# AN INFINITE COMPACT HAUSDORFF GROUP HAS UNCOUNTABLY MANY CONJUGACY CLASSES

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Abstract. We show that an infinite compact Hausdorff group has uncountably many conjugacy classes.

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## 1. Introduction

A compact group is a topological group whose topology is compact. All groups we consider are assumed to be Hausdorff spaces. The study of algebraic properties of compact groups is a well-established topic in Group Theory. Normal subgroups and homomorphic images of compact groups have been recently studied in [7, 8] (see also [6]). In this paper we consider properties of conjugacy classes of a compact group and we prove the following result.

**Theorem 1.1.** *The number of conjugacy classes in an infinite compact Hausdorff group is uncountable.*

The structure of an arbitrary compact group $G$ breaks into two pieces in a natural way. Let $G^0$ be the identity component of $G$. Then $G^0$ is normal and $G/G^0$ is a profinite group. The structure of $G^0$ is also well understood: $G^0 = ZP$, where $Z$ is the center of $G^0$ and $P \cong \prod S_i/D$ is a Cartesian product of compact connected simple Lie groups $S_i$ modulo a central subgroup $D$. Therefore, problems about a general compact group $G$ reduce to two cases, first dealing with Lie groups and second with profinite groups. The Lie group part of the proof of Theorem 1.1 is straightforward. However, the profinite part is more sophisticated. In particular, the proof relies, indirectly, on the Classification of the Finite Simple Groups via Hartley's generalization of the Brauer-Fowler theorem and it also uses the Hall-Higman theory and the theory of finite groups with almost regular automorphisms.

One of the first consequences for a compact group $G$ to have countably many conjugacy classes is to have an open conjugacy class. In our proof we use that, in fact, $G$ has many conjugacy classes with this property. It will be interesting to investigate what we can obtain if we assume this condition only on a single conjugacy class. Topological properties of conjugacy classes in totally disconnected locally compact groups have been studied in [11], in particular it is proved there that such groups cannot have a comeager conjugacy class. The following question is still open.

**Question 1.2** ( see also remark 3.5 (2) of [11].)**.** *Let $G$ be a compact Hausdorff group. Assume that $G$ has an open conjugacy class. Is it true that $G$ is virtually soluble?*

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This question is related to the following well-known problem in finite group theory.

**Question 1.3.** *Is there a function $f = f(m)$ such that a finite group $G$, having an element $g$ satisfying $|C_G(g)| \leq m$, contains a soluble subgroup of derived length and of index bounded by $f(m)$?*

More details about this problem can be found, for example, in [5].

After this paper was written John Wilson obtained a more general result for profinite groups: a profinite group $G$ has fewer than $2^{\aleph_0}$ conjugacy classes of $p$-elements (for a prime $p$) if and only if its Sylow pro-$p$ subgroups are finite.

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## 2. The proof of Theorem 1.1

Let $G$ be a compact group and assume that $G$ has countable number of conjugacy classes. We want to show that $G$ is finite. We divide our proof in several steps.

**Step 2.1.** *Let $H$ be an open subgroup of $G$ and let $N$ be a closed normal subgroup of $H$. Then $H/N$ has countably many conjugacy classes.*

*Proof.* Since $G$ is compact, $H$ is of finite index, say $n$ in $G$. Let $T = \{t_1, \dots, t_n\}$ be a left transversal of $G$ in $H$: $G = \cup_{i=1}^n t_i H$. Then for every $g \in H$, $g^G = \cup_{i=1}^n (g^{t_i})^H$ contains at most $n$ different $H$-conjugacy classes. Thus, $H$ and also $H/N$ have countably many conjugacy classes. □

**Step 2.2.** *Let $U$ be a closed subset of $G$. Then there exists $g \in U$ such that $g^G$ contains an open subset of $U$.*

*Proof.* Since $G$ is compact all conjugacy classes of $G$ are closed. Thus, $U = \cup_{i=1}^\infty (g_i^G \cap U)$ is a union of countable number of closed sets. Then the claim follows from the Baire category theorem. □

**Step 2.3.** *Assume that $G$ is profinite and let $U$ be a clopen subset of $G$. Then there exists $g \in U$ of finite order and $C > 0$, such that for any normal open subgroup $N$ of $G$, $|C_{G/N}(gN)| \leq C$ (see also Lemma 3.1 of* [11]*).*

*Proof.* By the previous step, there exists $g \in U$ such that $g^G$ contains an open subset of $G$. Hence $g^G$ contains $gH$ for some open subgroup $H$ of $G$ and so

$$|C_{G/N}(gN)| = \frac{|G/N|}{|g^G N/N|} \leq \frac{|G/N|}{|HN/N|} \leq |G : H|$$

for every normal open subgroup $N$ of $G$.

Since the sizes of $C_{G/N}(gN)$ are uniformly bounded when $N$ runs over all the open normal subgroups of $G$, the order of $g$ should be finite. $\square$

Recall that a prosolvable group is a profinite group isomorphic to an inverse limit of finite solvable groups. Any finite group $G$ has a unique maximal solvable normal subgroup $\mathrm{sol}(G)$, called the solvable radical of $G$. Similarly, a profinite group $G$ has a unique maximal normal prosolvable subgroup $\mathrm{sol}(G)$ called the prosolvable radical of $G$.

**Step 2.4.** *Assume that $G$ is profinite. Then the prosolvable radical of $G$ is open.*

*Proof.* By Step 2.3, there exists an element $g \in G$ of finite order and a constant $C$ such that for any normal open subgroup $N$ of $G$, $|C_{G/N}(gN)| \leq C$. Hence, by Hartley's generalization of the Brauer-Fowler theorem [2, Theorem A], there exists a constant $K$ depending only on $C$ such that

$$|G/N : \mathrm{sol}(G/N)| \leq K.$$

Let $K_0 = \max\{|G/N : \mathrm{sol}(G/N)| : \ N \trianglelefteq_o G\}$ and consider $\mathcal{S} = \{N \trianglelefteq_o G : \ |G/N : \mathrm{sol}(G/N)| = K_0\}$. For any $N \trianglelefteq_o G$ let $S_N$ be the subgroup of $G$, containing $N$, such that $S_N/N = \mathrm{sol}(G/N)$.

If $N_1 \leq N_2$ are open normal subgroups of $G$, then the canonical image of $S_{N_1}/N_1$ in $G/N_2$ is contained in $\mathrm{sol}(G/N_2)$. Hence $S_{N_1} \leq S_{N_1}N_2 \leq S_{N_2}$. Thus, if $N_2$ is in $\mathcal{S}$, then $S_{N_1} = S_{N_2}$ because $K_0 \geq |G : S_{N_1}| \geq |G : S_{N_2}| = K_0$. From this we also obtain that $N_1 \in \mathcal{S}$.

On one hand the argument from the previous paragraph implies that if $L \trianglelefteq_o G$ and $N \in \mathcal{S}$, then $N \cap L \in \mathcal{S}$. Therefore, $\mathcal{S}$ is a base of neighborhoods of 1 in $G$. On the other hand, we also have that if $N_1, N_2 \in \mathcal{S}$, then

$$S_{N_1} = S_{N_1 \cap N_2} = S_{N_2}.$$

Hence, if $N \in \mathcal{S}$, $S = S_N$ does not depend on $N$. Since $\mathcal{S}$ is a base of neighborhoods of 1 in $G$, we obtain that $S$ is isomorphic to the inverse limit of $\{S/N = \mathrm{sol}(G/N)\}_{N \in \mathcal{S}}$, and so, $S$ is prosolvable. Thus, $\mathrm{sol}(G)$ is open in $G$. $\square$

A (profinite) order is an abstract expression $\prod_i p_i^{k_i}$ where $p_i$ runs over all the primes and $k_i \in \{0, 1, 2, \ldots\} \cup \{\infty\}$. We say that $p_i$ divides $\prod_i p_i^{k_i}$ if $k_i \neq 0$. For any natural number $a$ and prime $p$ we denote by $\mathrm{ord}_p(a)$ the number $k$ such that the $p$-part of $a$ is equal to $p^k$.

Let $G$ be a profinite group, then the (profinite) order of $G$ is $\prod_i p_i^{k_i}$, where $k_i = \sup\{\mathrm{ord}_{p_i}(|G/N|) : \ N \trianglelefteq_o G\}$. The (profinite) order of an element $g \in G$ is the order of the cyclic profinite subgroup generated by $g$. Observe that the order of an element is a conjugacy invariant.

**Step 2.5.** *Assume that $G$ is profinite. Then the order of any element $g$ of $G$ is divisible by finitely many primes.*

*Proof.* Let $C$ be the cyclic profinite subgroup generated by $g$ and let $C_p$ be its Sylow pro-$p$ subgroup. Let $P$ be the set of primes $p$ such that $C_p \neq \{1\}$. For any $p \in P$, we choose some $a_p \in C_p \backslash \{1\}$. For any subset $S$ of $P$ we put $a_S = \prod_{p \in S} a_p$. Let $S_1$ and $S_2$ two subsets of $P$. Observe that the orders of $a_{S_1}$ and $a_{S_2}$ coincide if and only if $S_1 = S_2$. Any infinite set contains uncountably many subsets. Thus, since the order of an element is a conjugacy invariant, $P$ can not be infinite. $\square$

Let $G$ be a finite group. We denote by $F(G)$ the Fitting subgroup of $G$. This is the product of all normal nilpotent subgroups. We set $F_0(G) = \{1\}$ and if $i \geq 1$, $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$. If $G$ is a solvable group, the least number $h$ such that $G = F_h(G)$ is called Fitting height of $G$.

Recall that a pronilpotent group is a profinite group isomorphic to an inverse limit of finite nilpotent groups. If $G$ is profinite group, we put $F^0(G) = G$ and we denote by $F^1(G)$ the least normal closed subgroup of $G$ such that $G/F^1(G)$ is pronilpotent and if $i \geq 1$, $F^{i+1}(G) = F^1(F^i(G))$. If $G$ is finite solvable, $G$ has Fitting height $h$ if $F^h(G) = \{1\}$ and $F^{h-1}(G) \neq \{1\}$. Thus, we say that a prosolvable group $G$ has Fitting height $h$ if $F^h(G) = \{1\}$ and $h$ is the smallest number satisfying this property.

The following step can be obtained from Step 2.5, using a result of W. Herfort [3]. We include its proof for the completeness of exposition.

**Step 2.6.** *Assume that $G$ is prosolvable group. Then $G$ is a pro-$\pi$ group for a finite set of primes $\pi$.*

*Proof.* By Steps 2.2 and 2.3, there exists an element $g \in G$ of finite order and a constant $C$ such that for any normal open subgroup $L$ of $G$, $|C_{G/L}(gL)| \leq C$ and $gN \subseteq g^G$ for some open normal subgroup $N$ of $G$.

Let $\pi$ the set of primes dividing the order of $g$. Denote by $N_{\pi'}$ a pro-$\pi'$ Hall subgroup of $N$ (since $N$ is prosolvable, a pro-$\pi'$ Hall subgroup exists). Since all the pro-$\pi'$ Hall subgroups of $N$ are conjugated in $N$, there exists $n \in N$ such that $N_{\pi'}^{gn} = N_{\pi'}$. We put $h = gn$. Observe that $h \in gN \subseteq g^G$. Hence the order of $h$ is equal to the order of $g$.

Let $L \leq N$ be an open normal subgroup of $G$. Then $h$ acts on $N_{\pi'}L/L$. Take $aL \in C_{N_{\pi'}L/L}(h)$. The order of $haL$ is equal to the products of the orders of $hL$ and $aL$, because they are coprime. On the other hand, since $ha \in gN$, the order of $ha$ is equal to the order of $g$. These together imply that the order of $aL$ is one, i.e. $aL = L$ and we conclude that $h$ acts fixed point freely on $N_{\pi'}L/L$. Applying a result of J. Thompson [10, Corollary], we obtain that the Fitting heights of the quotients $N_{\pi'}L/L$ are uniformly bounded. Hence the Fitting height of $N_{\pi'}$ is finite (let say $f$).

Assume that for some $0 \leq i \leq f-1$, the profinite order of $F^i(N_{\pi'})/F^{i+1}(N_{\pi'})$ is divided by infinitely many primes $\{p_j\}$. Take $a_j \in G$ such that $a_jF^{i+1}(N_{\pi'})$ is not trivial element of the Sylow pro-$p_j$ subgroup of $F^i(N_{\pi'})/F^{i+1}(N_{\pi'})$. Then the order of $\prod a_j$ is divisible by infinitely many primes $\{p_j\}$. This contradicts Step 2.5. Thus for every $0 \leq i \leq f-1$, the profinite order of $F^i(N_{\pi'})/F^{i+1}(N_{\pi'})$ is divided by finitely many primes. Hence, the same is true for $N_{\pi'}$. Since $N$ is open in $G$ and $\pi$ is finite, we are done. □

Let $G$ be a profinite group. We say that $G$ has a finite $p$-length if there exists a series of normal closed subgroups of $G$

$$\{1\} = N_{-1} \leq P_0 < N_0 < P_1 < N_1 < \ldots < P_h \leq N_h = G,$$

such that $P_i/N_{i-1}$ is pro-$p$ and $N_i/P_i$ is pro-$p'$. The smallest possible $h$ is called the $p$-length of $G$. Clearly the $p$-length of a profinite group is equal to the supremum of the $p$-lengths of its finite quotients.

**Step 2.7.** *Assume that $G$ is prosolvable group. Let $p > 2$ be a prime. Then $G$ is of finite $p$-length.*

*Proof.* Let $S_p$ be the set of pro-$p$ elements of $G$. Observe that $S_p$ is a closed set. Hence, by Step 2.2, there exists $g \in S_p$ and an open normal subgroup $H$ of $G$ such that $gH \cap S_p \subset g^G$. First let us show that $g$ has finite order.

Applying Step 2.3 with $U = gH$, we obtain that there exists $h \in gH$ of finite order such that $h^G$ contains an open set. Let $h_p$ be the $p$-part of $h$. Now $hH = gH$ is a $p$-element of $G/H$ because we chose $g \in S_p$. Therefore $h_pH = gH$. Hence $h_p \in gH \cap S_p \subseteq g^G$. Thus $g$ is of finite order.

Let $P$ be a Sylow pro-$p$ subgroup of $G$ containing $g$. We have just shown that the order of elements of $g(P \cap H) = gH \cap P \subseteq g^G$ is finite and uniformly bounded. By [12, Theorem 3*], $H$ has finite $p$-length. Since $G$ is prosolvable and $H$ is open in $G$, $G$ also has finite $p$-length.

□

**Step 2.8.** *Let $p$ be a prime and assume that $G$ is a pro-$p$ group. Then $G$ is finite.*

*Proof.* By Step 2.3, there exists $g \in G$ and a constant $C$ such that for every an open normal subgroup $N$ of $G$, $|C_{G/N}(gN)| \leq C$. By a result of A. Shalev, [9, Theorem A'], the derived length of $G/N$ is bounded by some number which depends only on $C$. Hence $G$ is soluble. If $G$ was infinite, it would have an infinite virtually abelian quotient. Thus, Step 2.1 would imply that an infinite abelian profinite group has a countable number of conjugacy classes. But this is impossible, because an infinite profinite group is uncountable. Therefore, $G$ is finite. □

**Step 2.9.** *Assume that $G$ is prosolvable. Then $G$ is finite.*

*Proof.* By Step 2.6, only finitely many primes divide the order of $G$. We will prove the statement by induction on the number of primes dividing the order of $G$. The base of induction, when $G$ is pro-$p$, is considered in Step 2.8. Assume that we have proved that $G$ is finite if its order is divided by at most $n \geq 1$ primes. Let us now consider $G$ which order is divided by $n + 1$ primes.

Let $p$ be an odd prime dividing the order of $G$. By Step 2.7, $G$ has finite $p$-length. Thus, there exists a series of normal closed subgroups of $G$

$$\{1\} = N_{-1} \leq P_0 < N_0 < P_1 < N_1 < \ldots < P_h \leq N_h = G,$$

such that $P_i/N_{i-1}$ is pro-$p$ and $N_i/P_i$ is pro-$p'$. Now $G/P_h$ has countably many conjugacy classes and hence by the inductive assumption $G/P_h$ is finite. Step 2.1 gives that $P_h/N_{h-1}$ has countably many conjugacy classes and is therefore finite by Step 2.8. Continuing in the same way we conclude that $N_i/P_i$ and $P_i/N_{i-1}$ are finite for all $i = h, h-1, \ldots, 0$. Hence $G$ is finite. □

**Step 2.10.** *Assume that $G = G^0$ is connected. Then $G$ is trivial.*

*Proof.* By way of contradiction, we assume that $G$ is non-trivial. Therefore, since $G$ is connected, it is infinite. By [4, Corollary 2.43], $G$ has an infinite quotient isomorphic to a compact connected Lie group. Thus, without loss of generality we may assume that $G$ is a non-trivial compact connected Lie group.

Let $g \in G$. Then, by [4, Theorem 6.30], the centralizer $C_G(g)$ is of positive dimension (as a real manifold). Hence, the dimension of $g^G$ is less than the dimension of $G$, and so, the Haar measure of $g^G$ is 0. Thus, $G$ can not be the union of countable number of conjugacy classes. We have obtained a contradiction. Therefore, $G$ is trivial. □

**Step 2.11.** *$G$ is finite.*

*Proof.* Let $\bar{G} = G/G^0$. Then by van Dantzig's Theorem (see Appendix B5 in [1]), $\bar{G}$ is a profinite group with countable number of conjugacy classes. By Step 2.4, $\mathrm{sol}(\bar{G})$ is open in $\bar{G}$. Thus $\mathrm{sol}(\bar{G})$ is a prosolvable group with countable number of conjugacy classes. By Step 2.9, $\mathrm{sol}(\bar{G})$ is finite. Therefore, $\bar{G}$ is finite and so $G^0$ is open in $G$. Hence, $G^0$ is a connected compact group with countable number of conjugacy classes. By Step 2.10, $G^0$ is trivial and so $G$ is finite. □

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