PARAFREE FUNDAMENTAL GROUPS OF GRAPH OF GROUPS WITH CYCLIC EDGE SUBGROUPS

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Abstract. We determine when the fundamental group of a finite graph of groups with cyclic edge subgroups is parafree.

1. Introduction

A group is said to be *parafree* if it is residually nilpotent and its quotients by the terms of its lower central series are the same as those of a free group. These groups were introduced by Baumslag [3] who also constructed first examples of non-free parafree groups. In this paper we only consider finitely generated parafree groups and from now on we assume that to be finitely generated is a part of the definition of parafree groups.

Many known examples of parafree groups are isomorphic to amalgamated free products or HNN extensions of free groups [6]. The purpose of this paper is to characterize which amalgamated free products and HNN extensions with cyclic base groups are parafree.

The following statement describes exactly under which circumstances an amalgamated free product with cyclic amalgam is parafree.

Theorem 1.1. *Let U and V be finitely generated groups, $1 \neq u \in U$ and $1 \neq v \in V$. Consider the amalgamated free product $W = U \underset{u=v}{*} V$. Then W is parafree if and only if the following three conditions hold.*

1. *The groups U and V are parafree.*
2. *The element uv^{-1} is not a proper power in the abelianization of $U * V$.*
3. *At least one of u or v is not a proper power in U or V, respectively.*

A particular case of Theorem 1.1, where U and V are free, follows from [4, 1]. In the case of HNN extensions with cyclic base groups we obtain the following result.

Theorem 1.2. *Let U be a finitely generated group, $u \in U \setminus \{1\}$ and $\alpha : \langle u \rangle \to U$ a monomorphism. Put $v = \alpha(u)$. Consider the HNN extension $W = U*_{\alpha}$. Then W is parafree if and only if the following four conditions hold.*

1. *The group U is parafree.*
2. *The element uv^{-1} is not a proper power in the abelianization of U.*
3. *At least one of u or v is not a proper power in U.*

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(4) *The image of the element u is non-trivial in some finite nilpotent quotient of W.*

In the case where U is a free group on 2 generators the last condition can be replaced by a simpler one.

Corollary 1.3. *Let F_2 be a free group on two generators, $u \in F_2 \setminus \{1\}$ and $\alpha : \langle u \rangle \to F_2$ a monomorphism. Put $v = \alpha(u)$. Consider the HNN extension $W = F_2 *_\alpha$. Then W is parafree if and only if the following three conditions hold.*

(1) *The element uv^{-1} is not a proper power in the abelianization of F_2.*
(2) *At least one of u or v is not a proper power in F_2.*
(3) *The images of u and v in the abelianization of F_2 generate a subgroup isomorphic to $\mathbb{Z}^2$.*

It is tempting to try to formulate a general criterion for the fundamental group of a graph of groups with cyclic edge groups to be parafree. Combining Theorem 1.1 and Theorem 1.2 we obtain the following result.

Corollary 1.4. *Let $(\mathcal{G}, \Gamma)$ be a graph of groups over a finite graph Γ and $W = \pi(\mathcal{G}, \Gamma)$ be its fundamental group. Assume that all vertex subgroups $\mathcal{G}(v)$ $(v \in V(\Gamma))$ are finitely generated and all edge subgroups $\mathcal{G}(e)$ $(e \in E(\Gamma))$ are cyclic. Then W is parafree if and only if the following four conditions hold.*

(1) *All the vertex subgroups $\mathcal{G}(v)$ $(v \in V(\Gamma))$ are parafree.*
(2) *The abelianization of W is torsion-free of rank*
$$r_{\mathrm{ab}}(W) = \sum_{v \in V(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{e \in E(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(e)) - \chi(\Gamma),$$
where $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$-1.
(3) *All the centralizers of non-trivial elements in W are cyclic.*
(4) *For each non-trivial edge subgroup of $\mathcal{G}(e)$ $(e \in E(\Gamma))$ there is a finite nilpotent quotient of W where the image of this edge subgroup is non-trivial.*

It is relatively easy to see that the conditions presented in Theorem 1.1 and Theorem 1.2 are necessary for W to be parafree and they also imply that the quotients of W by the terms of its lower central series are the same as those of a free group. The difficult part in the proofs of both theorems is to show that W is residually nilpotent. We establish this as an application of recent methods developed in [12] which were used for constructing abstract subgroups of free pro-p groups.

The paper is organized as follows. In Section 2 we describe the preliminary results which we use in the paper. Section 3 is devoted to construction of new examples of $\mathcal{D}_{\mathbb{F}_p G}$-torsion-free $\mathbb{F}_p G$-modules. In Section 4 we prove Theorem 1.1 and Theorem 1.2 and their corollaries.

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2. Preliminaries

2.1. **General notation for rings, groups and pro-p groups.** All of our rings R will be associative and unitary. All ring homomorphisms will map $1 \mapsto 1$. All R-modules are left R-modules, unless we say that we consider right R-modules.

Given a group G, we will denote by $G' = [G, G]$ its commutator subgroup and by $G_{\mathrm{ab}} = G/G'$ its abelianization. The terms of the lower central series of G are defined as $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ if $n \geq 1$ and the terms of the p-lower central series of G are defined as $G_{1,p} = G$ and $G_{n+1,p} = [G_{n,p}, G]G_{n,p}^p$ if $n \geq 1$.

We denote by $d(G)$ the minimal number of generators of a group G and we put $r_{\mathrm{ab}}(G) = d(G_{\mathrm{ab}})$.

Let F be a pro-p group. In this case we denote by $d(F)$ the minimal number of topological generators of F. The *completed group algebra of F over* $\mathbb{F}_p$ is the inverse limit of the $\mathbb{F}_p$-group algebras $\mathbb{F}_p[F/U]$

$$\mathbb{F}_p[[F]] = \varprojlim_{U \trianglelefteq_o F} \mathbb{F}_p[F/U],$$

where U ranges over open normal subgroup of F.

We will denote the free product in the category of pro-p groups by $\coprod$.

2.2. **Parafree groups.** If G is a group we denote by $G_{\widehat{p}}$ the pro-p completion of G. In this paper we will use the following characterization of parafree groups.

Proposition 2.1. *Let G be finitely generated residually nilpotent group. Then G is parafree if and only if $G_{\widehat{p}}$ is a free pro-p group for every prime p. Moreover, a parafree group is residually-p for every prime p.*

Proof. A group is said to be weakly p-parafree if its quotients by the terms of its p-lower central series are the same as those of a free group. In these terms, a finitely generated G will be parafree if and only if it is residually nilpotent and it is weakly p-parafree for every prime p. We know from [14, Corollary 2.9] that a finitely generated G is weakly p-parafree if and only if it has free pro-p completion. This proves the first claim. The second claim of the proposition follows from the fact that finitely generated torsion-free nilpotent groups are residually-p for every prime p [10, Theorem 2.1]. □

The following result provides a useful necessary condition for a group to be parafree.

Proposition 2.2. *The centralizers of non-trivial elements of a parafree group are cyclic.*

Proof. Let A be the centralizer of a non-trivial element u of a parafree group G. By [5, Theorem 4.2], we know that

(a) a two-generated subgroup of G is free and
(b) an abelian subgroup of G is cyclic.

Thus, by (a), A is locally cyclic, and, by (b), A is cyclic. □

2.3. **Group algebra and the augmentation ideal.** Let G be a group and k a commutative ring. The group ring of G over k is denoted by kG. We denote by I_G the augmentation ideal of $\mathbb{Z}G$ and by $kI_G = k \otimes_{\mathbb{Z}} I_G$ the augmentation ideal of kG.

Given two groups $H \leq G$, we denote by kI_H^G the left ideal of kG generated by kI_H. Since kG is a natural free right kH-module, it follows that the canonical map of kG-modules

$$kG \underset{kH}{\otimes} kI_H \longrightarrow kI_H^G$$

is an isomorphism.

2.4. **The fundamental groups of graphs of groups.** We refer the reader to [8] for standard definitions and notions related to graph of groups and their fundamental groups.

By a *graph of groups* $(\mathcal{G}, \Gamma)$ we mean a connected graph $\Gamma = (V(\Gamma), E(\Gamma), \iota, \tau)$ together with a function $\mathcal{G}$ which assigns to each $v \in V(\Gamma)$ a group $\mathcal{G}(v)$, and to each $e \in E(\Gamma)$ a distinguished subgroup $\mathcal{G}(e)$ of $\mathcal{G}(\iota(e))$ and injective group homomorphism $t_e : \mathcal{G}(e) \to \mathcal{G}(\tau(e))$. The monomorphisms t_e are called the *edge functions*.

In this paper we work mostly with two particular cases: amalgamated free products and HNN extensions. The structure of the augmentation ideal of the group algebra of the fundamental group of a graph of groups is described in [7, Lemma 6]. We will describe this result in the cases that interest us.

Proposition 2.3. *Let k be a commutative ring.*

(1) *Let U and V be two groups, A a subgroup of U and $\alpha : A \to V$ a monomorphism. Put $W = U *_A V$. Then there exists an exact of sequence of kW-modules.*
$$0 \to kI_A^W \xrightarrow{\gamma} kI_U^W \oplus kI_V^W \xrightarrow{p} kI_W \to 0,$$
where $\gamma(a) = (a, -a)$ if $a \in kI_A$ and $p(b, c) = b+c$ if $b \in kI_U^W$ and $c \in kI_V^W$.

(2) *Let U be a group, A a subgroup of U and $\alpha : A \to U$ a monomorphism. Put $W = U*_\alpha = \langle U, t : tg = \alpha(g)t \text{ if } g \in A\rangle$. Then there exists an exact of sequence of kW-modules.*
$$0 \to kI_{\alpha(A)}^W \xrightarrow{\gamma} kI_U^W \oplus kW \xrightarrow{p} kI_W \to 0,$$
where $\gamma(a) = (a(1-t), a)$ if $a \in kI_{\alpha(A)}^W$ and $p(b, c) = b + c(t-1)$ if $b \in kI_U^W$ and $c \in kW$.

The following proposition will be used several times in the paper.

Proposition 2.4. [2, Corollary 1.14 and Proposition 1.20] *Let $(\mathcal{H}, \Delta)$ be a subgraph of groups of a graph of groups $(\mathcal{G}, \Gamma)$, i.e.*

(1) *Δ is a connected subgraph of a finite connected graph Γ;*
(2) *$\mathcal{H}(v) \leq \mathcal{G}(v)$ if $v \in V(\Delta)$, $\mathcal{H}(e) \leq \mathcal{G}(e)$ if $e \in E(\Delta)$ and the edge functions of $(\mathcal{H}, \Delta)$ are the restrictions of the edge functions of $(\mathcal{G}, \Gamma)$.*

Assume that
$$\mathcal{H}(e) = \mathcal{G}(e) \cap \mathcal{H}(\iota(e)) \text{ and } t_e(\mathcal{H}(e)) = t_e(\mathcal{G}(e)) \cap \mathcal{H}(\tau(e)) \text{ if } e \in E(\Delta).$$
Let Δ_0 be a maximal subtree in Δ and $\Delta_0 \subseteq \Gamma_0$ a maximal subtree in Γ. Then the canonical map between the fundamental groups of graphs of groups
$$\pi(\mathcal{H}, \Delta, \Delta_0) \to \pi(\mathcal{G}, \Gamma, \Gamma_0)$$
is injective.

2.5. **The induced map between the augmentation ideals.** Given a group homomorphism $\widetilde{G} \to G$, we obtain an induced map between the augmentation ideals $I_{\widetilde{G}} \to I_G$. Here we will explain how one can derive information from this induced map back to the initial group homomorphism.

Proposition 2.5. *Let $\phi : \widetilde{G} \to G$ be a surjective group homomorphism of kernel K. Suppose that the natural homomorphism of kG-modules*

$$\alpha : kG \underset{k\widetilde{G}}{\otimes} kI_{\widetilde{G}} \longrightarrow kI_G,$$

defined by the k-linear extension of $a \otimes b \mapsto a\phi(b)$, is an isomorphism. Then

$$k \otimes_{\mathbb{Z}} K_{\mathrm{ab}} = 0.$$

Proof. Observe that by Shapiro's lemma

$$k \otimes_{\mathbb{Z}} K_{\mathrm{ab}} \cong H_1(K, k) \cong H_1(\widetilde{G}, kG) = \mathrm{Tor}_1^{k\widetilde{G}}(kG, k).$$

Applying the right-exact functor $kG \otimes_{k\widetilde{G}}$ to the exact sequence of left $k\widetilde{G}$-modules

$$0 \to kI_{\widetilde{G}} \to k\widetilde{G} \to k \to 0,$$

we get the exact sequence

$$0 \to \mathrm{Tor}_1^{k\widetilde{G}}(kG, k) \to kG \otimes_{k\widetilde{G}} kI_{\widetilde{G}} \xrightarrow{\alpha} kG \to k \to 0.$$

Since α is injective, $\mathrm{Tor}_1^{k\widetilde{G}}(kG, k) = \{0\}$ and we are done. □

2.6. **$\mathcal{D}$-torsion-free modules.** Let $R \hookrightarrow \mathcal{D}$ be an embedding of the ring R into a division ring $\mathcal{D}$. Let M be an R-module. We say that M is *$\mathcal{D}$-torsion-free* if the canonical map $M \longrightarrow \mathcal{D} \underset{R}{\otimes} M$ is injective.

The following provides a more flexible criterion for verifying whether a module is torsion-free.

Lemma 2.6. [12, Lemma 4.1] *Let M be a R-module. Then M is $\mathcal{D}$-torsion-free if and only if there exists a $\mathcal{D}$-module N and an injective homomorphism of R-modules $M \hookrightarrow N$.*

Given a homomorphism $R \to \mathcal{D}$, where $\mathcal{D}$ is a division ring, we define *$\mathcal{D}$-dimension of M* to be the dimension of the $\mathcal{D}$-vector space $\mathcal{D} \otimes_R M$. We denote it by $\dim_{\mathcal{D}} M$.

Lemma 2.7. [12, Lemma 4.2] *Let $1 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. Assume that*

(1) *M_1 and M_3 are $\mathcal{D}$-torsion-free;*
(2) *$\dim_{\mathcal{D}} M_1$ and $\dim_{\mathcal{D}} M_3$ are finite; and*
(3) *$\dim_{\mathcal{D}} M_1 + \dim_{\mathcal{D}} M_3 = \dim_{\mathcal{D}} M_2$.*

Then M_2 is also $\mathcal{D}$-torsion-free.

The following lemma will be used in the computation of $\dim_{\mathcal{D}} M$.

Lemma 2.8. [12, Lemma 4.3] *Let M be a $\mathcal{D}$-torsion-free R-module of finite $\mathcal{D}$-dimension. Let L be a non-trivial R-submodule of M. Then $\dim_{\mathcal{D}}(M/L) < \dim_{\mathcal{D}}(M)$. Moreover, if $\dim_{\mathcal{D}} L = 1$, then $\dim_{\mathcal{D}}(M/L) = \dim_{\mathcal{D}} M - 1$.*

2.7. **The universal division rings of fraction of the group algebra of a subgroup of a free pro-p group.** Let $\mathbf{F}$ be a finitely generated free pro-p group. If G is an abstract subgroup of $\mathbf{F}$, then it turns out that $\mathbb{F}_pG$ has a universal division ring of fractions, denoted by $\mathcal{D}_{\mathbb{F}_pG}$. We will not give a formal definition of universal field of fractions which can be found in [13], but we will describe the main properties of the embeddings $\mathbb{F}_pG \subseteq \mathcal{D}_{\mathbb{F}_pG}$, which we will use in this paper.

If H is a subgroup of G, then the division closure of $\mathbb{F}_pH$ in $\mathcal{D}_{\mathbb{F}_pG}$ is the universal division ring of fractions of $\mathbb{F}_pH$, and, therefore, we will denote it by $\mathcal{D}_{\mathbb{F}_pH}$.

If N is a normal subgroup of G such that $G/N \cong \mathbb{Z}$, then $\mathbb{F}_pG$ is a free $\mathbb{F}_pN$-module with the basis $\{t^i : i \in \mathbb{Z}\}$, where $t \in G$ and Nt generates G/N. Thus $\mathbb{F}_pG$ is isomorphic to a skew-polynomial ring $\mathbb{F}_pN[t^{\pm 1}, \sigma]$ with the indeterminate t and coefficients in $\mathbb{F}_pN$. Here σ is an automorphism of the ring $\mathbb{F}_pN$, induced from the automorphism of conjugation-by-t.

Let R be a subring of $\mathcal{D}_{\mathbb{F}_pG}$ generated by $\mathcal{D}_{\mathbb{F}_pN}$ and t. It turns out that $\{t^i : i \in \mathbb{Z}\}$ is also a $\mathcal{D}_{\mathbb{F}_pN}$-basis of R (this is so called the *Hughes-free property* of the embedding $\mathbb{F}_pG \subseteq \mathcal{D}_{\mathbb{F}_pG}$, see [11, 9]). thus R is isomorphic to $\mathcal{D}_{\mathbb{F}_pN}[t^{\pm 1}, \sigma]$.

The following result gives a lower bound for the $\mathcal{D}_{\mathbb{F}_pG}$-dimension of $\mathbb{F}_pI_G$.

Proposition 2.9. [12, Corollary 3.7 and the discussion afterwards] *Let G be a finitely generated dense subgroup of a free pro-p group $\mathbf{F}$. Then $\dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_pI_G \geq d(\mathbf{F})$. Moreover, if G is parafree and $\mathbf{F} = G_{\widehat{p}}$, then $\dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_pI_G = d(\mathbf{F})$.*

The following result provides an important example of $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free-module.

Proposition 2.10. [12, Proposition 4.8] *Let $\mathbf{F}$ be a free pro-p group, $H \leq G \leq \mathbf{F}$ two subgroups of $\mathbf{F}$, and A a maximal abelian subgroup of H. Then the $\mathbb{F}_pG$-module $\mathbb{F}_pI_H^G \,/\, \mathbb{F}_pI_A^G$ is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free.*

2.8. **Embeddings of abstract groups into free pro-p groups.** In this subsection we detail a method introduced in [12] for ensuring that a map from an abstract group $\widetilde{G}$ to a free pro-p group is injective. We are particularly interested in the problem of producing families of parafree groups. Let $\widetilde{G}$ be a candidate to being parafree, meaning that $\widetilde{G}_{\widehat{p}}$ is free for every prime p. We want to study whether the canonical map $\widetilde{G} \longrightarrow \widetilde{G}_{\widehat{p}}$ is an embedding for some suitable prime p. This would establish the residual nilpotence of $\widetilde{G}$ and we could conclude that $\widetilde{G}$ is parafree.

Proposition 2.11. *Let $\widetilde{G}$ be a finitely generated group, $\mathbf{F}$ a finitely generated free pro-p group and $\phi : \widetilde{G} \to \mathbf{F}$ a group homomorphism. Suppose that we have the following conditions.*

1. *The image $G = \phi(\widetilde{G})$ is dense in $\mathbf{F}$.*
2. *The $\mathbb{F}_pG$-module $\mathbb{F}_pG \underset{\mathbb{F}_p\widetilde{G}}{\otimes} \mathbb{F}_pI_{\widetilde{G}}$ is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free and*
$$\dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_pG \underset{\mathbb{F}_p\widetilde{G}}{\otimes} \mathbb{F}_pI_{\widetilde{G}} = d(\mathbf{F}).$$
3. *The kernel of ϕ is free.*

Then the map ϕ is an embedding.

Proof. We will prove that the surjective map $\phi : \widetilde{G} \to G$ verifies the assumption of Proposition 2.5 to deduce that $\ker \phi$ has trivial p-abelianization. Since $\ker \phi$ is free, the latter would imply that $\ker \phi = 1$ and the conclusion would follow.

It is clear that the natural homomorphism of $\mathbb{F}_pG$-modules

$$\mathbb{F}_pG \underset{\mathbb{F}_p\widetilde{G}}{\otimes} I_{\widetilde{G}} \to \mathbb{F}_p I_G,$$

defined by $a \otimes b \mapsto a\phi(b)$, is surjective. If it was not injective, naming its kernel by L and naming $M = \mathbb{F}_pG \underset{\mathbb{F}_p\widetilde{G}}{\otimes} I_{\widetilde{G}}$, we would deduce, after applying Lemma 2.8,

$$d(\mathbf{F}) = \dim_{\mathcal{D}_{\mathbb{F}_pG}} M > \dim_{\mathcal{D}_{\mathbb{F}_pG}}(M/L) = \dim_{\mathcal{D}_{\mathbb{F}_pG}}(\mathbb{F}_p I_G),$$

which would contradict Proposition 2.9. □

In the setting of our problem, that is, taking some $\widetilde{G}$ with free pro-p completion and studying whether $\widetilde{G} \longrightarrow \widetilde{G}_{\widehat{p}}$ is injective, we make a few comments about the three conditions of Proposition 2.11. We take G to be the image of $\widetilde{G}$ inside $\widetilde{G}_{\widehat{p}}$.

(1) The first condition will be naturally ensured.
(2) The second condition is the hardest part and requires the most technical arguments presented in Section 3.
(3) The third condition is natural from the point of view of the Bass-Serre theory. If we take $\widetilde{G}$ to be the fundamental group of a graph of groups, and we ensure that $\ker\phi$ intersects trivially every vertex subgroup, then the $\ker\phi$ will necessarily be free.

3. Examples of $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free modules

In this section we construct two families of examples of $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free modules. The first result is a slight generalization of [12, Proposition 4.10].

Proposition 3.1. *Let H_1 and H_2 be two finitely generated subgroups of a finitely generated free pro-p group $\mathbf{F}$. Consider $A = H_1 \cap H_2$ and suppose that A is a maximal abelian subgroup of H_1. Let $G = \langle H_1, H_2\rangle$ and let*

$$J = \{(x, -x) : x \in \mathbb{F}_p I_A^G\} \leq \mathbb{F}_p I_{H_1}^G \oplus \mathbb{F}_p I_{H_2}^G.$$

Then the $\mathbb{F}_pG$-module

$$M = \frac{\mathbb{F}_p I_{H_1}^G \oplus \mathbb{F}_p I_{H_2}^G}{J}$$

is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free and

$$\dim_{\mathcal{D}_{\mathbb{F}_pG}} M = \dim_{\mathcal{D}_{\mathbb{F}_pH_1}} \mathbb{F}_p I_{H_1} + \dim_{\mathcal{D}_{\mathbb{F}_pH_2}} \mathbb{F}_p I_{H_2} - 1.$$

Before giving the proof, we shall make an observation. If A is an abelian subgroup of $\mathbf{F}$, as occurs in the previous proposition, the universal division $\mathbb{F}_pA$-ring of fractions $\mathcal{D}_{\mathbb{F}_pA}$ is the field of fractions $Q_{ore}(\mathbb{F}_pA)$ of the commutative domain $\mathbb{F}_pA$. Given a nontrivial ideal I of $\mathbb{F}_pA$, we see that the $Q_{\mathrm{ore}}(\mathbb{F}_pA)$-vector space $Q_{\mathrm{ore}}(\mathbb{F}_pA) \underset{\mathbb{F}_pA}{\otimes} I$ is one-dimensional. This implies that

$$\dim_{\mathcal{D}_{\mathbb{F}_pA}} I = \dim_{Q_{\mathrm{ore}}(\mathbb{F}_pA)} Q_{\mathrm{ore}}(\mathbb{F}_pA) \underset{\mathbb{F}_pA}{\otimes} I = 1. \quad (1)$$

Proof of Proposition 3.1. We consider the $\mathbb{F}_pG$-submodule of M defined by

$$L = (\mathbb{F}_p I_A^G \oplus \mathbb{F}_p I_{H_2}^G)/J.$$

We want to apply Lemma 2.7 to the short exact sequence

$$0 \to L \to M \to M/L \to 0$$

of $\mathbb{F}_pG$-modules. Since $L \cong \mathbb{F}_p I_{H_2}^G \leq \mathbb{F}_pG$, then it is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free and

$$\dim_{\mathcal{D}_{\mathbb{F}_pG}} L = \dim_{\mathcal{D}_{\mathbb{F}_pH_2}} \mathbb{F}_p I_{H_2}.$$

By the observation (1),

$$1 = \dim_{\mathcal{D}_{\mathbb{F}_pA}} \mathbb{F}_p I_A = \dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_p I_A^G = \dim_{\mathcal{D}_{\mathbb{F}_pG}} J.$$

So it is clear, by Lemma 2.8, that

$$\dim_{\mathcal{D}_{\mathbb{F}_pG}} M = \dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_p I_{H_1}^G + \dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_p I_{H_2}^G - 1 = \dim_{\mathcal{D}_{\mathbb{F}_pH_1}} \mathbb{F}_p I_{H_1} + \dim_{\mathcal{D}_{\mathbb{F}_pH_2}} \mathbb{F}_p I_{H_2} - 1.$$

On the other side, the quotient M/L is isomorphic to $\mathbb{F}_p I_{H_1}^G / \mathbb{F}_p I_A^G$, which is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free by Proposition 2.10. Again, by Lemma 2.8,

$$\dim_{\mathcal{D}_{\mathbb{F}_pG}} M/L = \dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_p I_{H_1}^G - \dim_{\mathcal{D}_{\mathbb{F}_pG}} \mathbb{F}_p I_A^G = \dim_{\mathcal{D}_{\mathbb{F}_pH_1}} \mathbb{F}_p I_{H_1} - 1.$$

Therefore, the short exact sequence $0 \to L \to M \to M/L \to 0$ of $\mathbb{F}_pG$-modules satisfies the requirements of Lemma 2.7; and the conclusion follows. □

We now turn to the study of torsion-free modules that have the form of an augmentation ideal of a cyclic HNN extension.

Our point is to prove that a certain module, which may have the form

$$R^m / R(u_1, \dots, u_m)$$

for some group ring R, is $\mathcal{D}_R$-torsion-free. When extending this R-module with coefficients in a bigger ring, some u_i may become invertible. The following elementary lemma simply studies this scenario, which we shall encounter many times.

Lemma 3.2. *Let R be a unital ring and M be a R-module. Let $m_0 \in M$ and let u be a unit of R. Then there is an isomorphism of R-modules*

$$\gamma : \frac{M \oplus R}{(m_0, u)} \longrightarrow M$$

given by

$$\gamma(m, r) = m - ru^{-1}m_0,$$

with inverse

$$\gamma^{-1}(m) = (m, 0).$$

We can now state the second result of this section.

Proposition 3.3. *Let $H \leq G$ be subgroups of $\mathbf{F}$. Suppose that we can write $G = N \rtimes (t) = \langle N, t \rangle$ for some $H \leq N \leq G$ and some $t \in G$. Let $u \in H$ be an element which generates a maximal abelian subgroup $\langle u \rangle$ in H and suppose that $v = tut^{-1} \in H$. Then the $\mathbb{F}_pG$-module M defined by*

$$M = \frac{\mathbb{F}_p I_H^G \oplus \mathbb{F}_pG}{\mathbb{F}_pG\,(v - 1 - t(u-1), v - 1)}$$

is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free.

Proof. The proof is relatively long and, for convenience, it is divided in several intermediate claims.

Let R be the subring of $\mathcal{D}_{\mathbb{F}_pG}$ generated by $\mathcal{D}_{\mathbb{F}_pN}$ and t. The structure of this ring is explained in Subsection 2.7.

Claim 3.4. *The natural map*

$$R \underset{\mathbb{F}_pG}{\otimes} M \to \mathcal{D}_{\mathbb{F}_pG} \underset{\mathbb{F}_pG}{\otimes} M$$

is injective.

Proof. Applying Lemma 3.2, we obtain that

$$R \underset{\mathbb{F}_pG}{\otimes} M \cong R \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_p I_H^G \text{ and } \mathcal{D}_{\mathbb{F}_pG} \underset{\mathbb{F}_pG}{\otimes} M \cong \mathcal{D}_{\mathbb{F}_pG} \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_p I_H^G.$$

Observe that

$$R \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_p I_H^G \cong R \underset{\mathbb{F}_pH}{\otimes} \mathbb{F}_p I_H \text{ and } \mathcal{D}_{\mathbb{F}_pG} \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_p I_H^G \cong \mathcal{D}_{\mathbb{F}_pG} \underset{\mathbb{F}_pH}{\otimes} \mathbb{F}_p I_H.$$

Since R is a direct $\mathcal{D}_{\mathbb{F}_pH}$-summand of $\mathcal{D}_{\mathbb{F}_pG}$ we obtain that the map

$$R \underset{\mathbb{F}_pH}{\otimes} \mathbb{F}_p I_H \to \mathcal{D}_{\mathbb{F}_pG} \underset{\mathbb{F}_pH}{\otimes} \mathbb{F}_p I_H$$

is injective. This proves the claim. □

Claim 3.5. *We have the following equality of subsets of $\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p I_H^N$,*

$$1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p I_H^N \bigcap \mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} (u-1) = 1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p N(u-1). \tag{2}$$

Proof. By assumption, $\langle u \rangle$ is a maximal abelian subgroup of H. By Proposition 2.10, the $\mathbb{F}_pN$-module

$$M_0 = \frac{\mathbb{F}_p I_H^N}{\mathbb{F}_p N(u-1)}$$

is $\mathcal{D}_{\mathbb{F}_pN}$-torsion-free. This implies that the canonical map

$$M_0 \longrightarrow \mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} M_0$$

is injective. This gives the claim. □

We denote $H^{t^n} = t^n H t^{-n}$, which are also subgroups of N having $\langle u^{t^n} \rangle$ as maximal abelian subgroup.

Claim 3.6. *For all $n \in \mathbb{Z}$, we have the following equality of subsets of $\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p I_G$,*

$$1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p t^n I_H^N \bigcap \mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} t^n(u-1) = 1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p t^n N(u-1). \tag{3}$$

Proof. The same way we had the equality (2), we can derive, for the same reasons,

$$1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p I_{H^{t^{-n}}}^N \bigcap \mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} (u^{t^{-n}}-1) = 1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p N(u^{t^{-n}}-1). \tag{4}$$

Notice that $\mathbb{F}_p I_G$ has a right $\langle t \rangle$-module structure by multiplication. This induces a right $\langle t \rangle$-module structure on $\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p I_G$. Since $N \trianglelefteq G$, then t normalises N. We have the equations of subsets of $\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p I_G$:

$$\left(1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_p N(u-1)\right) t^m = 1 \otimes \mathbb{F}_p\, t^m N(u^{t^{-m}}-1),$$

and

$$\left(1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p I_H^N\right) t^m = 1 \otimes \mathbb{F}_p\, t^m I_{H^{t^{-m}}}.$$

As a consequence, applying the multiplication-by-t^n automorphism of $\mathcal{D}_{\mathbb{F}_p N} \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p I_G$ to the equation (4), we get (3). □

Notice the following decomposition of $\mathbb{F}_p N$-modules

$$\mathbb{F}_p I_H^G = \bigoplus_{n\in\mathbb{Z}} \mathbb{F}_p\, t^n I_H^N,$$

which yields to the following decomposition of $\mathbb{F}_p$-vector spaces

$$\mathcal{D}_{\mathbb{F}_p N} \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p I_H^G \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_{\mathbb{F}_p N} \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p\, t^n I_H^N.$$

Claim 3.7. *We have the following equation of subsets of* $\mathcal{D}_{\mathbb{F}_p N} \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p I_H^G$,

$$1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p I_H^G \bigcap \mathcal{D}_{\mathbb{F}_p N} \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p G(v-1-t(u-1)) = 1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p G(v-1-t(u-1)). \quad (5)$$

Proof. Let us take an element w that belongs to the left-hand side. This element will have the form

$$w = \sum_{k=n_1}^{n_2} c_k \otimes t^k(v-1-t(u-1)), \text{ for some } c_k \in \mathcal{D}_{\mathbb{F}_p N},$$

and will also belong to $1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p I_H^G$. We rewrite

$$w = c_{n_1} \otimes t^{n_1}(v-1) + \sum_{k=n_1+1}^{n_2} \Big(c_k \otimes t^k(v-1) - c_{k-1} \otimes t^k(u-1)\Big) + c_{n_2} \otimes t^{n_2+1}(u-1).$$

Since $w \in 1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p I_H^G$, we can look at the highest power t^{n_2+1} to deduce that

$$c_{n_2} \otimes t^{n_2+1}(u-1) \in 1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p t^{n_2+1} I_H^N \bigcap \mathcal{D}_{\mathbb{F}_p N} \underset{\mathbb{F}_p N}{\otimes} t^{n_2+1}(u-1).$$

By (3), this implies that

$$c_{n_2} \otimes t^{n_2+1}(u-1) \in 1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p t^{n_2+1} N(u-1),$$

so $c_{n_2} \in \mathbb{F}_p N$. Let $n_1+1 \le k \le n_2$. Inspecting again the expression of w at the component with power t^k, we have that

$$c_k \otimes t^k(v-1) - c_{k-1} \otimes t^k(u-1) \in 1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p t^k I_H^N.$$

If we knew that $c_k \in \mathbb{F}_p N$, then it would follow that

$$c_{k-1} \otimes t^k(u-1) \in 1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p t^k I_H^N \bigcap \mathcal{D}_{\mathbb{F}_p N} \underset{\mathbb{F}_p N}{\otimes} t^k(u-1).$$

By (3), this means that

$$c_{k-1} \otimes t^k(u-1) \in 1 \underset{\mathbb{F}_p N}{\otimes} \mathbb{F}_p t^n N(u-1),$$

and this implies that $c_{k-1} \in \mathbb{F}_p N$.

We have proven that if $c_k \in \mathbb{F}_pN$, for $n_1 < k \leq n_2$, then $c_{k-1} \in \mathbb{F}_pN$. Since we also know that $c_{n_2} \in \mathbb{F}_pN$, an inductive argument gives that $c_k \in \mathbb{F}_pN$ for every k, meaning that

$$w \in 1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pN[t^{\pm}, \sigma]\,(v-1-t(u-1)) = 1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pG(v-1-t(u-1)).$$

This proves that

$$1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pI_H^G \bigcap \mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pG(v-1-t(u-1)) \subseteq 1 \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pG(v-1-t(u-1)),$$

one inclusion of (5). The reverse inclusion is trivial, so equation (5) is proven. ☐

Claim 3.8. *The natural map*

$$\alpha : M \to R \underset{\mathbb{F}_pG}{\otimes} M$$

is injective.

Proof. Observe that there is a canonical isomorphism of $\mathbb{F}_pN$-modules

$$R \cong \mathcal{D}_{\mathbb{F}_pN}[t^{\pm 1}, \sigma] \cong \mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pN[t^{\pm}, \sigma] = \mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pG.$$

This extends to a canonical isomorphism of $\mathbb{F}_pN$-modules

$$\psi : \frac{R \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_pI_H^G}{R \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_pG(v-1-t(u-1))} \longrightarrow \frac{\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pI_H^G}{\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pG(v-1-t(u-1))}.$$

Therefore , there is a commutative triangle of canonical $\mathbb{F}_pN$-homomorphisms

$$\begin{array}{ccc} \frac{\mathbb{F}_pI_H^G}{\mathbb{F}_pG(v-1-t(u-1))} & & \\ \downarrow \alpha' & \overset{\eta}{\searrow} & \\ \frac{R \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_pI_H^G}{R \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_pG(v-1-t(u-1))} & \xrightarrow{\psi} & \frac{\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pI_H^G}{\mathcal{D}_{\mathbb{F}_pN} \underset{\mathbb{F}_pN}{\otimes} \mathbb{F}_pG(v-1-t(u-1))}, \end{array}$$

The canonical map η is injective due to (5). Since ψ is an isomorphism, this implies that α' is injective.

Let $(x, y) \in \mathbb{F}_pI_H^G \oplus \mathbb{F}_pG$ be such that $(x, y) + \mathbb{F}_pG\,(v-1-t(u-1), v-1)$ belongs to the kernel of α. Then $1 \otimes (x, y) = c \otimes (v-1-t(u-1), v-1)$ for some $c \in \mathcal{D}_{\mathbb{F}_pG}$. Notice that

$$1 \otimes x = c \otimes (v-1-t(u-1)) \in 1 \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_pI_H^G \bigcap R \underset{\mathbb{F}_pG}{\otimes} (v-1-t(u-1)).$$

From the injectivity of α', this implies that

$$c \otimes (v-1-t(u-1)) \in 1 \underset{\mathbb{F}_pG}{\otimes} \mathbb{F}_pG(v-1-t(u-1)),$$

so $c \in \mathbb{F}_pG$ and then $(x, y) \in \mathbb{F}_pG(v-1-t(u-1), v-1)$. Thus α is injective and the claim is demonstrated.

☐

From Claims 3.8 and 3.4, we conclude that M is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free.

☐

4. Proofs of the main results

4.1. **Amalgamated products.** Now we are ready to prove Theorem 1.1.

Theorem 4.1. *Let $\widetilde{H_1}$ and $\widetilde{H_2}$ be finitely generated groups. Let $1 \neq u_1 \in \widetilde{H_1}$ and let $1 \neq u_2 \in \widetilde{H_2}$. Consider the following amalgamated product of cyclic amalgam*

$$\widetilde{G} = \widetilde{H_1} \underset{u_1 = u_2}{*} \widetilde{H_2} \cong \frac{\widetilde{H_1} * \widetilde{H_2}}{\langle\langle u_1 u_2^{-1} \rangle\rangle}.$$

Then $\widetilde{G}$ is parafree if and only if the three following conditions hold.

(1) *$\widetilde{H_1}$ and $\widetilde{H_2}$ are parafree.*
(2) *The element $u_1 u_2^{-1}$ of $\widetilde{H_1} * \widetilde{H_2}$ is not a proper power in the abelianization of $\widetilde{H_1} * \widetilde{H_2}$.*
(3) *There is at least one $i \in \{1, 2\}$ such that u_i is not a proper power in $\widetilde{H_i}$.*

Remark. *The condition (2) can be substituted by the condition*

(2') $r_{\text{ab}}(\widetilde{G}) = r_{\text{ab}}(\widetilde{H_1}) + r_{\text{ab}}(\widetilde{H_2}) - 1.$

The proof of the theorem shows that the condition (3) can be substituted by the condition

(3') *All the centralizers of non-trivial elements in W are cyclic.*

Proof. We first prove that the conditions are necessary. Let us assume that $\widetilde{G}$ is parafree.

Both $\widetilde{H_1}$ and $\widetilde{H_2}$ are subgroups of $\widetilde{G}$ and hence they are residually nilpotent. We want to show that, for all primes p, $\widetilde{H_1}_{\widehat{p}}$ and $\widetilde{H_2}_{\widehat{p}}$ are free. By Proposition 2.1, this would imply that $\widetilde{H_1}$ and $\widetilde{H_2}$ are parafree.

Fix a prime p. Observe that

$$d(\widetilde{G}_{\widehat{p}}) \geq d(\widetilde{H_1}_{\widehat{p}}) + d(\widetilde{H_2}_{\widehat{p}}) - 1. \tag{6}$$

Consider the closure $\overline{H_i}$ of the image of $\widetilde{H_i}$ under the canonical map $\widetilde{G} \to \widetilde{G}_{\widehat{p}}$. Since $\widetilde{G}$ is parafree, $\widetilde{G}_{\widehat{p}}$ is free pro-p, and so both $\overline{H_1}$ and $\overline{H_2}$ are free pro-p, because any closed subgroup of a free pro-p group is again free. Note that $\widetilde{H_1}$ and $\widetilde{H_2}$ generate $\widetilde{G}$. Hence the canonical map $f : \overline{H_1} \coprod \overline{H_2} \to \widetilde{G}_{\widehat{p}}$ is onto.

Since $\widetilde{G}$ is parafree, the images of u_1 and u_2 in $\widetilde{G}_{\widehat{p}}$ are non-trivial. Therefore, $\ker f \neq \{1\}$. We also recall that the groups $\overline{H_1} \coprod \overline{H_2}$ and $\widetilde{G}_{\widehat{p}}$ are free pro-p, so

$$d(\widetilde{G}_{\widehat{p}}) \leq d(\overline{H_1}) + d(\overline{H_2}) - 1. \tag{7}$$

Since $\overline{H_i}$ is a quotient of $\widetilde{H_i}_{\widehat{p}}$, then $d(\overline{H_i}) \leq d(\widetilde{H_i}_{\widehat{p}})$. The later observation in combination with (6) and (7) yields $d(\overline{H_i}) = d(\widetilde{H_i}_{\widehat{p}})$. The pro-$p$ group $\overline{H_i}$ is free, so the canonical map $\widetilde{H_i}_{\widehat{p}} \to \overline{H_i}$ is an isomorphism. Hence $\widetilde{H_i}_{\widehat{p}}$ is free pro-p and the first condition is proved.

The previous argument also shows that, for all primes p,

$$d(\widetilde{G}_{\widehat{p}}) = d(\widetilde{H_1}_{\widehat{p}}) + d(\widetilde{H_2}_{\widehat{p}}) - 1,$$

and this implies the second condition.

Finally, suppose that $u_1 = v_1^{n_1}$ in $\widetilde{H_1}$, and that $u_2 = v_2^{n_2}$ in $\widetilde{H_2}$ with $n_1, n_2 \geq 2$. Since $u_i \neq 1$ and $\widetilde{H_i}$ is torsion-free, then $\langle v_i \rangle$ are infinite cyclic. Consider the subgroup A generated by v_1 and v_2. By Proposition 2.4, A is isomorphic to $\langle v_1\rangle *_{v_1^{n_1}=v_2^{n_2}} \langle v_2\rangle$. This group is non-abelian and belongs to the centralizer of $u_1 = u_2$ in $\widetilde{G}$. This contradicts Proposition 2.2. This shows that the third condition holds.

We now verify that the three given conditions are sufficient. There is a canonical isomorphism

$$\widetilde{G}_{\mathrm{ab}} \cong \frac{\widetilde{H_1}_{\mathrm{ab}} \oplus \widetilde{H_2}_{\mathrm{ab}}}{u_1 - u_2}.$$

From the fact that $u_1 - u_2$ is not a proper power in $\widetilde{G}_{\mathrm{ab}}$, we see that $\widetilde{G}_{\mathrm{ab}}$ is torsion-free of rank $r_{\mathrm{ab}}(\widetilde{H_1}) + r_{\mathrm{ab}}(\widetilde{H_2}) - 1$. Therefore for any prime p,

$$d(\widetilde{G}_{\widehat{p}}) = d(\widetilde{H_1}_{\widehat{p}}) + d(\widetilde{H_2}_{\widehat{p}}) - 1.$$

Let us fix an arbitrary prime p from this point on. Consider the canonical map $\phi : \widetilde{G} \longrightarrow \widetilde{G}_{\widehat{p}}$.

Claim 4.2. *The pro-p group $\widetilde{G}_{\widehat{p}}$ is free.*

Proof. Since $\widetilde{H_1}$ and $\widetilde{H_2}$ are parafree, the pro-p groups $\widetilde{H_1}_{\widehat{p}}$ and $\widetilde{H_2}_{\widehat{p}}$ are free. thus $\widetilde{G}_{\widehat{p}}$ is a quotient of the free pro-p group $\widetilde{H_1}_{\widehat{p}} \coprod \widetilde{H_2}_{\widehat{p}}$ by the closed normal subgroup generated by $u_1 u_2^{-1}$. Observe that $u_1 u_2^{-1}$ is primitive in $\widetilde{H_1}_{\widehat{p}} \coprod \widetilde{H_2}_{\widehat{p}}$. Hence, $\widetilde{G}_{\widehat{p}}$ is free pro-p. □

Claim 4.3. *The restrictions of ϕ to each $\widetilde{H_i}$ are injective.*

Proof. To check this claim, we consider each restriction

$$\phi_i = \phi|_{\widetilde{H_i}} : \widetilde{H_i} \longrightarrow H_i,$$

where $H_i = \phi(\widetilde{H_i})$. The subgroups $\overline{H_1}$ and $\overline{H_2}$ of the free pro-p group $\widetilde{G}_{\widehat{p}}$ are closed. Hence they both are free pro-p groups.

Since the induced $\phi_{i\widehat{p}} : \widetilde{H}_{i\widehat{p}} \longrightarrow \overline{H_i}$ are surjective maps of free pro-p groups,

$$d(\widetilde{H}_{i\widehat{p}}) \geq d(\overline{H_i}), \quad \text{for all } i \in \{1, 2\}. \tag{8}$$

Furthermore, by the universal property of the coproduct, there is a continuous homomorphism

$$f : \overline{H_1} \coprod \overline{H_2} \longrightarrow \widetilde{G}_{\widehat{p}}$$

which sends $\overline{H_i}$ to each corresponding copy in $\widetilde{G}_{\widehat{p}}$. Notice that $\operatorname{Im} f$ contains both H_1 and H_2, so $G \leq \operatorname{Im} f$, implying that f is surjective.

We will show that $\phi_{i\widehat{p}}$ $(i = 1, 2)$ are isomorphisms. This will imply that ϕ_i are injective, because, since $\widetilde{H_i}$ are parafree, H_i is a subgroup of $\widetilde{H}_{i\widehat{p}}$.

Since

$$d(\overline{H_1}) + d(\overline{H_2}) \geq d(\widetilde{G}_{\widehat{p}}) = d(\widetilde{H_1}_{\widehat{p}}) + d(\widetilde{H_2}_{\widehat{p}}) - 1,$$

$d(\overline{H_1}) = d(\widetilde{H_1}_{\widehat{p}})$ or $d(\overline{H_2}) = d(\widetilde{H_2}_{\widehat{p}})$. Without loss of generality, we assume that $d(\overline{H_1}) = d(\widetilde{H_1}_{\widehat{p}})$. The fact that $\overline{H_1}$ and $\widetilde{H_1}_{\widehat{p}}$ are free pro-p implies that $\phi_{1\widehat{p}}$ is an isomorphism. Furthermore, since $\phi(u_1) \neq 1$ in $\overline{H_j}$ and $f(\phi(u_1)\phi(u_2)^{-1}) = 1$ in $\widetilde{G}_{\widehat{p}}$; then f has non-trivial kernel. Recall, in addition, that $\widetilde{G}_{\widehat{p}}$ is free pro-p, so

$$d(\overline{H_1}) + d(\overline{H_2}) - 1 \geq d(\widetilde{G}_{\widehat{p}}) = d(\widetilde{H_1}_{\widehat{p}}) + d(\widetilde{H_2}_{\widehat{p}}) - 1,$$

and hence $d(\overline{H_2}) = d(\widetilde{H_2}_{\hat{p}})$ and $\phi_{2\hat{p}}$ is also an isomorphism. □

Claim 4.4. *The map $\phi : \widetilde{G} \longrightarrow \widetilde{G}_{\hat{p}}$ is injective.*

Proof. We denote $G = \phi(\widetilde{G})$. We want to apply Proposition 2.11 to the map $\phi : \widetilde{G} \longrightarrow G \subseteq \widetilde{G}_{\hat{p}}$.

Notice first that, by Claim 4.3, the kernel of ϕ intersects trivially the subgroups $\widetilde{H_1}$ and $\widetilde{H_2}$. Therefore, by the Bass-Serre theory, the kernel acts freely on a tree, and so it is free.

We consider the corresponding $\mathbb{F}_pG$-module

$$M = \mathbb{F}_pG \underset{\mathbb{F}_p\widetilde{G}}{\otimes} \mathbb{F}_pI_{\widetilde{G}}.$$

Consider $A = \langle \phi(u_1) \rangle$ and

$$J = \{(x, -x) : x \in \mathbb{F}_pI_A^G\} \leq \mathbb{F}_pI_{H_1}^G \oplus \mathbb{F}_pI_{H_2}^G.$$

Then, by Proposition 2.3(1)

$$M \cong \frac{\mathbb{F}_pI_{H_1}^G \oplus \mathbb{F}_pI_{H_2}^G}{J}.$$

Without loss of generality, we suppose that u_1 is not a proper power in $\widetilde{H_1}$. By Claim 4.3, $\widetilde{H_i} \cong H_i$, so H_i $(i = 1, 2)$ are parafree and $\phi(u_1)$ is not a proper power in H_1. Hence $A = \langle u_1 \rangle$ is a maximal abelian subgroup of H_1.

By Proposition 3.1, M is $\mathcal{D}_{\mathbb{F}_pG}$-torsion-free with dimension

$$\dim_{\mathcal{D}_{\mathbb{F}_pG}} M = \dim_{\mathcal{D}_{\mathbb{F}_pH_1}} \mathbb{F}_pI_{H_1} + \dim_{\mathcal{D}_{\mathbb{F}_pH_2}} \mathbb{F}_pI_{H_2} - 1.$$

In addition, by Proposition 2.9,

$$\dim_{\mathcal{D}_{\mathbb{F}_pH_1}} \mathbb{F}_pI_{H_1} + \dim_{\mathcal{D}_{\mathbb{F}_pH_2}} \mathbb{F}_pI_{H_2} - 1 = d(H_{1\hat{p}}) + d(H_{2\hat{p}}) - 1 = d(\widetilde{G}_{\hat{p}}).$$

It follows that $\dim_{\mathcal{D}_{\mathbb{F}_pG}} M = d(\widetilde{G}_{\hat{p}})$. We can apply Proposition 2.11 to conclude that $\phi : \widetilde{G} \longrightarrow G$ is injective. □

The last claim implies that $\widetilde{G}$ is residually nilpotent. Moreover, we already know that each $\widetilde{G}_{\hat{p}}$ is free, so $\widetilde{G}$ is parafree by Proposition 2.1. □

4.2. **HNN extensions.** Now we prove Theorem 1.2.

Theorem 4.5. *Let $\widetilde{H}$ be a finitely generated group. Let $u, v \in \widetilde{H} \setminus \{1\}$. Consider the following cyclic HNN extension of $\widetilde{H}$*

$$\widetilde{G} = \frac{\widetilde{H} * \langle t \rangle}{\langle\langle tut^{-1}v^{-1} \rangle\rangle}.$$

Then $\widetilde{G}$ is parafree if and only if the four following conditions hold.

1. *The group $\widetilde{H}$ is parafree.*
2. *The element uv^{-1} is not a proper power in $\widetilde{H}_{\mathrm{ab}}$.*
3. *At least one of u or v is not a proper power in $\widetilde{H}$.*
4. *The image of the element u is non-trivial in some finite nilpotent quotient of $\widetilde{G}$.*

Remark. *The condition (2) can be substituted by the condition*

(2') $r_{\text{ab}}(\widetilde{G}) = r_{\text{ab}}(\widetilde{H})$.

The proof of the theorem shows that the condition (3) can be substituted by the condition

(3') *All the centralizers of non-trivial elements in W are cyclic.*

Proof. First let us show that he given conditions are necessary. Assume that $\widetilde{G}$ is parafree.

The group $\widetilde{H}$ is a subgroup of $\widetilde{G}$ and hence it is residually nilpotent. We want to show that, for all primes p, $\widetilde{H}_{\widehat{p}}$ is free. By Proposition 2.1, this would imply that $\widetilde{H}$ is parafree.

Fix a prime p. Observe that

$$d(\widetilde{G}_{\widehat{p}}) \geq d(\widetilde{H}_{\widehat{p}}).$$

Consider the closure $\overline{H}$ of the image of $\widetilde{H}$ under the canonical map $\widetilde{G} \to \widetilde{G}_{\widehat{p}}$. Since $\widetilde{G}$ is parafree, $\widetilde{G}_{\widehat{p}}$ is free pro-p, and so $\overline{H}$ is also free pro-p. Note that $\widetilde{H}$ and t generate $\widetilde{G}$. Hence, the canonical map $f : \overline{H} \coprod \mathbb{Z}_p \to \widetilde{G}_{\widehat{p}}$, which sends the generator 1 of $\mathbb{Z}_p$ to t, is onto.

Since $\widetilde{G}$ is parafree, the images of u and v in $\widetilde{G}_{\widehat{p}}$ are non-trivial. Therefore, $\ker f \neq \{1\}$. Thus, since the groups $\overline{H} \coprod \mathbb{Z}_p$ and $\widetilde{G}_{\widehat{p}}$ are free pro-p,

$$d(\widetilde{G}_{\widehat{p}}) \leq d(\overline{H}).$$

Since $\overline{H}$ is a quotient of $\widetilde{H}_{\widehat{p}}$, $d(\overline{H}) = d(\widetilde{H}_{\widehat{p}})$, and since, $\overline{H}$ is free pro-p, the canonical map $\widetilde{H}_{\widehat{p}} \to \overline{H}$ is an isomorphism. Hence, $\widetilde{H}_{\widehat{p}}$ is free pro-p, and the first condition is proved.

The previous argument also shows that for all prime p,

$$d(\widetilde{G}_{\widehat{p}}) = d(\widetilde{H}_{\widehat{p}}).$$

This implies the second condition.

Now suppose that $u = w^{n_1}$ and $v = w_2^{n_2}$ in $\widetilde{H}$ with $n_1, n_2 \geq 2$. Since $u, v \neq 1$ and $\widetilde{H}$ is torsion-free, then $\langle w_i \rangle$ are infinite cyclic. Consider the subgroup A of $\widetilde{G}$ generated by w_1, w_2 and t. By Proposition 2.4, A is isomorphic to the HNN extension $A' = \langle w_1, w_2, t | t w_1^{n_1} t^{-1} = w_2^{n_2} \rangle$. The centralizer of $w_1^{n_1} = (t^{-1} w_2 t)^{n_2}$ in A' contains w_1 and $t^{-1} w_2 t$, so it is not abelian. Thus the centralizer of u in $\widetilde{G}$ is not abelian. This contradicts Proposition 2.2. This shows that the third condition holds. The forth condition holds because $\widetilde{G}$ is parafree.

Now we are going to verify that these four conditions are sufficient for $\widetilde{G}$ to be parafree. First of all, it is clear that

$$\widetilde{G}_{\text{ab}} \cong \frac{\widetilde{H}_{\text{ab}}}{\langle u - v \rangle} \oplus \langle t \rangle.$$

From the fact that $u - v$ is not a proper power in $\widetilde{H}_{\text{ab}}$, we see that $\widetilde{G}_{\text{ab}}$ is torsion-free of the same rank as $\widetilde{H}_{\text{ab}}$. Therefore, $d(\widetilde{G}_{\widehat{p}}) = d(\widetilde{H}_{\widehat{p}})$. In addition, we also see that t is primitive in $\widetilde{G}_{\text{ab}}$. Let us fix a prime p such that the image of the element u is non-trivial in some finite p-quotient o $\widetilde{G}$. Consider the canonical map $\phi : \widetilde{G} \longrightarrow \widetilde{G}_{\widehat{p}}$.

Claim 4.6. *The pro-p group $\widetilde{G}_{\widehat{p}}$ is free and the element $\phi(t)$ is primitive in $\widetilde{G}_{\widehat{p}}$.*

Proof. The pro-p group $\widetilde{G}_{\widehat{p}}$ is the quotient of $\widetilde{H}_{\widehat{p}} \coprod \mathbb{Z}_p$ by the closed subgroup generated by $tut^{-1}v^{-1}$. Since $\widetilde{H}$ is parafree, $\widetilde{H}_{\widehat{p}}$, is free pro-$p$. Thus $\widetilde{H}_{\widehat{p}} \coprod \mathbb{Z}_p$ is free pro-p. Observe also that the element $tut^{-1}v^{-1}$ is primitive in $\widetilde{H}_{\widehat{p}} \coprod \mathbb{Z}_p$. Hence, $\widetilde{G}_{\widehat{p}}$ is free pro-p. □

We name $H = \phi(\widetilde{H})$.

Claim 4.7. *The restriction of ϕ to $\widetilde{H}$ is injective.*

Proof. To verify this, consider the closed subgroup $\overline{H} \leq \widetilde{G}_{\widehat{p}}$. Since $\widetilde{G}_{\widehat{p}}$ is free, the pro-p group $\overline{H}$ must be free. We notice that the epimorphism $\phi : \widetilde{H} \longrightarrow H$ induces a continuous epimorphism $\phi_{\widehat{p}} : \widetilde{H}_{\widehat{p}} \longrightarrow \overline{H}$. In particular,

$$d(\widetilde{H}_{\widehat{p}}) \geq d(\overline{H}). \tag{9}$$

Furthermore, by the universal property of the coproduct, there is a continuous homomorphism

$$f : \overline{H} \coprod \mathbb{Z}_p \longrightarrow \widetilde{G}_{\widehat{p}},$$

which sends $\overline{H}$ to $\widetilde{G}_{\widehat{p}}$, by inclusion; and $\mathbb{Z}_p$ to the cyclic pro-p group generated by $\phi(t)$. Since the image of f contains both H and $\phi(t)$, it follows that $\operatorname{Im} f$ contains $\phi(\widetilde{G})$, so f must be a surjective. In addition, it has a nontrivial kernel; since $\phi(t)\phi(u)\phi(t)^{-1}\phi(v)^{-1} = 1$ and $\phi(u) \neq 1$, by assumption. Here we have used that the canonical map $\iota : \overline{H} * \mathbb{Z}_p \longrightarrow \overline{H} \coprod \mathbb{Z}_p$ is injective.

This verifies that f is a surjective and non-injective continuous homomorphism between free pro-p groups. Hence,

$$d(\overline{H}) + 1 = d(\overline{H}) + d(\mathbb{Z}_p) = d\left(\overline{H} \coprod \mathbb{Z}_p\right) > d(\widetilde{G}_{\widehat{p}}).$$

This, in addition to (9), implies that $d(\widetilde{H}_{\widehat{p}}) = d(\overline{H})$. So $\phi_{\widehat{p}} : \widetilde{H}_{\widehat{p}} \to \overline{H}$ is a continuous epimorphism between free pro-p groups of the same rank, which implies that it is an isomorphism. Since $\widetilde{H}$ is parafree, it is residually-p. hence ϕ is also injective. □

Claim 4.8. *The map $\phi : \widetilde{G} \longrightarrow \widetilde{G}_{\widehat{p}}$ is injective.*

Proof. We denote $G = \phi(\widetilde{G})$. We want to apply Proposition 2.11 to the map $\phi : \widetilde{G} \longrightarrow G \subseteq \widetilde{G}_{\widehat{p}}$. We already know, from Claim 4.6, that $\widetilde{G}_{\widehat{p}}$ is free. Notice also that, by Claim 4.7, the kernel of ϕ intersects trivially the subgroup $\widetilde{H}$ of $\widetilde{G}$. The Bass-Serre theory implies that the kernel is free.

We define a continuous homomorphism $q : \widetilde{G}_{\widehat{p}} \longrightarrow \mathbb{Z}_p$ such that $q(\phi(t)) = 1$, and $q(\phi(h)) = 0$ if $h \in \widetilde{H}$. The restriction $q|_G$ verifies that its its kernel $\ker q|_G$ contains $H = \phi(\widetilde{H})$.

We now consider the $\mathbb{F}_p G$-module

$$M = \mathbb{F}_p G \underset{\mathbb{F}_p \widetilde{G}}{\otimes} \mathbb{F}_p I_{\widetilde{G}}.$$

By Proposition 2.3(2), this $\mathbb{F}_p G$-module is isomorphic to

$$M \cong \frac{\mathbb{F}_p I_H^G \oplus \mathbb{F}_p G}{\mathbb{F}_p G\,(\phi(v) - 1 - \phi(t)(\phi(u) - 1),\, \phi(v) - 1)}.$$

Without loss of generality, we suppose that u is not a proper power in $\widetilde{H}$. Since $\phi : \widetilde{H} \longrightarrow H$ is an isomorphism, H is parafree and $\phi(u)$ is not a proper power in H.

From Proposition 2.2, we deduce that that $\langle \phi(u) \rangle$ is a maximal abelian subgroup of H. By Proposition 3.3, this implies that M is $\mathcal{D}_{\mathbb{F}_p G}$-torsion-free.

Using Lemma 3.2, we have the following isomorphisms of $\mathcal{D}_{\mathbb{F}_p G}$-modules

$$\mathcal{D}_{\mathbb{F}_p G} \underset{\mathbb{F}_p G}{\otimes} M \cong \mathcal{D}_{\mathbb{F}_p G} \underset{\mathbb{F}_p G}{\otimes} \mathbb{F}_p I_H^G \cong \mathcal{D}_{\mathbb{F}_p G} \underset{\mathbb{F}_p H}{\otimes} \mathbb{F}_p I_H \cong \mathcal{D}_{\mathbb{F}_p G} \underset{\mathcal{D}_{\mathbb{F}_p H}}{\otimes} \left(\mathcal{D}_{\mathbb{F}_p H} \underset{\mathbb{F}_p H}{\otimes} \mathbb{F}_p I_H \right).$$

The combination of these isomorphisms with Proposition 2.9 yields to

$$\dim_{\mathcal{D}_{\mathbb{F}_p G}} M = \dim_{\mathcal{D}_{\mathbb{F}_p H}} \mathbb{F}_p I_H = d(\widetilde{H}_{\widehat{p}}) = d(\widetilde{G}_{\widehat{p}}).$$

We can apply Proposition 2.11 to conclude that $\phi : \widetilde{G} \longrightarrow G$ is injective. □

The last claim implies that $\widetilde{G}$ is residually nilpotent. Moreover, we already know that every pro-p completion $\widetilde{G}_{\widehat{p}}$ is free, so $\widetilde{G}$ is parafree by Proposition 2.1. □

Proof of Corollary 1.3. It is clear that the three conditions of the corollary imply the four conditions of Theorem 1.2. Therefore, they are sufficient.

Let us show that they are necessary. Assume that W is parafree. Again, in view of Theorem 1.2 we have to show only the third condition. Assume that the images of u and v in the abelianization of F_2 generate a subgroup not isomorphic to $\mathbb{Z}^2$.

Let p be a prime. Since W is parafree we consider W inside $W_{\widehat{p}}$. Let N be the normal closed subgroup of $(F_2)_{\widehat{p}}$ generated by uv^{-1}. Then in view of (1), $(F_2)_{\widehat{p}}/N \cong \mathbb{Z}_p$, and so $u \in N$. Observe that $uv^{-1} = utu^{-1}t^{-1}$ in $W_{\widehat{p}}$. Thus, u is contained in the closed normal subgroup of $W_{\widehat{p}}$ generated by $utu^{-1}t^{-1}$. Since, $W_{\widehat{p}}$ is a pro-p group, $u = 1$. This is a contradiction. Therefore, the images of u and v in the abelianization of F_2 generate a subgroup isomorphic to $\mathbb{Z}^2$. □

4.3. **The fundamental group of graph of groups.** In this subsection we prove Corollary 1.4.

Proof of Corollary 1.4. First assume that W is parafree. We want to show that the four condition of the corollary hold. The conditions (3) and (4) hold because W is parafree. Let us show the conditions (1) and (2). We will argue by induction on the number of edges $|E(\Gamma)|$. If $|E(\Gamma)| = 0$, the claim is obvious.

Now we assume that we have proved that (1) and (2) hold if $|E(\Gamma)| \leq n$ and we consider the case $|E(\Gamma)| = n + 1$. Let $e \in E(\Gamma)$.

If $\Gamma \setminus \{e\} = \Delta_1 \cup \Delta_2$ is disconnected then, by Proposition 2.4,

$$W \cong U *_{\mathcal{G}(e)} V, \text{ where } U \cong \pi(\mathcal{G}, \Delta_1) \text{ and } V \cong \pi(\mathcal{G}, \Delta_2).$$

If $\mathcal{G}(e) = \{1\}$, then $W \cong U * V$, and so, since W is parafree, U and V are also parafree. If $\mathcal{G}(e) \cong \mathbb{Z}$, then U and V are parafree by Theorem 4.1. Since $|E(\Delta_1)|, |E(\Delta_2)| \leq n$, we can apply the induction. Thus, the four conditions of Corollary 1.4 hold for U and V. This implies immediately that conditions (1) and (2) also hold for W.

If $\Gamma \setminus \{e\} = \Delta$ is connected, then, by Proposition 2.4,

$$W \cong U *_{t_e} \text{ where } U \cong \pi(\mathcal{G}, \Delta).$$

If $\mathcal{G}(e) = \{1\}$, then $W \cong U * \mathbb{Z}$, and so, since W is parafree, U is also parafree. If $\mathcal{G}(e) \cong \mathbb{Z}$, then U is parafree by Theorem 4.5. Since $|E(\Delta)| \leq n$, we can apply the induction. Thus, the four conditions of Corollary 1.4 holds for U. This implies immediately that the conditions (1) and (2) holds also for W.

Now we assume that the four conditions of the corollary hold. We want to show that W is parafree. We will argue by induction on the number of edges $|E(\Gamma)|$. If $|E(\Gamma)| = 0$, the claim is obvious.

Now we assume that we have proved that W is parafree if $|E(\Gamma)| \leq n$ and we consider the case $|E(\Gamma)| = n + 1$. Let $e \in E(\Gamma)$.

If $\Gamma \setminus \{e\} = \Delta_1 \cup \Delta_2$ is disconnected then, by Proposition 2.4,

$$W \cong U *_{\mathcal{G}(e)} V, \text{ where } U \cong \pi(\mathcal{G}, \Delta_1) \text{ and } V \cong \pi(\mathcal{G}, \Delta_2).$$

We have that

$$\begin{aligned} r_{\mathrm{ab}}(W) \geq r_{\mathrm{ab}}(U) + r_{\mathrm{ab}}(V) - r_{\mathrm{ab}}(\mathcal{G}(e)) \geq \\ \sum_{v \in V(\Delta_1)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Delta_1)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Delta_1) + \\ \sum_{v \in V(\Delta_2)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Delta_2)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Delta_2) - r_{\mathrm{ab}}(\mathcal{G}(e)) = \\ \sum_{v \in V(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Gamma) \end{aligned}$$

Since $r_{\mathrm{ab}}(W) = \sum_{v \in V(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Gamma)$, we obtain that

$$\begin{aligned} r_{\mathrm{ab}}(U) = \sum_{v \in V(\Delta_1)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Delta_1)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Delta_1), \\ r_{\mathrm{ab}}(V) = \sum_{v \in V(\Delta_2)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Delta_2)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Delta_2) \text{ and} \\ r_{\mathrm{ab}}(W) = r_{\mathrm{ab}}(U) + r_{\mathrm{ab}}(V) - r_{\mathrm{ab}}(\mathcal{G}(e)). \end{aligned}$$

Thus, U and V satisfy the four conditions of the corollary and, by the inductive hypothesis, they are parafree. If $\mathcal{G}(e) = \{1\}$, then $W \cong U * V$, and so, W is parafree. If $\mathcal{G}(e) \cong \mathbb{Z}$, then W is parafree by Theorem 4.1 and the remark afterwards.

If $\Gamma \setminus \{e\} = \Delta$ is connected then, by Proposition 2.4,

$$W \cong U *_{t_e} \text{ where } U \cong \pi(\mathcal{G}, \Delta).$$

We have that

$$\begin{aligned} r_{\mathrm{ab}}(W) \geq r_{\mathrm{ab}}(U) - r_{\mathrm{ab}}(\mathcal{G}(e)) + 1 \geq \\ \sum_{v \in V(\Delta)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Delta)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Delta) - r_{\mathrm{ab}}(\mathcal{G}(e)) + 1 = \\ \sum_{v \in V(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Gamma) \end{aligned}$$

Since $r_{\mathrm{ab}}(W) = \sum_{v \in V(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Gamma)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Gamma)$, we obtain that

$$\begin{aligned} r_{\mathrm{ab}}(U) = \sum_{v \in V(\Delta)} r_{\mathrm{ab}}(\mathcal{G}(v)) - \sum_{f \in E(\Delta)} r_{\mathrm{ab}}(\mathcal{G}(f)) - \chi(\Delta) \text{ and} \\ r_{\mathrm{ab}}(W) = r_{\mathrm{ab}}(U) - r_{\mathrm{ab}}(\mathcal{G}(e)) + 1. \end{aligned}$$

Thus, U satisfies the four conditions of the corollary and, by the inductive hypothesis, it is parafree. If $\mathcal{G}(e) = \{1\}$, then $W \cong U * \mathbb{Z}$, and so, W is parafree. If $\mathcal{G}(e) \cong \mathbb{Z}$, then W is parafree by Theorem 4.5 and the remark afterwards.

□

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