Cohomological properties of the profinite completion of Bianchi groups

F. Grunewald, A. Jaikin-Zapirain, P. A. Zalesskii *

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Abstract

We prove that the Bianchi groups, that is the groups $\mathrm{PSL}(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field, are good. This is a property introduced by J.P. Serre which relates the cohomology groups of a group to those of its profinite completion. We also develop properties of goodness to be able to show that certain natural central extensions of Fuchsian groups are residually finite. A result which contrasts examples of Deligne who shows that the analogous central extensions of $\mathrm{Sp}(4, \mathbb{Z})$ do not have this property.

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1 Introduction

Let G be a group and $\widehat{G}$ its profinite completion. The group G is called *good* if the homomorphism of cohomology groups

$$H^n(\widehat{G}, M) \longrightarrow H^n(G, M)$$

induced by the natural homomorphism $G \longrightarrow \widehat{G}$ of G to its profinite completion $\widehat{G}$ is an isomorphism for every finite G-module M. This important concept was introduced by J.P. Serre in [29].

It is known that finitely generated free groups and surface groups are good. From Lemma 3.2 it follows that finitely generated virtually free groups are good and also that a succession of extensions of finitely generated free groups is good. It is however in general very difficult to say which group is good and which is not. It is for example an important open question whether the mapping class groups are good.

In our paper we prove this property for a particularly important class of arithmetic Kleinian groups: Bianchi groups that are defined as $\mathrm{PSL}(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ ($d \in \mathbb{Z}$, $d \geq 1$). One of our main results is:

Theorem 1.1. *The Bianchi groups are good.*

Goodness is preserved by commensurability (see Section 3), so our Theorem 1.1 holds for all non-cocompact arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

Theorem 1.1 is designed to begin the study of the torsion cohomology of the Bianchi groups. In fact the goodness of these groups implies

Corollary 1.2. *The virtual cohomological dimension of the profinite completion $\widehat{\mathrm{PSL}}(2, \mathcal{O}_d)$ is equal to two. The congruence kernel $C_d \leq \widehat{\mathrm{PSL}}(2, \mathcal{O}_d)$ has cohomological dimension equal to one or two.*

The first statement comes from the fact that the Bianchi groups act discontinuously on 3-dimensional hyperbolic space with a finite volume quotient which is not compact (see Section 4.1). The proof of the second statement is contained in Section 4.2 where we also explain the construction of the congruence kernel. We where not able to decide whether C_d has cohomological dimension one or two.

We observe (in Section 4.2) that an arithmetic group having the congruence subgroup property is not good, since the profinite completion of it is not virtually torsion free and therefore its virtual cohomological dimension is infinite.

Another class of groups proved to be good in this paper are the so called limit groups, i.e. finitely generated fully residually free groups. Limit groups play a key role in the solution of the Tarski problems ([8], [9], [10], [21]-[26]) that asks whether the elementary theories of non-abelian free groups of different ranks are the same and whether this theory is decidable.

Theorem 1.3. *Limit groups are good.*

We also give a new way of applying the goodness of lattices in $\mathrm{PSL}(2,\mathbb{R})$. We show that an extension of a finitely generated residually finite good group with finitely generated residually finite kernel is residually finite. As a consequence it is deduced (in Section 3.2) that certain natural central extensions of Fuchsian groups are residually finite. A result which contrasts examples of Deligne [3] and Raghunathan [19] who show that the analogous central extensions of $\mathrm{Sp}(4,\mathbb{Z})$ and other arithmetic groups do not have this property.

The idea of the proof of Theorem 1.1 is to use that Bianchi groups admit a so called hierarchy, i.e. decomposition as a tower of free amalgamated products or HNN-extensions of finitely generated subgroups starting with the trivial subgroup. This hierarchy behaves well with respect to the profinite topology: we refer to this fact as the profinite topology being efficient. We give a general definition of a hierarchy for a group G in Section 3.1. The hierarchy of a group with efficient profinite topology is preserved in the profinite completion which allows us to use inductively the Mayer-Vietoris sequence. Thus in the most general form our result can be formulated as follows.

Theorem 1.4. *Let G be a group admitting a hierarchy such that the profinite topology on G is efficient (with respect to the given hierarchy). Then G is good.*

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2 Homology and cohomology of profinite groups

In this section we collect some notation and well known facts concerning the homology and cohomology of profinite groups.

Let G be a profinite group and A a discrete G-module. We define the cohomology group $H^q(G,A)$ $(q \in \mathbb{N} \cup \{0\})$ by

$$H^q(G,A) = \varinjlim H^q(G/U, A^U),$$

where U ranges over all open normal subgroups of G and A^U is the submodule of fixed points of U.

Similarly, let B be a profinite G-module. The homology group $H_q(G,B)$ $(q \in \mathbb{N} \cup \{0\})$ is defined as a projective limit

$$H_q(G,B) = \varprojlim H_q(G/U, B_U),$$

where U ranges over all open normal subgroups of G and

$$B_U = B/\overline{\langle\, ub - b \mid b \in B,\ u \in U\,\rangle}$$

is the largest quotient module of B with trivial action of U.

Let M be a locally compact abelian group. The dual group

$$M^* := \mathrm{Hom}(M, \mathbb{R}/\mathbb{Z})$$

is a locally compact topological group as well and M^{**} is topologically isomorphic to M. Note that if M is profinite then M^* is torsion discrete and vice versa (see Section 2.9 in [20] for details). The cohomology and homology of profinite groups are Pontryagin dual to each other, i.e. $H_q(G,B) = H^q(G,B^*)^*$ (see 6.3.6 in [20]).

The p-cohomological dimension of a profinite group G is the lower bound of the integers n such that for every discrete torsion G-module A, and for every $q > n$, the p-primary component of $H^q(G,A)$ is null. We shall use the standard notation $\mathrm{cd}_p(G)$ for p-cohomological dimension of the profinite group G. The cohomological dimension $\mathrm{cd}(G)$ of G is defined as the supremum $\mathrm{cd}(G) = \sup_p(\mathrm{cd}_p(G))$ where p varies over all primes p.

The next proposition gives a well-known characterization for cd_p.

Proposition 2.1. *Let G be a profinite group, p a prime and n an integer. The following properties are equivalent:*

1. $\mathrm{cd}_p(G) \leq n$,
2. $H^q(G,A) = 0$ *for all* $q > n$ *and every discrete* G*-module* A *which is a* p*-primary torsion module,*
3. $H^{n+1}(G,A) = 0$ *when* A *is simple discrete* G*-module annihilated by* p.
4. $H^{n+1}(H,\mathbb{F}_p) = 0$ *for any open subgroup* H *of* G.

Note that if G is pro-p then there is only one simple discrete G-module annihilated by p, namely the trivial module $\mathbb{F}_p$.

3 Goodness

Following [29] we say that a group G is *good* if the homomorphism of cohomology groups $H^n(\widehat{G},M) \to H^n(G,M)$ induced by the natural homomorphism $G \to \widehat{G}$ of G to its profinite completion $\widehat{G}$ is an isomorphism for all n and every finite G-module M.

We recall two simple facts concerning this concept which are already contained in [29]. We call two groups *commensurable* if they contain isomorphic normal subgroups of finite index.

Lemma 3.1. *Let G be a good group and H a group commensurable with G. Then H is good.*

Proof. By Exercise 1 in 2.6 [29] a group is good if and only if

$$\varinjlim_{N \lhd_f G} H^i(N,M) = 0$$

for all i and every finite module M, where N ranges over all normal subgroups of finite index. Since this limit can be started with any N of finite index, the result follows. $\square$

The following is exercise 2 (b) on page 16 of [29].

Lemma 3.2. *The group H is good if there is a short exact sequence*

$$\langle 1\rangle \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow \langle 1\rangle$$

such that G, N are good and if the cohomology groups $H^q(N, M)$ are finite for all q ($q \in \mathbb{N}$) and all finite H-modules M.

3.1 Amalgamated products and HNN-extensions

In this subsection we give sufficient conditions for an amalgamated free product and an HNN-extension of good groups to be good. We apply these results in the next section to show that the Bianchi groups are good. We would like to stress that it is natural to ask when an amalgamated free product or HNN-extension preserves goodness.

We remind the reader of two basic constructions of combinatorial group theory.

Let G_1, G_2 be groups, H a subgroup of G_1 and $f : H \to G_2$ an embedding. Then the amalgamated free product $G_1 *_H G_2$ is given by the presentation

$$G_1 *_H G_2 = \langle\, G_1,\, G_2 \mid \mathrm{rel}(G_1),\, \mathrm{rel}(G_2),\, h = f(h),\ h \in H\,\rangle.$$

By this notation we mean that $G_1 *_H G_2$ is generated by G_1, G_2 and defined by the relations $\mathrm{rel}(G_1)$, $\mathrm{rel}(G_2)$ of the groups G_1, G_2 together with the extra relations $h = f(h)$, $(h \in H)$.

Let G be a group, H a subgroup of G and $f : H \to G$ a monomorphism. Then the HNN-extension $\mathrm{HNN}(G, H, f)$ is given by the presentation

$$\mathrm{HNN}(G, H, f) = \langle\, G, t \mid \mathrm{rel}(G),\, tht^{-1} = f(h),\ h \in H\,\rangle.$$

Following [31] we say that the profinite topology on an amalgamated free product $G = G_1 *_H G_2$ is *efficient* if G is residually finite, the profinite topology on G induces the full profinite topology on G_1, G_2, H, and if G_1, G_2, H are closed in the profinite topology on G.

Similarly, we say that the profinite topology on an HNN-extension $\mathrm{HNN}(G, H, f)$ is *efficient* if G is residually finite, the profinite topology on G induces the full profinite topology on $G, H, f(H)$, and if G, H, $f(H)$ are closed in the profinite topology on G.

We have:

Proposition 3.3. *Let G be an amalgamated product or an HNN-extension of good groups and let the profinite topology on G be efficient. Then G is good.*

Proof. We start the proof with the case of HNN-extension.

Let $G = \mathrm{HNN}(K, A, f)$ be an HNN-extension of a good group K with an associated good subgroup A such that the profinite topology of G is efficient. First note that the efficiency implies that the profinite completion $\widehat{G}$ is a profinite

HNN-extension $\mathrm{HNN}(\widehat{K}, \widehat{A}, \hat{f})$, where $\hat{f} : \widehat{A} \to \widehat{K}$ is the continuous homomorphism of the completions induced by f. Moreover, this profinite HNN-extension is proper in sense of [20], i.e. $\widehat{K}, \widehat{A}$ are embedded in $\mathrm{HNN}(\widehat{K}, \widehat{A}, \hat{f})$ (cf. [31]). Consider the Mayer-Vietoris sequence associated to G and $\widehat{G}$:

$$\begin{array}{ccccccc} H^{n-1}(A, M) & \to & H^n(G, M) & \to & H^n(K, M) & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ H^{n-1}(\widehat{A}, M) & \to & H^n(\widehat{G}, M) & \to & H^n(\widehat{K}, M) & \to & \cdots \end{array}$$

where the vertical maps are induced by the natural embedding of the groups into their profinite completions. Since A and K are good the left vertical map and the right vertical map are isomorphisms, so the middle vertical map is an isomorphism as well. Since $H^0(G, M) = M^G = M^{\widehat{G}} = H^0(\widehat{G}, M)$ the result follows in case of HNN-extensions.

Next we consider the case of an amalgamated free product. Let $G = K_1 *_A K_2$ be an amalgamated free product of good groups K_1, K_2 with an amalgamated good subgroup A such that the profinite topology of G is efficient. First note that the efficiency implies that the profinite completion $\widehat{G}$ is a profinite amalgamated free product $\widehat{K}_1 \amalg_{\widehat{A}} \widehat{K}_2$. Moreover, this profinite amalgamated free product is proper in sense of [20], i.e. $\widehat{K}_1$, $\widehat{K}_2, \widehat{A}$ are embedded in $\widehat{K}_1 \amalg_{\widehat{A}} \widehat{K}_2$ (cf. [31]). Consider the Mayer-Vietoris sequence associated to G and $\widehat{G}$:

$$\begin{array}{ccccccc} H^{n-1}(A, M) & \to & H^n(G, M) & \to & H^n(K_1, M) \oplus H^n(K_2, M) & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ H^{n-1}(\widehat{A}, M) & \to & H^n(\widehat{G}, M) & \to & H^n(\widehat{K}_1, M) \oplus H^n(\widehat{K}_2, M) & \to & \cdots \end{array}$$

where the vertical maps are induced by the natural embedding of the groups to their profinite completions. Since A, K_1 and K_2 are good the left vertical map and the right vertical map are isomorphisms, so the middle vertical map is an isomorphism as well. Since $H^0(G, M) = M^G = M^{\widehat{G}} = H^0(\widehat{G}, M)$ the result follows. □

We shall now discuss an immediate application of the previous proposition.

Following [14] we shall call a group G to be an $\mathcal{F}$-group if G has a presentation of the form:

$$G = \langle\, a_1, b_1, \ldots, a_n, b_n,\ c_1, \ldots c_t,\ d_1, \ldots d_s \mid c_1^{e_1} = \ldots = c_t^{e_t} = 1,$$
$$d_1^{-1} \ldots d_s^{-1} c_1^{-1} \ldots c_t^{-1} [a_1, b_1] \ldots [a_n, b_n] = 1 \,\rangle$$

where n, s, $t \geq 0$, and $e_i > 1$ for $i = 1, \ldots, t$. $\mathcal{F}$-groups are exactly the groups which appear as lattices, i.e. discrete subgroups of finite covolume, in $\mathrm{PSL}(2, \mathbb{R})$. A torsion free $calF$-group Γ is called a surface group if it has $2g$ generators a_i, b_i $(i = 1, \ldots, g)$ subject to one relation $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1$. Surface groups are exactly the groups which appear as fundamental groups of closed surfaces. By Proposition III.7.4 in [14] any subgroup of finite index of an $\mathcal{F}$-group is again a $\mathcal{F}$-group and so a torsion free subgroup of a $\mathcal{F}$-group of finite index is a finitely generated free group or a surface group.

Proposition 3.4. *All $\mathcal{F}$-groups are good.*

Proof. By Lemma 3.1 it suffices to proof that surface groups Γ are good. Clearly, Γ admits a decomposition into a free product with amalgamation

$$\Gamma = \langle a_1, b_1 \rangle *_C \langle a_2, b_2, \dots a_g, b_g \rangle$$

of two free group with a cyclic amalgamation defined by

$$[a_1, b_1]^{-1} = [a_2, b_2] \cdots [a_g, b_g].$$

Using Proposition 3.3 it is clear that Γ is good (see Lemma 5.2 (iii) in [5] for a more detailed proof). $\square$

The following concept is useful in the next section

Definition 3.5. Let G be a group. A hierarchy for G is a finite collection $\mathcal{T}_0, \dots, \mathcal{T}_N$ of tuples of finitely generated subgroups

$$\mathcal{T}_r = (G_1^{[r]}, \dots, G_{n_r}^{[r]}) \qquad (r = 0, \dots, N,\ n_r \in \mathbb{N})$$

of G such that

- $\mathcal{T}_0 = (G)$,
- the coordinates of $\mathcal{T}_N$ are all trivial groups,
- for every $r \geq 0$ and $s = 1, \dots n_r$ there exists either $1 \leq i \leq n_{r+1}$ and a subgroup F of $G_i^{[r+1]}$ which is a $\mathcal{F}$-group such that

$$G_s^{[r]} = \mathrm{HNN}(G_i^{[r+1]}, F, t)$$

  or there exist $1 \leq i \neq j \leq n_{r+1}$ and a subgroup F of both $G_i^{[r+1]}$ and $G_j^{[r+1]}$ which is a $\mathcal{F}$-group such that

$$G_s^{[r]} = G_i^{[r+1]} *_F G_j^{[r+1]}.$$

We say that the profinite topology on a group G admitting such a hierarchy is *efficient* if G is residually finite and if the profinite topology on G induces the full profinite topology on all $G_s^{[r]}$ and F and if the groups $G_s^{[r]}$ and F are closed in the profinite topology of G.

3.2 An application of goodness

In this section we apply the concept of goodness in order to show that certain natural central extensions of Fuchsian groups are residually finite.

Proposition 3.6. *Let G be a residually finite good group and $\varphi : H \to G$ a surjective homomorphism with finite kernel K. Then H is residually finite.*

Proof. Lemma 3.2 shows that any extension of a good group by a finite group is good. Therefore, by induction, we may assume that K is a minimal normal subgroup of H. We distinguish two cases:

Case 1: K is abelian. The action of H on K by conjugation turns K into a finite G-module. The elements of $H^2(G,K)$ correspond to classes of extensions

$$\langle 1\rangle \longrightarrow K \to E \longrightarrow G \longrightarrow \langle 1\rangle$$

of G while the elements of $H^2(\widehat{G},K)$ correspond to classes of profinite extensions

$$\langle 1\rangle \longrightarrow K \to F \longrightarrow \widehat{G} \longrightarrow \langle 1\rangle$$

of $\widehat{G}$. Let $\omega : G\times G \to K$ be a 2-cocycle representing the extension $K \to H \to G$. Since the map $H^2(\widehat{G},K) \to H^2(G,K)$ induced by the inclusion $G \to \widehat{G}$ is an isomorphism we may choose a continuous 2-cocycle $\widehat{\omega} : \widehat{G} \times \widehat{G} \to K$ which restricts to ω on G. Let $\widehat{G}_\omega \to \widehat{G}$ be the corresponding group extension. There is a homomorphism $\psi : H \to \widehat{G}_\omega$ such that the following diagram is commutative:

$$\begin{tikzcd}
\langle 1\rangle \arrow[r] & K \arrow[r] & \widehat{G}_\omega \arrow[r] & \widehat{G} \arrow[r] & \langle 1\rangle \\
\langle 1\rangle \arrow[r] & K \arrow[r] \arrow[u, "="] & H \arrow[r] \arrow[u, "\psi"] & G \arrow[r] \arrow[u] & \langle 1\rangle .
\end{tikzcd}$$

It follows that ψ is injective and hence H is residually finite.

Case 2: K is not abelian. Here we again consider the action of H on its normal subgroup K by conjugation. This action gives rise to a homomorphism $H \to \mathrm{Aut}(K)$ of H to the automorphism group of K. Let N be the kernel of this homomorphism. Since K is finite N is a normal subgroup of finite index in H. Since K is minimally normal in H we have $N \cap K = \langle 1\rangle$. This implies that φ maps N injectively to a subgroup of finite index in G. We infer that N and hence H are residually finite. □

Corollary 3.7. *Let G be a residually finite good group and $H \to G$ a surjective homomorphism with residually finite and finitely generated kernel K. Then H is residually finite.*

Proof. Since K is finitely generated and residually finite, it has a sequence of characteristic subgroups K_n $(n \in \mathbb{N})$ of finite index such that the intersection of the K_n is trivial. The quotient groups H/K_n are residually finite for all $n \in \mathbb{N}$ by Proposition 3.6. This easily implies that H itself is residually finite. □

We shall now give an application of the preceding considerations.

Let $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ be the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$. The kernel Z of the covering homomorphism

$$\pi : \widetilde{\mathrm{PSL}}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$$

is infinite cyclic. Given a subgroup $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ we define

$$\Gamma_0 := \pi^{-1}(\Gamma), \qquad \Gamma_n := \pi^{-1}(\Gamma)/nZ \quad (n \in \mathbb{N}). \tag{3.1}$$

From Proposition 3.4 and Corollary 3.7 we get

Proposition 3.8. *Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a lattice (Fuchsian group) then Γ_0 and all Γ_n $(n \in \mathbb{N})$ are residually finite.*

Proposition 3.8 should be contrasted with examples of Deligne [3]. He considers subgroups of finite index in the integral symplectic group

$$\Gamma \leq \mathrm{Sp}(4,\mathbb{Z}) \leq \mathrm{Sp}(4,\mathbb{R}).$$

He shows that their inverse images in the universal cover of $\mathrm{Sp}(4,\mathbb{R})$ are not residually finite. In his arguments the congruence subgroup property of $\mathrm{Sp}(4,\mathbb{Z})$, i.e. the triviality of the congruence kernel, plays an essential role (compare Section 4.2). Similar results for cocompact discrete subgroups of $\mathrm{Spin}(2,n)$ are contained in [19]. For more on this theme see [2], [30].

4 Good and not good groups

This section contains our main result in its first subsection. In the second subsection we describe S-arithmetic groups which are not good.

4.1 Bianchi groups

In this section we prove that all Bianchi groups are good.

Fix a natural number d and let $\mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$ be the corresponding imaginary quadratic number field. Let $\mathcal{O}_d$ be its ring of integers. The groups $\mathrm{PSL}(2,\mathcal{O}_{\mathrm{d}})$ are traditionally called Bianchi groups. They are discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$, hence act discontinuously on the symmetric space

$$\mathbb{H}^3 := \mathrm{PSL}(2,\mathbb{C})/\mathrm{PSU}(2,\mathbb{C})$$

of $\mathrm{PSL}(2,\mathbb{C})$. For a detailed description and the basic properties of them see [4], [16]. Let $\Gamma \leq \mathrm{PSL}(2,\mathcal{O}_{\mathrm{d}})$ be a torsion free subgroup of finite index (such subgroups exist since $\mathrm{PSL}(2,\mathcal{O}_{\mathrm{d}})$ is finitely generated and linear). The quotient

$$X_\Gamma := \Gamma \backslash \mathbb{H}^3$$

inherits from $\mathbb{H}^3$ the structure of a 3-manifold. It is never compact, let $X_\Gamma \subset \hat{X}_\Gamma$ be its Borel-Serre compactification, see [27] for the construction. Important for us is that $\hat{X}_\Gamma$ is a compact 3-manifold with boundary consisting of a non zero but finite number of tori. We further need that the inclusion

$$X_\Gamma \subset \hat{X}_\Gamma$$

is a homotopy equivalence, see [27]. This implies that the fundamental groups of X_Γ and $\hat{X}_\Gamma$ are isomorphic. Identifying Γ with the fundamental group of X_Γ we obtain isomorphisms

$$\Gamma \cong \pi_1(X_\Gamma) \cong \pi_1(\hat{X}_\Gamma). \tag{4.1}$$

Theorem 4.1. *Every Bianchi group* $\mathrm{PSL}(2, \mathcal{O}_{\mathrm{d}})$ *has a subgroup of finite index which admits a hierarchy (see Definition 3.5).*

Proof. We choose any torsion free subgroup $\Gamma \leq \mathrm{PSL}(2, \mathcal{O}_{\mathrm{d}})$ of finite index and consider the Borel-Serre compactification $\hat{X}_\Gamma$. It is well known (see [4]) that every boundary torus T of $\hat{X}_\Gamma$ is incompressible. This is implied by the fact that the natural homomorphism $\pi_1(T) \to \pi_1(\hat{X}_\Gamma)$ is an inclusion. We conclude that $\hat{X}_\Gamma$ is a Haken-3-manifold. See [6] for explanation.

By [6, Chapter IV] there is a hierarchy for $\hat{X}_\Gamma$, i.e. a chain

$$(M_0, F_0),\ (M_1, F_1), \ldots, (M_n, \emptyset) \tag{4.2}$$

with $M_0 = \hat{X}_\Gamma$ and $F_0 = T$, where M_{i+1} is a (not necessarily connected) 3-manifold obtained by cutting M_i along an incompressible, non-boundary-parallel, 2-sided surface F_i, and where M_n is a union of 3-balls.

We infer from the Seifert-van Kampen theorem that Γ admits a hierarchy. Notice that incompressibility implies π_1-injectivity (i.e. embedding of the corresponding fundamental groups) for embeddings of 2-sided surfaces ($\neq S^2$) into the 3 manifolds M_i, see [6, Lemma III.8]. □

A group G is called subgroup separable (or LERF) if every finitely generated subgroup H of G is closed in the profinite topology of G, i.e. is the intersection of subgroups of finite index containing it. The subgroup separability of Bianchi groups has been recently established (see [13, Theorem 3.6.1]).

So Theorem 1.1 follows from the following

Theorem 4.2. *A subgroup separable group admitting a hierarchy (Definition 3.5) is good.*

Proof. We use induction on the level of the hierarchy of the decomposition of the preceding theorem. If the level is 1 the result is obvious. The inductive step follows from the following consideration. The subgroup separability implies that the profinite topology of our decomposition is efficient because a finitely generated subgroup of a subgroup separable group is subgroup separable. This allows to use Proposition 3.3. □

Remark. Some classes of cocompact Kleinian groups are also subgroup separable and hence can be proven to be good, see Theorem 5.1 in [13]. It is proved in [12] that the fundamental groups of compact 3-manifolds all of whose finite index subgroups have finite abelianizations (and so are not Haken) are good. On the other hand the fundamental group of a Haken manifold admits a hierarchy and so subgroup separability would imply its goodness. This gives the basis to conjecture that all Kleinian groups are good.

Analyzing the proof of 4.2 one can observe that we use only the fact that the profinite topology on a group G admitting a hierarchy is efficient. Thus we can claim the following

Theorem 4.3. *Let G be a group admitting a hierarchy such that the profinite topology on G is efficient. Then G is good.*

Using Lemma 3.1 we get

Corollary 4.4. *A group commensurable with Bianchi groups are good.*

From this corollary we infer that several groups given by generators and relations (such as some of the tetrahedral Coxeter groups) are good. For example

Corollary 4.5. *The tetrahedral hyperbolic Coxeter groups $\mathbf{CT}(1) - \mathbf{CT}(17)$ are good.*

Proof. All these groups are commensurable with Bianchi groups (see [4, Section 10.4]). $\square$

Theorem 4.3 can be applied to prove that so called limit groups, i.e. finitely generated fully residually free groups are good. A group G is called *fully residually free* if for any finite subset X of G there is an epimorphism $G \longrightarrow F$ onto a free group F whose restriction on X is injective.

Proof. of Theorem 1.3: The limit groups admit a hierarchy see [21] and [7], more precisely a hierarchy, where all amalgamated and associated subgroups in forming the free products with amalgamation and HNN-extensions are infinite cyclic. On the other hand, Henry Wilton [32] proved recently that limit groups are subgroup separable. Therefore, the result follows from Theorem 4.3. $\square$

4.2 Arithmetic groups with the congruence subgroup property

This subsection contains many examples of S-arithmetic groups which are not good.

We shall use the standard terminology concerning S-arithmetic groups which we introduce now. Let K be a number field and $\mathcal{O}$ its ring of integers. Let S be a finite set of places of K including the set S_∞ of archimedean places. We write

$$\mathcal{O}_S := \{\, a \in K \;\mid\; \nu(a) \geq 0 \text{ for all } \nu \notin S \,\}$$

for the subring of elements of K which are integral outside of S. If ν is a non-archimedean place of S we define $\mathcal{O}_\nu$ to be the completion of $\mathcal{O}$ at ν. The maximal ideal of $\mathcal{O}_\nu$ is denoted by $\mathfrak{m}_\nu$.

Let $\mathbf{G}$ be a semisimple and simply connected K-defined linear algebraic group. This means that $\mathbf{G}$ is a subgroup of $\mathrm{GL}(n,\mathbb{C})$ for some $n \in \mathbb{N}$ and is also the zero set of a bunch of polynomials with coefficients in K. Let R be a subring of the number field K. We write $\mathbf{G}(R) := \mathbf{G} \cap \mathrm{GL}(n,R)$ for the group of R-points of $\mathbf{G}$. Let $\mathfrak{a} \leq R$ be an ideal of finite index. Clearly the kernel of the entrywise reduction map

$$\mathbf{G}(R) \to \mathbf{G}(R/\mathfrak{a})$$

is a subgroup of finite index in $\mathbf{G}(R)$ (called a principal congruence subgroup). Taking the completion $\tilde{\mathbf{G}}(R)$ (the congruence completion) with respect to the topology defined by the principal congruence subgroups we obtain an exact sequence

$$\langle 1\rangle \to \mathrm{C}(\mathbf{G},R) \to \widehat{\mathbf{G}}(R) \to \bar{\mathbf{G}}(R) \to \langle 1\rangle. \tag{4.3}$$

The profinite group $\mathrm{C}(\mathbf{G},R)$ is traditionally called the congruence kernel.

Proof of Corollary 1.2: Since the goodness is preserved by commensurability, the goodness of Γ_d implies the goodness of $\mathrm{SL}(2,\mathcal{O}_d)$ an therefore the goodness of its every subgroup of finite index. Let H be a torsion free congruence subgroup of $\mathrm{SL}(2,\mathcal{O}_d)$. Then H has cohomological 2. It follows that the profinite completion $\widehat{H}$ has cohomological dimension 2 (as a profinite group). Since the congruence kernel of $\mathrm{SL}(2,\mathcal{O}_d)$ is contained in $\widehat{H}$, the congruence kernel $C \leq \widehat{\Gamma}_d$ has cohomological dimension at most 2.

Proposition 4.6. *Let $\mathbf{G}$ be a semisimple and simply connected K-defined algebraic group (K a number field). Let S be a finite set of places of K including the set S_∞ of archimedean places and let $\mathcal{O}_S$ be the corresponding ring of S-integers. Suppose that the congruence kernel $\mathrm{C}(\mathbf{G},\mathcal{O}_s)$ of $\mathbf{G}(\mathcal{O}_S)$ is finite. Then $\mathbf{G}(\mathcal{O}_S)$ (and any group commensurable to it) is not good.*

Examples of groups which are in the range of our proposition are

$$\mathrm{PSL}\left(2,\mathbb{Z}\left[\frac{1}{\mathrm{p}}\right]\right) \cong \mathrm{PSL}(2,\mathbb{Z}) *_{\Gamma_0(\mathrm{p})} \mathrm{PSL}(2,\mathbb{Z}) \tag{4.4}$$

where p is a prime number and

$$\Gamma_0(p) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, A \in \mathrm{PSL}(2,\mathbb{Z}),\ \text{p divides c} \right\}$$

embedded in the two obvious ways into $\mathrm{PSL}(2,\mathbb{Z})$ (see [28]). In fact a theorem of Mennicke [17] shows that this group has the congruence subgroup property which implies that it is not good. On the other hand the constituent groups in the amalgamated product (4.4) are good. This in turn implies that the decomposition (4.4) is not efficient.

For the proof of Proposition 4.6 we need

Lemma 4.7. *Let $\mathbf{G}$ be a semisimple and simply connected K-defined algebraic group. Let S be a finite set of places of K including the set S_∞ of archimedean places and let $\mathcal{O}_S$ be the corresponding ring of S-integers. Let $\Gamma \leq \mathbf{G}(K)$ be a subgroup commensurable with $\mathbf{G}(\mathcal{O}_S)$. Then the congruence completion $\bar{\Gamma}$ of Γ is not virtually torsion free.*

Proof. By the strong approximation Theorem (see Theorem 7.12 of [18]) $\bar{\Gamma}$ is an open subgroup of the product

$$\mathbf{G}(\hat{\mathcal{O}}_S) = \prod_{\nu \notin S} \mathbf{G}(\mathcal{O}_\nu).$$

Here $\bar{\mathcal{O}}_S$ stands for the completion of the ring $\mathcal{O}_S$ with respect to the topology defined by its ideals of finite index. We conclude that $\bar{\Gamma}$ contains the product

$$\prod_{\nu \notin S \cup S_0} \mathbf{G}(\mathcal{O}_\nu) \leq \prod_{\nu \notin S} \mathbf{G}(\mathcal{O}_\nu)$$

for some finite set of places S_0.

Thus it suffices to show that for infinitely many places ν we have non trivial torsion elements in $\mathbf{G}(\mathcal{O}_\nu)$. The norm (index) of the ideal $\mathfrak{m}_\nu$ is a power of the prime p, say.

Note that the kernel of the natural homomorphism

$$\mathbf{G}(\mathcal{O}_\nu) \to \mathbf{G}(\mathcal{O}_\nu/\mathfrak{m}_\nu)$$

is a pro-p group. The field $\mathbb{F}_\nu := \mathcal{O}_\nu/\mathfrak{m}_\nu$ is finite and therefore by [15, Proposition 14] $\mathbf{G}(\mathbb{F}_\nu)$ contains the multiplicative group $\mathbb{F}_\nu^*$, in particular an element of order $p-1$. Then $\mathbf{G}(\mathcal{O}_\nu)$ contains torsion elements of order prime to p, as required. □

Proof of Proposition 4.6: Let Γ be a group commensurable with $\mathbf{G}(\mathcal{O}_S)$. Since goodness is preserved by commensurability we may assume that Γ is torsion free and hence of finite cohomological dimension.

Since the congruence kernel $\mathrm{C}(\mathbf{G},\Gamma)$ of Γ is finite $\widehat{\Gamma}$ has torsion if and only if $\bar{\Gamma}$ has torsion and so has infinite cohomological dimension. The result follows. □

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Author's addresses

F. Grunewald
Mathematisches Institut
Heinrich-Heine-Universität Düsseldorf
Universitätsstr. 1
D-40225 Düsseldorf
email:fritz@math.uni-duesseldorf.de

A. Jaikin-Zapirain
Departamento de Matemáticas
Facultad de Ciencias Módulo C-XV
Universidad Autónoma de Madrid
Campus de Cantoblanco Ctra. de Colmenar Viejo
Km. 15 28049 Madrid
email:andrei.jaikin@uam.es

P. A. Zalesskii
Department of Mathematics, University of Brasília
70910-900 Brasília DF, Brazil,
email:pz@mat.unb.br