F}_pG})$. In particular, $\beta_k^{\mathrm{mod}-p}(G) = \dim_{\mathcal{D}_{\mathbb{F}_pG}} H_k(G; \mathcal{D}_{\mathbb{F}_pG})$.

The Lück approximation

Let G be a group of type FP_{k+1}, $G > G_1 > G_2 > \dots$ a chain of normal subgroup of G of finite index with trivial intersection.

W. Lück (1994): $\exists \lim_{i\to\infty} \frac{b_k(G_i;\mathbb{Q})}{|G:G_i|} = \beta_k^{(2)}(G)$.

Conjecture (The Lück approximation conjecture in characteristic p)

1. There exists $\lim_{i\to\infty} \frac{b_k(G_i;\mathbb{F}_p)}{|G:G_i|}$ and
2. the limit does not depend on the chain $\{G_i\}$.

Theorem B (A. Jaikin, 2019)

Let G be a FP_k subgroup of a finitely generated free pro-p group $\mathbf{F}$. Let $\mathbf{F} > N_1 > N_2 > \dots$ be a chain of open normal subgroups with trivial intersection. Let $G_i = N_i \cap G$. Then

$$\lim_{i\to\infty} \frac{b_k(G_i;\mathbb{F}_p)}{|G:G_i|} = \beta_k^{\mathrm{mod}-p}(G).$$

The main consequence of Theorem B

Theorem B (A. Jaikin, 2019)

Let G be a FP_k subgroup of a finitely generated free pro-p group $\mathbf{F}$. Let $\mathbf{F} > N_1 > N_2 > \dots$ be a chain of open normal subgroups with trivial intersection. Let $G_i = N_i \cap G$. Then $\lim_{i\to\infty} \frac{b_k(G_i;\mathbb{F}_p)}{|G:G_i|} = \beta_k^{\mathrm{mod}-p}(G)$.

Corollary

Let $\mathbf{F}$ be a free pro-p group of rank d and G a finitely generated dense subgroup of $\mathbf{F}$. Then $\beta_1^{\mathrm{mod}-p}(G) \geq d-1$.

Proof: The closure of G_j in $\mathbf{F}$ is equal to N_j. Therefore,

$$|G_j : G_j^p[G_j, G_j]| \geq |N_j : N_j^p[N_j, N_j]|.$$

Then we obtain

$$\dim_{\mathbb{F}_p} H_1(G_j;\mathbb{F}_p) = |G_j : G_j^p[G_j, G_j]| \geq |N_j : N_j^p[N_j, N_j]| = (d-1)|\mathbf{F} : N_j| + 1 = (d-1)|G : G_j| + 1.$$

Theorem B implies that $\beta_1^{\mathrm{mod}-p}(G) = \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{|G:G_i|} \geq d-1.$

A construction of subgroups of a free pro-p group

Theorem A (reformulation)

Let $H_0 = F$ be a finitely generated free group, $\mathbf{F}$ its pro-p completion and $H_0 \hookrightarrow \mathbf{F}$ the canonical embedding. We construct H_i inductively. For $i \geq 0$, let B_i be a finitely generated abelian subgroup of $\mathbf{F}$ such that $A_i = B_i \cap H_i$ is maximal abelian in H_i. Put $H_{i+1} = \langle H_i, B_i \rangle \leq \mathbf{F}$. Then for every $i \geq 1$, the canonical map $H_i *_{A_i} B_i \to H_{i+1}$ is an isomorphism.

Let G be a finitely generated dense subgroup of $\mathbf{F}$.

We have shown that Theorem B implies that $\beta_1^{\mathrm{mod}-p}(G) \geq d(\mathbf{F}) - 1$.

We say that a f.g. dense subgroup G of $\mathbf{F}$ is **strongly embeddable** (SE) if $\beta_1^{\mathrm{mod}-p}(G) = d(\mathbf{F}) - 1$.

Example: Let F be a free group and $\mathbf{F}$ its pro-p completion. Then F is a SE subgroup of $\mathbf{F}$.

Theorem A'

Let G be a SE subgroup of $\mathbf{F}$ and let B be a f.g. abelian subgroup of $\mathbf{F}$ such that $A = G \cap B$ is maximal abelian in G. Then the canonical map $G *_A B \to \langle G, B \rangle$ is an isomorphism and, moreover, $\langle G, B \rangle$ is SE.

The Swan-Lewin theorem

Let G be a group and H a subgroup of G. We denote by I_G the augmentation ideal of $\mathbb{F}_pG$ and by I_H^G the left ideal of $\mathbb{F}_p[G]$ generated by $\{h-1 : h \in H\}$.

R. Swan (1969), J. Lewin (1970)

Let $\widetilde{G}$ be a group and H_1 and H_2 two subgroups that generate $\widetilde{G}$ and have intersection $A = H_1 \cap H_2$. Then the canonical map $H_1 *_A H_2 \to \widetilde{G}$ is an isomorphism if and only if $I_{H_1}^{\widetilde{G}} \cap I_{H_2}^{\widetilde{G}} = I_A^{\widetilde{G}}$ in $\mathbb{F}_p[G]$.

The condition $I_{H_1}^{\widetilde{G}} \cap I_{H_2}^{\widetilde{G}} = I_A^{\widetilde{G}}$ is equivalent to the exactness of the sequence

$$\begin{array}{ccccccccc} 0 & \to & I_A^{\widetilde{G}} & \to & I_{H_1}^{\widetilde{G}} \oplus I_{H_2}^{\widetilde{G}} & \to & I_{\widetilde{G}} & \to & 0 \\ & & a & \mapsto & (a,-a) & & & & . \\ & & & & (b,c) & \mapsto b+c & & & \end{array}$$

or the triviality of K in the exact sequence

$$0 \to K \to (I_{H_1}^{\widetilde{G}} \oplus I_{H_2}^{\widetilde{G}})/I_A^{\widetilde{G}} \to I_{\widetilde{G}} \to 0.$$

Proof of Theorem A'

Theorem A'

Let G be a SE subgroup of $\mathbf{F}$ and let B be a f.g. abelian subgroup of $\mathbf{F}$ such that $A = G \cap B$ is maximal abelian in G. Then the canonical map $G *_A B \to \langle G, B \rangle$ is an isomorphism and, moreover, $\langle G, B \rangle$ is SE.

Proof: Let $\widetilde{G} = \langle G, B \rangle$. Consider the exact sequence $0 \to K \to (I_G^{\widetilde{G}} \oplus I_B^{\widetilde{G}})/I_A^{\widetilde{G}} \to I_{\widetilde{G}} \to 0$. We want to show that $K = 0$.

Our idea: Let R be a commutative domain and $\mathcal{D}_R$ its field of fractions. Let $0 \to L \to M_1 \to M_2 \to 0$ be an exact sequence such that $\dim_{\mathcal{D}_R} M_1 = \dim_{\mathcal{D}_R} M_2$. Does it mean that $L = 0$? NO. But the answer is YES if we know that M_1 is a submodule of $\mathcal{D}_R^n$ (M_1 is $\mathcal{D}_R$-torsion free).

We obtain that $K = 0$ in two steps:

Step 1: $\dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_G^{\widetilde{G}} \oplus I_B^{\widetilde{G}})/I_A^{\widetilde{G}} = \dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}} I_{\widetilde{G}}$.

Step 2: $(I_G^{\widetilde{G}} \oplus I_B^{\widetilde{G}})/I_A^{\widetilde{G}}$ can be embedded in $\mathcal{D}^n_{\mathbb{F}_p\widetilde{G}}$.

Proof of Step 1: Let $H \leq \tilde{G}$, then we have $\dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_H^{\widetilde{G}}) = \dim_{\mathcal{D}_{\mathbb{F}_pH}} I_H = \beta_1^{\mathrm{mod}-p}(H) + 1$.

Hence, $\dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_{\widetilde{G}}) \geq d$, $\dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_G^{\widetilde{G}}) = d$ and $\dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_A^{\widetilde{G}}) = \dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_B^{\widetilde{G}}) = 1$.

Therefore, $\dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_G^{\widetilde{G}} \oplus I_B^{\widetilde{G}})/I_A^{\widetilde{G}} = d$, and so,

$$d = \dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_G^{\widetilde{G}} \oplus I_B^{\widetilde{G}})/I_A^{\widetilde{G}} \geq \dim_{\mathcal{D}_{\mathbb{F}_p\widetilde{G}}}(I_{\widetilde{G}}) \geq d.$$

Consequences of Theorem A and further open questions

The Baumslag problem

The Magnus map $\Phi_{F^{\mathbb{Q}}} : F^{\mathbb{Q}}(X) \to 1 + \sum_{i\in I} y_i \mathbb{Q}\langle\langle Y \rangle\rangle$ (that sends x_i to $1 + y_i$) is an embedding.

Let X be a finite set. Then for any prime p, the free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}(X)$ is embedded into the pro-p completion of $F(X)$. Thus, any limit group can be embedded into a free pro-p group for every p.

Problem (A. Lubotzky)

Is a non-abelian free pro-p group linear over a field?

Can a non-abelian free pro-p group be embedded into $SL_2(\mathbb{C})$?

Let X be a finite set and p a prime. Then our proof shows that the free $\mathbb{Z}_p$-group $F^{\mathbb{Z}_p}(X)$ is embedded into the pro-p completion of $F(X)$.

Problem

Can every finitely generated subgroup of a free pro-p group be embedded into a free $\mathbb{Z}_p$-group.

Our results relate the problem of Lubotzky to another known problem about free $\mathbb{Q}$-groups.

Problem (I. Kapovich)

Are free $\mathbb{Q}$-groups linear over a field?

Thanks

THANK YOU FOR YOUR ATTENTION