UNITS OF GROUP RINGS, THE BOGOMOLOV MULTIPLIER, AND THE FAKE DEGREE CONJECTURE

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Abstract. Let π be a finite p-group and $\mathbb{F}_q$ a finite field with $q = p^n$ elements. Denote by $\mathrm{I}_{\mathbb{F}_q}$ the augmentation ideal of the group ring $\mathbb{F}_q[\pi]$. We have found a surprising relation between the abelianization of $1 + \mathrm{I}_{\mathbb{F}_q}$, the Bogomolov multiplier $\mathrm{B}_0(\pi)$ of π and the number of conjugacy classes $\mathrm{k}(\pi)$ of π:

$$\left|(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}\right| = q^{\mathrm{k}(\pi)-1}|\mathrm{B}_0(\pi)|.$$

In particular, if π is a finite p-group with a non-trivial Bogomolov multiplier, then $1 + \mathrm{I}_{\mathbb{F}_q}$ is a counterexample to the fake degree conjecture proposed by M. Isaacs.

1. Introduction

Let J be a finite dimensional nilpotent algebra over a finite field $\mathbb{F}_q$. Then the set $G = 1 + J$ is a finite group. The groups constructed in this way are called *algebra groups*. The group G naturally acts by conjugation on J. It is easy to see that this action is equivalent to the conjugation action of G on itself. The G-action on J induces a G-action on the dual space $J^* = \mathrm{Hom}_{\mathbb{F}_q}(J, \mathbb{F}_q)$, called the coadjoint action. It follows that for any of the three G-actions the number of orbits equals the number of conjugacy classes of G. Moreover, it can be shown that the sizes of coadjoint orbits are even powers of q. Consider the list of integers obtained by taking the square roots of the sizes of the coadjoint orbits of G on J^*. These numbers are q-powers, the sum of their squares is $|G|$ and the length of the list is the number of conjugacy classes of G. This precisely resembles the list of degrees of the irreducible characters of G. Indeed, if $J^p = 0$, there exists an explicit expression that gives a bijective correspondence between the characters of G and the orbits of J^* (see [17] and [21]). In particular, when $J^p = 0$, we obtain that the character degrees of G, counting multiplicities, are the square roots of the sizes of the G-orbits in J^*. It was conjectured by M. Isaacs that the same holds also in the general case:

Conjecture 1 (Fake degree conjecture)**.** *In every algebra group $G = 1 + J$ the character degrees coincide, counting multiplicities, with the square roots of the cardinals of the G-orbits in J^*.*

Note that an immediate corollary of this conjecture (see Lemma 9) is that the orders of $[J, J]_L$ and $[1 + J, 1 + J]_G$ have to be equal (in this work we write $[a, b] = a^{-1}b^{-1}ab$ for group commutators and $[a, b]_L = ab - ba$ for Lie brackets). Thus in order to understand Conjecture 1, one should first answer the following question.

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Question 2. *Is it true that the size of the abelianization of $1+J$ coincides with the index of $[J,J]_L$ in J?*

In [8] an example that provides a negative answer to Question 2 in characteristic 2 was constructed. However, in questions related to character correspondences for finite p-groups the prime $p=2$ always plays a special role (see, for example, [9]), and so one might hope that Conjecture 1 still holds in odd characteristic.

This was our motivation for looking at the following family of examples. Let π be a finite p-group. Given a ring R we will set I_R to be the augmentation ideal of the group ring $R[\pi]$. If we take $R=\mathbb{F}_q$, then $\mathrm{I}_{\mathbb{F}_q}$ is a nilpotent algebra and $1+\mathrm{I}_{\mathbb{F}_q}$ is the group of normalized units of the modular group ring $\mathbb{F}_q[\pi]$. It is not difficult to see that the index of $[\mathrm{I}_{\mathbb{F}_q},\mathrm{I}_{\mathbb{F}_q}]_L$ in $\mathrm{I}_{\mathbb{F}_q}$ is equal to $q^{\mathrm{k}(\pi)-1}$, where $\mathrm{k}(\pi)$ is the number of conjugacy classes of π (see Lemma 10). Our main result describes the size of the abelianization $(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$ of $1+\mathrm{I}_{\mathbb{F}_q}$.

Theorem 3. *Let π be a finite p-group. Then $|(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}| = q^{\mathrm{k}(\pi)-1}|\mathrm{B}_0(\pi)|$.*

The group $\mathrm{B}_0(\pi)$ that appears in the theorem is the *Bogomolov multiplier* of π. It is defined as the subgroup of the Schur multiplier $\mathrm{H}^2(\pi,\mathbb{Q}/\mathbb{Z})$ of π consisting of the cohomology classes vanishing after restriction to all abelian subgroups of π. The Bogomolov multiplier plays an important role in birational geometry of quotient spaces V/π as it was shown by Bogomolov in [2]. In a dual manner, one may view the group $\mathrm{B}_0(\pi)$ as an appropriate quotient of the homological Schur multiplier $\mathrm{H}_2(\pi,\mathbb{Z})$, see [14]. We were surprised to discover that, in this form, the Bogomolov multiplier had appeared in the literature much earlier in a paper of W. D. Neumann [15], as well as in the paper of B. Oliver [16] that plays an essential role in our proofs. The latter paper contains various results about Bogomolov multipliers that were only subsequently proved in the cohomological framework.

There are plenty of finite p-groups with non-trivial Bogomolov multipliers (see, for example, [12]). Thus we obtain a negative solution to the fake degree conjecture for all primes.

Corollary 4. *For every prime p there exists a finite dimensional nilpotent $\mathbb{F}_p$-algebra J such that the size of the abelianization of $1+J$ is greater than the index of $[J,J]_L$ in J. In particular, the fake degree conjecture is not valid in any characteristic.*

Our next result provides a conceptual explanation for the equality in Theorem 3. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. One can think of $\mathbf{G}=1+\mathrm{I}_{\mathbb{F}}$ as an algebraic group defined over $\mathbb{F}_p$. It is clear that $\mathbf{G}$ is a unipotent group. A direct calculation shows that the Lie algebra $\mathfrak{L}(\mathbf{G})$ of $\mathbf{G}$ is isomorphic to $\mathrm{I}_{\mathbb{F}}$. We write $\mathbf{G}(\mathbb{F}_q)$ for the $\mathbb{F}_q$-points of $\mathbf{G}$. The derived subgroup $\mathbf{G}'$ of $\mathbf{G}$ is also a unipotent algebraic group defined over $\mathbb{F}_p$ (see [3, Corollary I.2.3]), and so by [11, Remark A.3], $|\mathbf{G}'(\mathbb{F}_q)| = q^{\dim \mathbf{G}'}$. Note that in general we have only an inclusion

$$(1+\mathrm{I}_{\mathbb{F}_q})' = (\mathbf{G}(\mathbb{F}_q))' \subseteq \mathbf{G}'(\mathbb{F}_q),$$

but not the equality.

Theorem 5. *Let π be a finite p-group and $\mathbf{G}=1+\mathrm{I}_{\mathbb{F}}$.*

(1) *We have*

$$\dim \mathbf{G}' = \dim_{\mathbb{F}}[\mathfrak{L}(\mathbf{G}), \mathfrak{L}(\mathbf{G})]_L = |\pi| - \mathrm{k}(\pi).$$

In particular,

$$|\mathbf{G}(\mathbb{F}_q) : \mathbf{G}'(\mathbb{F}_q)| = q^{\mathrm{k}(\pi)-1}.$$

(2) *For every $q = p^n$, we have*

$$\mathbf{G}'(\mathbb{F}_q)/\mathbf{G}(\mathbb{F}_q)' \cong \mathrm{B}_0(\pi).$$

Our hope is that the second statement of the theorem would help better understand the structure of the Bogomolov multiplier. As an example of this reasoning, recall that a classical problem about the Schur multiplier asks what is the relation between the exponent of a finite group and of its Schur multiplier ([18]). Standard arguments reduce this question to the case of p-groups. It is known that the exponent of the Schur multiplier is bounded by some function that depends only on the exponent of the group ([13]), but this bound is obtained from the bounds that appear in the solution of the Restricted Burnside Problem and so it is probably very far from being optimal. Applying to the homological description of the Bogomolov multiplier, it is not difficult to see that the exponent of the Schur multiplier is at most the product of the exponent of the group by the exponent of the Bogomolov multiplier. Thus, we hope that the following theorem would help obtain a better bound on the exponent of the Schur multiplier.

Theorem 6. *Let π be a finite p-group and $\mathbf{G} = 1 + \mathrm{I}_{\mathbb{F}}$. For every $q = p^n$, we have*

$$\exp \mathrm{B}_0(\pi) = \min\{m \mid \mathbf{G}'(\mathbb{F}_q) \subseteq \mathbf{G}(\mathbb{F}_{q^m})'\}.$$

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2. Proofs of the results

2.1. **Proof of Theorem 3.** Let ζ_n denote a primitive n-root of unity. If p is a prime and q is a power of p, let $\mathrm{R}_q = \mathbb{Z}_p[\zeta_{q-1}]$ be a finite extension of the p-adic integers $\mathbb{Z}_p$. Note that $\mathrm{R}_q / p\,\mathrm{R}_q \cong \mathbb{F}_q$. Fix a $\mathbb{Z}_p$-basis $B_q = \{\lambda_j \mid 1 \leq j \leq n\}$ of R_q and let φ be a generator of $\mathrm{Aut}(\mathrm{R}_q \,|\mathbb{Z}_p) \cong \mathrm{Gal}(\mathbb{F}_q \,|\, \mathbb{F}_p)$ such that $\varphi(\lambda) \cong \lambda^p \pmod p$. Let us define

$$\bar{\mathrm{I}}_{\mathrm{R}_q} = \mathrm{I}_{\mathrm{R}_q} / \langle x - x^g \mid x \in \mathrm{I}_{\mathrm{R}_q},\, g \in \pi \rangle.$$

Set $\mathcal{C}$ to be a set of nontrivial conjugacy class representatives of π. Then $\bar{\mathrm{I}}_{\mathrm{R}_q}$ can be regarded as a free $\mathbb{Z}_p$-module with basis $\{\lambda\overline{(1-r)} \mid \lambda \in B_q,\, r \in \mathcal{C}\}$. Finally define the abelian group M_q to be

$$\mathrm{M}_q = \bar{\mathrm{I}}_{\mathrm{R}_q} / \langle p\lambda\overline{(1-r)} - \varphi(\lambda)\overline{(1-r^p)} \mid \lambda \in B_q,\ r \in \mathcal{C}\rangle.$$

The proof of Theorem 3 rests on the following structural description of the group $(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$.

Theorem 7. *Let π be a finite p-group. There is an exact sequence*

$$1 \longrightarrow \mathrm{B}_0(\pi)\times\pi_{\mathrm{ab}} \longrightarrow (1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}} \longrightarrow \mathrm{M}_q \longrightarrow \pi_{\mathrm{ab}} \longrightarrow 1.$$

Proof. Given a ring R, recall the first K-theoretical group $\mathrm{K}_1(\mathrm{R}) = \mathrm{GL}(\mathrm{R})_{\mathrm{ab}}$. When R is a local ring, there is an isomorphism $\mathrm{K}_1(\mathrm{R}) \cong \mathrm{R}^*_{\mathrm{ab}}$ (see [20, Corollary 2.2.6]). We therefore have $\mathrm{K}_1(\mathbb{F}_q[\pi]) \cong \mathbb{F}_q^* \times (1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$, and our proof relies on inspecting the connection between $\mathrm{K}_1(\mathbb{F}_q[\pi])$ and $\mathrm{K}_1(\mathrm{R}_q[\pi])$ by utilizing the results of [16].

Put Q_q to be the ring of fractions of R_q and let

$$\mathrm{SK}_1(\mathrm{R}_q[\pi]) = \ker\left(\mathrm{K}_1(\mathrm{R}_q[\pi]) \to \mathrm{K}_1(\mathrm{Q}_q[\pi])\right).$$

By [16, Theorem 3], we have that $\mathrm{SK}_1(\mathrm{R}_q[\pi]) \cong \mathrm{B}_0(\pi)$. Now set

$$\mathrm{Wh}'(\mathrm{R}_q[\pi]) = \mathrm{K}_1(\mathrm{R}_q[\pi])/(\mathrm{R}_q^* \times \pi_{\mathrm{ab}} \times \mathrm{SK}_1(\mathrm{R}_q[\pi])).$$

The crux of understanding the structure of the group $\mathrm{K}_1(\mathrm{R}_q[\pi])$ is in the short exact sequence (see [16, Theorem 2])

$$1 \longrightarrow \mathrm{Wh}'(\mathrm{R}_q[\pi]) \xrightarrow{\Gamma} \bar{\mathrm{I}}_{\mathrm{R}_q} \longrightarrow \pi_{\mathrm{ab}} \longrightarrow 1,$$

where the map Γ is defined by composing the p-adic logarithm with a linear automorphism of $\bar{\mathrm{I}}_{\mathrm{R}_q}\otimes\mathbb{Q}_p$. More precisely, there is a map $\mathrm{Log}\colon 1+\mathrm{I}_{\mathrm{R}_q} \to \mathrm{I}_{\mathrm{R}_q}\otimes\mathbb{Q}_p$, which induces an injection $\log\colon \mathrm{Wh}'(\mathrm{R}_q[\pi]) \to \bar{\mathrm{I}}_{\mathrm{R}_q}\otimes\mathbb{Q}_p$. Setting $\Phi\colon \mathrm{I}_{\mathrm{R}_q} \to \mathrm{I}_{\mathrm{R}_q}$ to be the map $\sum_{g\in\pi}\alpha_g g \mapsto \sum_{g\in\pi}\varphi(\alpha_g)g^p$, we define $\Gamma\colon \mathrm{Wh}'(\mathrm{R}_q[\pi]) \to \bar{\mathrm{I}}_{\mathrm{R}_q}\otimes\mathbb{Q}_p$ as the composite of log followed by the linear map $1-\frac{1}{p}\Phi$. It is shown in [16, Proposition 10] that $\operatorname{im}\Gamma \subseteq \bar{\mathrm{I}}_{\mathrm{R}_q}$, i.e., Γ is integer-valued. We thus have a diagram

(1)
$$\begin{tikzcd} & 1+\mathrm{I}_{R_q} \arrow[d] \arrow[dr, bend left, "(1-\frac{1}{p}\Phi)\circ\log"] & \\ 1 \arrow[r] & \mathrm{Wh}'(\mathrm{R}_q[\pi]) \arrow[r, "\Gamma"] & \bar{\mathrm{I}}_{R_q}. \end{tikzcd}$$

The group $\mathrm{Wh}'(\mathrm{R}_q[\pi])$ is torsion-free (cf. [19]), so we have an explicit description

(2)
$$\mathrm{K}_1(\mathrm{R}_q[\pi]) \cong \mathrm{R}_q^* \times \mathrm{SK}_1(\mathrm{R}_q[\pi]) \times \pi_{\mathrm{ab}} \times \mathrm{Wh}'(\mathrm{R}_q[\pi]).$$

To relate the above results to $\mathrm{K}_1(\mathbb{F}_q[\pi])$, we invoke a part of the K-theoretical long exact sequence for the ring $\mathrm{R}_q[\pi]$ with respect to the ideal generated by p,

(3)
$$\mathrm{K}_1(\mathrm{R}_q[\pi],p) \xrightarrow{\partial} \mathrm{K}_1(\mathrm{R}_q[\pi]) \xrightarrow{\mu} \mathrm{K}_1(\mathbb{F}_q[\pi]) \longrightarrow 1.$$

Note that $\mathrm{K}_1(\mathrm{R}_q[\pi],p) = (1+p\,\mathrm{R}_q)\times\mathrm{K}_1(\mathrm{R}_q[\pi],p\,\mathrm{I}_{\mathrm{R}_q})$ and $\mathrm{R}_q^*/(1+p\,\mathrm{R}_q) \cong \mathbb{F}_q^*$. Hence (2) and (3) give a reduced exact sequence

(4)
$$\mathrm{K}_1(\mathrm{R}_q[\pi],p\,\mathrm{I}_{\mathrm{R}_q}) \xrightarrow{\partial} \mathrm{Wh}'(\mathrm{R}_q[\pi]) \xrightarrow{\mu} \frac{(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}}{\mu(\mathrm{SK}_1(\mathrm{R}_q[\pi])\times\pi_{\mathrm{ab}})} \longrightarrow 1.$$

To determine the structure of the relative group $\mathrm{K}_1(\mathrm{R}_q[\pi],p\,\mathrm{I}_{\mathrm{R}_q})$ and its connection to the map ∂, we make use of [16, Proposition 2]. The restriction of the logarithm map Log to $1+p\,\mathrm{I}_{\mathrm{R}_q}$ induces an isomorphism $\log\colon \mathrm{K}_1(\mathrm{R}_q[\pi], p\,\mathrm{I}_{\mathrm{R}_q}) \to p\bar{\mathrm{I}}_{\mathrm{R}_q}$ such that the following diagram commutes:

(5)
$$\begin{tikzcd} & 1+p\,\mathrm{I}_{\mathrm{R}_q} \arrow[d] \arrow[dr, bend left, "\log"] & \\ 1 \arrow[r] & \mathrm{K}_1(\mathrm{R}_q[\pi], p\,\mathrm{I}_{\mathrm{R}_q}) \arrow[r, "\log"] & p\bar{\mathrm{I}}_{\mathrm{R}_q}. \end{tikzcd}$$

In particular, the group $\mathrm{K}_1(\mathrm{R}_q[\pi], p\,\mathrm{I}_{\mathrm{R}_q})$ is torsion-free, and so $\mu(\mathrm{SK}_1(\mathrm{R}_q[\pi]) \times \pi_{\mathrm{ab}}) \cong \mathrm{SK}_1(\mathrm{R}_q[\pi]) \times \pi_{\mathrm{ab}}$. Note that by [1, Theorem V.9.1], the vertical map $1+p\,\mathrm{I}_{\mathrm{R}_q} \to \mathrm{K}_1(\mathrm{R}_q[\pi], p\,\mathrm{I}_{\mathrm{R}_q})$ of the above diagram is surjective.

We now collect the stated results to prove the theorem. First combine the diagrams (1) and (5) into the following diagram:

(6)
$$\begin{tikzcd} 1+p\,\mathrm{I}_{\mathrm{R}_q} \arrow[rr] \arrow[d] \arrow[ddr, bend left, "\log"] & & 1+\mathrm{I}_{\mathrm{R}_q} \arrow[d] \arrow[ddr, bend left, "(1-\frac{1}{p}\Phi)\circ\log"] & \\ \mathrm{K}_1(\mathrm{R}_q[\pi], p\,\mathrm{I}_{\mathrm{R}_q}) \arrow[rr, "\partial"] \arrow[dr, "\log"] & & \mathrm{Wh}'(\mathrm{R}_q[\pi]) \arrow[dr, "\Gamma"] & \\ & p\bar{\mathrm{I}}_{\mathrm{R}_q} \arrow[rr, "1-\frac{1}{p}\Phi"] & & \bar{\mathrm{I}}_{\mathrm{R}_q}. \end{tikzcd}$$

Since the back and top rectangles commute and the left-most vertical map is surjective, it follows that the bottom rectangle also commutes. Whence $\operatorname{coker}\partial \cong \operatorname{coker}(1-\frac{1}{p}\Phi)$. Observing that the latter group is isomorphic to M_q, the exact sequence (4) gives an exact sequence

(7)
$$1 \longrightarrow \mathrm{B}_0(\pi)\times\pi_{\mathrm{ab}} \longrightarrow (1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}} \longrightarrow \mathrm{M}_q \longrightarrow \pi_{\mathrm{ab}} \longrightarrow 1.$$

The proof is complete. □

We now derive Theorem 3 from Theorem 7.

Proof of Theorem 3. The exact sequence of Theorem 7 implies that $|(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}| = |\mathrm{B}_0(\pi)|\cdot|\mathrm{M}_q|$. Hence it suffices to compute $|\mathrm{M}_q|$. To this end, we filter M_q by the series of its subgroups

$$\mathrm{M}_q \supseteq p\,\mathrm{M}_q \supseteq p^2\,\mathrm{M}_q \supseteq \cdots \supseteq p^{\log_p(\exp\pi)}\,\mathrm{M}_q\,.$$

Note that the relations $p\lambda\overline{(1-r)} - \varphi(\lambda)\overline{(1-r^p)} = 0$ imply $p^{\log_p(\exp\pi)}\,\mathrm{M}_q = 0$.

For each $0 \le i \le \log_p(\exp\pi)$, put

$$\pi_i = \{x^{p^i} \mid x\in\pi\} \text{ and } \mathcal{C}_i = \mathcal{C}\cap(\pi_i\setminus\pi_{i+1}).$$

Then

$$p^i\,\mathrm{M}_q/p^{i+1}\,\mathrm{M}_q = \langle\lambda\overline{(1-r)} : \lambda\in B_q,\ r\in\mathcal{C}_i\rangle \cong \bigoplus_{\mathcal{C}_i} C_p^n.$$

It follows that $|\mathrm{M}_q| = q^{|\mathcal{C}|} = q^{\mathrm{k}(\pi)-1}$ and the proof is complete. □

Example. Let π be the group given by the polycyclic generators $\{g_i \mid 1\le i\le 7\}$ subject to the power-commutator relations

$$g_1^2 = g_4,\ g_2^2 = g_5,\ g_3^2 = g_4^2 = g_5^2 = g_6^2 = g_7^2 = 1,$$
$$[g_2,g_1] = g_3,\ [g_3,g_1] = g_6,\ [g_3,g_2] = g_7,\ [g_4,g_2] = g_6,\ [g_5,g_1] = g_7,$$

where the trivial commutator relations have been omitted. The group π is of order 128 with $\pi_{\mathrm{ab}} \cong C_4 \times C_4$. Its Bogomolov multiplier is generated by the commutator relation $[g_3, g_2] = [g_5, g_1]$ of order 2, see [10, Family 39]. We have $\mathrm{k}(\pi) = 26$ and by inspecting the power structure of conjugacy classes, we see that $\mathrm{M}_q \cong C_2^{13} \times C_4^6$. On the other hand, using the available computational tool [5], it is readily verified that we have $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}} \cong C_2^{13} \times C_4^5 \times C_8$. Following the proof of Theorem 7, the embedding of $\mathrm{B}_0(\pi) \times \pi_{\mathrm{ab}}$ into $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$ maps the generating relation $[g_3, g_2] = [g_5, g_1]$ of $\mathrm{B}_0(\pi)$ into the element $\exp((1 - g_7)(g_3 - g_5))$, which belongs to $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}^4$. In particular, the embedding of $\mathrm{B}_0(\pi) \times \pi_{\mathrm{ab}}$ into $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$ may not be split.

2.2. **The fake degree conjecture.** In this subsection we explain in more detail how Corollary 4 follows from Theorem 3.

Given an algebra group $G = 1 + J$ where J is a finite dimensional nilpotent F-algebra, the fake degree conjecture establishes a bijection between degrees of irreducible characters of G and the square roots of the lengths of the coadjoint orbits in $J^* = \mathrm{Hom}_{\mathrm{F}}(J, \mathrm{F})$. The following result is well known and enables us to compute lengths of coadjoint orbits. We include its proof for the reader's convenience.

Lemma 8. *Let $\lambda \in J^*$. Define $B_\lambda : J \times J \mapsto \mathrm{F}$ to be the bilinear form which assigns to every pair $(u, v) \in J \times J$ the element $\lambda([u, v]) \in \mathrm{F}$. Then* $\mathrm{Stab}(\lambda) = 1 + \mathrm{Rad}\, B_\lambda$.

Proof. Let $g = 1 + u$ be an element of G. Then g fixes λ if and only if for every $v \in J$, $\lambda(gvg^{-1}) = \lambda(v)$ or equivalently $\lambda(gvg^{-1} - v) = 0$. Since multiplication by g acts bijectively on J this amounts to $\lambda(gv - vg) = \lambda(uv - vu) = \lambda([u, v]) = 0$ for every $v \in J$, i.e., $u \in \mathrm{Rad}\, B_\lambda$ and the result follows. □

We now focus on 1-dimensional characters. In this case, the fake degree conjecture would establish a bijection between linear characters of G and fixed points of J^* under the coadjoint action of G.

Lemma 9. *Let J be a finite dimensional nilpotent algebra over a finite field* F*. Put $G = 1 + J$. Then the number of fixed points in J^* under the coadjoint action of G equals the index of $[J, J]_L$ in J. In particular, if the fake degree conjecture holds, then*

$$|J/[J, J]_L| = |(1 + J)_{\mathrm{ab}}|.$$

Proof. By Lemma 8, $\lambda \in J^*$ is fixed under the coadjoint action of G if and only if $\mathrm{Rad}\, B_\lambda = J$, which amounts to $\lambda([J, J]_L) = 0$. The number of fixed points in J^* therefore equals the number of linear forms vanishing on $[J, J]_L$. Hence if the fake degree conjecture holds, then

$$|J/[J, J]_L| = |\{\text{fixed points of } J^*\}| = |\{\text{linear characters of } G\}| = |(1 + J)_{\mathrm{ab}}|. \quad \square$$

We now consider the case when J is an augmentation ideal of the group algebra $\mathrm{F}[\pi]$ of a finite p-group π over a finite field F of characteristic p. The ideal $\mathrm{I}_{\mathrm{F}} = \mathrm{Rad}\, \mathrm{F}[\pi]$ is nilpotent and hence $1 + \mathrm{I}_{\mathrm{F}}$ is an algebra group. The following result is well known.

Lemma 10. *Let π be a finite group and F a field. Then*

$$\dim_{\mathrm{F}} \mathrm{I_F} / [\mathrm{I_F}, \mathrm{I_F}]_L = \mathrm{k}(\pi) - 1.$$

Proof. It is clear that the set π is an F-basis for $\mathrm{F}[\pi]$. We first claim that

$$\dim_{\mathrm{F}} \mathrm{F}[\pi]/[\mathrm{F}[\pi], \mathrm{F}[\pi]]_L = \mathrm{k}(\pi).$$

Let $x_1, \ldots, x_{\mathrm{k}(\pi)}$ be representatives of conjugacy classes of π. Observe that for any $x, y, g \in \pi$ with $y = g^{-1}xg$, we have $x - y = [g, g^{-1}x]_L$. The elements $\bar{x}_1, \ldots, \bar{x}_{\mathrm{k}(\pi)}$ therefore span $\mathrm{F}[\pi]/[\mathrm{F}[\pi], \mathrm{F}[\pi]]_L$.

Set λ_i to be the linear functional on $\mathrm{F}[\pi]$ that takes the value 1 on the elements corresponding to the conjugacy class of x_i and vanishes elsewhere. Observe that for any $g, h \in \pi$, we have $[g, h]_L = g(hg)g^{-1} - hg$ and hence each λ_i induces a linear functional on $\mathrm{F}[\pi]/[\mathrm{F}[\pi], \mathrm{F}[\pi]]_L$. Now if $\sum_j \alpha_j \bar{x}_j = 0$ for some $\alpha_j \in \mathrm{F}$, then $\alpha_i = \lambda_i(\sum_j \alpha_j \bar{x}_j) = 0$ for each i. It follows that $\bar{x}_1, \ldots, \bar{x}_{\mathrm{k}(\pi)}$ are also linearly independent and hence a basis. This proves the claim.

Now, it is clear that $\{g - 1 : g \in \pi \setminus \{1\}\}$ is an F-basis for $\mathrm{I_F}$. Since for any $g, h \in \pi$, we have $[g, h]_L = [g - 1, h - 1]_L$, it follows that $[\mathrm{F}[\pi], \mathrm{F}[\pi]]_L = [\mathrm{I_F}, \mathrm{I_F}]_L$, whence the lemma. ∎

It follows readily from Theorem 2 and Lemma 10 that whenever π is a p-group with $\mathrm{B}_0(\pi) \neq 0$, the algebra $J = \mathrm{I_F}$ gives an example for the statement of Corollary 4. Since for each prime p there exist groups of order p^5 (resp. 2^6 for $p = 2$) with non-trivial Bogomolov multipliers (see [6, 4]), Corollary 4 follows.

2.3. **Proof of Theorem 5 and Theorem 6.**

Proof of Theorem 5 and 6. We will consider an extension $\mathbb{F}_l$ of $\mathbb{F}_q$ of degree m. The inclusion $\mathbf{G}(\mathbb{F}_q) \subseteq \mathbf{G}(\mathbb{F}_l)$ induces a map $f\colon \mathbf{G}(\mathbb{F}_q)_{\mathrm{ab}} \to \mathbf{G}(\mathbb{F}_l)_{\mathrm{ab}}$ with

$$\ker f = (\mathbf{G}(\mathbb{F}_q) \cap \mathbf{G}(\mathbb{F}_l)')/\mathbf{G}(\mathbb{F}_q)'.$$

Note that there exists a large enough m such that $\mathbf{G}'(\mathbb{F}_q) = \mathbf{G}(\mathbb{F}_q) \cap \mathbf{G}(\mathbb{F}_l)'$, and hence $\ker f = \mathbf{G}'(\mathbb{F}_q)/\mathbf{G}(\mathbb{F}_q)'$. For this reason we want to understand $\ker f$ for a given m.

The inclusion $\mathbb{F}_q \subseteq \mathbb{F}_l$ induces a map

$$\mathrm{incl}\colon\ \mathrm{K}_1(\mathbb{F}_q[\pi]) \to \mathrm{K}_1(\mathbb{F}_l[\pi]).$$

Note that f is just the restriction of incl to $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$. Recalling sequence (4) from the proof of Theorem 7, we set

$$\mathrm{SK}_1(\mathbb{F}_l[\pi]) = \mu(\mathrm{SK}_1(\mathrm{R}_l[\pi]) \subseteq (1 + \mathrm{I}_{\mathbb{F}_l[\pi]})_{\mathrm{ab}} = \mathbf{G}(\mathbb{F}_l)/\mathbf{G}(\mathbb{F}_l)'.$$

Commutativity of the diagram

$$\begin{tikzcd} \mathrm{K}_1(\mathrm{R}_q[\pi]) \arrow[r, "\mu"] \arrow[d, "\mathrm{incl}"] & \mathrm{K}_1(\mathbb{F}_q[\pi]) \arrow[d, "\mathrm{incl}"] \\ \mathrm{K}_1(\mathrm{R}_l[\pi]) \arrow[r, "\mu"] & \mathrm{K}_1(\mathbb{F}_l[\pi]) \end{tikzcd} \tag{8}$$

shows that incl restricts to a map $\mathrm{incl}\colon \mathrm{SK}_1(\mathbb{F}_q[\pi]) \to \mathrm{SK}_1(\mathbb{F}_l[\pi])$. Recalling that $\mathrm{SK}_1(\mathrm{R}_l[\pi]) \cong \mathrm{B}_0(\pi)$, we obtain from sequence (7) the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SK}_1(\mathbb{F}_q[\pi]) \times \pi_{\mathrm{ab}} \arrow[r] \arrow[d, "\mathrm{incl} \times \mathrm{id}"] & \mathbf{G}(\mathbb{F}_q)_{\mathrm{ab}} \arrow[r] \arrow[d, "f"] & \mathrm{M}_q \arrow[r] \arrow[d, "\iota"] & \pi_{\mathrm{ab}} \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{SK}_1(\mathbb{F}_l[\pi]) \times \pi_{\mathrm{ab}} \arrow[r] & \mathbf{G}(\mathbb{F}_l)_{\mathrm{ab}} \arrow[r] & \mathrm{M}_l \arrow[r] & \pi_{\mathrm{ab}} \arrow[r] & 1,
\end{tikzcd}$$

where ι is the map induced by the inclusion $\mathrm{I}_{\mathrm{R}_q} \subseteq \mathrm{I}_{\mathrm{R}_l}$.

We will now show that $\ker \iota = 0$. This will imply $\ker f \subseteq \mathrm{SK}_1(\mathbb{F}_q[\pi])$. Without loss of generality, we may assume that there is an inclusion of bases $B_q \subseteq B_l$. As in the proof of Theorem 3, let us consider the series

$$\mathrm{M}_l \supseteq p\,\mathrm{M}_l \supseteq p^2\,\mathrm{M}_l \supseteq \ldots \supseteq p^{\log_p(\exp \pi)}\,\mathrm{M}_l\,.$$

Observe again that for each $0 \le i \le \exp \pi - 1$ we have

$$p^i\,\mathrm{M}_q\,/p^{i+1}\,\mathrm{M}_q = \langle \lambda\overline{(1-r)} : \lambda \in B_q,\ r \in \mathcal{C}_i \rangle,$$
$$p^i\,\mathrm{M}_l\,/p^{i+1}\,\mathrm{M}_l = \langle \lambda\overline{(1-r)} : \lambda \in B_l,\ r \in \mathcal{C}_i \rangle.$$

If we consider the graded groups associated to the series above, we get an induced map

$$\mathrm{gr}(\iota)\colon \bigoplus_{i=0}^{\exp \pi - 1} p^i\,\mathrm{M}_q\,/p^{i+1}\,\mathrm{M}_q \to \bigoplus_{i=0}^{\exp \pi - 1} p^i\,\mathrm{M}_l\,/p^{i+1}\,\mathrm{M}_l\,.$$

By construction ι is induced by the assignments $\lambda\overline{(1-r_m)} \mapsto \lambda\overline{(1-r_m)}$, for every $\lambda \in B_q,\ r \in \mathcal{C}$. Hence $\mathrm{gr}(\iota)$ is injective in every component and therefore injective. This implies $\ker \iota = 0$, as desired. In particular, we obtain that

$$|\mathbf{G}(\mathbb{F}_q)/\mathbf{G}'(\mathbb{F}_q)| \ge |\,\mathrm{M}_q\,| = q^{\mathrm{k}(\pi)-1}. \tag{9}$$

We are now ready to show the first statement of Theorem 5. Observe that $\mathbf{G}$ is a unipotent connected algebraic group defined over $\mathbb{F}_p$ and so is $\mathbf{G}'$ ([3, Corollary I.2.3]). Hence $\mathbf{G}' \cong_{\mathbb{F}_p} \mathbb{A}^{\dim \mathbf{G}'}$ (c.f. [11, Remark A.3]) and so $|\mathbf{G}'(\mathbb{F}_p)| = p^{\dim \mathbf{G}'}$. By (9), we have $|\mathbf{G}(\mathbb{F}_p)/\mathbf{G}'(\mathbb{F}_p)| \ge p^{\mathrm{k}(\pi)-1}$, whence $\dim \mathbf{G}' \le |\pi| - \mathrm{k}(\pi)$. On the other hand we have $[\mathfrak{L}(G), \mathfrak{L}(G)]_L = [\mathrm{I}_{\mathbb{F}}, \mathrm{I}_{\mathbb{F}}]_L$, which, by Lemma 10, has dimension $|\pi| - \mathrm{k}(\pi)$. It is well known that for an algebraic group, $\dim \mathbf{G}' \ge \dim[\mathfrak{L}(\mathbf{G}), \mathfrak{L}(\mathbf{G})]_L$ (see [7, Corollary 10.5]). Thus $\dim \mathbf{G}' = |\pi| - \mathrm{k}(\pi)$.

Let us set $e = \exp \mathrm{B}_0(\pi)$. We now claim that $\ker f = \mathrm{SK}_1(\mathbb{F}_q[\pi])$ if and only if e divides $m = |\mathbb{F}_l : \mathbb{F}_q|$. This will imply the second statement of Theorem 5 and also Theorem 6.

Let us consider $\mathbb{F}_l[\pi] \cong \bigoplus_{i=1}^m \mathbb{F}_q[\pi]$ as a free $\mathbb{F}_q[\pi]$-module. This gives a natural inclusion $\mathrm{GL}_1(\mathbb{F}_l[\pi]) \to \mathrm{GL}_m(\mathbb{F}_q[\pi])$, which induces the transfer map

$$\mathrm{trf} : \mathrm{K}_1(\mathbb{F}_l[\pi]) \to \mathrm{K}_1(\mathbb{F}_q[\pi]).$$

Note that if $x \in \mathrm{K}_1(\mathbb{F}_q[\pi])$, then $(\mathrm{trf} \circ \mathrm{incl})(x) = x^m$. By commutativity of (8) the transfer map restricts to a map

$$\mathrm{trf} : \mathrm{SK}_1(\mathbb{F}_l[\pi]) \to \mathrm{SK}_1(\mathbb{F}_q[\pi]).$$

Moreover, by [16, Proposition 21] the transfer map is an isomorphism. It thus follows that $\mathrm{incl}(\mathrm{SK}_1(\mathbb{F}_q[\pi])) = 1$ if and only if e divides m. Hence $\ker f = \mathrm{SK}_1(\mathbb{F}_q[\pi])$ if and only if e divides m and we are done. □

References

[1] H. Bass, *Algebraic K-theory.* W. A. Benjamin, New York, Amsterdam, 1968.

[2] F. A. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), 485–516; English transl. in Math. USSR Izv. 30 (1988), 455–485.

[3] A. Borel, *Linear algebraic groups*, second edition. Graduate Texts in Mathematics **126**, Springer-Verlag, New York, 1991.

[4] H. Chu, S. Hu., M. Kang, and B. E. Kunyavskiĭ, *Noether's problem and the unramified Brauer group for groups of order* 64, Int. Math. Res. Not. **12** (2010), 2329–2366.

[5] V. Bovdi, A. Konovalov, R. Rossmanith and C. Schneider, LAGUNA – Lie AlGebras and UNits of group Algebras, Version 3.6.2; 2013 (`http://www.cs.st-andrews.ac.uk/ alexk/laguna/`).

[6] A. Hoshi, M. Kang and B. E. Kunyavskiĭ, *Noether's problem and unramified Brauer groups*, Asian J. Math. **17** (2013), 689–714.

[7] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics 21, Springer-Verlag, 1975.

[8] A. Jaikin-Zapirain, *A counterexample to the fake degree conjecture*, Chebyshevskii Sb. **5** (2004), no. 1(9), 188–192.

[9] A. Jaikin-Zapirain, *Zeta function of representations of compact p-adic analytic groups*, J. Amer. Math. Soc. **19** (2006), no. 1, 91–118.

[10] U. Jezernik and P. Moravec, *Bogomolov multipliers of groups of order* 128, Exp. Math. **23** (2014), no. 2, 174–180.

[11] T. Kambayashi, M. Miyanishi, M. Takeuchi, *Unipotent algebraic groups*, Lecture notes in Mathematics 144, Springer-Verlag, 1974.

[12] M. C. Kang, *Bogomolov multipliers and retract rationality for semidirect products*, J. Algebra **397** (2014), 407–425.

[13] P. Moravec, *Schur multipliers and power endomorphisms of groups*, J. Algebra **308** (2007), no. 1, 12–25.

[14] P. Moravec, *Unramified groups of finite and infinite groups*, Amer. J. Math. **134** (2012), no. 6, 1679–1704.

[15] W. D. Neumann, *Manifold cutting and pasting groups*, Topology **14** (1975), no. 3, 237–244.

[16] R. Oliver, SK_1 *for finite group rings: II*, Math. Scand. **47** (1980), 195–231.

[17] J. Sangroniz, *Clases de conjugación y caracteres en grupos finitos*, Ph.D. Thesis, Universidad del País Vasco, Bilbao, 2001.

[18] I. Schur, *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **127** (1904) 20–50.

[19] C. T. C. Wall, *Norms of units in group rings*, Proc. London Math. Soc. (3) **29** (1974), 593–632.

[20] J. Rosenberg, *Algebraic K-theory and its applications*, Graduate Texts in Mathematics 147, Springer-Verlag, 1994.

[21] D. Kazhdan, *Proof of Springer's Hypothesis*, Israel Journal of Mathematics **28** (1977) no. 4, 272–286.

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