THE BASE CHANGE IN THE ATIYAH AND THE LÜCK APPROXIMATION CONJECTURES

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Abstract. Let F be a free finitely generated group and $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F])$. For each quotient $G = F/N$ of F we can define a von Neumann rank function $\mathrm{rk}_G(A)$ associated with the l^2-operator $l^2(G)^n \to l^2(G)^m$ induced by right multiplication by A.

For example, in the case where G is finite, $\mathrm{rk}_G(A) = \frac{\mathrm{rk}_{\mathbb{C}}(\bar{A})}{|G|}$ is the normalized rank of the matrix $\bar{A} \in \mathrm{Mat}_{n\times m}(\mathbb{C}[G])$ obtained by reducing the coefficients of A modulo N.

One of the variations of the Lück approximation conjecture claims that the function $N \mapsto \mathrm{rk}_{F/N}(A)$ is continuous in the space of marked groups. The strong Atiyah conjecture predicts that if the least common multiple $\mathrm{lcm}(G)$ of the orders of finite subgroups of G is finite, then $\mathrm{rk}_G(A) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

In our first result we prove the sofic Lück approximation conjecture. In particular, we show that the function $N \mapsto \mathrm{rk}_{F/N}(A)$ is continuous in the space of sofic marked groups. Among other consequences we obtain that a strong version of the algebraic eigenvalue conjecture, the center conjecture and the independence conjecture hold for sofic groups.

In our second result we apply the sofic Lück approximation and we show that the strong Atiyah conjecture holds for groups from a class $\mathcal{D}$, virtually compact special groups, Artin's braid groups and torsion-free p-adic analytic pro-p groups.

Contents

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1. Introduction

1.1. **The strong Atiyah conjecture.** The strong Atiyah conjecture arose from a question of M. F. Atiyah [2, page 72] about whether L^2-Betti numbers of a manifold with a cocompact proper G-action can be irrational. In [8] J. Dodziuk reformulated the Atiyah question as a question about CW-complexes of finite type and this problem received the name of the Atiyah conjecture. T. Austin [3] showed that the set of L^2-Betti numbers arising from finitely generated groups is uncountable (hence it contains irrational values, but no explicit value was given). Explicit examples appear in [15, 24, 36, 16]. These examples also lead to constructions of closed Riemannian manifolds with irrational L^2-Betti numbers confirming the prediction of M. Atiyah. All these examples involve groups having finite subgroups of arbitrary large order. The formulation of the strong Atiyah conjecture for groups with a uniform bound of orders of finite subgroups is due to W. Lück and T. Schick. The case of torsion-free groups is of particular interest, because it generalizes the famous Kaplansky's zero divisor conjecture.

For a countable set X, let $l^2(X)$ denote the Hilbert space with Hilbert basis the elements of X; thus $l^2(X)$ consists of all square summable formal sums $\sum_{x\in X} a_x x$ with $a_x \in \mathbb{C}$ and the inner product is

$$\langle \sum_{x\in X} a_x x, \sum_{y\in X} b_y y\rangle = \sum_{x\in X} a_x \overline{b_x}.$$

Let G be a countable group. Then G acts by left and right multiplication on $l^2(G)$. The right action of G on $l^2(G)$ extends to an action of $\mathbb{C}[G]$ on $l^2(G)$ and so we obtain that the group algebra $\mathbb{C}[G]$ acts faithfully as bounded linear operators on $l^2(G)$. In what follows we will simply consider $\mathbb{C}[G]$ as a subalgebra of $\mathcal{B}(l^2(G))$, the algebra of bounded linear operators on $l^2(G)$.

A finitely generated **Hilbert** G-module is a closed subspace $V \le l^2(G)^n$, invariant under the left action of G. We denote by $\operatorname{proj}_V : l^2(G)^n \to l^2(G)^n$ the orthogonal projection onto V and we define

$$\dim_G V := \operatorname{Tr}_G(\operatorname{proj}_V) := \sum_{i=1}^n \langle (\mathbf{1}_i)\operatorname{proj}_V, \mathbf{1}_i\rangle_{(l^2(G))^n},$$

where $\mathbf{1}_i$ is the element of $l^2(G)^n$ having 1 in the ith entry and 0 in the rest of the entries. The number $\dim_G V$ is the **von Neumann dimension** of V.

Let $A \in \operatorname{Mat}_{n\times m}(\mathbb{C}[G])$ be a matrix over $\mathbb{C}[G]$. The action of A by right multiplication on $l^2(G)^n$ induces a bounded linear operator $\phi_G^A : l^2(G)^n \to l^2(G)^m$. We put

$$\operatorname{rk}_G(A) = \dim_G \overline{\operatorname{Im}\phi_G^A} = n - \dim_G \ker \phi_G^A.$$

Observe that if G is finite, then $\operatorname{rk}_G = \frac{\operatorname{rk}_{\mathbb{C}}}{|G|}$. If G is a quotient of a group F and $A \in \operatorname{Mat}_{n\times m}(\mathbb{C}[F])$ is a matrix over $\mathbb{C}[F]$, by abuse of notation, we will also write $\operatorname{rk}_G(A)$ instead of $\operatorname{rk}_G(\bar{A})$, where $\bar{A}$ is the image of A in $\operatorname{Mat}_{n\times m}(\mathbb{C}[G])$.

If G is not a countable group then rk_G is also well defined. Take a matrix A over $\mathbb{C}[G]$. Then the group elements that appear in A are contained in a finitely generated group H. We will put $\operatorname{rk}_G(A) = \operatorname{rk}_H(A)$. One easily checks that the value $\operatorname{rk}_H(A)$ does not depend on the subgroup H.

Now we are ready to formulate the strong Atiyah conjecture.

Conjecture 1 (The strong Atiyah conjecture over K for a group G)**.** *Let K be a subfield of $\mathbb{C}$. Assume that there exists an upper bound for the orders of finite subgroups of G and let $\operatorname{lcm}(G)$ be the least common multiple of these orders. Then for every $A \in \operatorname{Mat}_{n\times m}(K[G])$, we have that*

$$\operatorname{rk}_G(A) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

There is a considerable body of work to establish the strong Atiyah conjecture for different classes of groups and fields. The first important contribution is due to P. Linnell [25] who proved the strong Atiyah conjecture over an arbitrary subfield of $\mathbb{C}$ if G is a group from the class $\mathcal{C}$, where $\mathcal{C}$ is the smallest class of groups which

(1) contains all free groups;
(2) is closed under direct unions;
(3) satisfies $G \in \mathcal{C}$ whenever G contains a free subgroup N with G/N elementary amenable.

The free group case in Linnell's proof is handled by means of the Fredholm module associated with a group action on a tree. Then the general case is reduced to the free group case applying algebra techniques inspired by J. A. Moody's paper [33].

J. Dodziuk et al. [9] proved Conjecture 1 for groups from the class $\mathcal{D}$ over $\bar{\mathbb{Q}}$, the field of algebraic numbers. The class $\mathcal{D}$ is the smallest non-empty class of groups such that:

(1) If G is torsion-free and A is elementary amenable, and we have a projection $p: G \to A$ such that $p^{-1}(E) \in \mathcal{D}$ for every finite subgroup E of A, then $G \in \mathcal{D}$.
(2) $\mathcal{D}$ is subgroup closed.
(3) Let $G_i \in \mathcal{D}$ be a directed system of groups and G its (direct or inverse) limit. Then $G \in \mathcal{D}$.

Note that the class $\mathcal{D}$ contains all residually torsion-free solvable groups. A main new ingredient of the proof with respect to Linnell's paper [25] consists in the use of the Lück approximation which we will discuss below. This idea appeared first in a paper of T. Schick [39].

It is a standard fact that if G satisfies the strong Atiyah conjecture, then it also holds for every subgroup H of G satisfying $\operatorname{lcm}(H) = \operatorname{lcm}(G)$. The question whether the strong Atiyah Conjecture holds for a group G if it holds for a subgroup of finite index is a very delicate one. Some partial results were obtained in [26, 41]. Using these results the strong Atiyah conjecture over $\bar{\mathbb{Q}}$ is proved for Artin's braid groups [26] and for finite extensions of the fundamental groups of compact special cube complexes [41].

In [14, Theorem 1.1] the strong Atiyah conjecture over $\bar{\mathbb{Q}}$ was proved for torsion-free p-adic analytic pro-p groups. This, for example, implies that any finitely generated group, which is linear over a field of zero characteristic, contains a subgroup of finite index satisfying the strong Atiyah conjecture over $\bar{\mathbb{Q}}$. The proof also uses the Lück approximation.

In [9] it was shown that the Kaplansky zero-divisor conjeture for $\mathbb{C}[G]$ follows from the Kaplansky zero-divisor conjeture for $\bar{\mathbb{Q}}[G]$. In this paper we show that the strong Atiyah conjecture for sofic groups over an arbitrary field follows from the one over the field of algebraic numbers.

Theorem 1.1. *Let G be a sofic group. Assume that the strong Atiyah conjecture holds for G over $\bar{\mathbb{Q}}$. Then the strong Atiyah conjecture holds for G over $\mathbb{C}$.*

As a consequence we prove the strong Atiyah conjecture over an arbitrary subfield of $\mathbb{C}$ in all the cases that we have described above.

Corollary 1.2. *Let G be a group belonging to one of the following families*

(1) *the class $\mathcal{D}$;*
(2) *Artin's braid groups;*
(3) *finite extensions of fundamental groups of compact special cube complexes;*
(4) *torsion-free p-adic analytic pro-p groups.*

Then G satisfies the strong Atiyah conjecture over $\mathbb{C}$.

The proof of Theorem 1.1 depends on the solution of the sofic Lück approximation conjecture over arbitrary fields of zero characteristic that we present in Subsection 1.3.

1.2. **The Lück approximation conjecture.** First let us introduce the Lück approximation conjecture. We use the notation from the previous subsection.

Conjecture 2 (The Lück approximation conjecture over K for a group G)**.** *Let K be a subfield of $\mathbb{C}$, F a finitely generated free group and $F > N_1 > N_2 > \dots$ be a chain of normal subgroups of F with intersection $N = \cap N_i$. Put $G_i = F/N_i$ and $G = F/N$. Then for every $A \in \operatorname{Mat}_{n\times m}(K[F])$,*

$$\lim_{k\to\infty} \operatorname{rk}_{G_k}(A) = \operatorname{rk}_G(A).$$

W. Lück gave an elegant proof of this conjecture in the case where $\{G_k\}$ is a family of finite groups and $A \in \operatorname{Mat}_{n\times m}(\mathbb{Q}[F])$ ([29]). G. Elek and E. Szabó [13] observed that Lück's approach can be applied in a more general situation where the groups G_k are sofic. In [9] J. Dodziuk at al. developed a method that allowed to prove the Lück approximation conjecture over $\bar{\mathbb{Q}}$ when $\{G_k\}$ are in $\mathcal{D}$. Combining the ideas of [13] and [9] one can unify the results of these papers and show that the conjecture holds when the groups G_k are sofic and the coefficients of A are in $\bar{\mathbb{Q}}$ (a sketch of the proof of a slightly more general result can be found in [23]).

At this moment there are only two cases for which the Lück approximation conjecture with arbitrary coefficients is known to be true in full generality. The case where G is amenable is due to G. Elek ([10], see also [35]) and the case where G is free is due to the author ([22]).

Remark. *We have only formulated the Lück approximation conjecture for finitely generated groups G. If G is an arbitrary group, we say that it satisfies this conjecture if all its finitely generated subgroups do.*

1.3. **The sofic Lück approximation conjecture.** As before, let F be a free finitely generated group and assume that it is freely generated by a set S. Recall that an element w of F has **length** n if w can be expressed as a product of n elements from $S \cup S^{-1}$ and n is the smallest number with this property. We denote by $B_k(1)$ the set of elements of length at most k.

Let N be a normal subgroup of F. We put $G = F/N$. We say that G is **sofic** if there is a family $\{X_k : k \in \mathbb{N}\}$ of finite F-sets (F acts on the right) such that if we put

$$T_{k,s} = \{x \in X_k : \ x = x\cdot w \text{ if } w \in B_s(1)\cap N, \text{ and } \ x \neq x\cdot w \text{ if } w \in B_s(1)\setminus N\},$$

then for every s,

$$\lim_{k\to\infty} \frac{|T_{k,s}|}{|X_k|} = 1.$$

The family of F-sets $\{X_k\}$ is called a **sofic approximation** of G. This is one of many equivalent definitions of soficity for a finitely generated group; we have found this one in [43, Proposition 1.4]. For an arbitrary group G we say that G is **sofic** if every finitely generated subgroup of G is sofic. Amenable groups and residually finite groups are sofic. At this moment no nonsofic group is known.

Now, let us generalize slightly the notation introduced in Subsection 1.1. Let $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F])$ be a matrix over $\mathbb{C}[F]$. By multiplication on the right side, A induces a linear operator $\phi^A_{X_k} : l^2(X_k)^n \to l^2(X_k)^m$. We put

$$\dim_{X_k} \ker \phi^A_{X_k} = \frac{\dim_{\mathbb{C}} \ker \phi^A_{X_k}}{|X_k|} \quad \text{and} \quad \mathrm{rk}_{X_k}(A) = n - \dim_{X_k} \ker \phi^A_{X_k}.$$

Now we are ready to state the main result of this paper.

Theorem 1.3 (The sofic Lück approximation conjecture over K for a group G)**.** *Let K be a subfield of $\mathbb{C}$. Let $\{X_k\}$ be a sofic approximation of $G = F/N$. Then for every $A \in \mathrm{Mat}_{n\times m}(K[F])$,*

$$\lim_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A).$$

This result was known previously when $K = \bar{\mathbb{Q}}$ ([13], [9]). The proof of this particular case used the solution of the so-called determinant conjecture in the case of sofic groups when $K = \bar{\mathbb{Q}}$. This method cannot work for general K because the determinant conjecture is not valid over an arbitrary field K (see [23]). Our approach is completely different and it uses the theory of epic $*$-regular R-rings, which we develop in this paper. We will give a more detailed sketch of the proof of Theorem 1.3 in Section 2.

1.4. **The Lück approximation in the space of marked groups.** There are many different types of approximations that one may consider. In this subsection we describe the approximation in the space of marked groups. A more general kind of approximation is described in Subsection 6.3.

The **space of marked groups** $\mathrm{MG}(F,S)$ can be identified with the set of normal subgroups of F with the metric $d(N_1, N_2) = e^{-n}$, where n is the largest integer such that the balls of radius n in the Cayley graphs of F/N_1 and F/N_2 with respect to the generators S are simplicially isomorphic (with respect to an isomorphism respecting the labelings). In this setting the approximation conjecture may be stated in the following way.

Conjecture 3 (The Lück approximation conjecture in the space of marked groups over K for a group G)**.** *Let $\{N_k \in \mathrm{MG}(F,S)\}$ converge to $N \in \mathrm{MG}(F,S)$. Put $G = F/N$ and $G_k = F/N_k$. Then for every $A \in \mathrm{Mat}_{n\times m}(K[F])$,*

$$\lim_{k\to\infty} \mathrm{rk}_{G_k}(A) = \mathrm{rk}_G(A).$$

Clearly Conjecture 3 is a strong version of Conjecture 2. Applying Theorem 1.3 we obtain the following corollary.

Corollary 1.4. *Conjecture 3 holds over $\mathbb{C}$ if all G_i are sofic.*

1.5. **Some other applications of Theorem 1.3.** The algebraic eigenvalue conjecture was introduced in [9]. It claims that if $A \in \mathrm{Mat}_n(\bar{\mathbb{Q}}[F])$, then all the eigenvalues of the operator ϕ_G^A are algebraic. In [9] it was proved for the groups G from the Linnell class $\mathcal{C}$ and in [44] in the case where G is sofic, $A \in \mathrm{Mat}_n(\mathbb{Q}[F])$ and $A = A^*$.

Another consequence of Theorem 1.3 is the proof of the following strong version of the algebraic eigenvalue conjecture for sofic groups. In [26] P. Linnell and T. Schick showed that if K is a subfield of $\mathbb{C}$ closed under complex conjugation, then there exists the smallest $*$-regular subring of $\mathcal{U}(G)$ that contains $K[G]$. We denote this subring by $\mathcal{R}_{K[G]}$. The reader can look at Subsection 3.4 where we will give more details concerning the notion of $*$-regular closure.

Corollary 1.5 (The strong algebraic eigenvalue conjecture for sofic groups)**.** *Let G be a countable sofic group, K a subfield of $\mathbb{C}$ closed under complex conjugation and $A \in \mathrm{Mat}_n(\mathcal{R}_{K[G]})$. Then for any $\lambda \in \mathbb{C}$ which is not algebraic over K, the matrix $A - \lambda I_n$ is invertible over $\mathcal{U}(G)$.*

We denote by $\overline{\mathcal{R}_{K[G]}}$ the closure of $\mathcal{R}_{K[G]}$ in $\mathcal{U}(G)$ with respect to the rk_G-metric. Applying Theorem 1.3 we also obtain the following corollary.

Corollary 1.6 (The center conjecture for sofic groups)**.** *Let K be a subfield of $\mathbb{C}$ closed under complex conjugation and let G be a countable sofic group. Then*

$$\overline{\mathcal{R}_{K[G]}} \cap \mathbb{C} = K.$$

In particular, if G is an ICC group, then $Z(\overline{\mathcal{R}_{K[G]}}) = K$.

The next application of Theorem 1.3 shows that the von Neumann rank of a matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$ does not depend on the embedding of K into $\mathbb{C}$ if G is sofic.

Corollary 1.7 (The independence conjecture for sofic groups)**.** *Let G be a sofic group. Let K be a field and let $\phi_1, \phi_2 : K \to \mathbb{C}$ be two embeddings of K into $\mathbb{C}$. Then for every matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$*

$$\mathrm{rk}_G(\phi_1(A)) = \mathrm{rk}_G(\phi_2(A)).$$

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2. The description of the proofs of Theorems 1.3 and 1.1

In this section we give an overview of the proofs of Theorems 1.3 and 1.1. We hope that this section will help the reader to create a general picture of our argument. We start introducing the general notation used in the paper.

2.1. **General conventions and notations.**

In this paper all rings and homomorphisms are unital. The letter K is reserved for a field and by an algebra we will always mean a K-algebra.

If R is a ring, an R-module will usually mean a left R-module. The category of R-modules is denoted by R-Mod.

$R[x]$ is the ring of polynomials over R and $R[x^{\pm 1}]$ is the ring of Laurent polynomials.

A $*$-ring is a ring R with a map $*: R \to R$ that is an involution (i. e. $(x^*)^* = x,\ (x+y)^* = x^* + y^*,\ (xy)^* = y^*x^*\ (x, y \in R)$). If K is a $*$-ring, then a $*$-algebra is an algebra with an involution $*$ satisfying $(\lambda x)^* = \lambda^* x^*\ (\lambda \in K,\ x \in R)$.

An element of a $*$-ring e is called a **projection** if e is an **idempotent** ($e^2 = e$) and e is **self-adjoint** ($e^* = e$). A projection may be also defined as an element e satisfying $e = ee^*$.

For every subset S of a ring R,

$$\mathrm{Ann}_l^R(S) = \{r \in R : \ rx = 0 \text{ for every } \ x \in S\}$$

will denote the left annihilator of S in R. Similarly, $\mathrm{Ann}_r^R(S)$ will denote the right annihilator of S in R.

If $n \geq 1$ we denote by I_n the n by n identity matrix. For matrices A and B, $A \oplus B$ denotes the direct sum of A and B:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

2.2. **The theory of $*$-regular R-rings and a structural reformulation of the sofic Lück approximation conjecture.**

We start the paper by developing the theory of $*$-regular R-rings. In Section 3 we present the main results concerning von Neumann regular rings and $*$-regular rings and we also explain the construction of the $*$-regular closure. In Section 4 we recall the notion of epic homomorphisms and the Cohn theory of epic division R-algebras. This theory is our main inspiration to create the theory of epic $*$-regular R-rings. In Section 5 we explain the notion of Sylvester rank function and prove some of its properties. Finally, in Section 6 we will show that a $*$-regular closure of a $*$-subring R is an epic R-ring (Proposition 6.1) and using this result we will show that any $*$-regular Sylvester rank function has a canonical $*$-regular envelope (Theorem 6.3). There are two relevant examples which are related to the sofic Lück approximation conjecture.

Let K be a subfield of $\mathbb{C}$ closed under complex conjugation and let G be a countable group. Then rk_G is a $*$-regular Sylvester matrix rank function on $K[G]$. The $*$-regular algebra associated with rk_G is $\mathcal{R}_{K[G]}$. Now, let $\{X_k\}_{k\in\mathbb{N}}$ be a family of finite F-sets. Assume that $\{X_k\}$ approximates G. Fix a non-principal ultrafilter ω on $\mathbb{N}$. Then $\mathrm{rk}_\omega = \lim\limits_\omega \mathrm{rk}_{X_k}$ is another $*$-regular Sylvester matrix rank function on $K[G]$. We denote by $\mathcal{R}_{K[G],\omega}$ the $*$-regular $K[G]$-algebra associated with rk_ω.

A straightforward reformulation of the sofic Lück approximation conjecture over K is to say that for every non-principal ultrafilter ω on $\mathbb{N}$,

$$\mathrm{rk}_G = \mathrm{rk}_\omega \text{ as Sylvester matrix rank functions on } K[G].$$

Our structural reformulation of the sofic Lück approximation conjecture over K (Theorem 6.6) implies that it is equivalent to the existence of a $K[G]$-$*$-isomorphism

$$\alpha_K : \mathcal{R}_{K[G]} \to \mathcal{R}_{K[G],\omega} \text{ such that } \quad \mathrm{rk}_G = \mathrm{rk}_\omega \circ \alpha_K.$$

At first glance it seems that this reformulation cannot help us to prove Theorem 1.3, because to prove the existence of α_K is harder than to prove the equality between the Sylvester matrix rank functions rk_G and rk_ω. However, as we have already mentioned in Section 1, Theorem 1.3 is already known when $K = \mathbb{Q}$ (and in fact, when $K = \bar{\mathbb{Q}}$). Thus, we know that $\alpha_{\mathbb{Q}}$ exists! This will be the first brick in our construction of α_K for an arbitrary subfield K of $\mathbb{C}$.

It is clear that it is enough to prove the sofic Lück approximation conjecture over finitely generated subfields K of $\mathbb{C}$. Any finitely generated subfield K of $\mathbb{C}$ is a subfield of a field K_{2n}, where K_i are constructed inductively:

(1) $K_1 = \mathbb{Q}$;
(2) if $i \geq 1$, $K_{2i} = \overline{K_{2i-1}}$ is the algebraic closure of K_{2i-1} in $\mathbb{C}$;
(3) if $i \geq 1$, $K_{2i+1} = K_{2i}(\lambda_i)$ for some $\lambda_i \in \mathbb{C} \setminus K_{2i}$, satisfying $\overline{\lambda_i} = \lambda_i^{-1}$.

Observe that by construction, the subfields K_i are closed under complex conjugation. We will prove the sofic Lück approximation conjecture over K_i by induction on i.

2.3. **The inductive step for algebraic extensions.** Given a Sylvester matrix rank function rk on an algebra R and an algebraic extension E/K we will define in Subsection 7.5 a Sylvester matrix rank function $\widetilde{\mathrm{rk}}$ on the algebra $R \otimes_K E$ that we will call the natural algebraic extension of rk on $R \otimes_K E$.

Assume that K is closed under complex conjugation, G is sofic and the sofic Lück approximation holds over K (so α_K exists). Then we will show that

$$\mathcal{R}_{\bar{K}[G]} \cong \mathcal{R}_{K[G]} \otimes_K \bar{K}$$

and the restriction of rk_G to $\mathcal{R}_{\bar{K}[G]}$ is the natural algebraic extension of the restriction of rk_G to $\mathcal{R}_{K[G]}$ (Theorem 10.2). Similarly, we will obtain that

$$\mathcal{R}_{\bar{K}[G],\omega} \cong \mathcal{R}_{K[G],\omega} \otimes_K \bar{K}$$

and the restriction of rk_ω to $\mathcal{R}_{\bar{K}[G],\omega}$ is the natural algebraic extension of the restriction of rk_ω to $\mathcal{R}_{K[G],\omega}$. Therefore, there exists a unique $\bar{K}$-isomorphism $\alpha_{\bar{K}} : \mathcal{R}_{\bar{K}[G]} \to \mathcal{R}_{K[G],\omega}$ that extends α_K. Since the algebraic extension $\widetilde{\mathrm{rk}}$ of a Sylvester rank function rk is uniquely determined by rk, we obtain also that $\mathrm{rk}_G = \mathrm{rk}_\omega \circ \alpha_{\bar{K}}$ on $\bar{K}[G]$. In this way we prove that the sofic Lück approximation over K implies the sofic Lück approximation over $\bar{K}$ (Corollary 10.3).

2.4. **The inductive step for non-algebraic extensions.** Let us describe now the proof of the inductive step for transcendental extensions. We assume that the sofic Lück approximation holds over K_{2i}.

Given a regular Sylvester matrix rank function rk on an algebra R we will define in Section 7 a Sylvester matrix rank function $\widetilde{\mathrm{rk}}$ on the algebra $R \otimes_K K(t)$ that we will call the natural transcendental extension of rk.

If R is a von Neumann regular algebra then $\widetilde{\mathrm{rk}}$ is characterized by the condition that for every n by n matrix A over R, $\widetilde{\mathrm{rk}}(I_n + tA) = n$ (Proposition 7.7). Observe that the strong algebraic eigenvalue conjecture over K_{2i} (formulated in Subsection 1.5) implies that

$$\mathrm{rk}_G(I_n + \lambda_i A) = \mathrm{rk}_G(A + \lambda_i^{-1} I_n) = n$$

for every n by n matrix A over $\mathcal{R}_{K_{2i}[G]}$. From this observation we will obtain that the sofic Lück approximation and the strong algebraic eigenvalue conjecture over K_{2i} together imply that the restriction of rk_G on $K_{2i+1}[G]$ is the natural transcendental extension of the restriction of rk_G on $K_{2i}[G]$ and the restriction of rk_ω on $K_{2i+1}[G]$ is the natural transcendental extension of the restriction of rk_ω on $K_{2i}[G]$.

Recall that $\mathrm{rk}_G = \mathrm{rk}_\omega$ on $K_{2i}[G]$ by inductive hypothesis. Observe also that the natural transcendental extension $\widetilde{\mathrm{rk}}$ is uniquely determined by rk. Hence the sofic Lück approximation over K_{2i} together with the strong algebraic eigenvalue conjecture over K_{2i} imply the sofic Lück approximation over K_{2i+1} (the proof of Corollary 10.6).

Thus, we have to show that the sofic Lück approximation over K_{2i} also implies the strong algebraic eigenvalue conjecture over K_{2i}. We will do it by introducing a new tool which we present now.

2.5. **The centralizer of an operator in the space of Hilbert-Schmidt operators.** Let $\mathcal{H}$ be a separable Hilbert space. Consider the space $HS(\mathcal{H})$ of **Hilbert-Schmidt operators** on $\mathcal{H}$, i.e., linear operators $A : \mathcal{H} \to \mathcal{H}$ such that $\sum_i \|(e_i)A\|^2$ is finite where $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$. (For expository reasons the operators act on the right.) The space $HS(\mathcal{H})$ is endowed with the **Hilbert-Schmidt scalar product** given by

$$\langle A, B\rangle = \sum_i \langle (e_i)A, (e_i)B\rangle.$$

By a standard argument this definition does not depend on the choice of $\{e_i\}$. The associated norm on $HS(\mathcal{H})$ is called the **Hilbert-Schmidt norm** and we will always consider $HS(\mathcal{H})$ with respect to the topology induced by this norm. Observe that $HS(\mathcal{H})$ is an ideal in $\mathcal{B}(\mathcal{H})$. In what follows we will identify $HS(\mathcal{H}^n)$ and $\mathrm{Mat}_n(HS(\mathcal{H}))$.

In Section 9.1 we will define a structure of $(G\times G)$-Hilbert module on $HS(l^2(G)^n)$ in such a way that the left and right multiplications by an operator from $\mathrm{Mat}_n(\mathcal{N}(G))$ commute with the action of the elements from $G \times G$. For any operator A on a Hilbert space we denote by $\sigma_p(A)$ the set of eigenvalues of A.

If $A \in \mathrm{Mat}_n(\mathcal{U}(G))$, then the set $\sigma_p(A)$ is countable. For any $\lambda \in \mathbb{C}$ we put

$$n_{\lambda,i}(A) = \dim_G \ker(A - \lambda)^i.$$

We denote by $C_{HS(l^2(G)^n)}(A)$ the centralizer of A in $HS(l^2(G)^n)$. It turns out that $C_{HS(l^2(G)^n)}(A)$ is a $(G \times G)$-Hilbert submodule of $HS(l^2(G)^n)$.

Theorem 2.1. *Let K be a subfield of $\mathbb{C}$ closed under complex conjugation and let G be a countable group.*

(1) *Then for any operator $A \in \mathrm{Mat}_n(\mathcal{U}(G))$,*

$$\dim_{G\times G} C_{HS(l^2(G)^n)}(A) \geq \sum_{\lambda\in\sigma_p(A)} \sum_{i=0}^{\infty} (n_{\lambda,i+1}(A) - n_{\lambda,i}(A))^2.$$

(2) *(**The centralizer dimension property over** K) If, moreover, G is sofic and $A \in \mathrm{Mat}_n(\mathcal{R}_{K[G]})$, then*

$$\dim_{G\times G} C_{HS(l^2(G)^n)}(A) = \sum_{\lambda\in\bar{K}}\sum_{i=0}^{\infty}(n_{\lambda,i+1}(A) - n_{\lambda,i}(A))^2.$$

Since, $n_{\lambda,i}(A) \neq 0$ only if $\lambda \in \sigma_p(A)$ and $n_{\lambda,1} \neq 0$ if $\lambda \in \sigma_p(A)$, the strong algebraic eigenvalue conjecture over K for sofic groups is a consequence of the previous theorem. The first part of Theorem 2.1 will be proved in Proposition 9.9. The proof of the second part depends on a property of finite permutation representations described in the following subsection.

2.6. **The strict eigenvalue property.** Let F be a finitely generated free group and let X be a finite F-set. Denote by

$$f_X : \mathbb{C}[F] \to \mathrm{Mat}_{|X|}(\mathbb{C}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X])$$

the representation of $\mathbb{C}[F]$ associated with the permutation action of F on X. The **strict eigenvalue property** is the property that appears in the following theorem.

Theorem 2.2 (The strict eigenvalue property)**.** *Let $A, B \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F[)$ be two matrices over $\mathbb{C}[F]$. Then for any $\epsilon > 0$ the set*

$$S_\epsilon(A,B) = \{\lambda \in \mathbb{C} : \ \mathrm{rk}_{\mathbb{C}}(f_X(B)) - \mathrm{rk}_{\mathbb{C}}(f_X(B-\lambda A)) \geq \epsilon|X| \ \text{for some finite } F\text{-set } X\}$$

is finite.

Note that if A is an n by n matrix and $B = I_n$ then the theorem says that for any $\epsilon > 0$ there exists a finite set $S_\epsilon(A)$ such that if for some finite F-set X, λ is an eigenvalue of $f_X(A)$ with multiplicity greater than $\epsilon|X|$, then $\lambda \in S_\epsilon(A)$.

In Section 8 we will prove Theorem 2.2. In Subsection 9.4 we will show how the strict eigenvalue property and the sofic Lück approximation over $\bar{K}$ imply the second part of Theorem 2.1. This finishes the proof of Theorem 1.3. We resume the main steps of the proof of Theorem 1.3 in Table 1.

In Section 10 we will prove a more general version of Theorem 1.3, which, in particular, implies directly Corollary 1.4.

2.7. **The proofs of Theorem 1.1 and of other applications of Theorem 1.3.** Let us explain the proof of Theorem 1.1. Let G be a sofic group with finite $\mathrm{lcm}(G)$. If G satisfies the Strong Atiyah conjecture over $\bar{\mathbb{Q}}$, then $\mathcal{R}_{\bar{\mathbb{Q}}[G]}$ is semisimple Artinian. Hence there are division rings $D_1, \ldots, D_k$ and natural numbers $n_1 \ldots, n_k$ such that

$$\mathcal{R}_{\bar{\mathbb{Q}}[G]} \cong \mathrm{Mat}_{n_1}(D_1) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(D_k).$$

Using Theorem 10.2 and Theorem 10.5, we will show in Theorem 10.7 that

$$\mathcal{R}_{\mathbb{C}[G]} \cong \mathrm{Mat}_{n_1}(E_1) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(E_k),$$

where E_i is the division algebra isomorphic to the classical Ore ring of fractions of $D_i \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$. This will imply that G satisfies the strong Atiyah conjecture over $\mathbb{C}$.

Corollaries 1.5, 1.6 and 1.7 are proved in Subsection 10.4.

TABLE 1. The scheme of the proof of Theorem 1.3

The sofic Lück approximation over K_{2i-1}	*Corollary* 10.3 $\Longrightarrow$	The sofic Lück approximation over K_{2i}
		the proof of Theorem 9.12 $\Downarrow$
The strong algebraic eigenvalue conjecture over K_{2i}	*the proof of Corollary* 9.13 $\Longleftarrow$	The centralizer dimension property over K_{2i}
Subsection 10.2 $\Downarrow$		the proof of Theorem 9.12 $\Uparrow$
The sofic Lück approximation over K_{2i+1}		The strict eigenvalue property over K_{2i}

3. Von Neumann regular and $*$-regular rings

3.1. **Von Neumann regular rings.** An element x of a ring R is called **von Neumann regular** if there exists $y \in R$ satisfying $xyx = x$. A ring $\mathcal{U}$ is called **von Neumann regular** if all the elements of $\mathcal{U}$ are von Neumann regular. In the following proposition we collect the properties of von Neumann regular rings that we will need later.

Proposition 3.1. [17] *Let $\mathcal{U}$ be a von Neumann regular ring. Then the following statements hold:*

(1) *every finitely generated left ideal of $\mathcal{U}$ is generated by an idempotent;*
(2) *every finitely generated left submodule of a projective module P of $\mathcal{U}$ is a direct summand of P (and, in particular, it is projective);*
(3) *every finitely generated left projective module of $\mathcal{U}$ is a direct sum of left cyclic ideals of $\mathcal{U}$.*

3.2. **The ring of unbounded affiliated operators of a group.** The ring of unbounded affiliated operators $\mathcal{U}(G)$ of a countable group G is one of the main examples of von Neumann regular rings that appear in this paper. The Ph.D thesis of H. Reich [37] is a good source to learn basic facts about the ring $\mathcal{U}(G)$. In this subsection we briefly define this ring and also introduce additional notions that will motivate further definitions.

Recall that we consider $\mathbb{C}[G]$ as a subspace of $\mathcal{B}(l^2(G))$, the bounded linear operators on $l^2(G)$. The weak closure of $\mathbb{C}[G]$ in $\mathcal{B}(l^2(G))$ is the **group von Neumann algebra** $\mathcal{N}(G)$ of G. It is equal to the second centralizer of $\mathbb{C}[G]$ in $\mathcal{B}(l^2(G))$. The ring $\mathcal{N}(G)$ satisfies the left Ore condition (a result proved by S. K. Berberian in [5]) and its classical ring of fractions $Q_l(\mathcal{N}(G))$ is denoted by $\mathcal{U}(G)$ (see also Subsection 4.3). The ring $\mathcal{U}(G)$ can be also described as the ring of densely defined (unbounded) operators which commute with the left action of G. Therefore, $\mathcal{U}(G)$ is called the **ring of unbounded affiliated operators** of G. The ring $\mathcal{U}(G)$ is a $*$-regular ring. We will consider such rings in more detail in Subsection 3.4.

We can define a Sylvester matrix rank function rk_G on $\mathcal{U}(G)$ in the following way

$$\mathrm{rk}_G(s^{-1}A)=\mathrm{rk}_G(A)=\dim_G(\overline{l^2(G)^nA})=\sum_{i=1}^m\langle(\mathbf{1}_i)\operatorname{proj}_{\overline{l^2(G)^nA}},\mathbf{1}_i\rangle_{l^2(G)},\tag{1}$$

where $A\in \mathrm{Mat}_{n\times m}(\mathcal{N}(G))$ and $s\in\mathcal{N}(G)$ is a non-zero-divisor in $\mathcal{N}(G)$. Note that if $u\in\mathcal{U}(G)$, then

$$\mathrm{rk}_G(u)=1 \text{ if and only if } u \text{ is invertible in } \mathcal{U}(G).\tag{2}$$

The function rk_G is an example of a Sylvester matrix rank function on a $*$-regular ring. We will consider Sylvester rank functions in more detail in Section 5. The Sylvester matrix rank function rk_G induces a Sylvester dimension rank function $\dim_G$ on finitely presented left modules of $\mathcal{U}(G)$ (see Subsection 5.2 for more details) that satisfies

$$\dim_G(\mathcal{U}(G)u)=\mathrm{rk}_G(u),\ u\in\mathcal{U}(G).$$

3.3. **Von Neumann regular elements in a proper $*$-ring.** Let R be a $*$-ring. The involution $*$ is called **proper** if $x^*x=0$ implies $x=0$ and it is called **n-positive definite** if $\sum_{i=1}^n x_i^*x_i=0$ implies $x_1=\cdots=x_n=0$. Thus, the involution is proper if and only if it is 1-positive definite. If the involution is n-positive definite for all n, then we say that it is **positive definite**. We say that a $*$-ring is **proper** if its involution is proper. We say that $\mathcal{U}$ is a **positive definite** $*$-ring if its involution is positive definite. This is equivalent to $\mathrm{Mat}_n(\mathcal{U})$ being proper for every $n\in\mathbb{N}$. For example, $\mathrm{Mat}_n(\mathbb{C})$ and $\mathcal{U}(G)$ are positive definite $*$-rings.

In general if x is a von Neumann regular element there are several elements y satisfying $xyx=x$. However, if R is a proper $*$-ring there is a canonical one. We include the proof of the following proposition for convenience of the reader. Its variation can be found in [4].

Proposition 3.2. *Let R be a proper $*$-ring and let $x\in R$. Assume that x^*x and xx^* are von Neumann regular elements. Then the following holds.*

1. *$Rx=Rx^*x$.*
2. *x and x^* are von Neumann regular.*
3. *There exists a unique projection e in R such that $Re=Rx$ and there exists a unique projection f such that $fR=xR$ (we put $e=\mathrm{RP}(x)$ and $f=\mathrm{LP}(x)$).*
4. *There exists a unique $y\in eRf$ such that $yx=e$ and $xy=f$ (we put $x^{[-1]}=y$ and call it* **the relative inverse of** *x).*
5. *$\mathrm{RP}(x)=\mathrm{RP}(x^*x)=\mathrm{LP}(x^*)$ and $(x^*)^{[-1]}=(x^{[-1]})^*$.*
6. *$(x^*x)^{[-1]}=x^{[-1]}(x^*)^{[-1]}$ and $x^{[-1]}=(x^*x)^{[-1]}x^*$.*
7. *If x is self-adjoint, then x commutes with $x^{[-1]}$.*

Proof. Since x^*x is von Neumann regular there exists w such that $x^*xwx^*x=x^*x$. Hence $wx^*x-1\in\mathrm{Ann}_r^R(x^*x)$. Since R is proper,

$$\mathrm{Ann}_r^R(x^*x)=\mathrm{Ann}_r^R(x).$$

Thus, $xwx^*x=x$ and $x^*xw^*x^*=x^*$. This proves (1) and (2).

Let $f=xwx^*$. Then

$$f=xw(xwx^*x)^*=xwx^*xw^*x^*=ff^*.$$

Hence f is a projection. Observe also that

$$fR \geq fxR = xR \geq fR.$$

Thus $fR = xR$. If there exists another projection f' such that $f'R = fR$, then

$$f = f'f = (f'f)^* = ff' = f'.$$

The existence and the uniqueness of e is proved in the same way. Hence we obtain (3).

It is clear that there exists $y \in eRf$ such that $yx = e$. Hence $xy - f \in \mathrm{Ann}_l^R(x) = \mathrm{Ann}_l^R(f)$. Thus, $xy - f = (xy - f)f = 0$ and so $xy = f$. Now, if there exists another y' satisfying the same properties as y, then

$$y' = ey' = yxy' = yf = y.$$

This implies (4). The properties (5) and (6) follow from the uniqueness of $\mathrm{RP}(x)$, $(x^*)^{[-1]}$ and $(x)^{[-1]}$.

If x is self-adjoint, then

$$xx^{[-1]} = e = (xx^{[-1]})^* = x^{[-1]}x.$$

This proves (7). □

3.4. **Von Neumann ∗-regular rings.** A ∗-ring $\mathcal{U}$ is called **von Neumann ∗-regular** (or simply **∗-regular**) if it is von Neumann regular and its involution is proper. The ring $\mathrm{Mat}_n(\mathbb{C})$ is a ∗-regular ring. The ring $\mathbb{C}[G]$ is ∗-regular if and only if G is locally finite. However, we can embed $\mathbb{C}[G]$ in the ∗-regular ring $\mathcal{U}(G)$ for an arbitrary countable group G.

A direct product of ∗-regular rings is again ∗-regular. If $\mathcal{U}$ is a ∗-regular ring, then $\mathrm{Mat}_n(\mathcal{U})$ is again a ∗-ring and it is also von Neumann regular. However, recall that in general ∗ is not proper in $\mathrm{Mat}_n(\mathcal{U})$.

Although in the definition of a ∗-regular ring the properties to be von Neumann regular and to be proper do not interact, using them together we obtain many interesting consequences.

Proposition 3.3. *Let $\mathcal{U}$ be a ∗-regular ring and I an ideal of $\mathcal{U}$. Then I is ∗-closed and, moreover, ∗ is proper in $\mathcal{U}/I$.*

Proof. From Proposition 3.2(1), it follows that if $x^*x \in I$, then $x \in \mathcal{U}x^*x \subset I$ and $x^* \in x^*x\mathcal{U} \subset I$. Hence I is ∗-closed and ∗ is proper on $\mathcal{U}/I$.

□

The next proposition explains how to construct the minimal ∗-regular subring containing a given ∗-ring. This was proved first for positive definite ∗-regular rings in [27] and it appeared in the form that we present here in [1, Proposition 6.2].

Let R be a ∗-subring of a ∗-regular ring $\mathcal{U}$. We denote by $\mathcal{R}_1(R,\mathcal{U})$ the subring of $\mathcal{U}$ generated by R and all the relative inverses of the elements x^*x for $x \in R$. Observe that by Proposition 3.2(6), $x^{[-1]} \in \mathcal{R}_1(R,\mathcal{U})$ for every $x \in R$. Thus, $\mathcal{R}_1(R,\mathcal{U})$ can be also defined as the subring of $\mathcal{U}$ generated by R and the relative inverses of all the elements $x \in R$. Clearly $\mathcal{R}_1(R,\mathcal{U})$ is again a ∗-subring of $\mathcal{U}$. We put

$$\mathcal{R}_{n+1}(R,\mathcal{U}) = \mathcal{R}_1(\mathcal{R}_n(R,\mathcal{U}),\mathcal{U}).$$

Proposition 3.4. [1, Proposition 6.2] *Let $\mathcal{U}$ be a $*$-regular ring and let R be a $*$-subring of $\mathcal{U}$. Then there is a smallest $*$-regular subring $\mathcal{R}(R,\mathcal{U})$ of $\mathcal{U}$ containing R. Moreover,*

$$\mathcal{R}(R,\mathcal{U}) = \bigcup_{i=1}^{\infty} \mathcal{R}_i(R,\mathcal{U}).$$

The subring $\mathcal{R}(R,\mathcal{U})$ is called the **$*$-regular closure** of R in $\mathcal{U}$.

Corollary 3.5. *Let $\mathcal{U}$ be a $*$-regular ring and let R be a $*$-subring of $\mathcal{U}$. Let V be a subring of $\mathcal{U}$ such that*

(1) *R is a subring of V and*
(2) *for every self-adjoint element $v \in V$, $v^{[-1]} \in V$.*

Then $\mathcal{R}(R,\mathcal{U})$ is a subring of V.

If K is a subfield of $\mathbb{C}$ closed under complex conjugation and G is a countable group, then the $*$-regular closure of $K[G]$ in $\mathcal{U}(G)$ is denoted by $\mathcal{R}_{K[G]}$. For an arbitrary group G, $\mathcal{R}_{K[G]}$ is defined as the direct union of $\{\mathcal{R}_{K[H]}$: H is a finitely generated subgroup of $G\}$.

4. Epic homomorphisms

4.1. **Epic homomorphisms.** Let $f : R \to S$ be a ring homomorphism. We say that f is **epic** if for every ring Q and homomorphisms $\alpha, \beta : S \to Q$, the equality $\alpha \circ f = \beta \circ f$ implies $\alpha = \beta$. An **epic** R-ring is a pair (S, f) where $f : R \to S$ is epic. For simplicity we will write S instead of (S, f) when f is clear from the context.

We will say that two epic R-rings (S_1, f_1) and (S_2, f_2) are isomorphic if there exists an isomorphism $\alpha : S_1 \to S_2$ for which the following diagram is commutative:

$$\begin{array}{ccc} R & \to^{\mathrm{Id}} & R \\ \downarrow f_1 & & \downarrow f_2 \\ S_1 & \to^{\alpha} & S_2. \end{array}$$

We will use the following characterization of epic homomorphisms.

Proposition 4.1. [42, Proposition XI.1.2] *Let $f : R \to S$ be a ring homomorphism. Then f is epic if and only if the multiplication map*

$$m : S \otimes_R S \to S$$

is an isomorphism of S-bimodules.

More generally if $f : R \to S$ is a ring homomorphism, we say that $s \in S$ is **dominated** by f if for any ring Q and homomorphisms $\alpha, \beta : S \to Q$, the equality $\alpha \circ f = \beta \circ f$ implies $\alpha(s) = \beta(s)$. The set of elements of S dominated by f is a subring of S, called the **dominion** of f.

It is clear that every surjective homomorphism is epic. The following result implies that for von Neumann regular rings these two notions are equivalent.

Proposition 4.2. [42, Proposition XI.1.4] *Let $\mathcal{U}$ be a von Neumann regular ring. Then for every ring homomorphism $\gamma : \mathcal{U} \to S$, the dominion of γ is equal to $\gamma(\mathcal{U})$.*

Corollary 4.3. *Let R be an algebra and $\mathcal{U}_1$ and $\mathcal{U}_2$ von Neumann regular rings. Let $f_1 : R \to \mathcal{U}_1$ and $f_2 : R \to \mathcal{U}_2$ be two epic homomorphisms. Let $\gamma_1 : \mathcal{U}_1 \to S$ and $\gamma_2 : \mathcal{U}_2 \to S$ be two homomorphisms satisfying $\gamma_1 \circ f_1 = \gamma_2 \circ f_2$. Then $\operatorname{Im} \gamma_1 = \operatorname{Im} \gamma_2$.*

Proof. Let $\alpha, \beta : S \to Q$ be such that $\alpha \circ \gamma_1 \circ f_1 = \beta \circ \gamma_1 \circ f_1$. Since f_1 is epic $\alpha \circ \gamma_1 = \beta \circ \gamma_1$. Hence for every $u \in \mathcal{U}_1$, $\alpha(\gamma_1(u)) = \beta(\gamma_1(u))$. Therefore, the dominion of $\gamma_1 \circ f_1$ contains $\gamma_1(\mathcal{U}_1)$.

On the other hand, let $s \in S \setminus \gamma_1(\mathcal{U}_1)$. By Proposition 4.2, there are $\alpha, \beta : S \to Q$ such that $\alpha \circ \ gamma_1 = \beta \circ \gamma_1$, but $\alpha(s) \neq \beta(s)$. In particular, this implies that the dominion of $\gamma_1 \circ f_1 : R \to S$ is contained in $\gamma_1(\mathcal{U}_1)$. Thus, the dominion of $\gamma_1 \circ f_1 : R \to S$ is equal to $\gamma_1(\mathcal{U}_1)$.

Similarly, we obtain that the dominion of $\gamma_2 \circ f_2 : R \to S$ is equal to $\gamma_2(\mathcal{U}_2)$. Since $\gamma_1 \circ f_1 = \gamma_2 \circ f_2$, we conclude that $\gamma_1(\mathcal{U}_1) = \gamma_2(\mathcal{U}_2)$.

□

4.2. **Rational closures and universal localizations.** Let R be a subring of S. Denote by $\mathrm{GL}(R; S)$ the set of square matrices over R which are invertible over S. The **rational closure** of R in S is the subring of S generated by all the entries of the matrices M^{-1} for $M \in \mathrm{GL}(R; S)$. If $f : R \to S$ is a homomorphism and S is the rational closure of $f(R)$ in S, then f is epic.

Let $f : R \to S$ be a map. Let Σ be a set of matrices over R such that $f(\Sigma) \subset \mathrm{GL}(f(R); S)$. Then there exists the **universal localization of** R **with respect to** Σ. It is an R-ring $\lambda : R \to R_\Sigma$ such that $\lambda(\Sigma)$ are invertible over R_Σ and every Σ-inverting homomorphism from R to another ring can be factorized uniquely by λ (see [7, Theorem 4.1.3]).

4.3. **The Cohn theory of epic division R-algebras.** An **epic division R-ring** is an epic R-ring $f : R \to D$, where D is a division ring. Since the rational closure of $f(R)$ in D is a division subring of D, by [42, Proposition XI.1.4], D coincides with the rational closure of $f(R)$ in D.

If R is a commutative ring, then there exists a natural bijection between $\mathrm{Spec}(R)$ and the isomorphism classes of epic division R-rings: a prime ideal $P \in \mathrm{Spec}(R)$ corresponds to the field of fractions $Q(R/P)$ of R/P and $f : R \to Q(R/P)$ is defined as $f(r) = r + P$ for any $r \in R$.

The situation for an arbitrary ring R is much more complicated. Let us first recall the definition of the left Ore condition and the construction of the Ore ring of fractions. An element $r \in R$ is a **non-zero-divisor** if there exists no non-zero element $s \in R$ such that $rs = 0$ or $sr = 0$. Let T be a multiplicative subset of non-zero-divisors of R. We say that (T, R) satisfies the **left Ore condition** if for every $r \in R$ and every $t \in T$, the intersection $Tr \cap Rt$ is not trivial. If T consists of all the non-zero-divisors we simply say that R satisfies the **left Ore condition**.

If (T, R) satisfies the left Ore condition then we can construct the **left Ore ring of fractions** $T^{-1}R$ (for more details the reader may consult [32, Chapter 2]). The ring $T^{-1}R$ is isomorphic to the universal localization of R with respect to T, R is a subring of $T^{-1}R$ and any element of $T^{-1}R$ can be written in the form $t^{-1}r$ for some $t \in T$ and $r \in R$. When T consists of all the non-zero-divisors of R and (T, R) satisfies the left Ore condition, we denote $T^{-1}R$ by $Q_l(R)$ and we call it **the left classical ring of fractions** of R.

An important result in the theory of classical rings of quotients is Goldie's theorem [18, Theorem 6.15]. One of its consequences (see [18, Corollary 6.16]) is that every semiprime left Noetherian ring has a semisimple Artinian classical left ring of fractions.

If R is a (non-commutative) domain and it satisfies the left Ore condition then its classical left ring of fractions $Q_l(R)$ is a division ring. Moreover, as in the commutative case, the division R-ring $Q_l(R)$ is the unique (up to R-isomorphism) faithful epic division R-ring. Thus, if R is a left Noetherian ring, then there exists a natural bijection between the **strong prime** ideals of R (ideals P such that R/P is a domain) and the isomorphism classes of epic division R-rings.

For an arbitrary ring R, P. Cohn proposed the following approach to classify division R-rings. If D is a division ring, let $\mathrm{rk}_D(M)$ be the D-rank of a matrix M over D.

Proposition 4.4. [7, Theorem 4.4.1] *Let (D_1, f_1) and (D_2, f_2) be two epic division R-rings. Then (D_1, f_1) and (D_2, f_2) are isomorphic if and only if for each matrix M over R*

$$\mathrm{rk}_{D_1}(f_1(M)) = \mathrm{rk}_{D_2}(f_2(M)).$$

5. Sylvester rank functions

5.1. **Sylvester matrix rank functions.** Let R be an algebra. A **Sylvester matrix rank function** rk is a function that assigns a non-negative real number to each matrix over R and satisfies the following conditions.

(SMat1) $\mathrm{rk}(M) = 0$ if M is any zero matrix and $\mathrm{rk}(1) = 1$;

(SMat2) $\mathrm{rk}(M_1M_2) \le \min\{\mathrm{rk}(M_1), \mathrm{rk}(M_2)\}$ for any matrices M_1 and M_2 which can be multiplied;

(SMat3) $\mathrm{rk}\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for any matrices M_1 and M_2;

(SMat4) $\mathrm{rk}\begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \ge \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for any matrices M_1, M_2 and M_3 of appropriate sizes.

If $\phi : L_1 \to L_2$ is an R-homomorphism between two free finitely generated R-modules L_1 and L_2, then $\mathrm{rk}(\phi)$ is $\mathrm{rk}(A)$ where A is the matrix associated with ϕ with respect to some R-bases on L_1 and L_2. It is clear that $\mathrm{rk}(\phi)$ does not depend on the choice of the bases.

The following elementary properties of a Sylvester matrix rank function can be obtained from its definition.

Proposition 5.1. *Let R be an algebra and let* rk *be a Sylvester matrix rank function on R. Let $A, B \in \mathrm{Mat}_{n\times m}(R)$, $C \in \mathrm{Mat}_n(R)$, and $u, v, w \in R$. Then*

(1) $\mathrm{rk}(A + B) \le \mathrm{rk}(A) + \mathrm{rk}(B)$.
(2) *If $A = (a_{ij})$ then* $\mathrm{rk}(A) \le \sum_{i,j} \mathrm{rk}(a_{ij})$.
(3) $\mathrm{rk}(uw) \ge \mathrm{rk}(u) + \mathrm{rk}(w) - 1$. *In particular, if* $\mathrm{rk}(u) = 1$, *then* $\mathrm{rk}(uw) = \mathrm{rk}(w)$.
(4) *Assume that $Rv + Rw = R$ and $u \in Rv \cap Rw$. Then*

$$\mathrm{rk}(u) \le \mathrm{rk}(v) + \mathrm{rk}(w) - 1.$$

(5) *For every $\epsilon > 0$ there are at most $\frac{n}{\epsilon}$ values $\lambda \in K$, satisfying*

$$\mathrm{rk}(C - \lambda I_n) \le n - \epsilon.$$

Proof. The first statement is obtained as follows.

$$\operatorname{rk}(A+B) \overset{(\text{SMat3})}{=} \operatorname{rk}\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} =$$
$$\operatorname{rk}\left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) \overset{(\text{SMat2})}{\le}$$
$$\operatorname{rk}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \overset{(\text{SMat3})}{=} \operatorname{rk}(A)+\operatorname{rk}(B).$$

The second statement follows directly from the first one.
(3) We obtain(3) from the following series of inequalities.

$$\operatorname{rk}(uw) \overset{(\text{SMat3})}{=} \operatorname{rk}\begin{pmatrix} uw & 0 \\ 0 & 1 \end{pmatrix} - 1 =$$
$$\operatorname{rk}\left(\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 1 & w \end{pmatrix}\begin{pmatrix} w & 1 \\ -1 & 0 \end{pmatrix}\right) - 1 \overset{(\text{SMat4})}{\ge}$$
$$\operatorname{rk}(u)+\operatorname{rk}(w)-1.$$

(4) We can find $a, b, c, d \in R$ such that $av + bw = 1$ and $u = cv = dw$. Then

$$\operatorname{rk}(v)+\operatorname{rk}(w) \overset{(\text{SMat3})}{=} \operatorname{rk}\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \overset{(\text{SMat2})}{\ge}$$
$$\operatorname{rk}\left(\begin{pmatrix} a & b \\ -c & d \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) =$$
$$\operatorname{rk}\begin{pmatrix} 1 & bw \\ 0 & u \end{pmatrix} \overset{(\text{SMat4})}{\ge} 1+\operatorname{rk}(u).$$

(5) We will apply the part (4) for the ring $\operatorname{Mat}_n(R)$ and the Sylvester rank function $\frac{1}{n}\operatorname{rk}$. Assume $\lambda_1, \dots, \lambda_k$ are distinct and satisfy

$$\operatorname{rk}(C-\lambda_i I_n) \le n-\epsilon.$$

Put $v_i = C - \lambda_{i+1} I_n$ and $w_i = \prod_{j=1}^{i}(C-\lambda_j I_n)$. Let us prove that $\operatorname{rk}(w_i) \le n - i\epsilon$. We argue by induction on i. The case $i = 1$ is clear and the inductive step follows from the part (4):

$$\operatorname{rk}(w_{i+1}) \le \operatorname{rk}(v_i)+\operatorname{rk}(w_i)-n \le n-\epsilon+n-i\epsilon-n = n-(i+1)\epsilon.$$

Since $n - k\epsilon \ge \operatorname{rk}(w_k) \ge 0$, we obtain that $k \le \frac{n}{\epsilon}$.

□

For any algebra R we denote by $\mathbb{P}(R)$ the set of Sylvester matrix rank functions on R. The set $\mathbb{P}(R)$ is a compact convex subset of functions on matrices over R (with respect to the point convergence topology).

For a given homomorphism $f : R \to S$ of algebras, we define $f^{\#} : \mathbb{P}(S) \to \mathbb{P}(R)$ by

$$f^{\#}(\operatorname{rk})(M) = \operatorname{rk}(f(M)),\ \text{where } M \text{ is a matrix over } R.$$

5.2. **Sylvester module rank functions.** A **Sylvester module rank function** dim is a function that assigns a non-negative real number to each finitely presented R-module and satisfies the following conditions.

(SMod1) $\dim\{0\} = 0$, $\dim R = 1$;
(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;
(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \geq \dim M_2 \geq \dim M_3.$$

Given a matrix $A \in \mathrm{Mat}_{n\times m}(R)$ we put $M_A = R^m/(R^n A)$. It is clear that M_A is a finitely presented left R-module. Conversely, given a finitely presented left R-module M we can find a matrix $A \in \mathrm{Mat}_{n\times m}(R)$ such that $M_A \cong M$. This observation allows to construct a natural one-to-one correspondence between the Sylvester matrix rank functions and the Sylvester module rank functions.

Proposition 5.2. *(* [31], [40, Chapter 7]*) Let R be an algebra.*

(1) *Let* rk *be a Sylvester matrix rank function on R and let $A \in \mathrm{Mat}_{n\times m}(R)$. We put*
$$\dim(M_A) = m - \mathrm{rk}(A).$$
Then dim *is well defined and it is a Sylvester module rank function on R.*
(2) *Let* dim *be a Sylvester module rank function on R and let $A \in \mathrm{Mat}_{n\times m}(R)$. We put*
$$\mathrm{rk}(A) = m - \dim(M_A).$$
Then rk *is a Sylvester matrix rank function on R.*

Proof. Using [31, Lemma 2], we obtain that the definition from (1) is well-defined and does not depend on the choice of A. The rest of the proof is straightforward. □

If rk and dim are related as described in Proposition 5.2 we will say that they are **associated**.

5.3. **The pseudo-metric induced by a Sylvester matrix rank function.** Given a Sylvester matrix rank function rk on R, we define

$$\delta(x, y) = \mathrm{rk}(x - y),\ \ x, y \in R.$$

Proposition 5.1 implies that the function δ is a pseudo-metric on R.

Corollary 5.3. *Let $x, y, z \in R$. Then the following holds.*

(1) $\delta(x, y) = \delta(y, x)$.
(2) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

Even though δ is not always a metric, we refer to it as the rk**-metric** for convenient abbreviation. Observe that the set

$$\ker \mathrm{rk} = \{a \in R :\ \mathrm{rk}(a) = 0\}$$

is an ideal of R. We say that rk is **faithful** if $\ker \mathrm{rk} = 0$. By Proposition 5.1(1,2), rk may be seen as a faithful Sylvester matrix rank function on the quotient ring $R/\ker \mathrm{rk}$, and so, δ is a metric on $R/\ker \mathrm{rk}$. Since the multiplication and addition on R are uniformly continuous with respect to δ, the (Hausdorff) completion of $R/\ker \mathrm{rk}$, which we denote by $\overline{R_{\mathrm{rk}}}$ (or simply $\overline{R}$ when rk is clear from the context) is a ring. The kernel of the natural map $R \to \overline{R_{\mathrm{rk}}}$ is $\ker \mathrm{rk}$. The function rk can be extended by continuity on $\overline{R_{\text{rk}}}$ and on matrices over $\overline{R_{\text{rk}}}$ and one can easily check that this extension (denoted also by rk) is a Sylvester matrix rank function on $\overline{R_{\text{rk}}}$.

If G is a group and K a subfield of $\mathbb{C}$ closed under complex conjugation, then the completion of $\mathcal{R}_{K[G]}$ with respect to the rk_G-metric is denoted by $\overline{\mathcal{R}_{K[G]}}$.

5.4. **Sylvester matrix rank functions and rational closures.**

Proposition 5.4. *Let $f : R \to S$ be a homomorphism of algebras. Assume that S is a rational closure of $f(R)$. Then $f^\# : \mathbb{P}(S) \to \mathbb{P}(R)$ is injective.*

Moreover, if $S = R_\Sigma$ is a universal localization, then

$$\operatorname{Im} f^\# = \{\text{rk} \in \mathbb{P}(R) : \ \text{rk}(A) = n \textit{ if } \ A \in \Sigma \cap \text{Mat}_n(R)\}.$$

Proof. The first part of the proposition follows from Cramer's rule (see, for example, [40, Theorem 4.2]) and the second one is proved in [40, Theorem 7.4]. □

In view of the previous proposition if $f : R \to S$ is an algebra homomorphism and S is a rational closure of $f(R)$, then we will often consider $\mathbb{P}(S)$ as a subset of $\mathbb{P}(R)$.

Corollary 5.5. *Let R be an algebra and let T be a multiplicative set of non-zero-divisors. Assume that (T, R) satisfies the left Ore condition. Then*

$$\mathbb{P}(T^{-1}R) = \{\text{rk} \in \mathbb{P}(R) : \ \text{rk}(t) = 1 \textit{ for all } t \in T\}.$$

5.5. **Exact Sylvester rank functions.** We say that a Sylvester module rank function dim on R is **exact** if it satisfies the following additional condition

(SMod3′) given a surjection $\phi : M \twoheadrightarrow N$ between two finitely presented R-modules,

$$\dim M - \dim N = \inf\{\dim L : \ L \twoheadrightarrow \ker \phi \text{ and } L \text{ is finitely presented}\}.$$

The following result has been recently proved by Simone Virili.

Proposition 5.6. *([45]) Let R be an algebra and let* dim *be an exact Sylvester module rank function on R. For every finitely generated R-module M put*

$$\dim M = \inf\{\dim L : \ L \twoheadrightarrow M \textit{ and } L \textit{ is finitely presented}\},$$

and for every arbitrary R-module put

$$\dim M = \sup\{\dim L : \ L \le M \textit{ and } L \textit{ is finitely generated}\}.$$

Then the extended function $\dim : R\text{−Mod} \to \mathbb{R}_{\ge 0} \cup \{+\infty\}$ *satisfies the following conditions:*

(LF1) *if* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is exact then* $\dim M_1 + \dim M_3 = \dim M_2$.
(LF2) $\dim M = \sup\{\dim L : \ L \le M \textit{ and } L \textit{ is finitely generated}\}$.

A function on R-Mod satisfying (LF1) and (LF2) is called a **length function**. If a length function l satisfies $l(R) = 1$, then the restriction of l to finitely presented R-modules is an exact Sylvester module rank function on R. Moreover, l can be recovered from this restriction using the formulas which appear in Proposition 5.6.

5.6. **Sylvester rank functions on von Neumann regular algebras.** An arbitrary algebra may not have an exact Sylvester module rank function. However, if $\mathcal{U}$ is von Neumann regular, then, by Proposition 3.1(2), finitely presented $\mathcal{U}$-modules are projective, and so, all the exact sequences of finitely presented $\mathcal{U}$-modules split. Thus, every Sylvester module rank function on a regular algebra $\mathcal{U}$ is exact. Note also that, by Proposition 3.1(3), a Sylvester matrix rank function on a von Neumann regular algebra $\mathcal{U}$ is completely determined by its values on elements from $\mathcal{U}$. Thus, pseudo-rank functions studied in [17] are exactly our Sylvester matrix rank functions. We recall the results about Sylvester matrix rank functions on von Neumann regular algebras that we will use in this paper.

Proposition 5.7. *Let $\mathcal{U}$ be a von Neumann regular algebra and* rk *a Sylvester matrix rank function.*

(1) *The algebra $\overline{\mathcal{U}_{\mathrm{rk}}}$ is also von Neumann regular.*
(2) *The following conditions are equivalent:*
   (a) *$Z(\overline{\mathcal{U}_{\mathrm{rk}}})$ is a field;*
   (b) *$\overline{\mathcal{U}_{\mathrm{rk}}}$ is simple;*
   (c) rk *is the only Sylvester matrix rank function on $\overline{\mathcal{U}_{\mathrm{rk}}}$.*

Proof. The first statement is [17, Theorem 19.7].

The second statement follows from [17, Theorem 19.14] and [17, Theorem 19.13]. □

The conditions of the previous proposition hold in the following example. Recall that a group G is called **ICC group** if all the non-trivial conjugacy classes of G are infinite.

Proposition 5.8. *Let G be an ICC group and K a subfield of $\mathbb{C}$ closed under complex conjugation. Then $Z(\overline{\mathcal{R}_{K[G]}})$ is a subfield of $\mathbb{C}$.*

Proof. Since G is an ICC group, $Z(\mathcal{N}(G)) = \mathbb{C}$ and so by [28, Proposition 30], $Z(\mathcal{U}(G)) = \mathbb{C}$. Note that

$$Z(\mathcal{U}(G)) = C_{\mathcal{U}(G)}(G).$$

Hence $Z(\overline{\mathcal{R}_{K[G]}})$ is a subfield of $\mathbb{C}$. □

A Sylvester matrix rank function rk on an arbitrary algebra R is called **regular** if there exists an algebra homomorphism $f : R \to \mathcal{U}$ such that $\mathcal{U}$ is von Neumann regular and $\mathrm{rk} \in \operatorname{Im} f^{\#}$. In this case $\mathcal{U}$ is called **regular envelope** of rk. Clearly, rk may have many regular envelopes. We will see later that in some cases we can speak about the canonical regular envelope attached to rk. We denote by $\mathbb{P}_{reg}(R)$ the space of regular Sylvester matrix rank functions on R.

Observe that an n by n matrix A over a von Neumann regular ring $\mathcal{U}$ is invertible if and only if $\mathrm{rk}(A) = n$ for some (and therefore, for all) faithful Sylvester matrix rank function rk on $\mathcal{U}$. Thus, in view of Proposition 5.4, we obtain that if Σ is a set of square matrices over an algebra R, then

$$\mathbb{P}_{reg}(R_\Sigma) = \{\mathrm{rk} \in \mathbb{P}_{reg}(R) : \ \mathrm{rk}(A) = n \text{ if } \ A \in \Sigma \cap \mathrm{Mat}_n(R)\}. \tag{3}$$

5.7. **Ultraproducts of von Neumann regular rings.** Given a set X, an **ultrafilter** on X is a set ω consisting of subsets of X such that

(1) the empty set is not an element of ω;

(2) if A and B are subsets of X, A is a subset of B, and A is an element of ω, then B is also an element of ω;
(3) if A and B are elements of ω, then so is the intersection of A and B;
(4) if A is a subset of X, then either A or $X \setminus A$ is an element of ω.

If $a \in X$, we can define $\omega_a = \{A \subseteq X : \ a \in A\}$. It is an ultrafilter, called **principal**. It is a known fact that if X is infinite, the axiom of choice implies the existence of a non-principal ultrafilter.

Let ω be an ultrafilter on X and $\{a_i \in \mathbb{R}\}_{i\in X}$ a family of real numbers. We write $a = \lim\limits_{\omega} a_i$ if for any $\epsilon > 0$ the set $\{i \in X : \ |a - a_i| < \epsilon\}$ is an element of the ultrafilter w. It is not difficult to see that for any bounded sequence $\{a_i \in \mathbb{R}\}_{i\in X}$ there exists a unique $a \in \mathbb{R}$ such that $a = \lim\limits_{\omega} a_i$.

Let X be a set. Let $\{\mathcal{U}_i\}_{i\in X}$ be a family of von Neumann regular rings and for each $i \in X$ let rk_i be a Sylvester matrix rank function on $\mathcal{U}_i$. Then $\prod\limits_{i\in X} \mathcal{U}_i$ is a von Neumann regular ring. Let ω be an ultrafilter on X. We put

$$\mathrm{rk}_\omega(r) = \lim_{\omega} \mathrm{rk}_i(r_i), \text{ where } r = (r_i) \in \prod_{i\in X} \mathcal{U}_i.$$

One easily obtains that rk_ω is a Sylvester matrix rank function on $\prod\limits_{i\in X} \mathcal{U}_i$. We define

$$\prod_{\omega} \mathcal{U}_i = (\prod_{i\in X} \mathcal{U}_i)/\ker(\mathrm{rk}_\omega).$$

Then $\prod\limits_{\omega} \mathcal{U}_i$ is a von Neumann regular ring and rk_ω is a faithful Sylvester matrix rank function on $\prod\limits_{\omega} \mathcal{U}_i$. Observe that, if we start with $*$-regular rings $\mathcal{U}_i$, then, by Proposition 3.3, $\prod\limits_{\omega} \mathcal{U}_i$ is also $*$-regular.

The previous construction may be used to show that $\mathbb{P}_{reg}(R)$ is a closed convex subset of $\mathbb{P}(R)$. A similar result was observed by G. Elek in [12, Proposition 1.1]. We will give a different proof suggested to us by Hanfeng Li.

Proposition 5.9. *$\mathbb{P}_{reg}(R)$ is a closed convex subset of $\mathbb{P}(R)$.*

Proof. It is clear that $\mathbb{P}_{reg}(R)$ is convex. Let us show that it is closed.

For each $p \in \mathbb{P}_{reg}(R)$ let $\mathcal{U}_p$ be a regular envelope of p. Hence there are $f_p : R \to \mathcal{U}_p$ and a Sylvester matrix rank function rk_p on $\mathcal{U}_p$ satisfying $p = f_p^\#(\mathrm{rk}_p)$. We put $\mathcal{U} = \prod_{p\in\mathbb{P}_{reg}(R)} \mathcal{U}_p$ and let $f : R \to \mathcal{U}$ be $f = (f_p)_{p\in\mathbb{P}_{reg}(R)}$. It is clear that $\mathbb{P}_{reg}(R) = f^\#(\mathbb{P}(\mathcal{U}))$. Hence $\mathbb{P}_{reg}(R)$ is closed since $f^\#$ is continuous and $\mathbb{P}(\mathcal{U})$ is compact.

□

5.8. **The rank of a linear combination of two elements in a von Neumann regular algebra.** It is an interesting question whether every Sylvester matrix rank function on an algebra R is, in fact, regular. By a result of A. Schofield [40] a Sylvester matrix rank function taking its values in $\frac{1}{n}\mathbb{Z}$ comes from a map from R to $\mathrm{Mat}_n(D)$ for some division algebra D. The case $n = 1$ of this result was proved by P. Malcolmson [31] using the notion of prime matrix ideal introduced by

P. Cohn [7]. Thus, in view of Proposition 4.4, the Sylvester matrix rank functions on R taking integer values are in one-to-one correspondence with the epic division R-algebras.

In the proof of the following proposition we use regularity of the Sylvester matrix rank function in an essential way. Thus, it would be interesting to know whether the same statement holds for an arbitrary Sylvester matrix rank function.

Proposition 5.10. *Let R be an algebra and* rk *a regular Sylvester matrix rank function on R. Then for every $A, B \in \operatorname{Mat}_{n\times m}(R)$ and every $\epsilon > 0$ we have that*

$$|\{\lambda \in K : \operatorname{rk}(A) - \operatorname{rk}(A - \lambda B) \geq \epsilon\}| \leq \frac{\operatorname{rk}(A)}{\epsilon}.$$

Proof. Without loss of generality we may assume that $n = m$, R is von Neumann regular and rk is faithful. Let $C \in \operatorname{Mat}_n(R)$ be such that $ACA = A$. Then, since

$$\operatorname{rk}(CA) = \operatorname{rk}(A) \text{ and}$$
$$\operatorname{rk}(CA - \lambda CACBCA) = \operatorname{rk}(CAC(A - \lambda B)CA) \leq \operatorname{rk}(A - \lambda B),$$

we obtain that

$$\{\lambda \in K : \operatorname{rk}(A) - \operatorname{rk}(A - \lambda B) \geq \epsilon\} \subseteq$$
$$\{\lambda \in K : \operatorname{rk}(CA) - \operatorname{rk}(CA - \lambda CACBCA) \geq \epsilon\}.$$

Let $S = CA \operatorname{Mat}_n(R)CA$. By [17, Lemma 16.2], the function $\operatorname{rk}'(T) = \frac{\operatorname{rk}(T)}{\operatorname{rk}(CA)}$ defines a Sylvester rank function on S. Now, by Proposition 5.1(5).

$$\{\lambda \in K : \operatorname{rk}(CA) - \operatorname{rk}(CA - \lambda CACBCA) \geq \epsilon\} =$$
$$\{\lambda \in K : 1 - \operatorname{rk}'(1_S - \lambda CACBCA) \geq \frac{\epsilon}{\operatorname{rk}(CA)}\}$$

has at most $\frac{\operatorname{rk}(CA)}{\epsilon} = \frac{\operatorname{rk}(A)}{\epsilon}$ elements. □

5.9. **Sylvester rank functions on epic von Neumann regular R-rings.** Let R be an algebra and let $\mathcal{U}$ be an epic von Neumann regular R-ring. In the following proposition we see that any Sylvester matrix rank function on $\mathcal{U}$ is completely determined by its values on matrices over R.

Proposition 5.11. *Let R be a subalgebra of a von Neumann regular algebra $\mathcal{U}$. Assume that the embedding of R in $\mathcal{U}$ is epic. Then, for any $r_1, \ldots, r_k \in \mathcal{U}$, there exist a matrix M of size $a \times b$ over R and vectors $v_1, \ldots, v_k \in R^b$ such that for every $t_1, \ldots, t_k \in R$ and every Sylvester matrix rank function* rk *on $\mathcal{U}$,*

$$\operatorname{rk}(t_1 r_1 + \ldots + t_k r_k) = \operatorname{rk}\begin{pmatrix} M \\ t_1 v_1 + \ldots + t_k v_k \end{pmatrix} - \operatorname{rk}(M).$$

Proof. Any R-module can be written as a direct limit of finitely presented modules. Thus, there are a directed set $(J, \leq)$, a family of finitely presented R-modules $\{L_j : j \in J\}$ and a family of homomorphisms $\{\phi_{ji} : L_j \to L_i : j \leq i\}$, satisfying

(1) ϕ_{jj} is the identity map on L_j and
(2) $\phi_{jl} = \phi_{il} \circ \phi_{ji}$ for all $j \leq i \leq l$,

and such that $\mathcal{U} \cong \varinjlim_{j \in J} L_j$ as left R-modules. We denote by $\phi_j : L_j \to \mathcal{U}$ the associated homomorphisms.

Since the direct limit commutes with the tensor product, we also have

$$\mathcal{U}\otimes_R\mathcal{U}\cong\varinjlim_{j\in J}\mathcal{U}\otimes_R L_j$$

as left $\mathcal{U}$-modules, where the connecting homomorphisms in the direct limit are

$$\operatorname{Id}\otimes\phi_{ji}:\mathcal{U}\otimes_R L_j\to\mathcal{U}\otimes_R L_i.$$

Note also that $\mathcal{U}\otimes_R L_j$ is a finitely presented left $\mathcal{U}$-module, and so, $\mathcal{U}\otimes_R L_j$ and all its finitely generated $\mathcal{U}$-submodules are projective (see Proposition 3.1).

Let C be the kernel of the map $p:\mathcal{U}^k\to\sum_{i=1}^k\mathcal{U}r_i$ that sends $(c_1,\dots,c_k)\in\mathcal{U}^k$ to $p(c_1,\dots,c_k)=c_1r_1+\dots+c_kr_k$. By Proposition 3.1(2), p splits. Hence, there are k elements $s_l=(s_{l1},\dots,s_{lk})$ $(1\le l\le k)$ such that C is generated by $\{s_l\}$.

Let $j\in J$ be such that $\phi_j(L_j)$ contains $r_1,\dots,r_k$. Choose $r_{ij}\in L_j$ satisfying $\phi_j(r_{ij})=r_i$. Since the embedding of R in $\mathcal{U}$ is epic, by Proposition 4.1, for each $1\le l\le k$,

$$\sum_{i=1}^k s_{li}\otimes_R r_i=\sum_{i=1}^k s_{li}r_i\otimes_R 1=0,$$

and we obtain that

$$(\operatorname{Id}\otimes\phi_j)(\sum_{i=1}^k s_{li}\otimes_R r_{ij})=\sum_{i=1}^k s_{li}\otimes_R r_i=0.$$

If an element maps to 0 at the limit, then it must map to 0 in finite time. Hence, there exists $n\in J$ such that $\operatorname{Id}\otimes\phi_{jn}(\sum_{i=1}^k s_{li}\otimes_R r_{ij})=0$ for all $1\le l\le k$. Hence for every $(x_1,\dots,x_k)\in C$,

$$\sum_{i=1}^k x_i(1\otimes_R\phi_{jn}(r_{ij}))=\sum_{i=1}^k x_i\otimes_R\phi_{jn}(r_{ij})=\operatorname{Id}\otimes\phi_{jn}(\sum_{i=1}^k x_i\otimes_R r_{ij})=0. \tag{4}$$

By Proposition 4.1, the multiplication map gives the isomorphism

$$\sum_{i=1}^k\mathcal{U}(1\otimes_R r_i)\cong\sum_{i=1}^k\mathcal{U}r_i.$$

Therefore C coincides with

$$\{(x_1,\dots,x_k):\ x_1(1\otimes_R r_1)+\dots+x_k(1\otimes_R r_k)=0\}.$$

Observe that $\operatorname{Id}\otimes\phi_n$ sends $\sum_{i=1}^k\mathcal{U}(1\otimes_R\phi_{jn}(r_{ij}))$ onto $\sum_{i=1}^k\mathcal{U}(1\otimes_R r_i)$, with

$$\operatorname{Id}\otimes\phi_n(1\otimes_R\phi_{jn}(r_{ij}))=1\otimes_R r_i,$$

and also from (4) it follows that

$$C\le\{(x_1,\dots,x_k):\ x_1(1\otimes_R\phi_{jn}(r_{1j}))+\dots+x_k(1\otimes_R\phi_{jn}(r_{kj}))=0\}.$$

This means that, in fact,

$$C=\{(x_1,\dots,x_k):\ x_1(1\otimes_R\phi_{jn}(r_{1j}))+\dots+x_k(1\otimes_R\phi_{jn}(r_{kj}))=0\},$$

and so,

$$\sum_{i=1}^k\mathcal{U}(1\otimes_R\phi_{jn}(r_{ij}))\cong\sum_{i=1}^k\mathcal{U}(1\otimes_R r_i)\cong\sum_{i=1}^k\mathcal{U}r_i$$

and, moreover, this isomorphism can be realized by sending $1 \otimes_R \phi_{jn}(r_{ij})$ to r_i ($1 \le i \le k$). In particular

$$\mathcal{U} \sum_{i=1}^k t_i (1 \otimes_R \phi_{jn}(r_{ij})) \cong \mathcal{U} \sum_{i=1}^k t_i r_i.$$

Since L_n is finitely presented we have the following exact sequence of left R-modules:

$$R^a \stackrel{r_M}{\to} R^b \stackrel{\psi}{\to} L_n \to 0,$$

where r_M is realized as a multiplication by a matrix $M \in \mathrm{Mat}_{a\times b}(R)$:

$$r_M(c_1, \ldots, c_a) = (c_1, \ldots, c_a)M, \ c_i \in R.$$

Choose $v_1, \ldots, v_k \in R^b$ such that $\psi(v_i) = \phi_{jn}(r_{ij})$. We denote by dim the Sylvester module rank function associated with rk. Then we obtain

$$\mathrm{rk}(\sum_{i=1}^k t_i r_i) = \dim(\mathcal{U} \sum_{i=1}^k t_i r_i) = \dim(\mathcal{U}(1 \otimes_R \sum_{i=1}^k t_i \phi_{jn}(r_{ij}))) =$$

$$\dim(\mathcal{U} \otimes_R L_n) - \dim((\mathcal{U} \otimes_R L_n)/\mathcal{U}(1 \otimes_R \sum_{i=1}^k t_i \phi_{jn}(r_{ij}))) =$$

$$b - \mathrm{rk}(M) - (b - \mathrm{rk}\begin{pmatrix} M \\ t_1 v_1 + \ldots + t_k v_k \end{pmatrix}) = \mathrm{rk}\begin{pmatrix} M \\ t_1 v_1 + \ldots + t_k v_k \end{pmatrix} - \mathrm{rk}(M).$$

This finishes the proof. □

6. Epic $*$-regular R-rings and $*$-regular Sylvester rank functions

6.1. **Epic morphisms related to $*$-regular closures.** In this subsection we show that if R is a $*$-ring and $f : R \to \mathcal{U}$ is a $*$-homomorphism from R to a $*$-regular ring $\mathcal{U}$ such that $\mathcal{U}$ is equal to the $*$-regular closure of $f(R)$ in $\mathcal{U}$, then f is epic. This follows from the next proposition and Proposition 4.1.

Proposition 6.1. *Let $\mathcal{U}$ be a $*$-regular ring and R a $*$-subring of $\mathcal{U}$. Assume that $\mathcal{U} = \mathcal{R}(R,\mathcal{U})$. Then the multiplication map $m : \mathcal{U} \otimes_R \mathcal{U} \to \mathcal{U}$ is an isomorphism of $\mathcal{U}$-bimodules. In particular, $\mathcal{U}$ is an epic R-ring.*

Proof. We have to show that m is bijective. Observe that the restriction of the multiplication map m on $\mathcal{U} \otimes 1$ is bijective. Thus, it is enough to show that $\mathcal{U} \otimes_R 1 = \mathcal{U} \otimes_R \mathcal{U}$. Let

$$S = \{r \in \mathcal{U} : \ 1 \otimes_R r = r \otimes_R 1\}.$$

It is clear that $R \le S$. Note that if r_1 and $r_2 \in S$, then

$$1 \otimes_R (r_1 + r_2) = 1 \otimes_R r_1 + 1 \otimes_R r_2 = r_1 \otimes_R 1 + r_2 \otimes_R 1 = (r_1 + r_2) \otimes_R 1.$$

Using that $\mathcal{U} \otimes_R \mathcal{U}$ is a $\mathcal{U}$-bimodule, we also obtain that

$$1 \otimes_R r_1 r_2 = (1 \otimes_R r_1) r_2 = (r_1 \otimes_R 1) r_2 =$$
$$r_1 \otimes_R r_2 = r_1(1 \otimes_R r_2) = r_1(r_2 \otimes_R 1) = r_1 r_2 \otimes_R 1.$$

Thus, S is a subring of $\mathcal{U}$. Since $\mathcal{U} = \mathcal{R}(R,\mathcal{U})$, by Corollary 3.5, in order to prove that $S = \mathcal{U}$, we have to show that if $r \in S$ is a self-adjoint element and s is the relative inverse of r, then $s \in S$. By Proposition 3.2(7), $rs = sr$, $rsr = r$ and $srs = s$. Therefore we obtain

$$1 \otimes_R s = 1 \otimes_R srs = 1 \otimes_R rss = r \otimes_R ss = rsr \otimes_R ss + ss \otimes_R r(1 - sr) = \\ rs \otimes_R rss + ssr \otimes_R (1 - rs) = s \otimes_R rs + s \otimes_R (1 - rs) = s \otimes_R 1.$$

Now, by Proposition 4.1, $\mathcal{U}$ is an epic R-ring.

□

Corollary 6.2. *Let $\mathcal{U}$ be a $*$-regular ring and R a $*$-subring of $\mathcal{U}$. Assume that $\mathcal{U} = \mathcal{R}(R,\mathcal{U})$. Then, for any $r_1, \ldots, r_k \in \mathrm{Mat}_{n\times m}(\mathcal{U})$, there is a matrix M of size $a \times b$ over R and there are matrices $v_1, \ldots, v_k$ of size $n \times b$ over R such that for any $t_1, \ldots, t_k \in \mathrm{Mat}_n(R)$ and every Sylvester matrix rank function* rk *on $\mathcal{U}$,*

$$\mathrm{rk}(t_1r_1 + \ldots + t_kr_k) = \mathrm{rk}\begin{pmatrix} M \\ t_1v_1 + \ldots + t_kv_k \end{pmatrix} - \mathrm{rk}(M).$$

Proof. Without loss of generality we may assume that $m = n$. From Propositions 6.1 and 4.1 it follows that the embedding of $\mathrm{Mat}_n(R)$ into $\mathrm{Mat}_n(\mathcal{U})$ is epic. Hence the corollary follows from Proposition 5.11. □

6.2. **Epic $*$-regular R-rings.** Let R be a $*$-ring. An **epic $*$-regular R-ring** is a triple $(\mathcal{U}, \mathrm{rk}, f)$, such that

(1) $\mathcal{U}$ is a $*$-regular ring;
(2) rk is a faithful Sylvester matrix rank function on $\mathcal{U}$;
(3) $f : R \to \mathcal{U}$ is a $*$-homomorphism;
(4) $\mathcal{R}(f(R),\mathcal{U}) = \mathcal{U}$.

We will write simply $(\mathcal{U}, \mathrm{rk})$ or $\mathcal{U}$ instead of $(\mathcal{U}, \mathrm{rk}, f)$ if f or (rk, f) are clear from the context.

We will say that two epic $*$-regular R-rings $(\mathcal{U}_1, \mathrm{rk}_1, f_1)$ and $(\mathcal{U}_2, \mathrm{rk}_2, f_2)$ are isomorphic if there exists a $*$-isomorphism $\alpha : \mathcal{U}_1 \to \mathcal{U}_2$ for which the following diagram

$$\begin{array}{ccc} R & \to^{\mathrm{Id}} & R \\ \downarrow f_1 & & \downarrow f_2 \\ \mathcal{U}_1 & \to^{\alpha} & \mathcal{U}_2 \end{array}$$

is commutative and $\mathrm{rk}_2(\alpha(a)) = \mathrm{rk}_1(a)$ for every $a \in \mathcal{U}_1$.

In this subsection we will prove, that, as in the case of epic division R-rings (see Proposition 4.4), the values $\mathrm{rk}(f(M))$, where M is a matrix over R, determines the epic $*$-regular ring $(\mathcal{U}, f, \mathrm{rk})$ uniquely up to isomorphism.

Theorem 6.3. *Let $(\mathcal{U}_1, \mathrm{rk}_1, f_1)$ and $(\mathcal{U}_2, \mathrm{rk}_2, f_2)$ be two epic $*$-regular R-rings. Then $(\mathcal{U}_1, \mathrm{rk}_1, f_1)$ and $(\mathcal{U}_2, \mathrm{rk}_2, f_2)$ are isomorphic if and only if for every matrix M over R*

$$\mathrm{rk}_1(f_1(M)) = \mathrm{rk}_2(f_2(M)).$$

Proof. The "only if" part is clear. Let us prove the "if" part. Assume that for every matrix M over R

$$\mathrm{rk}_1(f_1(M)) = \mathrm{rk}_2(f_2(M)).$$

Let $f = (f_1, f_2) : R \to \mathcal{U}_1 \oplus \mathcal{U}_2$ and let $\mathcal{U} = \mathcal{R}(f(R), \mathcal{U}_1 \oplus \mathcal{U}_2)$. Let $\pi_1 : \mathcal{U} \to \mathcal{U}_1$ and $\pi_2 : \mathcal{U} \to \mathcal{U}_2$ be the corresponding projections. By Corollary 4.3, $\pi_1(\mathcal{U}) = \mathcal{U}_1$ and

$\pi_2(\mathcal{U}) = \mathcal{U}_2$. Note that $\mathrm{rk}_1 \circ \pi_1$ and $\mathrm{rk}_2 \circ \pi_2$ are Sylvester matrix rank functions of $\mathcal{U}$. Since for any matrix M over R,

$$\mathrm{rk}_1(\pi_1(f(M))) = \mathrm{rk}_1(f_1(M)) = \mathrm{rk}_2(f_2(M)) = \mathrm{rk}_2(\pi_2(f(M))), \tag{5}$$

Corollary 6.2 implies that for any $r = (r_1, r_2) \in \mathcal{U}$

$$\mathrm{rk}_1(r_1) = \mathrm{rk}_1(\pi_1(r)) = \mathrm{rk}_2(\pi_2(r)) = \mathrm{rk}_2(r_2).$$

Since rk_1 and rk_2 are faithful, we obtain that π_1 and π_2 are injective, and so, they are isomorphisms. Hence $\pi_2 \circ (\pi_1)^{-1} : \mathcal{U}_1 \to \mathcal{U}_2$ is an isomorphism of epic $*$-regular rings.

□

A Sylvester matrix rank function rk on a $*$-algebra R is called **$*$-regular** if there exists a $*$-algebra homomorphism $f : R \to \mathcal{U}$ such that $\mathcal{U}$ is $*$-regular and $\mathrm{rk} \in \mathrm{Im}\, f^{\#}$. The previous theorem shows that the epic $*$-regular R-ring $(\mathcal{R}(f(R), \mathcal{U}), \mathrm{rk}, f)$ is completely determined by rk. We say that $\mathcal{R}(f(R), \mathcal{U})$ is **the $*$-regular R-algebra associated with** rk. We denote by $\mathbb{P}_{*reg}$ the space of $*$-regular rank functions on R.

Proposition 6.4. *Let R be a $*$-algebra. Then $\mathbb{P}_{*reg}$ is a closed convex subset of $\mathbb{P}(R)$.*

Proof. The proof of Proposition 5.9 works in this case as well. □

6.3. **The general approximation.** Let H be a countable group and let X be a set on which H acts on the left side. Assume that H acts freely on X and $H\backslash X$ is finite. We denote by $\mathcal{U}_H(l^2(X))$ ($\mathcal{N}_H(l^2(X))$) the algebra of unbounded (bounded) operators on $l^2(X)$ commuting with the left H-action.

If we fix a set of H-representatives $\bar{X}$ in X, we obtain a $*$-isomorphism

$$\Psi_{\bar{X}} : \mathcal{U}_H(l^2(X)) \to \mathrm{Mat}_{|H\backslash X|}(\mathcal{U}(H)).$$

If A is a matrix over $\mathcal{U}_H(l^2(X))$, we put

$$\mathrm{rk}_X(A) = \frac{1}{|H\backslash X|} \mathrm{rk}_H(\Psi_{\bar{X}}(A)).$$

Then rk_X is a Sylvester matrix rank function on $\mathcal{U}_H(l^2(X))$. A different choice of $\bar{X}$ changes $\Psi_{\bar{X}}$ only by conjugation. Thus, rk_X does not depend on the choice of $\bar{X}$.

Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Let F be a finitely generated free group. If F acts on X on the right and this action commutes with the H-action, we obtain a $*$-homomorphism $f_X : K[F] \to \mathcal{U}_H(l^2(X))$. By an abuse of notation we also use rk_X to denote the induced Sylvester matrix rank function on $K[F]$.

Let S be a free set of generators of F. Let $\{H_k\}_{k\in\mathbb{N}}$ be a family of countable groups. For any $k \in \mathbb{N}$ let X_k be an (H_k, F)-set (i.e. H_k acts on the left, F acts on the right and these two actions commute) such that H_k acts freely on X_k and $H_k\backslash X_k$ is finite. Let N be a normal subgroup of F and $G = F/N$. We define the sets

$$T_{k,s} = \{x \in X_k : \ x = x \cdot w \text{ if } w \in B_s(1) \cap N, \text{ and } \ x \neq x \cdot w \text{ if } w \in B_s(1) \setminus N\}.$$

We say that $\{X_k\}$ **approximates** G if for every s,

$$\lim_{k\to\infty} \frac{|H_k\backslash T_{k,s}|}{|H_k\backslash X_k|} = 1.$$

The sofic approximation is a particular case of the general approximation and corresponds to the case when the groups H_k are trivial. The approximation in the space of marked groups arises from the general approximation in the case when H_k and F act transitively on X_k for every k. The reader may consult [23, Subsection 2.6], where we give a geometric interpretation of general approximation and [23, Subsection 12.1], where another example of general approximation is presented.

Now, we can generalize the previous notation and formulate the general Lück approximation conjecture.

Conjecture 4 (The general Lück approximation conjecture over K for a group G)**.** *Let K be a subfield of $\mathbb{C}$, F a finitely generated free group and N a normal subgroup of F. For each natural number k, let X_k be an (H_k, F)-set such that H_k is a countable group that acts freely on X_k and $H_k\backslash X_k$ is finite. Assume that $\{X_k\}$ approximates $G = F/N$. Then for every $A \in \mathrm{Mat}_{n\times m}(K[F])$,*

$$\lim_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A).$$

6.4. **A structural reformulation of Conjecture 4.** We use the notation of Subsection 6.3. Fix a set of H_k-representatives $\bar{X}_k$ in X_k. Put $n_k = |\bar{X}_k|$ and let

$$f_k = \Psi_{\bar{X}_k} \circ f_{X_k} : \mathbb{C}[F] \to \mathrm{Mat}_{n_k}(\mathcal{U}(H_k)).$$

Remark 6.5. *Note that if K is a subfield of $\mathbb{C}$ closed under complex conjugation and $A \in K[F]$, then $f_k(A) \in \mathrm{Mat}_{n_k}(K[H_k])$. Thus, the $*$-regular closure $\mathcal{R}(f_k(K[F]), \mathrm{Mat}_{n_k}(\mathcal{U}(H_k)))$ of $f_k(K[F])$ in $\mathrm{Mat}_{n_k}(\mathcal{U}(H_k))$ is contained in $\mathrm{Mat}_{n_k}(\mathcal{R}_{K[H_k]})$.*

Let ω be a non-principal ultrafilter on $\mathbb{N}$. We can define

$$f_\omega : \mathbb{C}[F] \to \prod_\omega \mathrm{Mat}_{n_k}(\mathcal{U}(H_k))$$

by sending $A \in \mathbb{C}[F]$ to

$$f_\omega(A) = (f_k(A)).$$

Then, since the family $\{X_k\}$ approximates $G = F/N$, $\ker f_\omega$ is the ideal of $\mathbb{C}[F]$ generated by $\{g - 1 : \ g \in N\}$. In particular, $f_\omega(K[F]) \cong K[G]$. We put

$$\mathcal{R}_{K[G],\omega} = \mathcal{R}(f_\omega(K[F]), \prod_\omega \mathrm{Mat}_{n_k}(\mathcal{U}(H_k))).$$

Now, we reformulate the general Lück approximation conjecture. In the case where G is amenable and the family $\{X_k\}$ comes from a Følner family of sets, this result was proven by G. Elek in [11].

Theorem 6.6. *Let K be a subfield of $\mathbb{C}$ closed under complex conjugation, F a finitely generated free group and N a normal subgroup of F. For each natural number k, let X_k be an (H_k, F)-set such that H_k is a countable group that acts freely on X_k and $H_k\backslash X_k$ is finite. Assume that $\{X_k\}$ approximates $G = F/N$. Then the following two conditions are equivalent:*

(1) *For any matrix A over $K[F]$,*

$$\lim_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A).$$

(2) *For every non-principal ultrafilter ω on $\mathbb{N}$,*

$$(\mathcal{R}_{K[G]}, \mathrm{rk}_G) \text{ and } (\mathcal{R}_{K[G],\omega}, \mathrm{rk}_\omega)$$

are isomorphic as $K[F]$-$*$-rings.

Proof. If (1) holds, then (2) follows from Theorem 6.3 if we put $R = K[F]$.

If (2) holds, then for any non-principal ultrafilter ω on $\mathbb{N}$ and for any matrix A over $K[F]$ of size $n \times m$

$$\lim_\omega \mathrm{rk}_{X_k}(A) = \lim_\omega \left(\frac{\mathrm{rk}_{H_k}(f_k(A))}{|H_k \backslash X_k|} \right) = \mathrm{rk}_\omega(f_\omega(A)) = \mathrm{rk}_G(A).$$

But this implies (1). □

7. The natural extension of regular Sylvester rank functions

Given an algebra R and a regular Sylvester rank function rk on R, we will introduce two notions of natural extensions of rk, which will be regular Sylvester rank functions on $R \otimes_K E$, where E is either $K(t)$ or an algebraic extension of K, respectively. These two constructions will play an essential role in our arguments in further sections.

7.1. **The natural transcendental extension of Sylvester matrix rank functions of von Neumann regular rings.** Let $\mathcal{U}$ be a von Neumann regular ring and let rk be a faithful Sylvester matrix rank function on $\mathcal{U}$. Denote by dim the Sylvester module rank function associated with rk. Then as we have explained in Subsection 5.6, dim can be uniquely extended as a length function on $\mathcal{U}$-modules.

The following proposition is a particular case of a result that has been proved recently by S. Virili.

Proposition 7.1. [46, Theorem B] *Let M be a $\mathcal{U}[t^{\pm 1}]$-module and let L be a $\mathcal{U}$-submodule of M satisfying $\dim L < \infty$. Put*

$$E_{M,L} = \lim_{i \to \infty} \frac{\dim(L + tL + \ldots + t^{i-1}L)}{i}$$

and let

$$\widetilde{\dim}\, M = \sup\{E_{M,N} : \ N \text{ is a } \mathcal{U}\text{-submodule of } M \text{ and } \ \dim N < \infty\}.$$

Then $E_{M,L}$ (and so $\widetilde{\dim}\, M$) are well-defined. Moreover, $\widetilde{\dim}$ is an exact Sylvester module rank function on $\mathcal{U}[t^{\pm 1}]$.

For each $i \geq 0$ let Q_i be the set of polynomials in $\mathcal{U}[t]$ of degree at most i. When M is a finitely generated $\mathcal{U}[t^{\pm 1}]$-module, then $\widetilde{\dim}\, M$ can be calculated in the following way.

Lemma 7.2. *Let M be a finitely generated $\mathcal{U}[t^{\pm 1}]$-module generated by $\{m_1, \ldots, m_k\}$ and let V be the $\mathcal{U}$-submodule generated by $\{m_1, \ldots, m_k\}$. Then*

$$\widetilde{\dim}\, M = E_{M,V} = \lim_{i \to \infty} \frac{\dim(V + tV + \ldots + t^{i-1}V)}{i}.$$

Proof. Let L be a $\mathcal{U}$-submodule of M satisfying $\dim L < \infty$. Since $\dim L < \infty$, for every $\epsilon > 0$, there exists k such that

$$\dim L - \dim(t^{-k} Q_{2k} V \cap L) < \epsilon.$$

Therefore,

$$\widetilde{\dim} M = \sup_k E_{M,t^{-k}Q_{2k}V} = \sup_k E_{M,Q_kV} = E_{M,V}.$$

□

We denote by $\widetilde{\mathrm{rk}}$ the Sylvester matrix rank function on $\mathcal{U}[t^{\pm 1}]$ associated with $\widetilde{\dim}$. The previous lemma can be reformulated in the following way.

Lemma 7.3. *Let $A \in \mathrm{Mat}_{n\times m}(\mathcal{U}[t])$. Then*

$$\widetilde{\mathrm{rk}}(A) = \lim_{i\to\infty} \frac{\dim((Q_{i-1})^n A)}{i}.$$

Proof. Since $\widetilde{\dim}$ is exact, we obtain

$$\widetilde{\mathrm{rk}}(A) = m - \widetilde{\dim}(\mathcal{U}[t^{\pm1}]^m/(\mathcal{U}[t^{\pm1}]^n A)) = \widetilde{\dim}(\mathcal{U}[t^{\pm1}]^n A) \overset{\text{Lemma 7.2}}{=} \lim_{i\to\infty} \frac{\dim((Q_{i-1})^n A)}{i}.$$

□

For any matrix $A \in \mathrm{Mat}_{n\times m}(\mathcal{U}[t])$ over the polynomial ring $\mathcal{U}[t]$ consider the maps

$$\phi^A_{\mathcal{U}[t]/(t^i)} : (\mathcal{U}[t]/(t^i))^n \to (\mathcal{U}[t]/(t^i))^m,\ (a_1,\ldots,a_n) \mapsto (a_1,\ldots,a_n)A.$$

These are maps between free finitely generated $\mathcal{U}$-modules, and so, we can consider $\mathrm{rk}(\phi^A_{\mathcal{U}[t]/(t^i)})$. We put

$$\widetilde{\mathrm{rk}}_i(A) = \frac{\mathrm{rk}(\phi^A_{\mathcal{U}[t]/(t^i)})}{i}.$$

It is clear that $\widetilde{\mathrm{rk}}_i$ is a regular Sylvester matrix rank function on $\mathcal{U}[t]$. Now we give an alternative formula for $\widetilde{\mathrm{rk}}$.

Proposition 7.4. *For every matrix A over $\mathcal{U}[t]$ we have that*

$$\widetilde{\mathrm{rk}}(A) = \lim_{i\to\infty} \widetilde{\mathrm{rk}}_i(A).$$

Moreover, $\widetilde{\mathrm{rk}}$ is regular (as a Sylvester matrix rank function on $\mathcal{U}[t^{\pm1}]$).

Proof. For simplicity of exposition let us assume that A is an 1 by 1 matrix. Let s be the degree of A as a polynomial over $\mathcal{U}$. Then for every $i \geq s+1$ we obtain

$$\frac{\dim(Q_{i-s-1}A)}{i} \leq \frac{\dim(Q_{i-1} \cap \mathcal{U}[t]A)}{i} \leq \widetilde{\mathrm{rk}}_i(A) \leq \frac{\dim(Q_{i-1}A)}{i}.$$

Therefore, by Lemma 7.3,

$$\widetilde{\mathrm{rk}}(A) = \lim_{i\to\infty} \widetilde{\mathrm{rk}}_i(A).$$

Since, by Proposition 5.9, $\mathbb{P}_{reg}(\mathcal{U}[t])$ is closed and $\widetilde{\mathrm{rk}}_i$ are regular, $\widetilde{\mathrm{rk}}$ is also regular (as a Sylvester matrix rank function on $\mathcal{U}[t]$). By (3), $\widetilde{\mathrm{rk}}$ is also regular as a Sylvester matrix rank function on $\mathcal{U}[t^{\pm1}]$.

□

7.2. **The definition of the natural transcendental extension.** Let R be an algebra and let rk be a Sylvester rank function on R. Consider a matrix $A \in \mathrm{Mat}_{n\times m}(R[t])$ over the polynomial ring $R[t]$ and let

$$\phi^A_{R[t]/(t^i)} : (R[t]/(t^i))^n \to (R[t]/(t^i))^m,\ (a_1,\dots,a_n) \mapsto (a_1,\dots,a_n)A.$$

As before we put

$$\widetilde{\mathrm{rk}}_i(A) = \frac{\mathrm{rk}(\phi^A_{R[t]/(t^i)})}{i}.$$

Proposition 7.5. *Let* rk *be a regular Sylvester matrix rank function. Then*

(1) *For every matrix A over $R[t]$ there exists $\lim_{i\to\infty} \widetilde{\mathrm{rk}}_i(A)$, which we denote by $\widetilde{\mathrm{rk}}(A)$.*
(2) $\widetilde{\mathrm{rk}}$ *is a regular Sylvester matrix rank function on $R[t^{\pm 1}]$.*
(3) *Moreover, if R is von Neumann regular, the associated Sylvester module rank function $\widetilde{\dim}$ on $R[t^{\pm 1}]$ is exact.*

Proof. Since rk is regular, (1) and (2) follow from Proposition 7.4.
(3) The last assertion of the proposition is a part of Proposition 7.1. □

Note that $\widetilde{\mathrm{rk}}(p) = 1$ for every $0 \neq p \in K[t]$. Thus, by Corollary 5.5, we can think about $\widetilde{\mathrm{rk}}$ as a Sylvester rank function on $R \otimes_K K(t)$. The Sylvester matrix rank function $\widetilde{\mathrm{rk}}$ on $R\otimes_K K(t)$ will be called **the natural transcendental extension** of rk.

7.3. **Another characterization of the natural transcendental extension.** In this subsection we will give alternative characterizations of the natural transcendental extension of a Sylvester matrix rank function on von Neumann regular rings.

First observe that there is an elegant way to calculate $\widetilde{\dim}\, I$ where I is a left ideal of $\mathcal{U}[t^{\pm 1}]$.

Proposition 7.6. *Let I be a left ideal of $\mathcal{U}[t^{\pm 1}]$. Then*

$$\widetilde{\dim}\, I = \sup\{\mathrm{rk}(a_0) \mid \exists n \geq 0,\ \exists a_0,\dots,a_n \in \mathcal{U} : a_0 + a_1 t + \ldots + a_n t^n \in I\}.$$

Proof. Let $P_i = Q_i \cap I$, $M = \mathcal{U}[t^{\pm 1}]/I$ and $\overline{Q_i} = (Q_i + I)/I$. Since $\widetilde{\dim}$ is exact,

$$\widetilde{\dim}\, I = 1 - \widetilde{\dim}\, M.$$

Hence, by Lemma 7.2,

$$\widetilde{\dim}\, I = 1 - \lim_{i\to\infty} \frac{\dim \overline{Q_{i-1}}}{i} = \lim_{i\to\infty} \frac{\dim P_{i-1}}{i}.$$

Note that multiplication by t sends P_i/P_{i-1} into P_{i+1}/P_i. Hence

$$\dim P_i - \dim P_{i-1} = \dim P_i/P_{i-1} \leq \dim P_{i+1}/P_i = \dim P_{i+1} - \dim P_i$$

and so

$$\widetilde{\dim}\, I = \lim_{i\to\infty} \frac{\dim P_i}{i+1} = \lim_{i\to\infty} \frac{\sum_{j=0}^{i} \dim P_j - \dim P_{j-1}}{i+1} = \lim_{i\to\infty} (\dim P_i - \dim P_{i-1}) = \lim_{i\to\infty} (\dim P_i - \dim tP_{i-1}) = \lim_{i\to\infty} \dim(P_i/tP_{i-1}).$$

Observe that

$$P_i/tP_{i-1} \cong T_i = \{b_0 \in \mathcal{U} \mid \exists b_0, \ldots, b_i \in \mathcal{U} : b_0 + b_1 t + \ldots + b_i t^i \in I\}.$$

Since any finitely generated $\mathcal{U}$-submodule of T_i is cyclic,

$$\dim T_i = \sup\{\operatorname{rk}(b_0) \mid \exists b_1, \ldots, b_i \in \mathcal{U} : b_0 + b_1 t + \ldots + b_i t^i \in I\}.$$

Therefore, we obtain that

$$\widetilde{\dim}\, I = \sup\{\operatorname{rk}(b_0) \mid \exists n \geq 0,\ \exists b_0, \ldots, b_n \in \mathcal{U} : b_0 + b_1 t + \ldots + b_n t^n \in I\}.$$

□

Now, we present a characterization of the natural transcendental extension which will help us to relate the sofic Lück approximation and the strong algebraic eigenvalue property.

Proposition 7.7. *Let $\mathcal{U}$ be a von Neumann regular algebra and let* rk *be a Sylvester rank function on $\mathcal{U}$. Let rk' be a Sylvester matrix rank function on $\mathcal{U}[t^{\pm 1}]$ that extends* rk*. Assume that for any n by n matrix A over $\mathcal{U}$,*

$$\operatorname{rk}'(I_n + tA) = n.$$

Then $\operatorname{rk}' = \widetilde{\operatorname{rk}}$.

Proof. Let $A = I_n + t(B + Ct)$ be an n by n matrix over $\mathcal{U}[t]$. Then

$$\operatorname{rk}'(A) = \operatorname{rk}'\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} - n.$$

Since multiplication by an invertible matrix does not change the Sylvester rank of a matrix, we obtain that

$$\operatorname{rk}'\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} = \operatorname{rk}'\begin{pmatrix} I_n + t(B+Ct) & B + tC \\ 0 & I_n \end{pmatrix} =$$
$$\operatorname{rk}'\begin{pmatrix} I_n & B + tC \\ -tI_n & I_n \end{pmatrix} = \operatorname{rk}'\begin{pmatrix} I_n + tB & tC \\ -tI_n & I_n \end{pmatrix} =$$
$$\operatorname{rk}'\left(I_{2n} + t\begin{pmatrix} B & C \\ -I_n & 0 \end{pmatrix}\right).$$

Let $s \geq 1$. If we apply the previous procedure $s-1$ times, we obtain that for any $A_1, \ldots, A_s \in \operatorname{Mat}_k(\mathcal{U})$, there exists $D \in \operatorname{Mat}_{k2^{s-1}}(\mathcal{U})$ such that

$$\operatorname{rk}'(I_k + A_1 t + \ldots + A_s t^s) = \operatorname{rk}'(I_{k2^{s-1}} + tD) - \sum_{i=0}^{s-2} k2^i = k. \tag{6}$$

We want to show that for every k by k matrix A over $\mathcal{U}[t^{\pm 1}]$, $\operatorname{rk}'(A) = \widetilde{\operatorname{rk}}(A)$. For simplicity of exposition we assume that A is a 1 by 1 matrix.

By Proposition 7.6, for every $\epsilon > 0$ there exists $b_0 + b_1 t + \ldots + b_m t^m \in \mathcal{U}[t^{\pm 1}]A$ and $c_0 + c_1 t + \ldots + c_m t^m \in \operatorname{Ann}_l^{\mathcal{U}[t^{\pm 1}]}(A)$ such that

$$\widetilde{\operatorname{rk}}(A) = \widetilde{\dim}(\mathcal{U}[t^{\pm 1}]A) \leq \operatorname{rk}(b_0) + \epsilon \text{ and}$$
$$\widetilde{\operatorname{rk}}(A) = 1 - \widetilde{\dim}(\operatorname{Ann}_l^{\mathcal{U}[t^{\pm 1}]}(A)) \geq 1 - \operatorname{rk}(c_0) - \epsilon.$$

Let $u\in\mathcal{U}$ be such that $b_0ub_0=b_0$. Then we obtain that

$$\mathrm{rk}'(A)\ge \mathrm{rk}'(b_0+b_1t+\ldots+b_mt^m)\ge \mathrm{rk}'(b_0u(b_0+b_1t+\ldots+b_mt^m))=$$
$$\mathrm{rk}'(b_0(1+ub_1t+\ldots+ub_mt^m))\overset{\text{by (6) and Proposition 5.1(3)}}{=}\mathrm{rk}'(b_0)=\mathrm{rk}(b_0)\ge\widetilde{\mathrm{rk}}(A)-\epsilon.$$

Since $c_0+c_1t+\ldots+c_mt^m\in \mathrm{Ann}_l^{\mathcal{U}[t^{\pm1}]}(A)$, by Proposition 5.1 (3),

$$\mathrm{rk}'(A)\le 1-\mathrm{rk}'(c_0+c_1t+\ldots+c_mt^m).$$

Let $v\in\mathcal{U}$ be such that $c_0vc_0=c_0$. Then

$$\mathrm{rk}'(A)\le 1-\mathrm{rk}'(c_0+c_1t+\ldots+c_lt^l)\le 1-\mathrm{rk}'(c_0v(c_0+c_1t+\ldots+c_lt^l))=$$
$$1-\mathrm{rk}'(c_0(1+vc_1t+\ldots+vc_lt^l))\overset{\text{by (6) and Proposition 5.1(3)}}{=}$$
$$1-\mathrm{rk}'(c_0)=1-\mathrm{rk}(c_0)\le\widetilde{\mathrm{rk}}(A)+\epsilon.$$

Since ϵ is arbitrary, $\mathrm{rk}'(A)=\widetilde{\mathrm{rk}}(A)$. □

An immediate consequence of Proposition 7.7 is the following corollary.

Corollary 7.8. *Let $\{\mathrm{rk}_i\}_{i\in\mathbb{N}}$ be a family of regular Sylvester matrix rank functions on an algebra R. For each $i\in\mathbb{N}$, let $\widetilde{\mathrm{rk}}_i\in\mathbb{P}(R[t])$ be the natural transcendental extension of rk_i. Let ω be an ultrafilter on $\mathbb{N}$. Then $\lim_\omega\widetilde{\mathrm{rk}}_i$ is the natural transcendental extension of $\lim_\omega \mathrm{rk}_i$.*

Let now $K=\bar K$ be algebraically closed. Let R be an algebra and rk a Sylvester matrix rank function on R. If $P=(t-\lambda)$ is a maximal ideal of $K[t]$ and A is a matrix over $R[t]$, we denote

$$\widetilde{\mathrm{rk}}_P(A)=\mathrm{rk}(\bar A),$$

where $\bar A$ is obtained by reducing the coefficients of A modulo P. It is clear that $\widetilde{\mathrm{rk}}_P$ is a Sylvester matrix rank function on $R[t]$.

Corollary 7.9. *Assume that $K=\bar K$ is algebraically closed. Let R be an algebra and rk a regular Sylvester matrix rank function on R. Let $\{P_i=(t-\lambda_i)\}_{i\in\mathbb{N}}$ be a family of distinct maximal ideals of $K[t]$. Then there exists $\lim_{i\to\infty}\widetilde{\mathrm{rk}}_{P_i}\in\mathbb{P}_{reg}(R[t])$ and it is equal to $\widetilde{\mathrm{rk}}$.*

Proof. We have to show that for every non-principal ultrafilter ω on $\mathbb{N}$, $\lim_\omega\widetilde{\mathrm{rk}}_{P_i}=\widetilde{\mathrm{rk}}$. Put $\mathrm{rk}'=\lim_\omega\widetilde{\mathrm{rk}}_{P_i}$. By Proposition 7.7, it is enough to prove that for every n by n matrix A over R,

$$\mathrm{rk}'(I_n+tA)=\lim_\omega\widetilde{\mathrm{rk}}_{P_i}(I_n+tA)=\lim_\omega\mathrm{rk}(I_n+\lambda_iA)=n.$$

But this follows from Proposition 5.1(5). □

7.4. **The natural transcendental extension of a $*$-regular Sylvester rank function.** Let R be a $*$-algebra and let rk be a $*$-regular Sylvester rank function on R. In what follows, we extend $*$ to $R[t^{\pm1}]$ by putting $t^*=t^{-1}$. In the next proposition we show that under some natural conditions $\widetilde{\mathrm{rk}}$ is also $*$-regular.

Proposition 7.10. *Let R be a $*$-algebra. Let* rk *be a $*$-regular Sylvester rank function on R having a positive definitive $*$-regular envelope. Then $\widetilde{\mathrm{rk}}$ is a $*$-regular Sylvester rank function on $R[t^{\pm 1}]$ (and so on $R \otimes_K K(t)$).*

Proof. There are a positive definite $*$-regular algebra $\mathcal{U}$, a Sylvester matrix rank function rk_1 on $\mathcal{U}$ and a $*$-algebra homomorphism $f : R \to \mathcal{U}$ such that $\mathrm{rk} = f^{\#}(\mathrm{rk}_1)$. Denote by rk_n the Sylvester matrix rank function $\frac{1}{n}\mathrm{rk}_1$ on $\mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n))$. Using that $\mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n))$ is canonically isomorphic to $\mathrm{Mat}_n(\mathcal{U})$, we can define an operation $*$ on $\mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n))$ using this canonical isomorphism. Therefore, $\mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n))$ is a $*$-regular algebra. Let ω be a non-principal ultrafilter on $\mathbb{N}$. Then $\prod_\omega \mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n))$ is also a $*$-regular algebra and $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_i$ is a Sylvester rank function on it.

Let $f_n : R[t] \to \mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n))$ be the natural map and let

$$f_\omega = (f_n) : R[t] \to \prod_\omega \mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n)).$$

Since $\mathrm{rk}_\omega(f_\omega(t)) = 1$, $f_\omega(t)$ is invertible. Thus, we can extend the homomorphism f_ω to $R[t^{\pm 1}]$, by sending t^{-1} to $f_\omega(t)^{-1}$. Although the ring $R[t]$ is not a $*$-ring, and so, the map f_n does not respect the $*$-operation, the map $f_\omega : R[t^{\pm 1}] \to \prod_\omega \mathrm{End}_{\mathcal{U}}(\mathcal{U}[t]/(t^n))$ is a $*$-homomorphism. Observe also that $\widetilde{\mathrm{rk}} = (f_\omega)^{\#}(\mathrm{rk}_\omega)$ by the definition of $\widetilde{\mathrm{rk}}$. Therefore, $\widetilde{\mathrm{rk}}$ is $*$-regular. □

Thus, if the hypotheses of the previous proposition hold we can speak about the $*$-regular $R[t^{\pm 1}]$-algebra associated with $\widetilde{\mathrm{rk}}$. When R is semisimple Artinian, we obtain the following description of the $*$-regular $R[t^{\pm 1}]$-algebra associated with $\widetilde{\mathrm{rk}}$.

Proposition 7.11. *Let R be a semisimple Artinian $*$-algebra and let* rk *be a faithful $*$-regular Sylvester rank function on R having a positive definite $*$-regular envelope. Then the $*$-regular $R[t^{\pm 1}]$-algebra associated with the natural transcendental extension $\widetilde{\mathrm{rk}}$ is isomorphic to $Q_l(R[t^{\pm 1}])$.*

Proof. Denote by $\mathcal{U}$ the $*$-regular $R[t^{\pm 1}]$-algebra associated with $\widetilde{\mathrm{rk}}$ and let $\widetilde{\dim}$ be the Sylvester module rank function on left $R[t^{\pm 1}]$-modules associated with $\widetilde{\mathrm{rk}}$. Let $a \in R[t^{\pm 1}]$ be a non-zero-divisor. Since $\widetilde{\dim}$ is exact,

$$\widetilde{\mathrm{rk}}(a) = 1 - \widetilde{\dim}(R[t^{\pm 1}]/R[t^{\pm 1}]a) = \widetilde{\dim}(R[t^{\pm 1}]a) = \widetilde{\dim}(R[t^{\pm 1}]) = 1.$$

Therefore, a is invertible in $\mathcal{U}$. Since R is a semisimple Artinian ring, $R[t^{\pm 1}]$ is a semiprime Noetherian ring. Hence, by Goldie's theorem, there exists $Q_l(R[t^{\pm 1}])$ and it is semisimple Artinian. Since any non-zero-divisor of $R[t^{\pm 1}]$ is invertible in $\mathcal{U}$, $Q_l(R[t^{\pm 1}])$ embeds in $\mathcal{U}$. Observe that $Q_l(R[t^{\pm 1}])$ is also a $*$-regular ring. Therefore $\mathcal{U} \cong Q_l(R[t^{\pm 1}])$.

□

7.5. **Algebraic extensions.** Let R be an algebra and rk a Sylvester matrix rank function on R. Let E/K be an algebraic extension of fields. Take a matrix $A \in \mathrm{Mat}_{n\times m}(R \otimes_K E)$. Then there exists a finite subextension E_0/K of E/K such that $A \in \mathrm{Mat}_{n\times m}(R \otimes_K E_0)$.

The action of $A \in \mathrm{Mat}_{n\times m}(R \otimes_K E_0)$ by right multiplication on $(R \otimes_K E_0)^n$ defines an R-homomorphism

$$\phi^A : (R \otimes_K E_0)^n \to (R \otimes_K E_0)^m$$

of free R-modules. We put

$$\widetilde{\mathrm{rk}}(A) = \frac{\mathrm{rk}(\phi^A)}{|E_0 : K|}.$$

Observe that $\widetilde{\mathrm{rk}}(A)$ does not depend on the choice of E_0. It is clear that $\widetilde{\mathrm{rk}}$ is a Sylvester matrix rank function on $R \otimes_K E$ and we call it **the natural (algebraic) extension** of rk on $R \otimes_K E$.

Proposition 7.12. *Let $\mathcal{U}$ be a von Neumann regular algebra and let E/K be an algebraic separable extension.*

1. *The algebra $\mathcal{U} \otimes_K E$ is von Neumann regular.*
2. *Let* rk *be a faithful Sylvester rank function on $\mathcal{U}$. Assume that $\mathcal{U}$ is complete with respect to the* rk*-metric, $Z(\mathcal{U})$ is a field and the algebraic elements of the extension $Z(\mathcal{U})/K$ are in K. Then*
   (a) *$\mathcal{U} \otimes_K E$ is simple and*
   (b) *$\widetilde{\mathrm{rk}}$ is the unique Sylvester matrix rank function on $\mathcal{U} \otimes_K E$.*

Proof. (1) Let $a \in \mathcal{U} \otimes_K E$. We want to show that a is von Neumann regular. Without loss of generality we may assume that E/K is finite.

The algebra $\mathcal{U} \otimes_K E$ is a subalgebra of

$$\mathcal{U} \otimes_K \mathrm{End}_K(E) \cong \mathcal{U} \otimes_K \mathrm{Mat}_{|E:K|}(K) \cong \mathrm{Mat}_{|E:K|}(\mathcal{U}).$$

Thus, there exists $b \in \mathcal{U} \otimes_K \mathrm{End}_K(E)$ such that $aba = a$.

Consider $\mathrm{End}_K(E)$ as E-bimodule. Since E/K is separable, $E \otimes_K E$ is semisimple, and so, there exists an E-subbimodule M of $\mathrm{End}_K(E)$ such that

$$\mathrm{End}_K(E) = E \oplus M.$$

Let $b = b_1 + b_2$ with $b_1 \in \mathcal{U} \otimes_K E$ and $b_2 \in \mathcal{U} \otimes_K M$. Hence

$$a = aba = a(b_1 + b_2)a = ab_1a + ab_2a.$$

Since $a, ab_1a \in \mathcal{U} \otimes_K E$ and $ab_2a \in \mathcal{U} \otimes_K M$, $ab_1a = a$. We are done.

(2) Without loss of generality we may again assume that E/K is finite. Then $\mathcal{U} \otimes_K E$ is complete with respect to the $\widetilde{\mathrm{rk}}$-metric. On the other hand, since the algebraic elements of the extension $Z(\mathcal{U})/K$ are in K,

$$Z(\mathcal{U} \otimes_K E) \cong Z(\mathcal{U}) \otimes_K E$$

is a field. Thus, the statement follows from Proposition 5.7(2). □

We finish this section with the following obvious analog of Corollary 7.8.

Proposition 7.13. *Let $\{\mathrm{rk}_i\}_{i\in\mathbb{N}}$ be a family of Sylvester matrix rank functions on an algebra R and let E/K be an algebraic extension. For each $i \in \mathbb{N}$, let $\widetilde{\mathrm{rk}}_i$ be the natural algebraic extension of rk_i on $R \otimes_K E$. Let ω be an ultrafilter on $\mathbb{N}$. Then $\lim\limits_{\omega} \widetilde{\mathrm{rk}}_i$ is the natural algebraic extension of $\lim\limits_{\omega} \mathrm{rk}_i$ on $R \otimes_K E$.*

8. The strict eigenvalue property

Let E be a subfield of $\mathbb{C}$ closed under complex conjugation. Let F be a finitely generated free group. Denote by $\mathcal{S}_F$ the set of finite F-sets. For every $X \in \mathcal{S}_F$, let

$$f_X : E[F] \to \mathrm{Mat}_{|X|}(E) \cong \mathrm{End}_E(E[X])$$

be the representation of $E[F]$ associated with the permutation action of F on X. We put

$$\mathcal{V} = \prod_{X\in \mathcal{S}_F} \mathrm{Mat}_{|X|}(E) \text{ and } f = (f_X) : E[F] \to \mathcal{V}.$$

We write an element $a \in \mathcal{V}$ in the form $a = (a_X)$, where $a_X \in \mathrm{Mat}_{|X|}(E)$. Since f is injective, we will identify the elements of $E[F]$ and their images in $\mathcal{V}$. In this section we prove the following result.

Theorem 8.1. *Assume the previous notation. Then for every two elements* $z = (z_X)$ *and* $w = (w_X)$ *of* $\mathrm{Mat}_{n\times m}(\mathcal{R}(f(E[F]),\mathcal{V}))$ *and for every* $\epsilon > 0$ *the set*

$$S_\epsilon(z,w) = \{\lambda \in \bar{E} : \text{ there exists } X \text{ such that } \mathrm{rk}_{\overline{E}}(w_X) - \mathrm{rk}_{\overline{E}}(w_X - \lambda z_X) \geq \epsilon |X|\}$$

is finite.

This theorem clearly implies Theorem 2.2. In the next subsection we will show that, in fact, Theorem 8.1 follows from Theorem 2.2.

8.1. **Reduction to the case** $z, w \in \mathrm{Mat}_{n\times m}(f(\bar{E}[F]))$**.** In this subsection we show that in order to prove Theorem 8.1 we can assume that $z, w \in \mathrm{Mat}_{n\times m}(f(E[F]))$ and $E = \bar{E}$ is algebraically closed.

Clearly we can assume that E is algebraically closed. Put $R = E[F]$ and $\mathcal{U} = \mathcal{R}(f(R),\mathcal{V})$. We denote by π_X the projection of $\mathcal{V}$ onto $\mathrm{Mat}_{|X|}(E)$ and we put

$$\mathrm{rk}_X = \mathrm{rk}_{\mathrm{Mat}_{|X|}(E)} \circ \pi_X \in \mathbb{P}(\mathcal{U}).$$

By an abuse of notation we use rk_X also to denote the induced Sylvester matrix rank function on R. By Corollary 6.2, there are a matrix M of size $n' \times m'$ over R and matrices v_1, v_2 over R of size $n'' \times m'$ such that for every Sylvester matrix rank function rk on $\mathcal{U}$ and every $\lambda \in E$

$$\mathrm{rk}(w - \lambda z) = \mathrm{rk}\begin{pmatrix} f(M) \\ f(v_1) - \lambda f(v_2) \end{pmatrix} - \mathrm{rk}(f(M)) = \mathrm{rk}(w' - \lambda z') - \mathrm{rk}(f(M)),$$

where

$$w' = f\begin{pmatrix} M \\ v_1 \end{pmatrix} \text{ and } z' = f\begin{pmatrix} 0 \\ v_2 \end{pmatrix}.$$

Let us put $z'_X = \pi_X(z')$ and $w'_X = \pi_X(w')$. Thus, we obtain

$$\mathrm{rk}_{\mathrm{Mat}_{|X|}(E)}(w_X) - \mathrm{rk}_{\mathrm{Mat}_{|X|}(E)}(w_X - \lambda z_X) = \mathrm{rk}_X(w) - \mathrm{rk}_X(w - \lambda z) =$$
$$\mathrm{rk}_X(w') - \mathrm{rk}_X(w' - \lambda z') = \mathrm{rk}_{\mathrm{Mat}_{|X|}(E)}(w'_X) - \mathrm{rk}_{\mathrm{Mat}_{|X|}(E)}(w'_X - \lambda z'_X).$$

Therefore, we have shown that it is enough to prove Theorem 8.1 assuming that

$$z, w \in \mathrm{Mat}_{n\times m}(f(E[F])) \text{ and } E = \bar{E}.$$

We will prove Theorem 8.1 in Subsection 8.8. Since its proof is quite technical and depends on several preliminary results, we consider first in the next subsection two very particular cases.

8.2. **Two particular cases of Theorem 8.1.** In this subsection we explain the proof of Theorem 8.1 in two particular cases. We will not use the results of this subsection later on.

Let us consider the first example. Here we assume that $z, w \in f(\mathbb{Z}[F])$. Let $a, b \in \mathbb{Z}$ be two coprime non-zero integers and assume that $\frac{a}{b} \in S_\epsilon(z, w)$. Thus, there exists an F-set X such that

$$\operatorname{rk}_{\mathbb{Q}}(w_X) - \operatorname{rk}_{\mathbb{Q}}(bw_X - az_X) \geq \epsilon|X|. \tag{7}$$

Let us show that $|a|$ and $|b|$ are bounded in terms of z, w and ϵ.

For any matrix $M \in \operatorname{Mat}_{n\times m}(\mathbb{Z})$, we define $\deg_+(M) = \log_2 |V_{tor}|$ to be the size of the torsion part V_{tor} of the $\mathbb{Z}$-module $V = \mathbb{Z}^m/\mathbb{Z}^n M$. Then Lemma 8.4 gives us the following two inequalities.

$$(m - \operatorname{rk}_{\mathbb{Q}}(M)) \log_2 |a| \leq \deg_+ \begin{pmatrix} M \\ aI_m \end{pmatrix} \leq (m - \operatorname{rk}_{\mathbb{Q}}(M)) \log_2 |a| + \deg_+(M). \tag{8}$$

Observe that since a and b are coprime,

$$\deg_+ \begin{pmatrix} bw_X - az_X \\ aI_m \end{pmatrix} = \deg_+ \begin{pmatrix} w_X \\ aI_m \end{pmatrix}.$$

Thus using (8) we obtain that

$$(m - \operatorname{rk}_{\mathbb{Q}}(bw_X - az_X)) \log_2 |a| \leq (m - \operatorname{rk}_{\mathbb{Q}}(w_X)) \log_2 |a| + \deg_+(w_X). \tag{9}$$

Therefore,

$$\log_2 |a| \overset{\text{by (7)}}{\leq} \log_2 |a| \frac{\operatorname{rk}_{\mathbb{Q}}(w_X) - \operatorname{rk}_{\mathbb{Q}}(bw_X - az_X)}{\epsilon|X|} \overset{\text{by (9)}}{\leq} \frac{\deg_+(w_X)}{\epsilon|X|}.$$

The proof of this case ends by showing that $\frac{\deg_+(w_X)}{|X|}$ is bounded in terms of w only (Lemmas 8.6 and 8.7). In the same way we bound $|b|$. Thus, we have shown that $S_\epsilon(z, w) \cap \mathbb{Q}$ is finite. A similar approach will be used in order to show that $S_\epsilon(z, w)$ is finite when w, z are arbitrary matrices from $f(\operatorname{Mat}_{n\times m}(\bar{\mathbb{Q}}[F]))$.

Now, let us consider the second example. Here we assume that Q is an algebraically closed subfield of $\mathbb{C}$ closed under complex conjugation and $x \in \mathbb{C} \setminus Q$. We also assume that Theorem 8.1 holds if $E = Q$. In this example we suppose that $w, z \in f(Q[x][F])$ and we will explain how to show that $S_\epsilon(z, w) \cap Q[x]$ is finite.

By way of contradiction we assume that $S_\epsilon(z, w) \cap Q[x]$ is infinite. Let $\{p_i\}_{i\in\mathbb{N}} \subseteq S_\epsilon(z, w) \cap Q[x]$ and assume that all p_i are different. For each $i \in \mathbb{N}$, let X_i be a finite F-set such that

$$\operatorname{rk}_{Q(x)}(w_{X_i}) - \operatorname{rk}_{Q(x)}(w_{X_i} - p_i z_{X_i}) \geq \epsilon|X_i|. \tag{10}$$

If M is a matrix over $Q[x][F]$ and $q \in Q$ we denote by $M(q)$ the matrix obtained from M by substituting x by q in the entries of M. Clearly, if M is a matrix over $Q[x]$, then $\operatorname{rk}_Q(M(q)) \leq \operatorname{rk}_{Q(x)}(M)$. Our idea consists in finding $q \in Q$ and a subset $J \subseteq \mathbb{N}$ such that

1. $\{p_i(q)\}_{i\in J}$ is an infinite subset of Q and
2. for every $i \in J$, $\operatorname{rk}_Q(w_{X_i}(q)) \geq \operatorname{rk}_{Q(x)}(w_{X_i}) - \frac{\epsilon}{2}|X_i|$.

Assume that such q exists. Observe that $w(q)_{X_i} = w_{X_i}(q)$. Thus, for every $i \in J$,

$$\operatorname{rk}_Q(w(q)_{X_i}) - \operatorname{rk}_Q(w(q)_{X_i} - p_i(q)z(q)_{X_i}) \geq$$
$$\operatorname{rk}_{Q(x)}(w_{X_i}) - \frac{\epsilon}{2}|X_i| - \operatorname{rk}_{Q(x)}(w_{X_i} - p_i z_{X_i}) \geq \frac{\epsilon}{2}|X_i|.$$

Hence $p_i(q) \in S_{\frac{\epsilon}{2}}(z(q), w(q))$. Since $\{p_i(q)\}_{i\in J}$ is infinite, we obtain a contradiction with our assumption that Theorem 8.1 holds if $E = Q$.

Observe that in general it is not true that if $\{f_i\}_{i\in\mathbb{N}} \subset Q[x]$ is infinite, then there exists $q \in Q$ such that $\{f_i(q)\}_{i\in\mathbb{N}}$ is infinite. A counterexample can be constructed when $Q = \{q_1, q_2, \ldots\}$ is countable. We simply put $f_i = \prod_{j=1}^{i}(x - q_j)$.

However, this situation does not occur in our case, since we can show that the degrees of polynomials p_i are bounded by a constant which depends only on ϵ and w. This is done in the following way.

For any matrix $M \in \mathrm{Mat}_{n\times m}(Q[x])$, we define $\deg_+(M) = \dim_Q V_{tor}$ to be the dimension of the torsion part V_{tor} of the $Q[x]$-module $V = (Q[x])^m/(Q[x])^n M$. Let $0 \neq p \in Q[x]$. Then Lemma 8.4 implies that

$$\deg(p)(m - \mathrm{rk}_{Q(x)}(M)) \le \deg_+\begin{pmatrix} M \\ pI_m \end{pmatrix} \le \deg(p)(m - \mathrm{rk}_{Q(x)}(M)) + \deg_+(M). \tag{11}$$

Observe that for every $i \in \mathbb{N}$,

$$\deg_+\begin{pmatrix} w_{X_i} - p_i z_{X_i} \\ p_i I_m \end{pmatrix} = \deg_+\begin{pmatrix} w_X \\ p_i I_m \end{pmatrix}.$$

Thus, using (11) we obtain that

$$\deg(p_i)(m - \mathrm{rk}_{Q(x)}(w_{X_i} - p_i z_{X_i})) \le \deg(p_i)(m - \mathrm{rk}_{Q(x)}(w_{X_i})) + \deg_+(w_{X_i}). \tag{12}$$

Therefore,

$$\deg(p_i) \overset{\text{by (10)}}{\le} \deg(p_i)\frac{\mathrm{rk}_{Q(x)}(w_{X_i}) - \mathrm{rk}_{Q(x)}(w_{X_i} - p_i z_{X_i})}{\epsilon|X_i|} \overset{\text{by (12)}}{\le} \frac{\deg_+(w_{X_i})}{\epsilon|X_i|}.$$

Now Lemmas 8.12 and 8.13 imply that $\frac{\deg_+(w_{X_i})}{|X_i|}$ is bounded in terms of w only. Thus, we obtain that the degrees of p_i are uniformly bounded in terms of ϵ and w. This implies that there exists a constant $C \in \mathbb{N}$ such that for any $C+1$ elements $q_1, \ldots, q_{C+1}$ from Q and for any infinite subset J of $\mathbb{N}$ there exists $1 \le j \le C+1$ such that the set $\{p_k(q_j)\}_{k\in J}$ is infinite.

Let us now show how to construct $q \in Q$ and a subset $J \subseteq \mathbb{N}$ such that

(1) $\{p_k(q)\}_{k\in J}$ is an infinite subset of Q and
(2) for every $k \in J$, $\mathrm{rk}_Q(w_{X_k}(q)) \ge \mathrm{rk}_{Q(x)}(w_{X_k}) - \frac{\epsilon}{2}|X_k|$.

Let ω be a non-principal ultrafilter on $\mathbb{N}$. We put $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_{X_k} \in \mathbb{P}(Q[x][F])$. By Corollary 7.8, rk_ω is the natural transcendental extension of the restriction of rk_ω on $Q[F]$. Therefore, by Corollary 7.9, for almost all $q \in Q$

$$\mathrm{rk}_\omega(w(q)) \ge \mathrm{rk}_\omega(w) - \frac{\epsilon}{4}.$$

Fix $C+1$ elements $q_1, \ldots, q_{C+1}$ from Q satisfying this property. Since $\mathrm{rk}_\omega(w) = \lim_\omega \mathrm{rk}_{X_k}(w)$ and for each i, $\mathrm{rk}_\omega(w(q_i)) = \lim_\omega \mathrm{rk}_{X_i}(w(q_i))$ we conclude that the set

$$J = \{k \in \mathbb{N} : \mathrm{rk}_{X_k}(w(q_i)) - \mathrm{rk}_{X_k}(w) \ge \mathrm{rk}_\omega(w(q_i)) - \mathrm{rk}_\omega(w) - \frac{\epsilon}{4}\ \forall 1 \le i \le C+1\}$$

is in ω, and so infinite. Thus, we obtain that for every $k \in J$ and $1 \le i \le C+1$

$$\frac{\mathrm{rk}_Q(w_{X_k}(q_i)) - \mathrm{rk}_{Q(x)}(w_{X_k})}{|X_k|} = \mathrm{rk}_{X_k}(w(q_i)) - \mathrm{rk}_{X_k}(w) \ge$$

$$\mathrm{rk}_\omega(w(q_i)) - \mathrm{rk}_\omega(w) - \frac{\epsilon}{4} \ge -\frac{\epsilon}{2}.$$

But, recall that we have shown before that there exists q_j such that the set $\{p_k(q_j)\}_{k\in J}$ is infinite. This contradicts our assumption that $S_{\frac{\epsilon}{2}}(z(q_j), w(q_j))$ is finite. Thus, we have shown that $S_\epsilon(z,w) \cap Q[x]$ is finite. A similar approach will be used in order to show that $S_\epsilon(z,w)$ is finte when w, z are arbitrary matrices from $f(\mathrm{Mat}_{n\times m}(\overline{Q(x)}[F]))$.

8.3. **Dedekind domains.** In this subsection $\mathcal{O}$ is a **Dedekind** domain, that is, a commutative domain, which is not a field, and in which every nonzero ideal is a product of maximal ones. The facts concerning Dedekind domains which appear in this subsection may be found in [34, Chapter 1].

Let l be a length function on $\mathcal{O}$-mod (see Subsection 5.5) such that $l(M) = +\infty$ if M is a finitely generated non-Artinian $\mathcal{O}$-module and $l(M) < +\infty$ if M is Artinian.

The main two examples that will appear in this paper are the following.

(1) Let K be a number field. We denote by $\mathcal{O}_K$ the ring of integers of K. Then $\mathcal{O}_K$ is a Dedekind domain. For any $\mathcal{O}_K$-module M we put $l(M) = \log_2 |M|$.
(2) Let Q be an algebraically closed field and let K be a finite extension of $Q(x)$. Denote by $\mathcal{O}_K$ the elements of K which are integral over $Q[x]$. Then $\mathcal{O}_K$ is a Dedekind domain. For any $\mathcal{O}_K$-module M we put $l(M) = \dim_Q M$.

In this subsection the nth power $I^n = \langle a_1 \dots a_n : a_i \in I\rangle$ of an ideal I and the n-fold cartesian product $I^n = \{(a_1,\dots,a_n) : a_i \in I\}$ have the same notation. We hope that the reader will not be confused by this ambiguity.

Let K be the ring of fractions of $\mathcal{O}$. For each maximal ideal P of $\mathcal{O}$, denote by

$$\mathcal{O}_P = \{\frac{a}{b} \in K : a, b \in \mathcal{O}, b \notin P\}$$

the localization of $\mathcal{O}$ at P. For any $\mathcal{O}_P$-module V, we put $l_P(V) = l(V)$. Then l_P is a length function on $\mathcal{O}_P$. Clearly we have

$$l(V) = \sum_P l_P(\mathcal{O}_P \otimes_{\mathcal{O}} V), \tag{13}$$

where the sum is over all maximal ideals of $\mathcal{O}$.

Observe that $\mathcal{O}$ is not a PID, in general. However, $\mathcal{O}_P$ is a PID, and so, it is easier to work with l_P than with l. The formula (13) will help us to extend the results, obtained first for l_P, to l.

We denote by $\mathcal{I}_{\mathcal{O}}$ the ideal group of $\mathcal{O}$, i.e., $\mathcal{I}_{\mathcal{O}}$ is the set of non-zero finitely generated $\mathcal{O}$-submodules of K (called also fractional ideals of $\mathcal{O}$). Any element $I \in \mathcal{I}_{\mathcal{O}}$ may be represented in a unique way as a product $I = \prod_{i=1}^s P_i^{n_i}$, where P_i are different maximal ideals of $\mathcal{O}$ and n_i are non-zero integers. We put $I_+ = \prod_{n_i>0} P_i^{n_i}$ and $I_- = \prod_{n_i<0} P_i^{-n_i}$. Let us define

$$\deg_+(I) = l(\mathcal{O}/I_+),\ \deg_{P,+}(I) = l_P(\mathcal{O}_P/\mathcal{O}_P I_+)$$

and for simplicity if $0 \ne \alpha \in K$, we put $\deg_+(\alpha) = \deg_+(\mathcal{O}\alpha)$ and $\deg_{P,+}(\alpha) = \deg_{P,+}(\mathcal{O}\alpha)$.

Let $M = (m_{ij}) \in \mathrm{Mat}_{n\times m}(K)$. Since $\mathcal{O}$ is a Dedekind domain, finitely generated torsion-free $\mathcal{O}$-modules are projective ([34, Theorem 1.10, Corollary 1.11]). Therefore,

$$V = (\mathcal{O}^m + \mathcal{O}^n M)/\mathcal{O}^n M \cong (V/V_{tor}) \oplus V_{tor},$$

where V_{tor} is the torsion part of $\mathcal{O}$-module V. Generalizing the previous notation we put

$$\deg_+(M) = l(V_{tor}) \text{ and } \deg_{P,+}(M) = l_P(\mathcal{O}_P \otimes_{\mathcal{O}} V_{tor}).$$

Remark 8.2. *Observe that if $\mathcal{O} = \mathcal{O}_P$, then $\deg_+(M) = \deg_{P,+}(M)$ for any matrix M over K.*

Ler $\mathrm{rk}_K(M)$ denote the ordinary rank of a matrix over K. Then

$$K \otimes_{\mathcal{O}} V/V_{tor} \cong K^{m-\mathrm{rk}_K(M)}.$$

Observe that for non-zero ideals I, J of $\mathcal{O}$,

$$I/IJ \cong \mathcal{O}/J.$$

Since every finitely generated projective $\mathcal{O}$-module is isomorphic to a direct sum of ideals of $\mathcal{O}$, we obtain that

$$(V/V_{tor})/J \cdot (V/V_{tor}) \cong (\mathcal{O}/J)^{m-\mathrm{rk}_K(M)}. \tag{14}$$

If $M \in \mathrm{Mat}_{n\times m}(\mathcal{O})$, then the invariant $\deg_+(M)$ can be computed using the following lemma.

Lemma 8.3. *Let $M \in \mathrm{Mat}_{n\times m}(\mathcal{O})$ be a non-zero matrix of K-rank k. Let I be the ideal of $\mathcal{O}$ generated by all $k \times k$ minors of the matrix M. Then $\deg_+(M) = l(\mathcal{O}/I)$.*

Proof. This lemma is well-known if $\mathcal{O}$ is a principal ideal domain and follows from the existence of the Smith normal form. Thus, we obtain that for any maximal ideal P of $\mathcal{O}$,

$$\deg_{P,+}(M) = l_P(\mathcal{O}_P/\mathcal{O}_P I).$$

Now, using (13), we conclude that

$$\deg_+(M) = \sum_P \deg_{P,+}(M) = \sum_P l_P(\mathcal{O}_P/\mathcal{O}_P I) = l(\mathcal{O}/I).$$

□

Given $M \in \mathrm{Mat}_{n\times m}(\mathcal{O})$ and $\alpha \in \mathcal{O}$ we want to estimate $\deg_+\begin{pmatrix} M \\ \alpha I_m \end{pmatrix}$.

Lemma 8.4. *Let $M \in \mathrm{Mat}_{n\times m}(\mathcal{O})$ and $0 \neq \alpha \in \mathcal{O}$. Then*

$$\deg_+(\alpha)(m - \mathrm{rk}_K(M)) \le \deg_+\begin{pmatrix} M \\ \alpha I_m \end{pmatrix} \le \deg_+(\alpha)(m - \mathrm{rk}_K(M)) + \deg_+(M).$$

Proof. Let $V = \mathcal{O}^m/\mathcal{O}^n M \cong (V/V_{tor}) \oplus V_{tor}$. Therefore, by (14),

$$\mathcal{O}^m/(\mathcal{O}^n M + \mathcal{O}^m \alpha) \cong (\mathcal{O}/(\alpha))^{m-\mathrm{rk}_K(M)} \oplus V_{tor}/(\alpha V_{tor}). \tag{15}$$

Thus, we obtain the following

$$\deg_+\begin{pmatrix} M \\ \alpha I_m \end{pmatrix} = l(\mathcal{O}^m/(\mathcal{O}^n M + \mathcal{O}^m \alpha)) \overset{\text{by (15)}}{=} \deg_+(\alpha)(m - \mathrm{rk}_K(M)) + l(V_{tor}/(\alpha V_{tor})).$$

Since $\deg_+(M) = l(V_{tor})$, we are done. □

Proposition 8.5. *Let $M_1, M_2 \in \mathrm{Mat}_{n\times m}(\mathcal{O})$, and let $0 \neq \alpha \in K$. We put*

$$m_1 = \mathrm{rk}_K(M_2 - \alpha M_1) \text{ and } m_2 = \mathrm{rk}_K(M_2).$$

Assume that $m_2 > m_1$. Then

$$\deg_+(\alpha) \le \frac{\deg_+(M_2)}{m_2 - m_1}.$$

Proof. Let P be a maximal ideal of $\mathcal{O}$. If $\alpha \notin \mathcal{O}_P$, then $\deg_{P,+}(\alpha) = 0$ and so

$$\deg_{P,+}(\alpha) = 0 \le \frac{\deg_{P,+}(M_2)}{m_2 - m_1}.$$

Assume now that $\alpha \in \mathcal{O}_P$. Applying Lemma 8.4 to matrices over $\mathcal{O}_P$ and $\deg_{P,+}$ (see Remark 8.2), we obtain that

$$\deg_{P,+}(\alpha)(m - m_1) \overset{\text{by Lemma 8.4}}{\le} \deg_{P,+}\begin{pmatrix} M_2 - \alpha M_1 \\ \alpha I_m \end{pmatrix} =$$

$$\deg_{P,+}\begin{pmatrix} M_2 \\ \alpha I_m \end{pmatrix} \overset{\text{by Lemma 8.4}}{\le} \deg_{P,+}(\alpha)(m - m_2) + \deg_{P,+}(M_2).$$

Therefore, since $m_1 < m_2$, we obtain again the inequality

$$\deg_{P,+}(\alpha) \le \frac{\deg_{P,+}(M_2)}{m_2 - m_1}.$$

Thus, we conclude that

$$\deg_+(\alpha) = \sum_P \deg_{P,+}(\alpha) \le \sum_P \frac{\deg_{P,+}(M_2)}{m_2 - m_1} = \frac{\deg_+(M_2)}{m_2 - m_1}.$$

□

8.4. **The number field case.** In this subsection we apply the results of Subsection 8.3 in the case where K is a number field and $\mathcal{O} = \mathcal{O}_K$ is the ring of integers of K. The length function on $\mathcal{O}_K$-mod is defined as follows:

$$l(M) = \log_2 |M|, \text{ where } M \text{ is an } \mathcal{O}_K\text{-module}.$$

We will write $\deg_+^K(\alpha)$ instead of $\deg_+(\alpha)$ if we want to emphasize that $\deg_+$ proceeds from a length function on $\mathcal{O}_K$. If $\alpha \in K_1 \le K_2$ then

$$\deg_+^{K_2}(\alpha) = |K_2 : K_1| \deg_+^{K_1}(\alpha).$$

In the case $0 \neq \alpha \in \mathcal{O}_K$ we have that

$$2^{\deg_+(\alpha)} = |\mathcal{O}_K/\mathcal{O}_K\alpha| = |N_{K/\mathbb{Q}}(\alpha)| = (\prod_{i=1}^n |\alpha_i|)^{\frac{|K:\mathbb{Q}|}{n}},$$

where $\alpha_1, \ldots, \alpha_n$ are the roots of the minimal polynomial of α over $\mathbb{Q}$. Let $\lceil \alpha \rceil = \max_i |\alpha_i|$. Then we obtain

$$\deg_+(\alpha) = \log_2 |N_{K/\mathbb{Q}}(\alpha)| \le |K : \mathbb{Q}| \log_2 \lceil \alpha \rceil. \tag{16}$$

If $M = (m_{ij}) \in \mathrm{Mat}_{n\times m}(K)$ is a non-zero matrix, we put

$$\lceil M \rceil = \max_j \sum_i \lceil m_{ij} \rceil.$$

For a zero matrix M we put $\lceil M \rceil = 1$. Now we can estimate $\deg_+(M)$ for $M \in \mathrm{Mat}_{n\times m}(\mathcal{O}_K)$.

Lemma 8.6. *Let $M \in \mathrm{Mat}_{n\times m}(\mathcal{O}_K)$. Then the following inequality holds:*
$$\deg_+(M) \le m|K:\mathbb{Q}|\log_2\lceil M\rceil.$$

Proof. Let $k = \mathrm{rk}_K(M)$ and let I be the ideal of $\mathcal{O}_K$ generated by all $k\times k$ minors of the matrix M. Then by Lemma 8.3, $\deg_+(M) = l(\mathcal{O}_K/I)$. In particular, if y is a non-zero $k\times k$ minor of M, then
$$\deg_+(M) = l(\mathcal{O}_K/I) \le l(\mathcal{O}_K/\mathcal{O}_K y) = \deg_+(y).$$
Since for any $a, b \in K$ we have that $\lceil a+b\rceil \le \lceil a\rceil + \lceil b\rceil$ and $\lceil ab\rceil \le \lceil a\rceil\lceil b\rceil$, we obtain that
$$\lceil y\rceil \le \lceil M\rceil^k \le \lceil M\rceil^m.$$
Therefore,
$$\deg_+(M) \le \deg_+(y) \overset{\text{by (16)}}{\le} |K:\mathbb{Q}|\log_2\lceil y\rceil \le m|K:\mathbb{Q}|\log_2\lceil M\rceil$$
and we are done.

□

For any element $f = \sum_{h\in F} a_h h$ ($a_h \in \bar{\mathbb{Q}}$) of the group algebra $\bar{\mathbb{Q}}[F]$ we put
$$\lceil f\rceil = \sum_{h\in F}\lceil a_h\rceil.$$
If $M = (m_{ij})$ is a matrix over $\bar{\mathbb{Q}}[F]$, then we define
$$\lceil M\rceil = \max_j \sum_i \lceil m_{ij}\rceil.$$
The following lemma is a direct consequence of the definitions.

Lemma 8.7. *Let X be a finite F-set and let $f_X : \bar{\mathbb{Q}}[F] \to \mathrm{Mat}_{|X|}(\bar{\mathbb{Q}})$ be the map induced by the action of F on X. Let $M \in \mathrm{Mat}_{n\times m}(\bar{\mathbb{Q}}[F])$ (and so $f_X(M) \in \mathrm{Mat}_{n|X|\times m|X|}(\bar{\mathbb{Q}})$). Then*
$$\lceil f_X(M)\rceil \le \lceil M\rceil.$$

Let $\alpha \in \bar{\mathbb{Q}}$. Let us put
$$N_+(\alpha) = 2^{\deg_+^{\mathbb{Q}(\alpha)}(\alpha)} = |\mathcal{O}_{\mathbb{Q}(\alpha)}/(\alpha)_+|.$$

Corollary 8.8. *Let F be a free finitely generated group and let X be a finite F-set. Let K be a finite extension of $\mathbb{Q}$. Assume that $M_1, M_2 \in \mathrm{Mat}_{n\times m}(\mathcal{O}_K[F])$ and $\alpha \in K$. We put*
$$m_1 = \mathrm{rk}_K(f_X(M_2 - \alpha M_1)) \text{ and } m_2 = \mathrm{rk}_K(f_X(M_2)).$$
Assume that $m_2 > m_1$. Then
$$N_+(\alpha) \le \lceil M_2\rceil^{\frac{m|X||\mathbb{Q}(\alpha):\mathbb{Q}|}{m_2-m_1}}.$$

Proof. Putting together the previous results, we obtain
$$\deg_+^{\mathbb{Q}(\alpha)}(\alpha) = \frac{\deg_+^K(\alpha)}{|K:\mathbb{Q}(\alpha)|} \overset{\text{Proposition 8.5}}{\le} \frac{\deg_+(f_X(M_2))}{(m_2-m_1)|K:\mathbb{Q}(\alpha)|} \overset{\text{Lemma 8.6}}{\le}$$
$$\frac{m|X||K:\mathbb{Q}|\log_2\lceil f_X(M_2)\rceil}{(m_2-m_1)|K:\mathbb{Q}(\alpha)|} \overset{\text{Lemma 8.7}}{\le} \frac{m|X||\mathbb{Q}(\alpha):\mathbb{Q}|\log_2\lceil M_2\rceil}{(m_2-m_1)}.$$

□

8.5. **The function field case.** In this subsection we apply the results of Subsection 8.3 in the case where K is a finite extension of $Q(x)$, where Q is an algebraically closed subfield of $\mathbb{C}$ and $\mathcal{O} = \mathcal{O}_K$ is the integral closure of $Q[x]$ in K. The length function on $\mathcal{O}_K$-mod is defined as follows:

$$l(M) = \dim_Q M, \text{ where } M \text{ is a } \mathcal{O}_K\text{-module.}$$

As in the number field case we will write $\deg_+^{\mathcal{O}_K}(\alpha)$ if we want to emphasize that $\deg_+$ proceeds from a length function on $\mathcal{O}_K$ (since the construction of $\mathcal{O}_K$ is not canonical and depends on the choice of x, we write $\deg_+^{\mathcal{O}_K}(\alpha)$ and not $\deg_+^K(\alpha)$).

Let L be an algebraic extension of $Q(x)$. A **Q-valuation on** L is a homomorphism $v : L \setminus \{0\} \to (\mathbb{R}, +)$ such that

1. $v(q) = 0$ for every $q \in Q$ and
2. $v(a + b) \geq \min\{v(a), v(b)\}$.

We say that a valuation is **trivial** if $v(a) = 0$ for every $a \in L$. In this paper a valuation will always mean a non-trivial valuation. We will also put $v(0) = \infty$. Two valuations v_1 and v_2 are equivalent if there are two positive real numbers r_1 and r_2 such that $r_1 v_1 = r_2 v_2$. Denote by C_L the set of equivalence classes of Q-valuations on L.

If K is a finite extension of $Q(x)$, then an equivalence class $v \in C_K$ will be always represented by a valuation whose image is equal to $\mathbb{Z}$ (there is exactly one such valuation in each class). It is a standard fact that C_K may be thought of as a non-singular projective curve over Q and K is the field of rational functions on C_K.

In general, we will say that $v \in C_L$ is **a zero** of $\alpha \in L$ if $v(\alpha) > 0$ and v is **a pole** of α if $v(\alpha) < 0$. If $v(\alpha) \geq 0$, there exists a unique $q \in Q$ such that $v(\alpha - q) > 0$. Then we put $\alpha(v) = q$. If v is a pole, then we put $\alpha(v) = \infty$. If we denote by $\mathcal{O}_v$ the set of elements α of L such that v is not a pole of α, then the map $\pi_v : \alpha \mapsto \alpha(v)$ is the unique Q-algebra homomorphism that sends $\mathcal{O}_v$ onto Q.

The following equality is well-known.

Lemma 8.9. [20, Proposition II.6.9] *Let Q be an algebraically closed field, $K/Q(x)$ a finite extension and $0 \neq \alpha \in K$. Then*

$$|K : Q(\alpha)| = \sum_{v \in C_K, v(\alpha) > 0} v(\alpha) = -\sum_{v \in C_K, v(\alpha) < 0} v(\alpha).$$

Corollary 8.10. *Let Q be an algebraically closed field, $K/Q(x)$ a finite extension and $\alpha_1, \alpha_2 \in K$. Assume that $\alpha_1 \neq \alpha_2$. Then*

$$|K : Q(\alpha_1 - \alpha_2)| \leq |K : Q(\alpha_1)| + |K : Q(\alpha_2)|.$$

Proof. Let $v \in C_K$. If $v(\alpha_1-\alpha_2)<0$, then $v(\alpha_1)<0$ or $v(\alpha_2)<0$ and $v(\alpha_1-\alpha_2)\ge \min\{v(\alpha_1),v(\alpha_2)\}$. Therefore,

$$|K:Q(\alpha_1-\alpha_2)| \overset{\text{Lemma 8.9}}{=} -\sum_{v\in C_K, v(\alpha_1-\alpha_2)<0} v(\alpha_1-\alpha_2) \le$$
$$-\sum_{v\in C_K,\ v(\alpha_1)<0 \text{ or } v(\alpha_2)<0} \min\{v(\alpha_1),v(\alpha_2)\} \le$$
$$-\sum_{v\in C_K, v(\alpha_1)<0} v(\alpha_1) - \sum_{v\in C_K, v(\alpha_2)<0} v(\alpha_2) \overset{\text{Lemma 8.9}}{=}$$
$$|K:Q(\alpha_1)|+|K:Q(\alpha_2)|.$$

□

Let us denote by $\mathcal{O}'_K$ the elements of K that are integral over $Q[x^{-1}]$.

Lemma 8.11. *We have that*

$$\deg_+^{\mathcal{O}_K}(\alpha) = \sum_{v\in C_K, v(\alpha)\ge 0, v(x)\ge 0} v(\alpha) \text{ and } \deg_+^{\mathcal{O}'_K}(\alpha) = \sum_{v\in C_K, v(\alpha)\ge 0, v(x)\le 0} v(\alpha).$$

In particular

$$\max\{\deg_+^{\mathcal{O}_K}(\alpha), \deg_+^{\mathcal{O}'_K}(\alpha)\} \le |K:Q(\alpha)| \le \deg_+^{\mathcal{O}_K}(\alpha)+\deg_+^{\mathcal{O}'_K}(\alpha).$$

Proof. First observe that

$$\mathcal{O}_K = \{a\in K : v(a)\ge 0 \text{ for every } v\in C_K \text{ such that } v(x)\ge 0\}$$

and any maximal ideal of $\mathcal{O}_K$ is of the form

$$P_v = \{a\in \mathcal{O}_K :\ v(a)>0\}$$

for some $v\in C_K$ satisfying $v(x)\ge 0$. Then we obtain that

$$(\alpha)_+ = \prod_{v(x)\ge 0, v(\alpha)>0} P_v^{v(\alpha)}.$$

Therefore,

$$\deg_+^{\mathcal{O}_K}(\alpha) = \sum_{v\in C_K, v(\alpha)\ge 0, v(x)\ge 0} v(\alpha).$$

The formula for $\deg_+^{\mathcal{O}'_K}(\alpha)$ is obtained in the same way. □

Now let us estimate $\deg_+^{\mathcal{O}_K}(M)$ for $M\in \mathrm{Mat}_{n\times m}(\mathcal{O}_K)$. If $M=(m_{ij})\in \mathrm{Mat}_{n\times m}(K)$, we put

$$D_K(M) = -\sum_{v\in C_K} \min\{\{v(m_{ij})\},0\}.$$

Lemma 8.12. *Let $M\in \mathrm{Mat}_{n\times m}(\mathcal{O}_K)$. Then the following inequality holds:*

$$\deg_+^{\mathcal{O}_K}(M) \le mD_K(M).$$

Proof. Let $k=\mathrm{rk}_K(M)$ and let I be the ideal of $\mathcal{O}_K$ generated by all $k\times k$ minors of the matrix M. Then by Lemma 8.3, $\deg_+^{\mathcal{O}_K}(M)=l(\mathcal{O}_K/I)$. In particular, if y is a non-zero $k\times k$ minor of M, then

$$\deg_+^{\mathcal{O}_K}(M) = l(\mathcal{O}_K/I) \le l(\mathcal{O}_K/\mathcal{O}_K y) = \deg_+^{\mathcal{O}_K}(y).$$

Let $v \in C_K$. Since for any $a, b \in K$ we have that $v(a+b) \geq \min\{v(a), v(b)\}$ and $v(ab) = v(a) + v(b)$, we obtain that

$$v(y) \geq k(\min\{v(m_{ij})\}).$$

Therefore,

$$\deg_+^{\mathcal{O}_K}(M) \leq \deg_+^{\mathcal{O}_K}(y) \overset{\text{Lemma 8.11}}{\leq} |K : Q(y)| \overset{\text{Lemma 8.9}}{=} - \sum_{v \in C_K, v(y)<0} v(y) \leq -k \sum_{v \in C_K, v(y)<0} \min\{v(m_{ij})\} \leq m D_K(M).$$

□

For any element $f = \sum_{h \in F} a_h h$ ($a_h \in K$) of the group algebra $K[F]$ we put $v(f) = \min_{h \in F}\{v(a_h)\}$. If $M = (m_{ij})$ is a matrix over $K[F]$, then we define

$$D_K(M) = - \sum_{v \in C_K} \min\{\{v(m_{ij}) : \ i, j\}, 0\}.$$

The following lemma is a direct consequence of the definitions.

Lemma 8.13. *Let X be a finite F-set and let $f_X : K[F] \to \mathrm{Mat}_{|X|}(K)$ be the map induced by the action of F on X. Let $M \in \mathrm{Mat}_{n\times m}(K[F])$ (and so $f(M) \in \mathrm{Mat}_{n|X|\times m|X|}(K)$). Then*

$$D_K(f_X(M)) \leq D_K(M).$$

Let $Q(x) \leq K \leq L$ be two finite extensions of $Q(x)$. Then the restriction map

$$\mathrm{res}_{L/K} : C_L \to C_K$$

is onto. Moreover for any $v \in C_K$ and $\tilde{v} \in C_L$ satisfying $\mathrm{res}_{L/K}(\tilde{v}) = v$ there exists a number $e_{\tilde{v}}$ such that

$$|L : K| = \sum_{\tilde{v} \in C_L, \mathrm{res}_{L/K}(\tilde{v}) = v} e_{\tilde{v}} \text{ and for every } \ \alpha \in K, \ \tilde{v}(\alpha) = e_{\tilde{v}} v(\alpha). \tag{17}$$

From (17) it follows that

$$D_L(M) = |L : K| D_K(M). \tag{18}$$

Corollary 8.14. *Let F be a free finitely generated group and let X be a finite F-set. Let Q be an algebraically closed field and let E be an algebraic closure of $Q(x)$. Let $K/Q(x)$ be a finite subextension of $E/Q(x)$ and $\alpha \in E$. Denote by $\mathcal{O}_K$ the integral closure of $Q[x]$ in K and by $\mathcal{O}'_K$ the integral closure of $Q[x^{-1}]$ in K. Assume that $M_1, M_2 \in \mathrm{Mat}_{n\times m}(\mathcal{O}_K[F])$ and let $0 \neq d \in K$ be such that $dM_1, dM_2 \in \mathrm{Mat}_{n\times m}(\mathcal{O}'_K[F])$. We put*

$$m_1 = \mathrm{rk}_E(f_X(M_2 - \alpha M_1)) \text{ and } m_2 = \mathrm{rk}_E(f_X(M_2)).$$

Suppose that $m_2 > m_1$. Then

$$|K(\alpha) : Q(\alpha)| \leq \frac{m|X||K(\alpha) : K|(D_K(M_2) + D_K(dM_2))}{m_2 - m_1}.$$

Proof. Set $L = K(\alpha)$ and let $\mathcal{O}_L$ be the integral closure of $Q[x]$ in L and $\mathcal{O}'_L$ the integral closure of $Q[x^{-1}]$ in L. Putting together the previous results, we obtain that

$$\deg_+^{\mathcal{O}_L}(\alpha) \overset{\text{Proposition 8.5}}{\leq} \frac{\deg_+^{\mathcal{O}_L}(f_X(M_2))}{m_2 - m_1} \overset{\text{Lemma 8.12}}{\leq} \frac{m|X|D_L(f_X(M_2))}{m_2 - m_1} \overset{\text{Lemma 8.13}}{\leq} \frac{m|X|D_L(M_2)}{m_2 - m_1}.$$

Similarly, we have that

$$\deg_+^{\mathcal{O}'_L}(\alpha) \leq \frac{m|X|D_L(dM_2)}{m_2 - m_1}.$$

Hence by Lemma 8.11,

$$|L : Q(\alpha)| \leq \deg_+^{\mathcal{O}_L}(\alpha) + \deg_+^{\mathcal{O}'_L}(\alpha) \leq \frac{m|X|(D_L(M_2) + D_L(dM_2))}{m_2 - m_1} \overset{\text{by (18)}}{=} \frac{m|X||L : K|(D_K(M_2) + D_K(dM_2))}{m_2 - m_1}.$$

□

8.6. **A number theory result.** In this subsection we will show the following result.

Proposition 8.15. *For given $k \in \mathbb{N}$ and $C \in \mathbb{R}$ there are only finitely many algebraic numbers α satisfying*

(1) $|\mathbb{Q}(\alpha) : \mathbb{Q}| = k$ *and*
(2) $N_+(\alpha - i) \leq C$ *for* $i = 0, \dots, k$.

Proof. Let $K = \mathbb{Q}(\alpha)$. Write $\alpha = \frac{a}{b}$ with $a, b \in \mathcal{O}_K$. Let $\sigma_j : K \to \bar{\mathbb{Q}}$ be k different embeddings of K into $\bar{\mathbb{Q}}$. We put $a_j = \sigma_j(a)$ and $b_i = \sigma_i(b)$. Consider the polynomial

$$f(x) = \prod_{j=1}^{k}(a_j - b_j x) = \sum_{j=0}^{k} c_j x^j \in \mathbb{Z}[x].$$

Put $A = |\mathcal{O}_K/(\mathcal{O}_K a + \mathcal{O}_K b)|$. Then we obtain

$$|f(i)| = |\mathcal{O}_K/\mathcal{O}_K(a - ib)| = A \cdot |(\mathcal{O}_K a + \mathcal{O}_K b)/\mathcal{O}_K(a - ib)| = A \cdot |(\mathcal{O}_K(a - ib) + \mathcal{O}_K b)/\mathcal{O}_K(a - ib)| = A \cdot |(\mathcal{O}_K + \mathcal{O}_K(\alpha - i))/\mathcal{O}_K(\alpha - i)| = A \cdot N_+(\alpha - i).$$

Let $M = (m_{ij})$ be a $(k+1) \times (k+1)$ matrix with $m_{ij} = (j-1)^{i-1}$ (as usual $0^0 = 1$). Since M is invertible,

$$(c_0, \dots, c_k) = (f(0), \dots, f(k))M^{-1} = A \cdot (\pm N_+(\alpha), \dots, \pm N_+(\alpha - k))M^{-1}.$$

Put $c'_i = \frac{c_i}{A}$. Then

$$(c'_0, \dots, c'_k) = (\pm N_+(\alpha), \dots, \pm N_+(\alpha - k))M^{-1}.$$

Since we assume that all $N_+(\alpha - i)$ are bounded by C, there are only finitely many possibilities for $(c'_0, \dots, c'_k)$. Now, note that α is a root of the polynomial $\sum_{j=0}^{k} c'_j x^j$. Hence we are done.

□

8.7. **A result on rational functions of algebraic curves.** Let Q be an algebraically closed field. Let E be an algebraic closure of $Q(x)$. The set of Q-valuations on E is denoted by C_E. Since E is the union of finite subextensions $K/Q(x)$, we have the following equality

$$C_E = \varprojlim_{K/Q(x) \text{ is finite } \ K\le E} C_K.$$

Since $\operatorname{res}_{L/K} : C_L \to C_K$ is onto when $L/Q(x)$ and $K/Q(x)$ are finite, for every finite subextension $L/Q(x)$ of $E/Q(x)$ the restriction map $\operatorname{res}_{E/L} C_E \to C_L$ is also onto. Observe that if $\alpha \in L$ and $\tilde{v} \in C_E$, then $\alpha(\tilde{v}) = \alpha(\operatorname{res}_{E/L}(\tilde{v}))$ (here α in the first place is considered as an element of E and in the second as an element of L).

Now we are ready to present the main result of this subsection.

Proposition 8.16. *Let $C \in \mathbb{R}$ and let $K/Q(x)$ be a finite subextension of $E/Q(x)$. For every $i \in \mathbb{N}$ let $\alpha_i \in E$ be such that*

(1) $|K(\alpha_i) : Q(\alpha_i)| \le C$ *and*
(2) $|K(\alpha_i) : K| \le C$.

Assume that all α_i are different. Let $\mathcal{S} = \{\tilde{v} \in C_E : \{\alpha_i(\tilde{v})\}_{i\in\mathbb{N}}$ is finite$\}$. Then $\operatorname{res}_{E/K}(\mathcal{S})$ contains at most $C(2C+1)$ valuations.

Proof. Let $\tilde{v}_1, \dots \tilde{v}_N \in \mathcal{S}$ such that $\{\operatorname{res}_{E/K}(\tilde{v}_i)\}$ consists of N different valuations. Since $\tilde{v}_1 \in \mathcal{S}$, there exists $\beta_1 \in Q \cup \{\infty\}$ such that $J_1 = \{j \in \mathbb{N} : \alpha_j(\tilde{v}_1) = \beta_1\}$ is infinite. Similarly, since $\tilde{v}_2 \in \mathcal{S}$, there exists $\beta_2 \in Q \cup \{\infty\}$ such that $J_2 = \{j \in J_1 : \alpha_j(\tilde{v}_1) = \beta_2\}$ is infinite. Repeating this procedure, we obtain that for every $1 \le i \le N$ there are $\beta_1, \dots, \beta_i \in Q \cup \{\infty\}$ and an infinite subset $J_i \subseteq \mathbb{N}$ such that

$$\alpha_j(\tilde{v}_s) = \beta_s \text{ for every } \ j \in J_i, \ 1 \le s \le i.$$

Without loss of generality we may assume that

$$1, 2 \in J_N, \{\beta_i : \ 1 \le i \le N_1\} \subset Q \text{ and } \ \beta_i = \infty \text{ if } \ i = N_1 + 1, \dots, N.$$

Hence we obtain that $\tilde{v}_1, \dots, \tilde{v}_{N_1}$ are zeros of the function $\alpha_1 - \alpha_2 \in E$. Let $L = K(\alpha_1, \alpha_2)$. The following holds.

$$\begin{aligned} |L : Q(\alpha_1 - \alpha_2)| &\overset{\text{Corollary 8.10}}{\le} |L : Q(\alpha_1)| + |L : Q(\alpha_2)| \le \\ &|L : K(\alpha_1)||K(\alpha_1) : Q(\alpha_1)| + |L : K(\alpha_2)||K(\alpha_2) : Q(\alpha_2)| \le \\ &|K(\alpha_2) : K||K(\alpha_1) : Q(\alpha_1)| + |K(\alpha_1) : K||K(\alpha_1) : Q(\alpha_2)| \le 2C^2. \end{aligned}$$

Thus, by Lemma 8.9, $\alpha_1 - \alpha_2$ has at most $2C^2$ zeros in C_L. Since

$$\{\operatorname{res}_{E/L}(\tilde{v}_i) : \ 1 \le i \le N_1\}$$

consists of N_1 different valuations, $N_1 \le 2C^2$.

Since $|K(\alpha_1) : Q(\alpha_1)| \le C$, by Lemma 8.9, α_1 has at most C different poles in $C_{K(\alpha_1)}$. Hence $N - N_1 \le C$, and so $N \le 2C^2 + C$. □

8.8. **Proof of Theorem 8.1.** We maintain the notation from the beginning of this section and Subsection 8.1. As we have explained in Subsection 8.1 we can assume that $z = f(A)$ and $w = f(B)$, where A and B are n by m matrices over $E[F]$ and E is algebraically closed. Clearly we can also assume that $E/\mathbb{Q}$ has finite transcendental degree, and so, it will be enough to consider the following two situations.

(1) $E = \bar{\mathbb{Q}}$ (in this case we put $T = \mathbb{Q}$ and $\mathcal{O}_T = \mathbb{Z}$) or
(2) Q is an algebraically closed subfield of $\mathbb{C}$, $E = \overline{Q(x)}$ is the algebraic closure of $Q(x)$ in $\mathbb{C}$, where $x \in \mathbb{C} \setminus Q$, and Theorem 8.1 holds over Q (in this case we put $T = Q(x)$ and $\mathcal{O}_T = Q[x]$).

Let K be a finite extension of T such that A and B are matrices over $K[F]$. Let $\mathcal{O}_K$ be the integral closure of $\mathcal{O}_T$ in K. Multiplying A and B by a non-zero element from $\mathcal{O}_K$ we may also assume that A and B are matrices over $\mathcal{O}_K[F]$.

Assume that $S_\epsilon(z, w)$ is infinite. Then we can find different

$$\{\lambda_j \in E : j \in \mathbb{N}\} \subseteq S_\epsilon(z, w).$$

For each $j \in \mathbb{N}$ let X_j be such that

$$\mathrm{rk}_{X_j}(w) - \mathrm{rk}_{X_j}(w - \lambda_j z) \geq \epsilon. \tag{19}$$

For simplicity, we will write w_j instead of w_{X_j}, z_j instead of z_{X_j} and rk_j instead of rk_{X_j}.

Denote by L_j the extension of K generated by λ_j. Let $\sigma \in \mathrm{Gal}(E/K)$. Since $\sigma(w_j) = w_j$ and $\sigma(z_j) = z_j$, we obtain that

$$\mathrm{rk}_j(w) - \mathrm{rk}_j(w - \lambda_j z) = \mathrm{rk}_{\mathrm{Mat}_{|X_j|}(L_j)}(w_j) - \mathrm{rk}_{\mathrm{Mat}_{|X_j|}(L_j)}(w_j - \lambda_j z_j) =$$
$$\mathrm{rk}_{\mathrm{Mat}_{|X_j|}(L_j)}(w_j) - \mathrm{rk}_{\mathrm{Mat}_{|X_j|}(\sigma(L_j))}(w_j - \sigma(\lambda_j) z_j) = \mathrm{rk}_j(w) - \mathrm{rk}_j(w - \sigma(\lambda_j) z)).$$

Thus, by Proposition 5.10, $|L_j : K| \leq \frac{n}{\epsilon}$. Therefore,

$$|T(\lambda_j) : T| \leq |L_j : T| \leq \frac{|K : T| n}{\epsilon}.$$

Put $C_1 = \frac{|K:T|n}{\epsilon}$.

Let us fix a non-principal ultrafilter ω on $\mathbb{N}$. We define $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_j \in \mathbb{P}(E[F])$.

8.8.1. *Case 1.* First we assume that we are in the case $E = \bar{\mathbb{Q}}$. By Proposition 5.10, there exists $C_2 \in \mathbb{Z}_{\geq 0}$, such that

$$\mathrm{rk}_\omega(w) - \mathrm{rk}_\omega(w + (C_2 - i)z) \leq \frac{\epsilon}{4} \text{ for every } \ i = 0, \ldots, C_1.$$

Therefore we have that the set

$$J = \{j : \mathrm{rk}_j(w) - \mathrm{rk}_j(w + (C_2 - i)z) \leq \frac{\epsilon}{2} \text{ for every } \ i = 0, \ldots, C_1\} \tag{20}$$

belongs to ω. In particular, J is infinite.

Combining (19) and (20), we obtain that for every $j \in J$ and $i = 0, \ldots, C_1$,

$$\mathrm{rk}_{L_j}(w_j + (C_2 - i)z_j) - \mathrm{rk}_{L_j}(w_j + (C_2 - i)z_j - (\lambda_j + C_2 - i)z_j) =$$
$$|X_j|(\mathrm{rk}_j(w + (C_2 - i)z) - \mathrm{rk}_j(w - \lambda_j z)) \geq \frac{\epsilon}{2}|X_j|. \tag{21}$$

Applying Corollary 8.8, we obtain that for every $j \in J$ and $i = 0, \ldots, C_1$,

$$N_+(\lambda_j + C_2 - i) \overset{\text{Corollary 8.8}}{\leq} (\lceil B + (C_2 - i)A \rceil)^{\frac{m|X_j||\mathbb{Q}(\lambda_j):\mathbb{Q}|}{\mathrm{rk}_{L_j}(w_j + (C_2 - i)z_j) - \mathrm{rk}_{L_j}(w_j - \lambda_j z_j)}} \overset{\text{by (21)}}{\leq} (\lceil B + (C_2 - i)A \rceil)^{\frac{2m|\mathbb{Q}(\lambda_j):\mathbb{Q}|}{\epsilon}} \leq (\lceil B \rceil + (C_1 + C_2)\lceil A \rceil)^{\frac{2mC_1}{\epsilon}}.$$

Observe that the last expression does not depend on $j \in J$. Since $\mathbb{Q}(\lambda_j + C_2) = \mathbb{Q}(\lambda_j)$ has degree at most C_1 over $\mathbb{Q}$, we have only finitely many possibilities for λ_j by Proposition 8.15. Hence we have obtained a contradiction.

8.8.2. *Case 2.* Now we assume that we are in the second case and $E = \overline{Q(x)}$.

Let $L/Q(x)$ be a subextension of $E/Q(x)$. If $\tilde{v} \in C_L$, then we put $\mathcal{O}_{\tilde{v}} = \{a \in L : \tilde{v}(a) \geq 0\}$. There is a unique Q-algebra homomorphism

$$\phi_{\tilde{v}} : \mathcal{O}_{\tilde{v}} \to Q.$$

For every Sylvester matrix rank function rk on $\mathcal{O}_{\tilde{v}}[F]$, we define

$$\mathrm{rk}_{\tilde{v}}(M) = \mathrm{rk}(\phi_{\tilde{v}}(M)).$$

Denote by $\mathrm{Spec}(\mathcal{O}_K)$ the set

$$\mathrm{Spec}(\mathcal{O}_K) = \{v \in C_K : \mathcal{O}_K \leq \mathcal{O}_v\} = \{v \in C_K : v(x) \geq 0\}.$$

Then $C_K \setminus \mathrm{Spec}(\mathcal{O}_K)$ is finite. First let us show the following lemma.

Lemma 8.17. *For infinitely many* $v \in \mathrm{Spec}(\mathcal{O}_K)$

$$\mathrm{rk}_\omega(B) - \mathrm{rk}_{\omega,v}(B) \leq \frac{\epsilon}{4}.$$

Proof. Consider the map

$$(\mathcal{O}_K[F])^n \to (\mathcal{O}_K[F])^m, \ (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n)B,$$

as a map of $Q[x][F]$-modules and let $\tilde{B}$ be a matrix over $Q[x][F]$ associated with this map. Since, by Proposition 7.13, rk_ω is the natural algebraic extension of the restriction of rk_ω on $Q(x)[F]$,

$$\mathrm{rk}_\omega(B) = \frac{\mathrm{rk}_\omega(\tilde{B})}{|K : Q(x)|}.$$

By Corollary 7.8, the restriction of rk_ω on $Q(x)[F]$ is the natural transcendental extension of the restriction of rk_ω on $Q[F]$. Thus, Corollary 7.9 implies that for almost all $p \in \mathrm{Spec}(Q[x])$,

$$\mathrm{rk}_\omega(\tilde{B}) - \mathrm{rk}_{\omega,p}(\tilde{B}) \leq \frac{\epsilon}{4}|K : Q(x)|. \tag{22}$$

For almost all $p \in \mathrm{Spec}(Q[x])$, p unramifies over K, i.e.

$$|(\mathrm{res}_{K/Q(x)})^{-1}(p)| = |K : Q(x)|. \tag{23}$$

For these p,

$$\mathrm{rk}_{\omega,p}(\tilde{B}) = \sum_{v \in (\mathrm{res}_{K/Q(x)})^{-1}(p)} \mathrm{rk}_{\omega,v}(B),$$

and so

$$|K : Q(x)|\, \mathrm{rk}_\omega(B) - \sum_{v \in (\mathrm{res}_{K/Q(x)})^{-1}(p)} \mathrm{rk}_{\omega,v}(B) \leq \frac{\epsilon}{4}|K : Q(x)|.$$

Thus, if $p \in \mathrm{Spec}(Q[x])$ satisfies (22) and (23), then there exists $v \in (\mathrm{res}_{K/Q(x)})^{-1}(p)$ such that

$$\mathrm{rk}_\omega(B) - \mathrm{rk}_{\omega,v}(B) \le \frac{\epsilon}{4}.$$

Since there are infinitely many $p \in \mathrm{Spec}(Q[x])$ satisfying (22) and (23), we are done. □

Let

$$C_2 = \max\{C_1, \frac{mC_1(D_K(B)+D_K(dB))}{\epsilon}\} \text{ and } \ C_3 = C_2(2C_2+1)+1.$$

By Lemma 8.17, we can choose C_3 different $v_i \in \mathrm{Spec}(\mathcal{O}_K)$ $(i = 1, \dots, C_3)$ such that

$$\mathrm{rk}_\omega(B) - \mathrm{rk}_{\omega,v_i}(B) \le \frac{\epsilon}{4}.$$

Since $\mathrm{rk}_w = \lim_\omega \mathrm{rk}_{X_i}$ and $\mathrm{rk}_{\omega,v} = \lim_\omega \mathrm{rk}_{X_i,v}$, the set

$$J = \{j \in \mathbb{N} : \mathrm{rk}_{X_j}(B) - \mathrm{rk}_{X_j,v_i}(B) \le \frac{\epsilon}{2} \text{ for } i = 1, \dots, C_3\}$$

belongs to ω and so it is infinite. For each $i = 1, \dots, C_3$, let $\tilde{v}_i \in C_E$ be such that $\mathrm{res}_{E/K}(\tilde{v}_i) = v_i$.

Lemma 8.18. *There exists $1 \le i \le C_3$ such that the set*

$$\{\lambda_j(\tilde{v}_i) : \ j \in J\}$$

is infinite.

Proof. Let us check that $\{\lambda_j : \ j \in J\}$ satisfies the hypothesis of Proposition 8.16. We have already shown that

$$|K(\lambda_j) : K| \le |K(\lambda_j) : Q(x)| \le C_1.$$

On the other hand for every $j \in J$ we obtain that

$$|K(\lambda_j) : Q(\lambda_j)| \overset{\text{Corollary 8.14}}{\le} \frac{m|X_j||K(\lambda_j) : K|(D_K(B)+D_K(dB))}{\mathrm{rk}_E(f_{X_j}(B)) - \mathrm{rk}_E(f_{X_j}(B-\lambda_j A))} \overset{\text{by (19)}}{\le} \frac{mC_1(D_K(B)+D_K(dB))}{\epsilon}.$$

By Proposition 8.16, $\mathrm{res}_{E/K}(\mathcal{S})$ has at most $C_3 - 1$ elements, where

$$\mathcal{S} = \{\tilde{v} \in C_E : \{\lambda_j(\tilde{v})\}_{j\in J} \text{ is finite}\}.$$

Therefore, there exists $1 \le i \le C_3$ such that the set $\{\lambda_j(\tilde{v}_i) : \ j \in J\}$ is infinite. □

Now we will finish the proof of the second case. Let us put $\tilde{v} = \tilde{v}_i$ (where $\tilde{v}_i$ is as in the previous lemma). Let

$$I = \{j \in J : \lambda_j(\tilde{v}) \ne \infty\}.$$

The rank of a finite matrix over $\mathcal{O}_{\tilde{v}}$ cannot increase after the reduction modulo $\ker \phi_{\tilde{v}}$. Hence for every $j \in I$,

$$\mathrm{rk}_Q(f_{X_j}(\phi_{\tilde{v}}(B) - \lambda_j(\tilde{v})\phi_{\tilde{v}}(A))) = \mathrm{rk}_Q(f_{X_j}(\phi_{\tilde{v}}(B-\lambda_j A))) \le \mathrm{rk}_E(f_{X_j}(B-\lambda_j A)).$$

Therefore, we obtain

$$\operatorname{rk}_Q(f_{X_j}(\phi_{\tilde v}(B))) - \operatorname{rk}_Q(f_{X_j}(\phi_{\tilde v}(B) - \lambda_j(\tilde v)\phi_{\tilde v}(A))) \geq \operatorname{rk}_E(f_{X_j}(B)) - \frac{\epsilon}{2}|X_j| - \operatorname{rk}_E(f_{X_j}(B - \lambda_j A)) \overset{\text{by } (19)}{\geq} \frac{\epsilon}{2}|X_j|.$$

This means that $S_{\frac{\epsilon}{2}}(\phi_{\tilde v}(A), \phi_{\tilde v}(B))$ is infinite. Now, since Theorem 8.1 holds over Q, we obtain a contradiction.

9. The centralizer of an operator in the space of Hilbert-Schmidt operators

9.1. **The space $HS(l^2(X))$ of Hilbert-Schmidt operators as an $H \times H$-Hilbert module.** Let H be a countable group. A (left) finitely generated **Hilbert** H-module is a Hilbert space V together with a linear isometric H-action such that there exists an isometric linear embedding of V into $l^2(H)^n$ with the obvious left H-action. A **morphism** between two finitely generated Hilbert H-modules U and V is a bounded H-equivariant map $\alpha : U \to V$.

Let V be a finitely generated Hilbert H-module and consider a linear H-equivariant embedding of $V \hookrightarrow l^2(H)^n$. Let $P_V : l^2(H)^n \to l^2(H)^n$ be the orthogonal projection onto V. We put

$$\dim_H V := \operatorname{Tr}_H(P_V) := \sum_{i=1}^n \langle (\mathbf{1}_i)P_V, \mathbf{1}_i\rangle_{(l^2(H))^n},$$

where $\mathbf{1}_i$ is the element of $l^2(H)^n$ having 1 in the ith entry and 0 in the rest of the entries. The number $\dim_H V$ is the **von Neumann dimension** of V and it does not depend on the linear H-equivariant embedding of V into $l^2(H)^n$.

We call a sequence of Hilbert H-modules $U \to V \to W$ **weakly exact** if the kernel of the second map coincides with the closure of the image of the first one. Recall the following important fact about the von Neumann dimension.

Proposition 9.1. [30, Theorem 1.12] *If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of Hilbert H-modules, then*

$$\dim_H U + \dim_H W = \dim_H V.$$

Let X be a set on which H acts on the left side. Assume that H acts freely on X and $H\backslash X$ is finite. The group $H \times H$ acts on $X \times X$ freely with finite number of orbits and so $l^2(X \times X)$ is isometric to the $H \times H$-Hilbert module $l^2(H \times H)^{|H\backslash X|^2}$.

If $v \in l^2(X)$ we denote by $\delta_v \in l^2(X)^*$ the functional on $l^2(X)$ that sends $u \in l^2(X)$ to $\langle u, v\rangle$. For every $(x, y) \in X \times X$ let

$$\Psi_1(x, y) = \delta_x \otimes y \in (l^2(X))^* \otimes l^2(X).$$

This map uniquely extends to an $H \times H$-isometry between the Hilbert space $l^2(X \times X)$ and the Hilbert tensor product $(l^2(X))^* \otimes l^2(X)$.

For $v^* \in (l^2(X))^*$ and $u \in l^2(X)$, let $\Psi_2(v^* \otimes u)$ be the Hilbert-Schmidt operator that sends $w \in l^2(X)$ to $(w)v^*u$. This maps extends uniquely to an isometry between $(l^2(X))^* \otimes l^2(X)$ and $HS(l^2(X))$.

We define a left $H \times H$ action on $HS(l^2(X))$ in the following way:

$$(v)[(g, h) \cdot A] = h((g^{-1}v)A),$$

where $A \in HS(l^2(X))$, $g, h \in H$, $v \in l^2(X)$. It is straightforth to check that Ψ_2 commutes with the $H \times H$ action. Thus, $HS(l^2(X))$ is a left $H \times H$-Hilbert module isometric to $l^2(X \times X)$. We collect the main properties of the maps δ and Ψ_2 in the following lemma.

Lemma 9.2. *The following properties hold.*

(1) *Let $v \in l^2(X)$ and $A \in \mathcal{B}(l^2(X))$. Assume that $(v)A = \sum_{x\in X} \alpha_x x$. Then*
$$\delta_{(v)A} = \sum_{x\in X} \overline{\alpha_x}\delta_x.$$

(2) *If $v, u \in l^2(X)$ and $A \in \mathcal{B}(l^2(X))$, then*
$$\Psi_2(\delta_{(v)A^*} \otimes u) = A\Psi_2(\delta_v \otimes u) \text{ and } \Psi_2(\delta_v \otimes (u)A) = \Psi_2(\delta_v \otimes u)A.$$

We denote by $\mathcal{U}_H(l^2(X))$ ($\mathcal{N}_H(l^2(X))$) the algebra of unbounded (bounded) operators on $l^2(X)$ commuting with the left H-action. Note that $\mathcal{N}_H(l^2(X))$ is isomorphic to $\mathrm{Mat}_{|H\backslash X|}(\mathcal{N}(H))$, and so, $\mathcal{U}_H(l^2(X))$ is isomorphic to the left and right classical ring of fractions of $\mathcal{N}_H(l^2(X))$.

If $A \in \mathcal{N}_H(l^2(X))$, then the centralizer of A in $HS(l^2(X))$ is defined as
$$C_{HS(l^2(X))}(A) = \{D \in HS(l^2(X)) : AD = DA\}.$$
We would like to define in the same way the centralizer of A in $HS(l^2(X))$ for $A \in \mathcal{U}_H(l^2(X))$. However, we cannot do it because in this case AD and DA are not always defined if $D \in HS(l^2(X))$. Therefore, we will proceed differently.

Since $A \in \mathcal{U}_H(l^2(X))$, we can write $A = B_1^{-1}C_1 = C_2B_2^{-1}$ with $B_1, B_2, C_1, C_2 \in \mathcal{N}_H(l^2(X))$ (B_1, B_2 are invertible in $\mathcal{U}_H(l^2(X))$). We put
$$C_{HS(l^2(X))}(A) = \{D \in HS(l^2(X)) : C_1DB_2 = B_1DC_2\}.$$
It is clear that $C_{HS(l^2(X))}(A)$ is an $H \times H$-Hilbert submodule of $HS(l^2(X))$. Observe also that the definition of $C_{HS(l^2(X))}(A)$ does not depend on the choice of representations of A as $A = B_1^{-1}C_1 = C_2B_2^{-1}$.

Let $\tau_i : \mathcal{N}_H(l^2(X)) \to \mathcal{N}_{H\times H}(l^2(X \times X))$ ($i = 1, 2$) be two embeddings such that for every $A \in \mathcal{N}_H(l^2(X))$ and $x, y \in X$
$$(x, y)\tau_1(A) = \sum_{z\in X} \overline{\alpha_z}(z, y) \text{ and } (y, x)\tau_2(A) = \sum_{z\in X} \alpha_z(y, z) \tag{24}$$
if $(x)A = \sum_{z\in X} \alpha_z z$.

Now let us put $\Psi = \Psi_2 \circ \Psi_1$.

Proposition 9.3. *Let $A \in \mathcal{N}_H(l^2(X))$. Then for every $f \in l^2(X \times X)$,*
$$\Psi((f)\tau_1(A^*)) = A\Psi(f) \text{ and } \Psi((f)\tau_2(A)) = \Psi(f)A.$$

Proof. Since Ψ is linear and continuous, it is enough to consider $f = (x, y) \in X \times X$. Write
$$(x)A^* = \sum_{z\in X} \beta_z z.$$
Then, using Lemma 9.2, we obtain that for any $w \in X$,
$$(w)\Psi((x, y)\tau_1(A^*)) = (w)\Psi(\sum_{z\in X} \overline{\beta_z}(z, y)) = (w)(\Psi_2(\sum_{z\in X} \overline{\beta_z}\delta_z \otimes y) =$$
$$(w)(\Psi_2(\delta_{(x)A^*} \otimes y)) = (w)(A\Psi_2(\delta_x \otimes y)) = (w)A\Psi(x, y).$$

Hence we obtain that $\Psi((x,y)\tau_1(A^*)) = A\Psi(x,y)$. The equality $\Psi((x,y)\tau_2(A)) = \Psi(x,y)A$ is proved similarly. □

Clearly we can extend τ_i to the $*$-homomorphism

$$\tau_i : \mathcal{U}_H(l^2(X)) \to \mathcal{U}_{H\times H}(l^2(X\times X))(i=1,2).$$

Corollary 9.4. *Let* $A \in \mathcal{U}_H(l^2(X))$. *Then*

$$\Psi(\ker(\tau_1(A^*) - \tau_2(A))) = C_{HS(l^2(X))}(A).$$

In particular,

$$\dim_{H\times H} C_{HS(l^2(X))}(A) = \dim_{H\times H} \ker(\tau_1(A^*) - \tau_2(A)).$$

Proof. Represent A as $A = B_1^{-1}C_1 = C_2B_2^{-1}$ with $B_1, B_2, C_1, C_2 \in \mathcal{N}_H(l^2(X))$. Then, since the operators $\tau_1(B_1^*)$ and $\tau_2(B_2)$ are injective and they commute, we obtain that

$$\begin{aligned}\ker(\tau_1(A^*) - \tau_2(A)) = \ker(\tau_1((B_1^{-1}C_1)^*) - \tau_2(C_2B_2^{-1})) = \\ \ker((\tau_1(C_1^*(B_1^*)^{-1}) - \tau_2(C_2B_2^{-1}))\tau_1(B_1^*)\tau_2(B_2)) = \\ \ker(\tau_1(C_1^*)\tau_2(B_2) - \tau_1(B_1^*)\tau_2(C_2)).\end{aligned}$$

Thus, from Proposition 9.3 and the definition of $C_{HS(l^2(X))}(A)$ we conclude that

$$\begin{aligned}\Psi(\ker(\tau_1(A^*) - \tau_2(A))) = \Psi(\ker(\tau_1(C_1^*)\tau_2(B_2) - \tau_1(B_1^*)\tau_2(C_2))) = \\ C_{HS(l^2(X))}(A).\end{aligned}$$

□

9.2. **An application of the Lück approximation.** Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Let G be a finitely generated sofic group. Represent G as $G = F/N$ where F is a finitely generated free group and let $\{X_i\}$ be a collection of finite F-sets which approximates G. Put $R = K[F]$.

The action of F on X_i induces a $*$-homomorphism f_{X_i} of R:

$$f_{X_i} : R \to \mathcal{V}_i = \mathcal{B}(l^2(X_i)) \cong \mathrm{Mat}_{|X_i|}(\mathbb{C}).$$

Consider $X_0 = G$ as a (G,F)-set. We put $\mathcal{V}_0 = \mathcal{U}_G(l^2(X_0)) \cong U(G)$. Let $f_{X_0} : R \to \mathcal{V}_0$ be the natural $*$-homomorphism of R in $\mathcal{V}_0$. As usual rk_{X_i} is the canonical Sylvester matrix rank function on $\mathcal{V}_i$.

Now define $\mathcal{V} = \prod_{i=0}^{\infty} \mathcal{V}_i$ and let $f = (f_{X_i})$ be a representation of R in $\mathcal{V}$. Let $\mathcal{U} = \mathcal{R}(f(R), \mathcal{V})$. We denote by π_i the projection of $\mathcal{V}$ on the ith coordinate and put $\mathrm{rk}_i = \mathrm{rk}_{X_i} \circ \pi_i$.

Proposition 9.5. *The following holds.*

1. *The projection* $\pi_0 : \mathcal{V} \to \mathcal{U}(G)$ *restricts to a surjective $*$-homomorphism* $\pi_0|_{\mathcal{U}} : \mathcal{U} \to \mathcal{R}_{K[G]}$.
2. *For* $i \geq 1$, *the projection* $\pi_i : \mathcal{V} \to \mathrm{Mat}_{|X_i|}(\mathbb{C})$ *satisfies* $\pi_i(\mathcal{U}) \subseteq \mathrm{Mat}_{|X_i|}(K)$.
3. *Moreover, if the sofic Lück approximation holds over* K, *then for any* $z \in \mathrm{Mat}_n(\mathcal{U})$,
$$\lim_{i\to\infty} \mathrm{rk}_i(z) = \mathrm{rk}_0(z).$$

Proof. (1) and (2) follow from Corollary 4.3 and Remark 6.5. (3) is a consequence of Corollary 6.2, because we assume that the sofic Lück approximation holds over K. □

The main result of this subsection is the following.

Proposition 9.6. *Assume that the sofic Lück approximation holds over K. Let $z = (z_i)_{i\ge 0} \in \mathrm{Mat}_n(\mathcal{U})$. Then the following equality holds.*

$$\dim_{G\times G} C_{\mathrm{Mat}_n(HS(l^2(G)))}(z_0) = \lim_{j\to\infty} \frac{\dim_{\mathbb{C}} C_{\mathrm{Mat}_n(\mathcal{V}_j)}(z_j)}{|X_j|^2}.$$

Proof. For simplicity of exposition let us assume that $n = 1$.

Let F_1 and F_2 be two groups isomorphic to F. Put $\tilde F = F_1 * F_2$ and let $\tilde R = K[\tilde F]$. The sets $X_j \times X_j$ are $\tilde F$-sets where F_1 acts on the first factor and F_2 acts on the second. Since $\{X_j\}$ approximates $G = F/N$, $\{X_j \times X_j\}$ approximate $\tilde G \cong G \times G$.

The action of $\tilde F$ on $X_j \times X_j$ induces a representation $\tilde f_j$ of $\tilde R$ in

$$\tilde{\mathcal{V}}_j = \mathcal{B}(l^2(X_j \times X_j)) \cong \mathrm{Mat}_{|X_j|^2}(\mathbb{C}).$$

We denote by $\mathrm{rk}_{\tilde{\mathcal{V}}_j}$ the unique rank function on $\tilde{\mathcal{V}}_j$.

We put

$$\tilde{\mathcal{V}}_0 = \mathcal{U}_{G\times G}(l^2(G \times G)) \cong \mathcal{U}(G \times G)$$

and let $f_0 : \tilde R \to \tilde{\mathcal{V}}_0$ be the natural representation of $\tilde R$ in $\tilde{\mathcal{V}}_0$. Let $\mathrm{rk}_{\tilde{\mathcal{V}}_0}$ be the rank function $\mathrm{rk}_{G\times G}$ on $\tilde{\mathcal{V}}_0$.

Now define $\tilde{\mathcal{V}} = \prod_{i=0}^{\infty} \tilde{\mathcal{V}}_i$ and let $\tilde f = (\tilde f_i)$ be a representation of $\tilde R$ in $\tilde{\mathcal{V}}$. Let $\tilde{\mathcal{U}} = \mathcal{R}(\tilde f(\tilde R), \tilde{\mathcal{V}})$. We denote by π_i the projection of $\tilde{\mathcal{V}}$ on the ith coordinate and put $\mathrm{rk}_i = \mathrm{rk}_{\tilde{\mathcal{V}}_i} \circ \pi_i$.

If $j \ge 0$ and $i = 1, 2$ we define $\tau_i^j : \mathcal{V}_j \to \tilde{\mathcal{V}}_j$ as we have done in Section 9.1: first it is defined for bounded operators as in (24) and then it is extended to unbounded operators. For $i = 1, 2$ we put

$$\tau_i = (\tau_i^j) : \mathcal{V} \to \tilde{\mathcal{V}}.$$

Note that $\tau_1(f(R))$ and $\tau_2(f(R))$ are subalgebras of $\tilde f(\tilde R)$. Hence, by Corollary 4.3,

$$\tilde z = \tau_1(z^*) - \tau_2(z) \in \mathcal{R}(\tilde f(\tilde R), \tilde{\mathcal{V}}).$$

Since $\{X_j \times X_j\}$ approximates $G \times G$, the argument that we have used in order to prove Proposition 9.5 can be applied again here to show that

$$\mathrm{rk}_0(\tilde z) = \lim_{k\to\infty} \mathrm{rk}_k(\tilde z).$$

By Corollary 9.4

$$\dim_{G\times G} C_{HS(l^2(G))}(z_0) = 1 - \mathrm{rk}_0(\tilde z) \text{ and}$$

$$\frac{1}{|X_j|^2} \dim_{\mathbb{C}} C_{HS(l^2(X_j))}(z_j) = 1 - \mathrm{rk}_j(\tilde z) \text{ if } j \ge 1.$$

This implies the desired result. □

9.3. **An estimation of the size of the centralizer of an operator in the space of Hilbert-Schmidt operators.** For any operator A on a Hilbert space we denote by $\sigma_p(A)$ the set of eigenvalues of A. Let H be a countable group acting freely on the left on a set X with finite number of orbits and let $A \in \mathrm{Mat}_n(\mathcal{U}_H(l^2(X)))$. For any $\lambda \in \sigma_p(A)$ we put

$$n_{\lambda,i}(A) = \dim_H \ker(A - \lambda)^i.$$

If A is a matrix over $\mathbb{C}$ (that is $H = 1$), we have the following formula, which can be found in [6, Theorem 6.13].

Proposition 9.7. *Let A be a complex n by n matrix over $\mathbb{C}$. Then we have the following equality*

$$\dim_{\mathbb{C}} C_{\mathrm{Mat}_n(\mathbb{C})}(A) = \sum_{\lambda\in\sigma_p(A)} \sum_{i=0}^{\infty} (n_{\lambda,i+1}(A) - n_{\lambda,i}(A))^2.$$

First observe the following consequence.

Corollary 9.8. *Let A be an n by n matrix over a subfield K of $\mathbb{C}$ and let $\epsilon < \frac{1}{2}$. Put $\delta = \frac{1+\sqrt{1-2\epsilon}}{2}$. Assume that $\dim_{\mathbb{C}} C_{\mathrm{Mat}_n(\mathbb{C})}(A) \geq n^2(1-\epsilon)$. Then there exists exactly one $\lambda \in \sigma_p(A)$ satisfying $n_{\lambda,1}(A) \geq \delta n$. Moreover $\lambda \in K$.*

Proof. The existence and uniqueness of λ follow from the equality in Proposition 9.7. It is clear that λ is algebraic over K and every Galois conjugate λ' of λ also satisfies $n_{\lambda',1}(A) \geq \delta n$. Hence $\lambda = \lambda'$, and so, $\lambda \in K$. □

Now we shall give a constructive proof of the first part of Theorem 2.1.

Proposition 9.9. *Let H be a countable group acting freely on the left on a set X with finite number of orbits. Let $A \in \mathrm{Mat}_n(\mathcal{U}_H(l^2(X)))$. Then*

$$\dim_{H\times H} C_{\mathrm{Mat}_n(HS(l^2(X)))}(A) \geq \sum_{\lambda\in\sigma_p(A)} \sum_{i=0}^{\infty} (n_{\lambda,i+1}(A) - n_{\lambda,i}(A))^2.$$

Let $U_{A,\lambda,j}$ be the space $\ker(A-\lambda)^{j+1} \cap (\ker(A-\lambda)^j)^{\perp}$. For any $\epsilon > 0$, let $U^{\epsilon}_{A,\lambda,j}$ be an H-invariant closed subspace of $U_{A,\lambda,j}$ such that

(1) $\dim_H U^{\epsilon}_{A,\lambda,j} \geq \dim_H U_{A,\lambda,j} - \epsilon$ and
(2) A is bounded when restricted to $U^{\epsilon}_{A,\lambda,j}$, $(U^{\epsilon}_{A,\lambda,j})A$, …, $(U^{\epsilon}_{A,\lambda,j})A^j$.

In the same way we define a space $U^{\epsilon}_{A^*,\bar{\lambda},j}$. Then we have that

$$\dim_H U^{\epsilon}_{A,\lambda,j}, \dim_H U_{A^*,\bar{\lambda},j} \geq n_{\lambda,j+1}(A) - n_{\lambda,j}(A) - \epsilon. \tag{25}$$

For every $v \in U^{\epsilon}_{A^*,\bar{\lambda},j}$ and $u \in U^{\epsilon}_{A,\lambda,j}$ we define

$$w_j(v,u) = (\delta_{(v)(A^*-\bar{\lambda})^j} \otimes u) + (\delta_{(v)(A^*-\bar{\lambda})^{j-1}} \otimes (u)(A-\lambda)) + \cdots + (\delta_v \otimes (u)(A-\lambda)^j)$$

and let $W^{\epsilon}_{A,\lambda,j}$ be the closed subspace of $(l^2(X))^* \otimes l^2(X)$ generated by the vectors

$$\{w_j(v,u) : \ v \in U^{\epsilon}_{A^*,\bar{\lambda},j}, \ u \in U^{\epsilon}_{A,\lambda,j}\}.$$

Lemma 9.10. *Let $A \in \mathcal{U}_H(l^2(X))$. Then the following holds.*

(1) *For every $\epsilon > 0$, $\Psi_2(W^{\epsilon}_{A,\lambda,j}) \leq C_{HS(l^2(X))}(A)$.*
(2) *Let $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k \in \sigma_p(A)$ and $n_1, \ldots, n_k \in \mathbb{N}$. Then*

$$\lim_{\epsilon\to 0^+} \dim_{H\times H} \left(\sum_{i=1}^{k} \sum_{j=0}^{n_i} \Psi_2(W^{\epsilon}_{A,\lambda_i,j}) \right) = \sum_{i=1}^{k} \sum_{j=0}^{n_i} (n_{\lambda_i,j+1}(A) - n_{\lambda_i,j}(A))^2.$$

Proof. (1) For simplicity we assume that $\lambda = 0$. Let $v \in U^{\epsilon}_{A^*,0,j}$ and $u \in U^{\epsilon}_{A,0,j}$. Represent A as $A = B_1^{-1}C_1 = C_2B_2^{-1}$ with $B_1, B_2, C_1, C_2 \in \mathcal{N}_H(l^2(X))$. Applying Lemma 9.2, we obtain that

$$C_1\Psi_2(w_j(v,u))B_2 - B_1\Psi_2(w_j(v,u))C_2 =$$

$$C_1\Psi_2\left(\sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i}}\otimes(u)A^i\right)B_2 - B_1\Psi_2\left(\sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i}}\otimes(u)A^i\right)C_2 \overset{\text{by Lemma 9.2(2)}}{=}$$

$$\Psi_2\left(\sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i}C_1^*}\otimes(u)A^i\right)B_2 - B_1\Psi_2\left(\sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i}}\otimes(u)A^iC_2\right) =$$

$$\Psi_2\left(\sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i+1}B_1^*}\otimes(u)A^i\right)B_2 - B_1\Psi_2\left(\sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i}}\otimes(u)A^{i+1}B_2\right) \overset{\text{by Lemma 9.2(2)}}{=}$$

$$B_1\Psi_2\left(\sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i+1}}\otimes(u)A^i - \sum_{i=0}^{j}\delta_{(v)(A^*)^{j-i}}\otimes(u)A^{i+1}\right)B_2 =$$

$$B_1\Psi_2\left(\delta_{(v)(A^*)^{j+1}}\otimes u - \delta_v\otimes(u)A^{j+1}\right)B_2 = 0.$$

This proves (1).

(2) Fix $\epsilon > 0$. Let ρ be a continuous linear map

$$\rho: \quad \oplus_{i=1}^{k}\oplus_{j=0}^{n_i}\delta_{U^{\epsilon}_{A^*,\overline{\lambda_i},j}}\otimes U^{\epsilon}_{A,\lambda_i,j} \quad\to\quad \textstyle\sum_{i=1}^{k}\sum_{j=0}^{n_i}\Psi_2(W^{\epsilon}_{A,\lambda_i,j})$$

defined on the generating set $\{\delta_{v_{i,j}}\otimes u_{i,j} : v_{i,j}\in U^{\epsilon}_{A^*,\overline{\lambda_i},j}, u_{i,j}\in U^{\epsilon}_{A,\lambda_i,j}\}$ by means of

$$\rho: \quad \delta_{v_{i,j}}\otimes u_{i,j} \quad\mapsto\quad \Psi_2(w_j(v_{i,j},u_{i,j}))\ .$$

Observe that ρ is bounded. Since the sums

$$U^{\epsilon} = \sum_{i=1}^{k}\sum_{j=0}^{n_i}U^{\epsilon}_{A,\lambda_i,j} \text{ and } V^{\epsilon} = \sum_{i=1}^{k}\sum_{j=0}^{n_i}\delta_{U_{A^*,\overline{\lambda_i},j}}$$

are direct sums of closed subspaces of $l^2(X)$ and $l^2(X)^*$ respectively, the obvious map

$$\tilde{\rho}: \oplus_{i=1}^{k}\oplus_{j=0}^{n_i}\delta_{U^{\epsilon}_{A^*,\overline{\lambda_i},j}}\otimes U^{\epsilon}_{A,\lambda_i,j}\to V^{\epsilon}\otimes U^{\epsilon},$$

which sends $\delta_{v_{i,j}}\otimes u_{i,j}$ to $w_j(v_{i,j},u_{i,j})$, is a weak monomorphism. Therefore, $\rho = \Phi_2\circ\tilde{\rho}$ is a weak isomorphism. Applying Proposition 9.1 and (25), we conclude that (2) holds .

□

It is clear that Proposition 9.9 follows from the previous lemma. We finish this section by proving the following result which we will use later.

Proposition 9.11. *Let A be a complex n by n matrix and let $k \in \mathbb{N}$. Then we have the following inequality*

$$\dim_{\mathbb{C}} C_{\mathrm{Mat}_n(\mathbb{C})}(A) - \sum_{\lambda\in\sigma_p(A),n_{\lambda,1}(A)\geq\frac{n}{k}}\ \sum_{i=0}^{k-1}(n_{\lambda,i+1}(A)-n_{\lambda,i}(A))^2 \leq \frac{n^2}{k}.$$

Proof. Let $\mu \in \mathbb{C}$ and $s \geq 0$. Then

$$\sum_{i=s}^{\infty}(n_{\mu,i+1}(A)-n_{\mu,i}(A))^2 \leq (n_{\mu,s+1}(A)-n_{\mu,s}(A))\sum_{i=s}^{\infty}(n_{\mu,i+1}(A)-n_{\mu,i}(A)) \leq \min\{n_{\mu,1}(A), \frac{n}{s+1}\}\cdot n_{\mu,n}(A). \tag{26}$$

Hence, using Proposition 9.7, we obtain that

$$\dim_{\mathbb{C}} C_{\mathrm{Mat}_n(\mathbb{C})}(A) - \sum_{\lambda\in\sigma_p(A), n_{\lambda,1}(A)\geq\frac{n}{k}}\sum_{i=0}^{k-1}(n_{\lambda,i+1}(A)-n_{\lambda,i}(A))^2 \overset{\text{Proposition 9.7}}{=} \sum_{\lambda\in\sigma_p(A), n_{\lambda,1}(A)<\frac{n}{k}}\sum_{i=0}^{\infty}(n_{\lambda,i+1}(A)-n_{\lambda,i}(A))^2 + \sum_{\lambda\in\sigma_p(A), n_{\lambda,1}(A)\geq\frac{n}{k}}\sum_{i=k}^{\infty}(n_{\lambda,i+1}(A)-n_{\lambda,i}(A))^2 \overset{\text{by (26)}}{\leq} \frac{n}{k}\sum_{\lambda\in\sigma_p(A)} n_{\lambda,n}(A) = \frac{n^2}{k}.$$

□

9.4. **From the strict eigenvalue property to the centralizer dimension property.** In this subsection we will show that the sofic Lück approximation over an algebraically closed subfield K of $\mathbb{C}$ implies the centralizer dimension property over K for sofic groups. As we have already mentioned in Subsection 2.5, the main ingredient of the proof is the strict eigenvalue property.

Theorem 9.12. *Let K be an algebraically closed subfield of $\mathbb{C}$ closed under complex conjugation. Assume that the sofic Lück approximation holds over K. Then for every sofic group G and $A \in \mathrm{Mat}_n(\mathcal{R}_{K[G]})$,*

$$\dim_{G\times G} C_{\mathrm{Mat}_n(HS(l^2(G)))}(A) = \sum_{\lambda\in K}\sum_{i=0}^{\infty}(n_{\lambda,i+1}(A)-n_{\lambda,i}(A))^2.$$

Proof. We will use the notation of Subsection 9.2. By Proposition 9.5, there exists $z = (z_i) \in \mathrm{Mat}_n(\mathcal{U})$ such that $z_0 = A$. Moreover, $z_j \in \mathrm{Mat}_n(\mathrm{Mat}_{|X_i|}(K))$ for all $j \geq 1$.

For every $\epsilon > 0$, denote by S_ϵ the following set

$$S_\epsilon = \{\lambda \in K : \text{ there exists } j \geq 1 \text{ such that } \mathrm{rk}_j(z-\lambda) \leq n(1-\epsilon)\}.$$

Observe that, by Theorem 8.1, S_ϵ is finite.

From Proposition 9.6 and Proposition 9.11 we obtain that for every $k \in \mathbb{N}$,

$$|\dim_{G\times G} C_{\mathrm{Mat}_n(HS(l^2(G)))}(z_0) - \lim_{j\to\infty} \sum_{\lambda\in S_{\frac{1}{k}}} \frac{1}{|X_j|^2} \sum_{i=0}^{k-1} (n_{\lambda,i+1}(z_j) - n_{\lambda,i}(z_j))^2| \overset{\text{by Proposition 9.11}}{\leq} \lim_{j\to\infty} |\dim_{G\times G} C_{\mathrm{Mat}_n(HS(l^2(G)))}(z_0) - \frac{\dim_{\mathbb{C}} C_{\mathrm{Mat}_n(\mathcal{V}_j)}(z_j)}{|X_j|^2}| + \frac{n^2}{k} \overset{\text{by Proposition 9.6}}{=} \frac{n^2}{k}.$$

By Proposition 9.5, for every $\lambda \in \mathbb{C}$ and every $k \in \mathbb{N}$,

$$n_{\lambda,k}(z_0) = \lim_{j\to\infty} \frac{n_{\lambda,k}(z_j)}{|X_j|}. \tag{27}$$

Thus, we obtain that

$$\dim_{G\times G} C_{\mathrm{Mat}_n(HS(l^2(G)))}(z_0) = \lim_{k\to\infty}\lim_{j\to\infty} \sum_{\lambda\in S_{\frac{1}{k}}} \frac{1}{|X_j|^2} \sum_{i=0}^{k-1} (n_{\lambda,i+1}(z_j) - n_{\lambda,i}(z_j))^2 \overset{\text{since } S_{1/k} \text{ is finite}}{=} \lim_{k\to\infty} \sum_{\lambda\in S_{\frac{1}{k}}} \sum_{i=0}^{k-1} \lim_{j\to\infty} \frac{1}{|X_j|^2} (n_{\lambda,i+1}(z_j) - n_{\lambda,i}(z_j))^2 \overset{\text{by (27)}}{=} \lim_{k\to\infty} \sum_{\lambda\in S_{\frac{1}{k}}} \sum_{i=0}^{k-1} (n_{\lambda,i+1}(z_0) - n_{\lambda,i}(z_0))^2 \leq \sum_{\lambda\in K} \sum_{i=0}^{\infty} (n_{\lambda,i+1}(z_0) - n_{\lambda,i}(z_0))^2.$$

Combining this inequality with the one from Proposition 9.9, we obtain the theorem. □

Corollary 9.13. *Let K be an algebraically closed subfield of $\mathbb{C}$ closed under complex conjugation. Assume that the sofic Lück approximation holds over K. Let G be a sofic group and $A \in \mathrm{Mat}_n(\mathcal{R}_{K[G]})$. Then for any $\lambda \in \mathbb{C} \setminus K$, the matrix $A - \lambda I_n$ is invertible over $\mathcal{R}_{\mathbb{C}[G]}$.*

Proof. From Theorem 9.12 and Proposition 9.9, we obtain that $n_{\lambda,1}(A) = 0$ since $\lambda \notin K$. Hence $\mathrm{rk}_G(A - \lambda I_n) = n$, and so, $A - \lambda I_n$ is invertible over $\mathcal{R}_{\mathbb{C}[G]}$. □

Remark 9.14. *It is not necessary to assume in Theorem 9.12 and Corollary 9.13 that K is closed under complex conjugation if the matrix A has entries in $K[G]$.*

10. The proof of Theorem 1.3 and of its corollaries

In this section we will finish the proof of Theorem 1.3 and its corollaries. In fact we will prove the following strong version of Theorem 1.3 which resolves Conjecture 4 over an arbitrary subfield of $\mathbb{C}$ in the case where all H_i are sofic.

Theorem 10.1. *Let K be a subfield of $\mathbb{C}$, F a finitely generated free group and N a normal subgroup of F. For each natural number k, let X_k be an (H_k, F)-set* *such that H_k is a countable sofic group that acts freely on X_k and $H_k\backslash X_k$ is finite. Assume that $\{X_k\}$ approximates $G = F/N$. Then for every $A \in \mathrm{Mat}_{n\times m}(K[F])$,*

$$\lim_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A).$$

As we have explained in Subsection 2.2, without loss of generality, we can assume that K in Theorem 10.1 is finitely generated and so we only have to consider K of finite transcendental degree over $\mathbb{Q}$. For these K we prove Theorem 10.1 by induction on the transcendental degree of $K/\mathbb{Q}$. The base of induction $K = \mathbb{Q}$ is known and it can be proved using the original ideas of W. Lück [29] (see also [13] and [23]). There are two types of inductive steps:

(1) Assuming that Theorem 10.1 holds over a subfield K of $\mathbb{C}$ closed under complex conjugation we will show that it holds over $\bar{K}$.
(2) Assuming that Theorem 10.1 holds over Q, where Q is an algebraically closed subfield of $\mathbb{C}$ closed under complex conjugation, we will show that it holds over $Q(\lambda)$, where $\lambda \in \mathbb{C} \setminus Q$ and $|\lambda| = 1$.

10.1. **The Lück approximation over the algebraic closure.** Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. In this subsection we prove that if Theorem 10.1 holds over K, then it holds over $\bar{K}$. This will be obtained as a corollary of the following theorem.

Theorem 10.2. *Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Assume that the sofic Lück approximation holds over K. Let G be a sofic group. Then the restriction of rk_G on $\bar{K}[G]$ is the natural algebraic extension of the restriction of rk_G on $K[G]$.*

Corollary 10.3. *Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. If the general sofic Lück approximation conjecture holds over K, then it holds over $\bar{K}$.*

Proof. We will use the notation of Subsection 6.4, assuming that H_i are sofic. Since the general sofic Lück approximation conjecture holds over K, rk_G and $\mathrm{rk}_{G,\omega}$ are equal on the matrices over $K[F]$. By Theorem 10.2, the restriction of rk_G on $\bar{K}[G]$ is the natural algebraic extension of the restriction of rk_G on $K[G]$. On the other hand, applying again Theorem 10.2 we obtain that for each i, the restriction of rk_{H_i} on $\bar{K}[H_i]$ is the natural algebraic extension of the restriction of rk_{H_i} on $K[H_i]$, and so, by Proposition 7.13, the restriction of $\mathrm{rk}_{G,\omega}$ on $\bar{K}[F]$ is the natural algebraic extension of the restriction of $\mathrm{rk}_{G,\omega}$ on $K[F]$. Therefore, rk_G and $\mathrm{rk}_{G,\omega}$ also are equal on the matrices over $\bar{K}[G]$. □

Now let us prove Theorem 10.2. The main step of the proof is the following weak version of the center conjecture.

Proposition 10.4. *Let G be a finitely generated sofic ICC group and let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Assume that the sofic Lück approximation holds over K. Then any element in $Z(\overline{\mathcal{R}_{K[G]}}) \setminus K$ is not algebraic over K.*

Proof. Represent G as $G = F/N$ where F is a free group and let $\{X_i\}$ be a collection of finite F-sets which approximates G. Put $R = K[F]$. The action of F on X_i induces a representation f_i of R in $\mathrm{Mat}_{|X_i|}(K)$ (see Remark 6.5).

We denote by $\mathrm{rk}_{\mathcal{W}_i}$ the unique rank function on $\mathcal{W}_i = \mathrm{Mat}_{|X_i|}(K)$. We put $\mathcal{W}_0 = \mathcal{U}(G)$. Let $f_0 : R \to \mathcal{W}_0$ be the natural representation of R in $\mathcal{W}_0$ and let $\mathrm{rk}_{\mathcal{W}_0} = \mathrm{rk}_G$.

Now, we define $\mathcal{W} = \prod_{i=0}^{\infty} \mathcal{W}_i$ and let $f = (f_i)$ be a representation of R in $\mathcal{W}$. We put $\mathcal{U} = \mathcal{R}(f(R), \mathcal{W})$. We denote by π_i the projection of $\mathcal{W}$ on the ith coordinate and put $\mathrm{rk}_i = \mathrm{rk}_{\mathcal{W}_i} \circ \pi_i$.

Let $c \in Z(\overline{\mathcal{R}_{K[G]}})$. By Proposition 5.8, $c \in \mathbb{C}$. Assume that c is algebraic over K. Hence there exists a polynomial p irreducible over K such that $p(c) = 0$. Let $z_0 \in \mathcal{R}_{K[G]} = \mathcal{R}(f_0(R), \mathcal{W}_0)$ be such that $\mathrm{rk}_G(c - z_0) \le \frac{1}{5}$. In particular,

$$\mathrm{rk}_G(p(z_0)) \le \frac{1}{5}.$$

Since $\mathrm{rk}_G(c - z_0) \le \frac{1}{5}$, by Proposition 9.9,

$$\dim_{G\times G} C_{HS(l^2(G))}(z_0) \ge \frac{16}{25}.$$

By Corollary 4.3, there exists $z \in \mathcal{U}$ with $\pi_0(z) = z_0$. For $i \ge 1$, we put $z_i = \pi_i(z)$. By Proposition 9.5 and Proposition 9.6, there exists i such that

$$\mathrm{rk}_{\mathcal{W}_i}(p(z_i)) \le \frac{1}{4} \text{ and } \dim_{\mathbb{C}} C_{\mathcal{W}_i}(z_i) \ge \frac{3}{5}|X_i|^2.$$

By Corollary 9.8, there exists $k \in K$ such that

$$\mathrm{rk}_{\mathcal{W}_i}(z_i - k) \le \frac{1}{2}.$$

Therefore

$$\mathrm{rk}_{\mathcal{W}_i}(p(k)) \le \mathrm{rk}_{\mathcal{W}_i}(p(z_i) - p(k)) + \mathrm{rk}_{\mathcal{W}_i}(p(z_i)) \le \mathrm{rk}_{\mathcal{W}_i}(z_i - k) + \mathrm{rk}_{\mathcal{W}_i}(p(z_i)) \le \frac{3}{4}.$$

Since $p(k) \in K$, $p(k) = 0$. Thus p has degree 1, and so, $c \in K$. □

Now we are ready to prove the main result of this subsection.

Proof of Theorem 10.2. If G is sofic, then $\mathbb{Z} \wr G$ is sofic by [19]. If G is infinite, then $\mathbb{Z} \wr G$ is an ICC group. Thus, without loss of generality we may assume that G is an ICC group.

Since G is an ICC group, $Z(\overline{\mathcal{R}_{K[G]}})$ is a field. By Proposition 10.4, the algebraic elements of the extension $Z(\overline{\mathcal{R}_{K[G]}})/K$ are in K. Hence by Proposition 7.12(2a), we obtain that $\overline{\mathcal{R}_{K[G]}} \otimes_K \bar{K}$ is simple.

Let $\mathcal{U}$ be the subring of $\mathcal{U}(G)$ generated by $\overline{\mathcal{R}_{K[G]}}$ and $\bar{K}$. Since $\overline{\mathcal{R}_{K[G]}} \otimes_K \bar{K}$ is simple,

$$\mathcal{U} \cong \overline{\mathcal{R}_{K[G]}} \otimes_K \bar{K}.$$

Moreover, by Proposition 7.12(2b), the restriction of rk_G on $\mathcal{U}$ is the unique Sylvester matrix rank function on $\mathcal{U}$, and so, it coincides with the natural algebraic extension of the restriction of rk_G on $\overline{\mathcal{R}_{K[G]}}$. Since $\bar{K}[G]$ is a subalgebra of $\mathcal{U}$, we are done. □

10.2. **The proof of the inductive step for transcendental extensions.** In this subsection we prove that if the general sofic Lück approximation conjecture holds over an algebraically closed subfield Q of $\mathbb{C}$ closed under complex conjugation, then it holds over $Q(\lambda)$. This will be obtained as a corollary of the following theorem.

Theorem 10.5. *Let Q be an algebraically closed subfield of $\mathbb{C}$ closed under complex conjugation. Assume that the sofic Lück approximation holds over Q and let $\lambda \in \mathbb{C} \setminus Q$. Let G be a sofic group. Then the restriction of rk_G on $Q(\lambda)[G]$ is the natural transcendental extension of the restriction of rk_G on $Q[G]$.*

Proof. Let $A \in \operatorname{Mat}_n(\mathcal{R}_{Q[G]})$. Since the sofic Lück approximation holds over Q, by Corollary 9.13, $-\lambda^{-1}$ is not an eigenvalue of A and so,

$$\operatorname{rk}_G(I_n + \lambda A) = n.$$

Therefore, Proposition 7.7 implies that the restriction of rk_G on $\mathcal{R}_{Q[G]}[\lambda]$ is the natural transcendental extension of the restriction of rk_G on $\mathcal{R}_{Q[G]}$. □

Corollary 10.6. *Let Q be an algebraically closed subfield of $\mathbb{C}$ closed under complex conjugation. If the general sofic Lück approximation conjecture holds over Q, then it holds over $Q(\lambda)$ for every $\lambda \in \mathbb{C} \setminus Q$.*

Proof. The argument of the proof of Corollary 10.3 works in this case as well. Instead of Theorem 10.2 one should use Theorem 10.5 and instead of Proposition 7.13, Corollary 7.8. □

10.3. **The proof of Theorem 1.1.** Theorem 1.1 follows from the following result.

Theorem 10.7. *Let G be a countable sofic group such that $\operatorname{lcm}(G)$ is finite. Assume that G satisfies the strong Atiyah conjecture over $\bar{\mathbb{Q}}$. Then the following holds.*

1. *There are division rings $D_1, \dots, D_k$ and natural numbers $n_1 \dots, n_k$ such that*
$$\mathcal{R}_{\bar{\mathbb{Q}}[G]} \cong \operatorname{Mat}_{n_1}(D_1) \oplus \dots \oplus \operatorname{Mat}_{n_k}(D_k).$$
2. *The ring $\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ satisfies the left Ore condition and*
$$\mathcal{R}_{\mathbb{C}[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}) \cong \operatorname{Mat}_{n_1}(Q_l(D_1 \otimes_{\bar{\mathbb{Q}}} \mathbb{C})) \oplus \dots \oplus \operatorname{Mat}_{n_k}(Q_l(D_k \otimes_{\bar{\mathbb{Q}}} \mathbb{C})).$$
Moreover, the rings $E_i = Q_l(D_i \otimes_{\bar{\mathbb{Q}}} \mathbb{C})$ are division rings.
3. *G satisfies the strong Atiyah conjecture over $\mathbb{C}$.*

Proof. Let K be a subfield of $\mathbb{C}$ and let Dim_G^K be the additive subgroup of $\mathbb{R}$ generated by

$$\{\operatorname{rk}_G(A) : \ A \text{ is a matrix over } \ K[G]\}.$$

It is clear that in general $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z} \le \operatorname{Dim}_G^K$. On the other hand, the strong Atiyah conjecture for G over K is equivalent to the equality $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z} = \operatorname{Dim}_G^K$. By Proposition 5.11, we have that

$$\operatorname{Dim}_G^K = \langle \operatorname{rk}_G(r) : r \in \mathcal{R}_{K[G]} \rangle. \tag{28}$$

(1) Since $\operatorname{Dim}_G^{\bar{\mathbb{Q}}} = \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$, using (28) we obtain that $1 \in \mathcal{R}_{\bar{\mathbb{Q}}[G]}$ can be decomposed as a sum of at most $\operatorname{lcm}(G)$ orthogonal idempotents. Therefore, $\mathcal{R}_{\bar{\mathbb{Q}}[G]}$ is equal to a direct sum of at most $\operatorname{lcm}(G)$ irreducible left $\mathcal{R}_{\bar{\mathbb{Q}}[G]}$-modules. Hence,

$\mathcal{R}_{\bar{\mathbb{Q}}[G]}$ is semisimple Artinian, and so by the Artin-Wedderburn theorem, there are division rings $D_1, \ldots, D_k$ and natural numbers $n_1 \ldots, n_k$ such that

$$\mathcal{R}_{\bar{\mathbb{Q}}[G]} \cong \mathrm{Mat}_{n_1}(D_1) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(D_k).$$

(2) Let K be a finitely generated extension of $\bar{\mathbb{Q}}$. First observe that $\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K$ is Noetherian by the Hilbert Basis Theorem. The ring $\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K$ is an **almost simple** ring, that is a ring where every non-trivial ideal has a non-trivial intersection with the center (see [21, Lemma 2.15]). Thus, since the center of $\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K$ is semiprime, $\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K$ is semiprime as well. Thus, by Goldie's theorem, $\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K$ satisfies the left Ore condition. This implies that $\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ satisfies the left Ore condition.

Also, by [38, Lemma 1.1], for every field extension $L/\bar{\mathbb{Q}}$, $D_i \otimes_{\bar{\mathbb{Q}}} L$ does not have non-trivial zero-divisors, and so $Q_l(D_i \otimes_{\bar{\mathbb{Q}}} L)$ are division rings.

Now we will show that

$$\mathcal{R}_{L[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} L)$$

for every finitely generated extension $L/\bar{\mathbb{Q}}$ such that L is a subfield of $\mathbb{C}$ closed under complex conjugation. This also implies that $\mathcal{R}_{\mathbb{C}[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} \mathbb{C})$.

Observe that a finitely generated subextension $L/\bar{\mathbb{Q}}$ of $\mathbb{C}/\bar{\mathbb{Q}}$ is a subfield of a field K_{2n}, where K_i are constructed inductively:

(1) $K_1 = \bar{\mathbb{Q}}$;
(2) if $i \geq 1$, $K_{2i} = K_{2i-1}(\lambda_i)$ for some $\lambda_i \in \mathbb{C}$ which is not algebraic over K_{2i-1} and has complex norm 1.
(3) if $i \geq 1$, $K_{2i+1} = \overline{K_{2i}}$ is the algebraic closure of K_{2i} in $\mathbb{C}$;

We will show by induction on i that

$$\mathcal{R}_{K_i[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K_i).$$

The base of induction is clear.

Assume that

$$\mathcal{R}_{K_{2i-1}[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K_{2i-1}).$$

In particular, $\mathcal{R}_{K_{2i-1}[G]}$ is semisimple Artinian.

Let rk be the Sylvester matrix rank function induced by the restriction of rk_G on $\mathcal{R}_{K_{2i-1}[G]}$ and let $\widetilde{\mathrm{rk}}$ be its natural transcendental extension to $\mathcal{R}_{K_{2i-1}[G]}[\lambda_i^{\pm 1}]$. We have

$$Q_l(\mathcal{R}_{K_{2i-1}[G]}[\lambda_i^{\pm 1}]) \cong Q_l(\mathcal{R}_{K_{2i-1}[G]} \otimes_{K_{2i-1}} K_{2i}).$$

By Proposition 7.11, the left-hand side is the $*$-regular $\mathcal{R}_{K_{2i-1}[G]}[\lambda_i^{\pm 1}]$-algebra associated with $\widetilde{\mathrm{rk}}$. Observe that the inclusion of $K_{2i}[G]$ into $Q_l(\mathcal{R}_{K_{2i-1}[G]} \otimes_{K_{2i-1}} K_{2i})$ is epic. Hence $Q_l(\mathcal{R}_{K_{2i-1}[G]} \otimes_{K_{2i-1}} K_{2i})$ is isomorphic to the $*$-regular $K_{2i}[G]$-algebra associated with the restriction of $\widetilde{\mathrm{rk}}$ on $K_{2i}[G]$. By Theorem 10.5, this restriction coincides with rk_G. Since $\mathcal{R}_{K_{2i}[G]}$ is the $*$-regular $K_{2i}[G]$-algebra associated with the restriction of rk_G on $K_{2i}[G]$, we obtain that

$$\mathcal{R}_{K_{2i}[G]} \cong Q_l(\mathcal{R}_{K_{2i-1}[G]} \otimes_{K_{2i-1}} K_{2i})$$

as $K_{2i}[G]$-rings, and so,

$$\mathcal{R}_{K_{2i}[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K_{2i}).$$

Assume now that

$$\mathcal{R}_{K_{2i}[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K_{2i}).$$

Let $\phi : \mathcal{R}_{K_{2i}[G]} \otimes_{K_{2i}} K_{2i+1} \to \mathcal{U}(G)$ be the canonical homomorphism induced by the inclusions $\mathcal{R}_{K_{2i}[G]} \subset \mathcal{U}(G)$ and $K_{2i+1}[G] \subset \mathcal{U}(G)$. By Proposition 7.12(1), $\mathcal{R}_{K_{2i}[G]} \otimes_{K_{2i}} K_{2i+1}$ is von Neumann regular. Therefore, $\operatorname{Im}\phi$ is also von Neumann regular, and so, $\operatorname{Im}\phi = \mathcal{R}_{K_{2i+1}[G]}$. Let rk be the Sylvester matrix rank function induced by the restriction of rk_G on $\mathcal{R}_{K_{2i}[G]}$ and let $\widetilde{\operatorname{rk}}$ be its natural algebraic extension to $\mathcal{R}_{K_{2i}[G]} \otimes_{K_{2i}} K_{2i+1}$. Observe that the inclusion of $K_{2i+1}[G]$ into $\mathcal{R}_{K_{2i}[G]} \otimes_{K_{2i}} K_{2i+1}$ is epic. Therefore, by Proposition 5.11, any Sylvester matrix rank function on $\mathcal{R}_{K_{2i}[G]} \otimes_{K_{2i}} K_{2i+1}$ is completely determined by its values on matrices over $K_{2i+1}[G]$. Thus, by Theorem 10.2, $\widetilde{\operatorname{rk}} = \phi^{\#}(\operatorname{rk}_G)$. This implies that $\ker\phi = \{0\}$. Thus, we obtain that

$$\mathcal{R}_{K_{2i+1}[G]} \cong \mathcal{R}_{K_{2i}[G]} \otimes_{K_{2i}} K_{2i+1}$$

as $K_{2i+1}[G]$-rings, and so,

$$\mathcal{R}_{K_{2i+1}[G]} \cong Q_l(\mathcal{R}_{\bar{\mathbb{Q}}[G]} \otimes_{\bar{\mathbb{Q}}} K_{2i+1}).$$

(3) Let $\phi : \mathcal{R}_{\mathbb{C}[G]} \to \operatorname{Mat}_{n_1}(E_1) \oplus \ldots \oplus \operatorname{Mat}_{n_k}(E_k)$ be an isomorphism that sends $\mathcal{R}_{\bar{\mathbb{Q}}[G]}$ isomorphically onto $\operatorname{Mat}_{n_1}(D_1) \oplus \ldots \oplus \operatorname{Mat}_{n_k}(D_k)$. Let e_i denote a minimal idempotent of $\operatorname{Mat}_{n_i}(E_i)$. Then

$$\frac{1}{\operatorname{lcm}(G)}\mathbb{Z} = \operatorname{Dim}_G^{\bar{\mathbb{Q}}} = \langle \operatorname{rk}_G(\phi^{-1}(e_i)) : \ i = 1, \ldots, k\rangle = \operatorname{Dim}_G^{\mathbb{C}}.$$

Hence G satisfies the strong Atiyah conjecture over $\bar{\mathbb{C}}$. □

10.4. **The proof of other applications of Theorem 1.3.** Corollary 1.4 can be obtained as a consequence of Theorem 1.3, but it is also a particular case of Theorem 10.1.

Corollary 1.5 follows from Corollary 9.13 and Theorem 1.3.

Observe that since every sofic group is embedded in an ICC sofic group, it is enough to prove Corollary 1.6 for ICC groups. Assume that G is an ICC sofic group. Let $c \in Z(\overline{\mathcal{R}_{KG}})$. By Proposition 5.8, $c \in \mathbb{C}$. By Corollary 1.5, c is algebraic over K. By Proposition 10.4, $c \in K$. This proves Corollary 1.6.

Corollary 1.7 is a straightforward consequence of the sofic Lück approximation.

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