FREE $\mathbb{Q}$-GROUPS ARE RESIDUALLY TORSION-FREE NILPOTENT

ANDREI JAIKIN-ZAPIRAIN

Abstract. We develop a method to show that some (abstract) groups can be embedded into a free pro-p group. In particular, we show that a finitely generated subgroup of a free $\mathbb{Q}$-group can be embedded into a free pro-p group for almost all primes p. This solves an old problem raised by G. Baumslag: free $\mathbb{Q}$-groups are residually torsion-free nilpotent.

1. Introduction

A group G is called a **$\mathbb{Q}$-group** if for any $n \in \mathbb{N}$ and $g \in G$ there exists exactly one $h \in G$ satisfying $h^n = g$. These groups were introduced by G. Baumslag in [2] under the name of $\mathcal{D}$-groups. He observed that $\mathbb{Q}$-groups may be viewed as universal algebras, and as such they constitute a variety. Every variety of algebras contains free algebras (in that variety). In the variety of $\mathbb{Q}$-groups we call such free algebras **free $\mathbb{Q}$-groups**. G. Baumslag dedicated several papers to the study of residual properties of free $\mathbb{Q}$-groups [3, 5, 7]. For example, in [3] he showed that a free $\mathbb{Q}$-group is residually periodic-soluble and locally residually finite-soluble. He wrote in [3] "It is, of course, still possible that, locally, free $\mathcal{D}$-groups are, say, residually finite p-groups" or in [5] "In particular it seems likely that free $\mathcal{D}$-groups are residually torsion-free nilpotent. However the complicated nature of free $\mathcal{D}$-groups makes it difficult to substantiate such a remark." This conjecture is part of two main collections of problems in Group Theory ([8, Problem F12] and [35, Problem 13.39 (a),(c)]), and in addition to mentioned works of Baumslag, it was also studied in [13, 20]. In this paper we solve Baumslag's conjecture.

Theorem 1.1. *A free $\mathbb{Q}$-group is residually torsion-free nilpotent.*

Let I be a set and $X = \{x_i : i \in I\}$ and $Y = \{y_i : i \in I\}$ two sets indexed by elements of I. We denote by $F^{\mathbb{Q}}(X)$ the free $\mathbb{Q}$-group on X and by $\mathbb{Q}\langle\langle Y\rangle\rangle$ the ring of non-commutative power series in Y with coefficients in $\mathbb{Q}$. There is a unique group homomorphism $\phi : F^{\mathbb{Q}}(X) \to \mathbb{Q}\langle\langle Y\rangle\rangle^*$ (the Magnus homomorphism), that sends x_i to $1+y_i$. G. Baumslag observed that $F^{\mathbb{Q}}(X)$ is residually torsion-free nilpotent if and only if ϕ is a monomorphism. Thus, in order to prove Theorem 1.1 we have to show that the restriction of ϕ on

Date: August 2021.

2010 *Mathematics Subject Classification.* Primary: 20E06, Secondary: 16K40, 20C07, 20E18, 20E26.

Key words and phrases. Free $\mathbb{Q}$-groups, free pro-p groups, mod-p L^2-Betti numbers, Lück approximation, universal division ring of fractions.

any finitely generated subgroup of $F^{\mathbb{Q}}(X)$ is injective. The structure of finitely generated subgroups of $F^{\mathbb{Q}}(X)$ was studied already in [2] (see also [45, Section 8]). It was shown that $F^{\mathbb{Q}}(X)$ is the end result of repeatedly freely adjoining nth roots to the free group $F(X)$. The key point of our proof of Theorem 1.1 is to show that any finitely generated subgroup of $F^{\mathbb{Q}}(X)$ can be embedded into a finitely generated free pro-p group for some prime p. In fact, we prove the following stronger result.

Theorem 1.2. *Let p be a prime. Let $H_0 = F(X)$ be a free group on generators $X = \{x_1, \dots, x_d\}$ and let $H_0 \hookrightarrow \mathbf{F}$ be the canonical embedding of H_0 into its pro-p completion $\mathbf{F}$. Construct subgroups H_i of $\mathbf{F}$ inductively in the following way. Let A_i be a maximal abelian subgroup of H_i and let B_i be a finitely generated abelian subgroup of $\mathbf{F}$ which contains A_i. Put $H_{i+1} = \langle H_i, B_i\rangle$. Then for every $k \geq 1$, the canonical map*

$$H_{k-1} *_{A_{k-1}} B_{k-1} \to H_k$$

is an isomorphism.

Let us make few remarks about the groups A_i and B_i. It is relatively easy to describe abelian subgroups of amalgamated products. In particular, the conclusion of the theorem implies that all abelian subgroups of H_i are finitely generated. Thus, an implicit hypothesis, which appears in the theorem, that A_i are finitely generated, is automatically fulfilled. A maximal abelian subgroup of $\mathbf{F}$ is isomorphic to the additive group of the ring of p-adic numbers $(\mathbb{Z}_p, +)$. Therefore, for any finitely generated (abstract) abelian subgroup A of $\mathbf{F}$ and any finitely generated torsion-free abelian group B which contains A and such that B/A has no p-torsion, it is posible to extend the embedding $A \hookrightarrow \mathbf{F}$ to an embedding $B \hookrightarrow \mathbf{F}$. This extension is unique if and only if B/A is finite.

Let H be a group and A the centralizer of a non-trivial element. Then the group $G = H *_A (A \times \mathbb{Z}^k)$ is said to be obtained from H by **extension of a centralizer**. A group is called an **ICE** group if it can be obtained from a free group using iterated centralizer extensions. A group G is a limit group if and only if it is a finitely generated subgroup of an ICE group (see [34, 12]). All centralizers of non-trivial elements of an ICE group are abelian. Thus, Theorem 1.1 provides explicit realizations of ICE groups (and so limit groups) as subgroups of a non-abelian free pro-p-groups (for this application we only need the case where all B_i/A_i are torsion-free). Non-explicit realizations of limit groups as subgroups of a compact group containing a non-abelian free group was obtained in [1] (see also [10]).

The most interesting part of Theorem 1.2 corresponds to the case where B_i/A_i have non-trivial torsion. In the proof of Theorem 1.1 we only use the case where all B_i and A_i are infinite cyclic.

A residually nilpotent group G is called **parafree** if its quotients by the terms of its lower central series are the same as those of a free group. Baumslag introduced this family of groups and produced many examples of them [4]. In [32] we apply the methods of the proof of Theorem 1.2 in order to construct new examples of finitely generated parafree groups.

The paper is organized as follows. In Section 2 we give basic preliminaries. The proof of Theorem 1.2 uses the theory of mod-p L^2-Betti numbers. In Section 3 we explain how to define them for subgroups G of a free pro-p group. In Section 4 we introduce a technical notion of $\mathcal{D}$-torsion-free modules and show that some relevant $\mathbb{F}_p[G]$-modules are $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free (see Proposition 4.10). In Section 5 we prove Theorem 1.1 and Theorem 1.2. In the last section we discuss two well-known problems concerning linearity of free pro-p groups and free $\mathbb{Q}$-groups.

Acknowledgments

I am very grateful to Pavel Zalesski who explained to me how a surface group can be embedded into a free pro-p group. This was a starting point of Theorem 1.2. I would like to thank Warren Dicks for providing the references needed in the proof of Proposition 2.4, Yago Antolin and Ashot Minasyan for explaining to me the proof of Theorem 6.1, Emmanuel Breulliard for pointing out the paper [1] and Ismael Morales for useful suggestions and comments.

This paper is partially supported by the grants MTM2017-82690-P and PID2020-114032GB-I00 of the Ministry of Science and Innovation of Spain and by the ICMAT Severo Ochoa project CEX2019-000904-S4.

2. Preliminaries

2.1. **R-rings.** All rings in this paper have the identity element and ring homomorphisms send the identity to the identity. We denote the invertible elements of a ring R by R^*. A R-module means a left R-module. By an **R-ring** we understand an homomorphism $\varphi : R \to S$. We will often refer to S as R-ring and omit the homomorphism φ if φ is clear from the context. Two R-rings $\varphi_1 : R \to S_1$ and $\varphi_2 : R \to S_2$ are said to be **isomorphic** if there exists an isomorphism $\alpha : S_1 \to S_2$ such that $\alpha \circ \varphi_1 = \varphi_2$.

2.2. **Left ideals in group algebras.** Let G be a group and k a commutative ring. We denote by I_G the augmentation ideal of $k[G]$. If H is a subgroup of G we denote by I_H^G the left ideal of $k[G]$ generated by I_H. The following lemma gives an alternative description of the $k[G]$-module I_H^G.

Lemma 2.1. *Let $H \le T$ be subgroups of G. Then the following holds.*
(a) *The canonical map*

$$k[G] \otimes_{k[H]} I_H \to I_H^G$$

sending $a \otimes b$ to ab, is an isomorphism.
(b) *The canonical map $k[G] \otimes_{k[T]} (I_T/I_H^T) \to I_T^G/I_H^G$, sending $a \otimes (b + I_H^T)$ to $ab + I_H^G$, is an isomorphism.*

Proof. (a) Consider an exact sequence

$$0 \to I_H \to k[H] \to k \to 0.$$

The freeness of $k[G]$ as $k[H]$-module implies that the sequence

$$0 \to k[G] \otimes_{k[H]} I_H \xrightarrow{\alpha} k[G] \xrightarrow{\beta} k[G] \otimes_{k[H]} k \to 0$$

is also exact. Here α sends $a\otimes b$ to ab and β sends a to $a\otimes 1$. Thus, α establishes an isomorphisms between $k[G]\otimes_{k[H]} I_H$ and $\ker\beta = I_H^G$. This proves the lemma.

(b) Consider now the exact sequence

$$0\to I_H^T\to I_T\to I_T/I_H^T\to 0.$$

Applying $k[G]\otimes_{k[T]}$ and taking again into account that $k[G]$ is a free $k[T]$-module, we obtain the exact sequence

$$0\to I_H^G\to I_T^G\to k[G]\otimes_{k[T]}(I_T/I_H^T)\to 0.$$

This prove the second claim. □

2.3. **Left ideals in completed group algebras.** In this paper the letters $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, etc. will denote pro-p groups. Almost all pro-p groups that we consider are free pro-p groups. Recall that a closed subgroup of a free pro-p group is also free pro-p.

Let $\mathbf{G}$ be pro-p group. We denote by $\mathbb{F}_p[[\mathbf{G}]]$ the inverse limit of $\mathbb{F}_p[\mathbf{G}/\mathbf{U}]$, where the limit is taken over all open normal subgroups $\mathbf{U}$ of $\mathbf{G}$. $\mathbb{F}_p[[\mathbf{G}]]$ is called the **completed group algebra of G over** $\mathbb{F}_p$.

We denote by $I_{\mathbf{G}}$ the augmentation ideal of $\mathbb{F}_p[[\mathbf{G}]]$. If $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$, then $I_{\mathbf{H}}^{\mathbf{G}}$ denotes the closed left ideal of $\mathbb{F}_p[[\mathbf{G}]]$ generated by $I_{\mathbf{H}}$.

A **profinite** $\mathbb{F}_p[[\mathbf{G}]]$**-module** is an inverse limit of finite topological $\mathbb{F}_p[[\mathbf{G}]]$-modules. If $M=\varprojlim M_i$ and $N=\varprojlim N_i$ are right and left, respectively, profinite $\mathbb{F}_p[[\mathbf{G}]]$-modules ($M_i$ and N_i are finite topological $\mathbb{F}_p[[\mathbf{G}]]$-modules), then the profinite tensor product is denoted by $\widehat{\otimes}$ and it is defined as the inverse limit of $M_i\otimes_{\mathbb{F}_p[[\mathbf{G}]]} N_i$.

Lemma 2.2. *Let* $\mathbf{H}$ *be a closed subgroup of* $\mathbf{G}$ *and let* M *be a finitely presented* $\mathbb{F}_p[[\mathbf{H}]]$*-module. Then* M *is a profinite module and the canonical map*

$$\mathbb{F}_p[[\mathbf{G}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]} M\to \mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{H}]]} M$$

is an isomorphism.

Proof. Since M is finitely presented, there exists an exact sequence of $\mathbb{F}_p[[\mathbf{H}]]$-modules

$$\mathbb{F}_p[[\mathbf{H}]]^r\xrightarrow{\alpha}\mathbb{F}_p[[\mathbf{H}]]^d\to M\to 0.$$

Thus, we can write $M\cong \mathbb{F}_p[[\mathbf{H}]]^d/I$, where $I=\operatorname{Im}\alpha$. Since α is continuous and $\mathbb{F}_p[[\mathbf{H}]]^r$ is compact, I is closed in $\mathbb{F}_p[[\mathbf{H}]]^d$. Hence $\mathbb{F}_p[[\mathbf{H}]]^d/I$, and so M, are profinite.

In order to see the second claim of the lemma, we have to check that $\mathbb{F}_p[[\mathbf{G}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]} M$ is profinite as a $\mathbb{F}_p[[\mathbf{G}]]$-module. This follows from the same argument as before using the exact sequence

$$\mathbb{F}_p[[\mathbf{G}]]^r\to \mathbb{F}_p[[\mathbf{G}]]^d\to \mathbb{F}_p[[\mathbf{G}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]} M\to 0.$$

□

Let us cite few applications of the previous lemma which we use in this paper.

Lemma 2.3. *Let $\mathbf{G}$ be a pro-p group and let $\mathbf{H} \leq \mathbf{T}$ be closed subgroups of $\mathbf{G}$. Then the following holds.*

(a) *The continuous map*
$$\mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{H}]]} I_{\mathbf{H}} \to I_{\mathbf{H}}^{\mathbf{G}}$$
that sends $a \otimes b$ to ab, is an isomorphism.

(b) *If $\mathbf{H}$ is finitely presented, the map*
$$\mathbb{F}_p[[\mathbf{G}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} I_{\mathbf{H}} \to I_{\mathbf{H}}^{\mathbf{G}}$$
that sends $a \otimes b$ to ab, is an isomorphism.

(c) *If $\mathbf{T}$ is finitely presented and $\mathbf{H}$ is finitely generated, the map*
$$\mathbb{F}_p[[\mathbf{G}]] \otimes_{\mathbb{F}_p[[\mathbf{T}]]} (I_{\mathbf{T}}/I_{\mathbf{H}}^{\mathbf{T}}) \to I_{\mathbf{T}}^{\mathbf{G}}/I_{\mathbf{H}}^{\mathbf{G}}$$
sending $a \otimes (b + I_{\mathbf{H}}^{\mathbf{T}})$ to $ab + I_{\mathbf{H}}^{\mathbf{G}}$, is an isomorphism.

Proof. (a) Consider an exact sequence
$$0 \to I_{\mathbf{H}} \to \mathbb{F}_p[[\mathbf{H}]] \to \mathbb{F}_p \to 0.$$
The freeness of $\mathbb{F}_p[[\mathbf{G}]]$ as profinite $\mathbb{F}_p[[\mathbf{H}]]$-module implies that
$$0 \to \mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{H}]]} I_{\mathbf{H}} \xrightarrow{\alpha} \mathbb{F}_p[[\mathbf{G}]] \xrightarrow{\beta} \mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{H}]]}\mathbb{F}_p \to 0$$
is also exact. Thus, $\mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{H}]]} I_{\mathbf{H}}$ is isomorphic to $\ker \beta = I_{\mathbf{H}}^{\mathbf{G}}$.

(b) Since $\mathbf{H}$ is finitely presented $I_{\mathbf{H}}$ is finitely presented as $\mathbb{F}_p[[\mathbf{H}]]$-module, and so, by Lemma 2.2, $\mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{H}]]} I_{\mathbf{H}} \cong \mathbb{F}_p[[\mathbf{G}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} I_{\mathbf{H}}$. Thus we conclude that the natural map from $\mathbb{F}_p[[\mathbf{G}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} I_{\mathbf{H}}$ to $I_{\mathbf{H}}^{\mathbf{G}}$ is also an isomorphism by (a).

(c) Consider now the exact sequence
$$0 \to I_{\mathbf{H}}^{\mathbf{T}} \to I_{\mathbf{T}} \to I_{\mathbf{T}}/I_{\mathbf{H}}^{\mathbf{T}} \to 0.$$
Applying $\mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{T}]]}$ and taking again into account that $\mathbb{F}_p[[\mathbf{G}]]$ is a free profinite $\mathbb{F}_p[[\mathbf{T}]]$-module, we obtain the exact sequence
$$0 \to I_{\mathbf{H}}^{\mathbf{G}} \to I_{\mathbf{T}}^{\mathbf{G}} \to \mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{T}]]}(I_{\mathbf{T}}/I_{\mathbf{H}}^{\mathbf{T}}) \to 0.$$
Since $\mathbf{T}$ is finitely presented and $\mathbf{H}$ is finitely generated, $I_{\mathbf{T}}/I_{\mathbf{H}}^{\mathbf{T}}$ is finitely presented as $\mathbb{F}_p[[\mathbf{T}]]$-module. Thus, by Lemma 2.2, $\mathbb{F}_p[[\mathbf{G}]]\widehat{\otimes}_{\mathbb{F}_p[[\mathbf{T}]]}(I_{\mathbf{T}}/I_{\mathbf{H}}^{\mathbf{T}}) \cong \mathbb{F}_p[[\mathbf{G}]] \otimes_{\mathbb{F}_p[[\mathbf{T}]]} (I_{\mathbf{T}}/I_{\mathbf{H}}^{\mathbf{T}})$. This prove the last claim. □

2.4. **On amalgamated products of groups.** Let G be a group and H_1 and H_2 two subgroups that generate G and have intersection $A = H_1 \cap H_2$. The following result gives an algebraic condition for G to be isomorphic to the amalgamated product of H_1 and H_2 over A.

Proposition 2.4. [49, 36] *Let k be a commutative ring. Then the canonical map $H_1 *_A H_2 \to G$ is an isomorphism if and only if $I_{H_1}^G \cap I_{H_2}^G = I_A^G$ in $k[G]$.*

Proof. The "if" part follows from [49, Lemma 2.1] and the "only if" part is proved in [36, Theorem 1] (the proof is given for $k = \mathbb{Z}$, but the same argument works also for an arbitrary ring k). □

2.5. **On convergence of Sylvester rank functions.** Let R be a ring. A **Sylvester matrix rank function** rk on R is a function that assigns a non-negative real number to each matrix over R and satisfies the following conditions.

(SMat1) $\operatorname{rk}(M) = 0$ if M is any zero matrix and $\operatorname{rk}(1) = 1$;

(SMat2) $\operatorname{rk}(M_1 M_2) \leq \min\{\operatorname{rk}(M_1), \operatorname{rk}(M_2)\}$ for any matrices M_1 and M_2 which can be multiplied;

(SMat3) $\operatorname{rk}\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \operatorname{rk}(M_1) + \operatorname{rk}(M_2)$ for any matrices M_1 and M_2;

(SMat4) $\operatorname{rk}\begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \geq \operatorname{rk}(M_1) + \operatorname{rk}(M_2)$ for any matrices M_1, M_2 and M_3 of appropriate sizes.

We denote by $\mathbb{P}(R)$ the set of Sylvester matrix rank functions on R, which is a compact convex subset of the space of functions on matrices over R considered with pointwise convergence.

Many problems can be reinterpreted in terms of convergence in $\mathbf{P}(R)$. For example, if G is group and $G > G_1 > G_2 > \dots$ is a chain of normal subgroups of G of finite index with trivial intersection, then the Lück approximations over $\mathbb{Q}$ and $\mathbb{C}$ is equivalent to the convergence of $\operatorname{rk}_{G/G_i}$ to rk_G in $\mathbf{P}(\mathbb{Q}[G])$ and $\mathbf{P}(\mathbb{C}[G])$ respectively (see [40, 27]).

An alternative way to introduce Sylvester rank functions is via Sylvester module rank functions. A **Sylvester module rank function** dim on R is a function that assigns a non-negative real number to each finitely presented R-module and satisfies the following conditions.

(SMod1) $\dim\{0\} = 0$, $\dim R = 1$;

(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;

(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \geq \dim M_2 \geq \dim M_3.$$

There exists a natural bijection between Sylvester matrix and module rank functions over a ring. Given a Sylvester matrix rank function rk on R and a finitely presented R-module $M \cong R^n/R^m A$ (A is a matrix over R), we define the corresponding Sylvester module rank function dim by means of $\dim M = n - \operatorname{rk}(A)$. By a recent result of Li [37], any Sylvester module rank function dim on R can be extended to a function (satisfying some natural conditions) on arbitrary modules over R. We will call this extension, the **extended Sylvester module rank function** and denote it also by dim. For example, if M is finitely generated module, then $\dim M$ is defined as

(1)
$$\dim M = \inf\{\dim \widetilde{M} : \ \widetilde{M} \text{ is finitely presented and } M \text{ is a quotient of } \widetilde{M}\}.$$

If rk, $\operatorname{rk}_i \in \mathbb{P}(R)$ ($i \in \mathbb{N}$) are Sylvester matrix rank functions corresponding to Sylvester module rank functions dim, $\dim_i$, respectively, then $\operatorname{rk} = \lim\limits_{i\to\infty} \operatorname{rk}$ if and only if for any finitely presented R-module M, $\dim M = \lim\limits_{i\to\infty} \dim_i M$.

However, the existence of the limit $\mathrm{rk} = \lim_{i\to\infty} \mathrm{rk}$ does not imply that $\dim M = \lim_{i\to\infty} \dim_i M$ for any finitely generated R-module M. This phenomena is well-known. For example, it explains why the Lück approximation of the first L^2-Betti numbers is valid for finitely presented groups but not always valid for finitely generated groups (see [41]).

If M is a finitely generated R-module, we only have that

$$\dim M \geq \limsup_{i\to\infty} \dim_i M. \tag{2}$$

Indeed, let $\mathcal{F}$ be the set of all finitely presented R-modules $\tilde{M}$ such that M is a quotient of $\tilde{M}$. Then

$$\dim M = \inf_{\tilde{M}\in\mathcal{F}} \dim \tilde{M} = \inf_{\tilde{M}\in\mathcal{F}} \lim_{i\to\infty} \dim_i \tilde{M} \geq$$
$$\limsup_{i\to\infty} \inf_{\tilde{M}\in\mathcal{F}} \dim_i \tilde{M} = \limsup_{i\to\infty} \dim_i M.$$

In this subsection we will explain how to overcome this problem in some situations. For two Sylvester rank functions rk_1 and $\mathrm{rk}_2 \in \mathbf{P}(R)$ we write $\mathrm{rk}_1 \geq \mathrm{rk}_2$ if $\mathrm{rk}_1(A) \geq \mathrm{rk}_2(A)$ for any matrix A over R. If $\dim_1$ and $\dim_2$ are the Sylvester module rank functions on R corresponding to rk_1 and rk_2, then the condition $\mathrm{rk}_1 \geq \mathrm{rk}_2$ is equivalent to the condition $\dim_1 \leq \dim_2$, meaning that $\dim_1 M \leq \dim_2 M$ for any finitely presented R-module M.

Proposition 2.5. *Let R be a ring and let $\mathrm{rk}, \mathrm{rk}_i \in \mathbb{P}(R)$ $(i \in \mathbb{N})$ be Sylvester matrix rank functions on R corresponding to (extended) Sylvester module rank functions $\dim, \dim_i$ respectively. Assume that $\mathrm{rk} = \lim_{i\to\infty} \mathrm{rk}$ and for all i, $\mathrm{rk} \geq \mathrm{rk}_i$. Then, for any finitely generated R-module M,*

$$\dim M = \lim_{i\to\infty} \dim_i M.$$

Proof. Fix $\epsilon > 0$. Let k be such that

$$\liminf_{i\to\infty} \dim_i M \geq \dim_k M - \epsilon.$$

There exists a finitely presented R-module $\tilde{M}$ such that M is a quotient of $\tilde{M}$ and $\dim_k M \geq \dim_k \tilde{M} - \epsilon$. Since $\mathrm{rk} \geq \mathrm{rk}_k$, we have that $\dim_k \tilde{M} \geq \dim \tilde{M}$. Thus, we obtain

$$\liminf_{i\to\infty} \dim_i M \geq \dim_k M - \epsilon \geq \dim_k \tilde{M} - 2\epsilon \geq \dim \tilde{M} - 2\epsilon \geq \dim M - 2\epsilon.$$

Since ϵ is arbitrary, we conclude that $\liminf_{i\to\infty} \dim_i M \geq \dim M$. In view of (2), we are done. □

2.6. **Epic division R-rings.** Let R be a ring. An **epic** division R-ring is a R-ring $\phi : R \to \mathcal{D}$ where $\mathcal{D}$ is a division ring generated by $\phi(R)$. Moreover, we say that $\mathcal{D}$ is a **division R-ring of fractions** if ϕ is injective. In this case we will normally omit ϕ and see R as a subring of $\mathcal{D}$.

Each epic division R-ring $\mathcal{D}$ induces an (extended) Sylvester module rank function $\dim_{\mathcal{D}}$ on R (see, for example, [29]): for every R-module M we define $\dim_{\mathcal{D}} M$ to be equal to the dimension of $\mathcal{D} \otimes_R M$ as a $\mathcal{D}$-module. We will use $\dim_{\mathcal{D}}$ for the Sylvester module rank function on R and for the $\mathcal{D}$-dimension of $\mathcal{D}$-spaces. This is a coherent notation because, since $\mathcal{D}$ is epic, $\mathcal{D}\otimes_R\mathcal{D}$ is isomorphic to $\mathcal{D}$ as R-bimodule (see [28, Section 4]).

The following result of P. M. Cohn will be used several times in the paper.

Proposition 2.6. [16, Theorem 4.4.1] *Two epic division R-rings $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic if and only if $\dim_{\mathcal{D}_1} M = \dim_{\mathcal{D}_2} M$ for every finitely presented R-module M.*

2.7. **Natural extensions of Sylvester rank functions.** Let G be a group with trivial element e. We say that a ring R is **G-graded** if R is equal to the direct sum $\oplus_{g\in G} R_g$ and $R_gR_h\subseteq R_{gh}$ for all g and h in G. If for each $g\in G$, R_g contains an invertible element u_g, then we say that R is a **crossed product** of R_e and G and we will write $R=S*G$ if $R_e=S$.

Let $R=S*G$ be a crossed product. Let rk be a Sylvester matrix rank function on S and dim its associated Sylvester module rank function. We say that rk (and dim) are **R-compatible** if for any $g\in G$ and any matrix A over S, $\mathrm{rk}(A)=\mathrm{rk}(u_gAu_g^{-1})$. If G is finite and dim is R-compatible, we define

$$\widetilde{\dim}\ M=\frac{\dim M}{|G|}, \tag{3}$$

where M is a finitely presented R-module. This defines a Sylvester module rank function on R, called **the natural extension** of dim. This notion was studied, for example, in [33]. We notice that the same formula (3) holds also for extended Sylvester module rank functions (that is, when M is an arbitrary R-module). In this subsection we prove the following result.

Proposition 2.7. *Let $R=S*G$ be a crossed product with G finite and let $R\hookrightarrow\mathcal{D}$ be a division R-ring of fractions. Denote by $\mathcal{D}_e$ the division closure of S in $\mathcal{D}$. Then the following are equivalent.*

(a) $\widetilde{\dim}_{\mathcal{D}_e}=\dim_{\mathcal{D}}$ *as Sylvester functions on R.*
(b) $\dim_{\mathcal{D}_e}\mathcal{D}=|G|$.
(c) *$\mathcal{D}$ is isomorphic to a crossed product $\mathcal{D}_e*G$.*
(d) *$\mathcal{D}$ is isomorphic to $\mathcal{D}_e\otimes_S R$ as $(\mathcal{D}_e,R)$-bimodule.*
(e) *$\mathcal{D}$ is isomorphic to $R\otimes_S\mathcal{D}_e$ as $(R,\mathcal{D}_e)$-bimodule*

Proof. For any $h,g\in G$, define $\alpha(g,h)=u_gu_h(u_{gh})^{-1}\in R$. Since the conjugation by u_g fixes S, it also fixes $\mathcal{D}_e$. Therefore, we can define a ring structure on $T=\oplus_{g\in G}\mathcal{D}_e v_g$ defining the multiplication on homogenous elements by means of

$$(d_1v_g)\cdot(d_2v_h)=(d_1u_gd_2(u_g)^{-1}\alpha(g,h))v_{gh},\ d_1,d_2\in\mathcal{D}_e,\ g,h\in G.$$

It is clear that $T=\mathcal{D}_e*G$ is a crossed product, it contains R as a subring, and T is isomorphic to $\mathcal{D}_e\otimes_S R$ as $(\mathcal{D}_e,R)$-bimodule.

There exists a natural map $\gamma: T\to\mathcal{D}$, sending $\sum_{g\in G} d_gv_g$ ($d_g\in D_e$) to $\sum_{g\in G} d_gu_g$. Observe that $\gamma(T)$ is a domain and of finite dimension over $\mathcal{D}_e$. Thus, $\gamma(T)$ is a division subring of $\mathcal{D}$. Since T contains R, $\mathcal{D}=\gamma(T)$. This implies that (c) and (d) are equivalent.

Since $\dim_{D_e}T=|G|$, (b) implies that γ is an isomorphism, and so, (b) implies (c).

Now, let us assume (d). Let M be an R-module. We have the following.

$$\widetilde{\dim}_{\mathcal{D}_e} M = \frac{\dim_{\mathcal{D}_e}(\mathcal{D}_e \otimes_S M)}{|G|} = \frac{\dim_{\mathcal{D}_e}(\mathcal{D}_e \otimes_S (R \otimes_R M))}{|G|} = \frac{\dim_{\mathcal{D}_e}((\mathcal{D}_e \otimes_S R) \otimes_R M))}{|G|} \overset{(c)}{=} \frac{\dim_{\mathcal{D}_e}(\mathcal{D} \otimes_R M)}{|G|} = \dim_{\mathcal{D}}(\mathcal{D} \otimes_R M) = \dim_{\mathcal{D}} M.$$

This proves (a).

Now, we assume that (a) holds. Since $\mathcal{D}_e$ es an epic S-ring $\mathcal{D}_e \otimes_S \mathcal{D}_e$ is isomorphic to $\mathcal{D}_e$ as $\mathcal{D}_e$-bimodule and by the same reason, $\mathcal{D} \otimes_R \mathcal{D}$ is isomorphic to $\mathcal{D}$ as $\mathcal{D}$-bimodule. Consider $M = \mathcal{D}$ as an R-module and $N = \mathcal{D}_e$ as a S-module. Then

$$1 = \dim_{\mathcal{D}}(\mathcal{D} \otimes_R M) = \dim_{\mathcal{D}} M \overset{(a)}{=} \widetilde{\dim}_{\mathcal{D}_e} M = \frac{\dim_{\mathcal{D}_e}(\mathcal{D}_e \otimes_S M)}{|G|} = \frac{\dim_{\mathcal{D}_e}(\mathcal{D}_e \otimes_S (N^{\dim_{\mathcal{D}_e} \mathcal{D}}))}{|G|} = \frac{\dim_{\mathcal{D}_e} \mathcal{D}}{|G|}$$

This implies (b).

Since the condition (c) is symmetric, the conditions (d) and (e) are equivalent. □

3. On mod-p L^2-Betti numbers of subgroups of a free pro-p groups

3.1. **Universal division ring of fractions.** Given two epic division R-rings $\mathcal{D}_1$ and $\mathcal{D}_2$ the condition $\dim_{\mathcal{D}_1} \le \dim_{\mathcal{D}_2}$ is equivalent to the existence of a specialization from $\mathcal{D}_1$ to $\mathcal{D}_2$ in the sense of P. Cohn ([16, Subsection 4.1]). We say that an epic division R-ring $\mathcal{D}$ is **universal** if for every division R-ring $\mathcal{E}$, $\dim_{\mathcal{D}} \le \dim_{\mathcal{E}}$. If a universal epic division R-ring exists, it is unique up to R-isomorphism. We will denote it by $\mathcal{D}_R$ and instead of $\dim_{\mathcal{D}_R}$ we will simply write $\dim_R$.

We say that a ring R is a **semifir** if every finitely generated left ideal of R is free of fixed rank. For example, if K is a field, the ring $K\langle\langle X \rangle\rangle$ of non-commutative power series is a semifir ([15, Theorem 2.9.4]). By a theorem of P. M. Cohn [14] a semifir R has a universal division R-ring. P. M. Cohn proved that in this case $\mathcal{D}_R$ can be obtained from R by formally inverting all full matrices over R and $\dim_R$ is the smallest Sylvester module rank function among all the Sylvester module rank functions on R. We will need the following result.

Proposition 3.1. [29, Proposition 2.2] *Let R be a semifir. Then*

$$\operatorname{Tor}_1^R(\mathcal{D}_R, M) = 0$$

for any R-submodule of a $\mathcal{D}_R$-module.

Let G be a residually torsion-free nilpotent group (for example, G is a subgroup of a free pro-p group). Let K be a field. Then the universal division ring of fractions $\mathcal{D}_{K[G]}$ exists (see [30]). It can be constructed in the following way. Since G is residually torsion free nilpotent, G is bi-orderable. Fix a bi-invariant order $\preceq$ on G. A. Malcev [43] and B. Neumann [46] (following an idea of H. Hahn [22]) showed independently that the set $K((G, \preceq))$ of formal power series over G with coefficients in K having well-ordered support has a natural structure of a ring and, moreover, it is a division ring. $\mathcal{D}_{K[G]}$ can be defined as the division closure of $K[G]$ in $K((G, \preceq))$. The universality of this division ring is shown in [30, Theorem 1.1].

If A is a torsion-free abelian group, then $\mathcal{D}_{K[A]}$ coincides with the classical ring of fractions $\mathcal{Q}(K[A])$ of $K[A]$.

3.2. **The division ring $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$.** If $\mathbf{F}$ is a free pro-p group freely generated by $f_1, \ldots, f_d$, then the continuous homomorphism $\mathbb{F}_p\langle\langle x_1, \ldots, x_d\rangle\rangle \to \mathbb{F}_p[[\mathbf{F}]]$ that sends x_i to $f_1 - 1$, is an isomorphism. Thus, there exists a universal division ring of fraction $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$. Using results of [29] we establish the following formula for $\dim_{\mathbb{F}_p[[\mathbf{F}]]}$ which is one of main ingredients of the proof of Theorem 1.2.

Proposition 3.2. *Let $\mathbf{F} = \mathbf{N}_1 > \mathbf{N}_2 > \mathbf{N}_3 > \ldots$ be a chain of open normal subgroups of a finitely generated free pro-p group $\mathbf{F}$ with trivial intersection. Let M be a finitely generated $\mathbb{F}_p[[\mathbf{F}]]$-module. Then*

$$\dim_{\mathbb{F}_p[[\mathbf{F}]]} M = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i] \otimes_{\mathbb{F}_p[[\mathbf{F}]]} M)}{|\mathbf{F} : \mathbf{N}_i|}.$$

Proof. Let $\mathbf{N}$ be a normal open subgroup of $\mathbf{F}$ and let $\dim_{\mathbb{F}_p[\mathbf{F}/\mathbf{N}]}$ be a Sylvester module rank function on $\mathbb{F}_p[[\mathbf{F}]]$ defined by

$$\dim_{\mathbb{F}_p[\mathbf{F}/\mathbf{N}]} L = \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[\mathbf{F}/\mathbf{N}] \otimes_{\mathbb{F}_p[[\mathbf{F}]]} L)}{|\mathbf{F} : \mathbf{N}|},$$

where L is a finitely presented $\mathbb{F}_p[[\mathbf{F}]]$-module.

Let M be a finitely generated $\mathbb{F}_p[[\mathbf{F}]]$-module. Since $\mathbb{F}_p[\mathbf{F}/\mathbf{N}] \otimes_{\mathbb{F}_p[[\mathbf{F}]]} M$ is finite, there exists a finitely presented $\mathbb{F}_p[[\mathbf{F}]]$-module $\widetilde{M}$ such that
(1) M is a quotient of $\widetilde{M}$ and
(2) $\mathbb{F}_p[\mathbf{F}/\mathbf{N}] \otimes_{\mathbb{F}_p[[\mathbf{F}]]} \widetilde{M} \cong \mathbb{F}_p[\mathbf{F}/\mathbf{N}] \otimes_{\mathbb{F}_p[[\mathbf{F}]]} M$.
Therefore, from (1) we obtain that

$$\dim_{\mathbb{F}_p[\mathbf{F}/\mathbf{N}]} M = \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[\mathbf{F}/\mathbf{N}] \otimes_{\mathbb{F}_p[[\mathbf{F}]]} M)}{|\mathbf{F} : \mathbf{N}|}.$$

In the case where M is finitely presented, [29, Theorem 1.4] implies that

$$\dim_{\mathbb{F}_p[[\mathbf{F}]]} M = \lim_{i\to\infty} \dim_{\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i]} M. \tag{4}$$

Since we also have $\dim_{\mathbb{F}_p[[\mathbf{F}]]} M \le \dim_{\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i]} M$ for all i, Proposition 2.5 implies that (4) holds also when M is finitely generated. □

In the following proposition we collect some basic properties of $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$.

Proposition 3.3. *Let $\mathbf{H}$ be a finitely generated closed subgroup of $\mathbf{F}$. The following holds.*
(a) *Let $\mathcal{D}_{\mathbf{H}}$ be the division closure of $\mathbb{F}_p[[\mathbf{H}]]$ in $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$. Then $\mathcal{D}_{\mathbf{H}}$ is isomorphic to $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$ as a $\mathbb{F}_p[[\mathbf{H}]]$-ring.*

(b) *If M is a finitely generated $\mathbb{F}_p[[\mathbf{H}]]$-module, then*

$$\dim_{\mathbb{F}_p[[\mathbf{H}]]}(M) = \dim_{\mathbb{F}_p[[\mathbf{F}]]}(\mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M).$$

(c) *If $\mathbf{H}$ is open then,*

$$\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]} \cong \mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} \mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$$

as $(\mathbb{F}_p[[\mathbf{F}]], \mathbb{F}_p[[\mathbf{H}]])$-bimodules.

Proof. (a) Fix a normal chain $\mathbf{F} = \mathbf{N}_1 > \mathbf{N}_2 > \mathbf{N}_3 > \dots$ of open normal subgroups of $\mathbf{F}$, and let $\mathbf{H}_i = \mathbf{N}_i \cap \mathbf{H}$. Let M be a finitely generated $\mathbb{F}_p[[\mathbf{H}]]$-module. Observe first, that by Proposition 3.2,

$$\dim_{\mathbb{F}_p[[\mathbf{H}]]} M = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[\mathbf{H}/\mathbf{H}_i] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M)}{|\mathbf{H} : \mathbf{H}_i|}. \tag{5}$$

Considering $\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i]$ as a right $\mathbb{F}_p[[\mathbf{H}]]$-module, we obtain that

$$\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i] \cong \mathbb{F}_p[\mathbf{H}/\mathbf{H}_i]^{|\mathbf{F}:\mathbf{N}_i\mathbf{H}|}$$

as right $\mathbb{F}_p[[\mathbf{H}]]$-modules. Thus,

$$\dim_{\mathbb{F}_p}(\mathbb{F}_p[\mathbf{H}/\mathbf{H}_i] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M) = \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M)}{|\mathbf{F} : \mathbf{N}_i\mathbf{H}|}. \tag{6}$$

Therefore, from (5, (6) and Proposition 3.2, we conclude that

$$\dim_{\mathbb{F}_p[[\mathbf{H}]]} M = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i] \otimes_{\mathbb{F}_p[[\mathbf{F}]]} (\mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M)}{|\mathbf{F} : \mathbf{N}_i|} = \dim_{\mathbb{F}_p[[\mathbf{F}]]}(\mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M). \tag{7}$$

On the other hand, we have

$$\dim_{\mathbb{F}_p[[\mathbf{F}]]}(\mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M) = \dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]} \otimes_{\mathbb{F}_p[[\mathbf{F}]]} (\mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M)) = \dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]} \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M)) = \dim_{\mathcal{D}_{\mathbf{H}}}(\mathcal{D}_{\mathbf{H}} \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M).$$

Thus, we conclude that

$$\dim_{\mathbb{F}_p[[\mathbf{H}]]} M = \dim_{\mathcal{D}_{\mathbf{H}}}(\mathcal{D}_{\mathbf{H}} \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M),$$

and so, by Proposition 2.6, $\mathcal{D}_{\mathbf{H}}$ is isomorphic to $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$ as a $\mathbb{F}_p[[\mathbf{H}]]$-ring.

(b) This is the equality (7).

(c) First assume that $\mathbf{H}$ is normal in $\mathbf{F}$. Observe that for large i, $\mathbf{N}_i \le \mathbf{H}$. Let M be a finitely presented $\mathbb{F}_p[[F]]$-module. Then we obtain that

$$\dim_{\mathbb{F}_p[[\mathbf{F}]]} M = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{F}_p[[\mathbf{N}_i]]} M)}{|\mathbf{F} : \mathbf{N}_i|} = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{F}_p[[\mathbf{N}_i]]} M)}{|\mathbf{F} : \mathbf{H}||\mathbf{H} : \mathbf{N}_i|} = \frac{\dim_{\mathbb{F}_p[[\mathbf{H}]]} M}{|\mathbf{F} : \mathbf{H}|}.$$

Therefore, $\dim_{\mathbb{F}_p[[\mathbf{F}]]} = \widetilde{\dim}_{\mathbb{F}_p[[\mathbf{H}]]}$. Now, the result follows from Proposition 2.7.

Now we assume that $\mathbf{H}$ is arbitrary, We argue by induction on $|\mathbf{F}:\mathbf{H}|$. If $\mathbf{H}$ has index p in $\mathbf{F}$, then it is normal. If $|\mathbf{F}:\mathbf{H}|>p$, we find $\mathbf{H}_1$ of index p in $\mathbf{F}$ containing $\mathbf{H}$. Then by induction,

$$\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]} \cong \mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}_1]]} \mathcal{D}_{\mathbb{F}_p[[\mathbf{H}_1]]} \cong \mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}_1]]} (\mathbb{F}_p[[\mathbf{H}_1]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} \mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}) \cong \mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} \mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}\,.$$

□

In view of the previous proposition, we will identify $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$ and the division closure of $\mathbb{F}_p[[\mathbf{H}]]$ in $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$, and see $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$ as a subring of $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$.

3.3. **The division rings $\mathcal{D}(\mathbb{F}_p[G],\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]})$.** Let G be a subgroup of $\mathbf{F}$. As we have explained in Subsection 3.1 there exists the universal division $\mathbb{F}_p[G]$-ring of fractions $\mathcal{D}_{\mathbb{F}_p[G]}$. Let $\mathcal{D}_G=\mathcal{D}(\mathbb{F}_p[G],\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]})$ be the division closure of $\mathbb{F}_p[G]$ in $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$. In this subsection we will show that $\mathcal{D}_{\mathbb{F}_p[G]}$ and $\mathcal{D}_G$ are isomorphic as $\mathbb{F}_p[G]$-rings. In the case $G=F$ is a finitely generated free group and $\mathbf{F}$ is the pro-p completion of F, this result follows from [15, Corollary 2.9.16].

Proposition 3.4. *Let $\mathbf{F}$ be a finitely generated free pro-p group and let G be a finitely generated subgroup of $\mathbf{F}$. Let $\mathbf{F}=\mathbf{N}_1>\mathbf{N}_2>\mathbf{N}_3>\dots$ be a chain of open normal subgroups of $\mathbf{F}$ with trivial intersection. We put $G_j=G\cap\mathbf{N}_j$. Let $\mathcal{D}_G=\mathcal{D}(\mathbb{F}_p[G],\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]})$ be the division closure of $\mathbb{F}_p[G]$ in $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$. Then for every finitely generated $\mathbb{F}_p[G]$-module M,*

$$\dim_{\mathcal{D}_G}(\mathcal{D}_G\otimes_{\mathbb{F}_p[G]}M)=\lim_{i\to\infty}\frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p\otimes_{\mathbb{F}_p[G_i]}M)}{|G:G_i|}= \dim_{\mathcal{D}_{\mathbb{F}_p[G]}}(\mathcal{D}_{\mathbb{F}_p[G]}\otimes_{\mathbb{F}_p[G]}M).$$

In particular, the divison rings $\mathcal{D}_G$ and $\mathcal{D}_{\mathbb{F}_p[G]}$ are isomorphic as $\mathbb{F}_p[G]$-rings.

Proof. First observe that

$$\dim_{\mathcal{D}_G}(\mathcal{D}_G\otimes_{\mathbb{F}_p[G]}M)=\dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[G]}M)= \dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}(\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[G]}M))=\dim_{\mathbb{F}_p[[\mathbf{F}]]}(\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[G]}M). \tag{8}$$

Observe also that $|\mathbf{F}:\mathbf{N}_i|=|G:G_i|$ and

$$\mathbb{F}_p\otimes_{\mathbb{F}_p[G_i]}M\cong\mathbb{F}_p[G/G_i]\otimes_{\mathbb{F}_p[G]}M\cong \mathbb{F}_p[\mathbf{F}/\mathbf{N}_i]\otimes_{\mathbb{F}_p[G]}M\cong\mathbb{F}_p[\mathbf{F}/\mathbf{N}_i]\otimes_{\mathbb{F}_p[[\mathbf{F}]]}(\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[G]}M)).$$

Thus, Proposition 3.2 implies that

$$\dim_{\mathcal{D}_G}(\mathcal{D}_G\otimes_{\mathbb{F}_p[G]}M)=\lim_{i\to\infty}\frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p\otimes_{\mathbb{F}_p[G_i]}M)}{|G:G_i|}. \tag{9}$$

Let $\mathbf{F}_i=\gamma_i(\mathbf{F})$ and we put $H_i=G\cap\mathbf{F}_i$. The ring $\mathbb{F}_p[G/H_i]$ is a Noetherian domain and its classical field of fractions $\mathcal{Q}(\mathbb{F}_p[G/H_i])$ is universal. Moreover, by [30, Theorem 1.2], we have that

$$\dim_{\mathcal{D}_{\mathbb{F}_p[G]}}(\mathcal{D}_{\mathbb{F}_p[G]}\otimes_{\mathbb{F}_p[G]}M)=\lim_{i\to\infty}\dim_{\mathcal{D}_{\mathbb{F}_p[G/H_i]}}(\mathcal{D}_{\mathbb{F}_p[G/H_i]}\otimes_{\mathbb{F}_p[G]}M). \tag{10}$$

Observe that $\mathbb{F}_p[[\mathbf{F}/\mathbf{F}_i]]$ is also a Noetherian domain, and so the division closure of $\mathbb{F}_p[G/H_i]$ in $\mathcal{Q}(\mathbb{F}_p[[\mathbf{F}/\mathbf{F}_i]])$ is isomorphic to $\mathcal{D}_{\mathbb{F}_p[G/H_i]}$ (as a $\mathbb{F}_p[G]$-ring). Therefore,

(11)
$$\dim_{\mathcal{Q}(\mathbb{F}_p[[\mathbf{F}/\mathbf{F}_i]])}(\mathcal{Q}(\mathbb{F}_p[[\mathbf{F}/\mathbf{F}_i]])\otimes_{\mathbb{F}_p[G]}M) = \dim_{\mathcal{D}_{\mathbb{F}_p[G/H_i]}}(\mathcal{D}_{\mathbb{F}_p[G/H_i]}\otimes_{\mathbb{F}_p[G]}M).$$

Using Proposition 3.2 and arguing as in the proof of [26, Theorem 2.3], we obtain that

(12)
$$\dim_{\mathbb{F}_p[[\mathbf{F}]]}(\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[G]} M) = \lim_{i\to\infty}\dim_{\mathcal{Q}(\mathbb{F}_p[[\mathbf{F}/F_i]])}(\mathcal{Q}(\mathbb{F}_p[[\mathbf{F}/\mathbf{F}_i]])\otimes_{\mathbb{F}_p[G]} M).$$

Therefore, putting together (10), (11), (12), (8) and (9), we obtain that

$$\dim_{\mathcal{D}_{\mathbb{F}_p[G]}}(\mathcal{D}_{\mathbb{F}_p[G]}\otimes_{\mathbb{F}_p[G]}M) = \dim_{\mathcal{D}_G}(\mathcal{D}_G\otimes_{\mathbb{F}_p[G]}M) = \lim_{i\to\infty}\frac{\dim_{\mathbb{F}_p}(\mathbb{F}_p\otimes_{\mathbb{F}_p[G_i]} M)}{|G:G_i|}.$$

Applying Proposition 2.6, we obtain that the divison rings $\mathcal{D}_G$ and $\mathcal{D}_{\mathbb{F}_p[G]}$ are isomorphic as $\mathbb{F}_p[G]$-rings. □

An alternative approach of proving that $\mathcal{D}_G$ is isomorphic to $\mathcal{D}_{\mathbb{F}_p[G]}$ as $\mathbb{F}_p[G]$-ring is taken in [44, Lemma 7.5.5], where the result is proved by using a variation from [47, Theorem 6.3] of the uniqueness of Hughes-free division rings [24] (see also [17]).

3.4. **Mod-p L^2-Betti numbers.** L^2-Betti numbers play an important role in the solution of many problems in Group Theory. In the last years there was an attempt to develop a theory of mod-p L^2-Betti numbers for different families of groups (see [28]). If G is torsion-free and satisfies the Atiyah conjecture, P. Linnell [38] showed that L^2-Betti numbers of G can be defined as

$$b_i^{(2)}(G) = \dim_{\mathcal{D}(G)} H_i(G;\mathcal{D}(G)),$$

where $\mathcal{D}(G)$ is the division ring obtained as the division closure of $\mathbb{Q}[G]$ in the ring of affilated operators $\mathcal{U}(G)$. It turns out that if G is residually torsion-free nilpotent, $\mathcal{D}(G)$ is isomorphic to the universal division ring of fractions $\mathcal{D}_{\mathbb{Q}[G]}$ (see, for example, [31]). Therefore, we have

$$b_i^{(2)}(G) = \dim_{\mathcal{D}_{\mathbb{Q}[G]}} H_i(G;\mathcal{D}_{\mathbb{Q}[G]}).$$

Thus, by analogy, if G is a residually torsion-free nilpotent group, we define the ith mod-p L^2-Betti number of G as

$$\beta_i^{\mathrm{mod}\ p}(G) = \dim_{\mathcal{D}_{\mathbb{F}_p[G]}} H_i(G;\mathcal{D}_{\mathbb{F}_p[G]}).$$

In the case, where G is a subgroup of a free pro-p group, we also obtain the following formula, which can be seen as a mod-p analogue of the Lück approximation theorem [40].

Proposition 3.5. *Let $\mathbf{F}$ be a finitely generated free pro-p group and let G be a finitely generated subgroup of $\mathbf{F}$ of type FP_k for some $k\geq 1$. Let*

$\mathbf{F} = \mathbf{N}_1 > \mathbf{N}_2 > \mathbf{N}_3 > \ldots$ *be a chain of open normal subgroups of* $\mathbf{F}$ *with trivial intersection. We put* $G_j = G \cap \mathbf{N}_j$. *Then*

$$\beta_k^{mod\ p}(G) = \lim_{j\to\infty} \frac{\dim_{\mathbb{F}_p} H_k(G_j;\mathbb{F}_p)}{|G : G_j|}.$$

Proof. There exists a resolution of the $\mathbb{F}_p[G]$-module $\mathbb{F}_p$

$$0 \to R_k \to \mathbb{F}_p[G]^{n_{k-1}} \xrightarrow{\phi} \ldots \to \mathbb{F}_p[G]^{n_0} \to \mathbb{F}_p \to 0$$

with R_k finitely generated. The relevant part of the sequence for calculation of $H_k(G;*)$ is the following exact sequence

$$0 \to R_k \to \mathbb{F}_p[G]^{n_{k-1}} \to S_k \to 0,$$

where $S_k = \operatorname{Im}\phi$. Then we obtain that

$\beta_k^{\mathrm{mod}\ p}(G) = \dim_{\mathcal{D}_{\mathbb{F}_p[G]}} H_k(G;\mathcal{D}_{\mathbb{F}_p[G]}) = \dim_{\mathbb{F}_p[G]} R_k - n_{k-1} + \dim_{\mathbb{F}_p[G]} S_k$ and

$\dim_{\mathbb{F}_p} H_k(G_j;\mathbb{F}_p) = \dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{F}_p[G_j]} R_k) - n_{k-1}|G : G_j| + \dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{F}_p[G_j]} S_k)$.

Thus, Proposition 3.4 implies the proposition. □

In this paper we will work only with $\beta_1^{\mathrm{mod}\ p}(G)$. Observe that in this case, if G is infinite, we have the formula

$$\beta_1^{\mathrm{mod}\ p}(G) = \dim_{\mathbb{F}_p[G]} I_G - 1.$$

Also observe that if A is a non-trivial torsion-free abelian group, then since $\mathcal{D}_{\mathbb{F}_p[A]} = \mathcal{Q}(\mathbb{F}_p[A])$ is the field of fractions of $\mathbb{F}_p[A]$,

$$\beta_1^{\mathrm{mod}\ p}(A) = \dim_{\mathbb{F}_p[A]} I_A - 1 = \dim_{\mathcal{Q}(\mathbb{F}_p[A])}(\mathcal{Q}(\mathbb{F}_p[A]) \otimes_{\mathbb{F}_p[A]} I_A) - 1 = 0 \tag{13}$$

3.5. **Strong embeddings into free pro-p groups.** Assume that a finitely generated group G is a subgroup of a free pro-p group $\mathbf{F}$. Changing $\mathbf{F}$ by the closure of G in $\mathbf{F}$, we may assume that G is dense in $\mathbf{F}$. Let $\mathbf{F} = \mathbf{N}_1 > \mathbf{N}_2 > \mathbf{N}_3 > \ldots$ be a chain of open normal subgroups of $\mathbf{F}$ with trivial intersection and put $G_j = G \cap \mathbf{N}_j$. Observe that the closure of G_j in $\mathbf{F}$ is equal to $\mathbf{N}_j$, and so

$$|G_j : G_j^p[G_j, G_j]| \geq |\mathbf{N}_j : \mathbf{N}_j^p[\mathbf{N}_j, \mathbf{N}_j]|. \tag{14}$$

Denote by $d(\mathbf{F})$ the number of profinite generators of $\mathbf{F}$. Then we obtain

$$\begin{aligned}\dim_{\mathbb{F}_p} H_1(G_j;\mathbb{F}_p) = \log_p |G_j : G_j^p[G_j, G_j]| \geq \log_p |\mathbf{N}_j : \mathbf{N}_j^p[\mathbf{N}_j, \mathbf{N}_j]| = \\ d(\mathbf{N}_j) = (d(\mathbf{F}) - 1)|\mathbf{F} : \mathbf{N}_j| + 1 = (d(\mathbf{F}) - 1)|G : G_j| + 1.\end{aligned}$$

Thus, Proposition 3.5 implies the following corollary.

Corollary 3.6. *Let* G *be a finitely generated subgroup of a free pro-p group* $\mathbf{F}$. *Then*

$$\dim_{\mathbb{F}_p G} I_G = \beta_1^{mod\ p}(G) + 1 \geq d(\mathbf{F}) \tag{15}$$

We say that an embedding $G \hookrightarrow \mathbf{F}$ of a finitely generated group G into a free pro-p group $\mathbf{F}$ is **strong** if G is dense in $\mathbf{F}$ and $\dim_{\mathbb{F}_p G} I_G = \beta_1^{\text{mod } p}(G) + 1 = d(\mathbf{F})$. A finitely generated group G is called **strongly embeddable in a free pro-p group** (SE(p) for simplicity) if there exists a strong embedding $G \hookrightarrow \mathbf{F}$.

Let G be a parafree group. Observe that G is residually-p for every prime p. Thus, if G is finitely generated, its pro-p completion $G_{\widehat{p}}$ is a finitely generated free pro-p group and G is a subgroup of $G_{\widehat{p}}$. In this case the inequality (14) is an equality, and so in the same way as we obtained the inequality (15), we obtain that $\beta_1^{\text{mod } p}(G) = d(G_{\widehat{p}}) - 1$. Therefore, the embedding $G \hookrightarrow G_{\widehat{p}}$ is strong. Thus, all finitely generated parafree groups are SE(p). On the other hand, not every subgroup of a parafree group is SE(p). For example, the fundamental group of an oriented surface of genus greater than 1 is not SE(p). However the fundamental group of an oriented surface of genus greater than 2 can be embedded in a parafree group (see [6, Section 4.1]).

By [11, Proposition 7.5], if G is a finitely generated dense subgroup of a finitely generated free pro-p group $\mathbf{F}$, then $b_1^{(2)}(G) \geq d(\mathbf{F}) - 1$. On the other hand, by [19, Theorem 1.6] and Proposition 3.5 , $b_1^{(2)}(G) \leq \beta_1^{\text{mod } p}(G)$. Thus, if $G \hookrightarrow \mathbf{F}$ is a strong embedding, we have $b_1^{(2)}(G) = \beta_1^{\text{mod } p}(G) = d(\mathbf{F}) - 1$.

4. On $\mathcal{D}$-torsion-free modules.

4.1. **General results.** Let R be a ring and let $R \hookrightarrow \mathcal{D}$ be an embedding of R into a division ring $\mathcal{D}$. Let M be a left R-module. We say that M is **$\mathcal{D}$-torsion-free** if the canonical map

$$M \to \mathcal{D} \otimes_R M, \ m \mapsto 1 \otimes m,$$

is injective. The following lemma describes several equivalent definitions of $\mathcal{D}$-torsion-free modules.

Lemma 4.1. *The following statements for a left R-module M are equivalent.*
(a) *M is $\mathcal{D}$-torsion-free.*
(b) *There are a $\mathcal{D}$-module N and an injective homomorphism $\varphi : M \to N$ of R-modules.*
(c) *For any $0 \neq m \in M$, there exists a homomorphism of R-modules $\varphi : M \to \mathcal{D}$, such that $\varphi(m) \neq 0$.*

Proof. The proof is straightforward and we leave it to the reader. □

Let M be a left R-module. Recall that we use $\dim_{\mathcal{D}} M$ to denote the dimension of $\mathcal{D} \otimes_R M$ as a $\mathcal{D}$-module. Observe that if $\dim_{\mathcal{D}} M$ is finite, it is also equal to the dimension of $\operatorname{Hom}_R(M, \mathcal{D})$ as a right $\mathcal{D}$-module.

Lemma 4.2. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. Assume that*
(1) *M_1 and M_3 are $\mathcal{D}$-torsion-free,*
(2) *$\dim_{\mathcal{D}} M_1$ and $\dim_{\mathcal{D}} M_3$ are finite and*
(3) *$\dim_{\mathcal{D}} M_2 = \dim_{\mathcal{D}} M_1 + \dim_{\mathcal{D}} M_3$.*

Then M_2 is also $\mathcal{D}$-torsion-free.

Proof. We are going to use the third characterization of $\mathcal{D}$-torsion-free modules from Lemma 4.1. Consider the following exact sequence of right $\mathcal{D}$-modules.

$$0 \to \operatorname{Hom}_R(M_3, \mathcal{D}) \to \operatorname{Hom}_R(M_2, \mathcal{D}) \to \operatorname{Hom}_R(M_1, \mathcal{D}).$$

Since $\dim_{\mathcal{D}} M_2 = \dim_{\mathcal{D}} M_1 + \dim_{\mathcal{D}} M_3$, the last map is surjective.

Let $m \in M_2$. If $m \notin M_1$, then since M_3 is $\mathcal{D}$-torsion-free, there exists $\varphi \in \operatorname{Hom}_R(M_2/M_1, \mathcal{D})$ such that $\varphi(m + M_1) \neq 0$. Hence there exists $\widetilde{\varphi} \in \operatorname{Hom}_R(M_2, \mathcal{D})$, such that $\widetilde{\varphi}(m) \neq 0$.

If $m \in M_1$, then since M_1 is $\mathcal{D}$-torsion-free, there exists $\varphi \in \operatorname{Hom}_R(M_1, \mathcal{D})$ such that $\varphi(m) \neq 0$. Using that the restriction map

$$\operatorname{Hom}_R(M_2, \mathcal{D}) \to \operatorname{Hom}_R(M_1, \mathcal{D})$$

is surjective, we obtain again that there exists $\widetilde{\varphi} \in \operatorname{Hom}_R(M_2, \mathcal{D})$, such that $\widetilde{\varphi}(m) \neq 0$. □

In the calculations of $\dim_{\mathcal{D}}$ the following elementary lemma will be useful.

Lemma 4.3. *Let $\mathcal{D}$ be a division R-ring and M be a $\mathcal{D}$-torsion-free R-module of finite $\mathcal{D}$-dimension. Let L be a non-trivial R-submodule of M. Then $\dim_{\mathcal{D}}(M/L) < \dim_{\mathcal{D}} M$. Moreover, if $\dim_{\mathcal{D}} L = 1$, then $\dim_{\mathcal{D}}(M/L) = \dim_{\mathcal{D}} M - 1$.*

Proof. Since M is $\mathcal{D}$-torsion-free, $\mathcal{D} \otimes_R (M/L)$ is a proper quotient of $\mathcal{D} \otimes_R M$. Hence $\dim_{\mathcal{D}} M/L < \dim_{\mathcal{D}} M$.

Now assume that $\dim_{\mathcal{D}} L = 1$. In this case $\dim_{\mathcal{D}}(M/L) \geq \dim_{\mathcal{D}} M - 1$. Therefore, $\dim_{\mathcal{D}}(M/L) = \dim_{\mathcal{D}} M - 1$. □

4.2. **$\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-torsion-free modules.** Let $\mathbf{F}$ be a finitely generated free pro-p group. The main purpose of this subsection is to prove the following result.

Proposition 4.4. *Assume that $1 \neq z \in \mathbf{F}$ is not a proper p-power of an element of $\mathbf{F}$. Denote by $\mathbf{Z}$ the closed subgroup of $\mathbf{F}$ generated by z. Then the module $I_{\mathbf{F}}/I_{\mathbf{Z}}^{\mathbf{F}}$ is $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-torsion-free.*

Before proving the proposition we have to establish several preliminary results.

Lemma 4.5. *Let $\mathbf{H}$ be an open subgroup of $\mathbf{F}$ Let M be a $\mathbb{F}_p[[\mathbf{H}]]$-module. Then $\mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M$ is $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-torsion-free if and only if M is $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$-torsion-free.*

Proof. Assume that M is $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$-torsion-free. We have that the map $M \to \mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]} \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M$ is injective. Then, since $\mathbb{F}_p[[\mathbf{F}]]$ is a free right $\mathbb{F}_p[[\mathbf{H}]]$-module, the map

$$\mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M \xrightarrow{\alpha} \mathbb{F}_p[[\mathbf{F}]] \otimes_{\mathbb{F}_p[[\mathbf{H}]]} (\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]} \otimes_{\mathbb{F}_p[[\mathbf{H}]]} M)$$

is also injective.

Consider the canonical isomorphism between tensor products

$$\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}\otimes_{\mathbb{F}_p[[\mathbf{H}]]}M)\xrightarrow{\beta}(\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]}\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]})\otimes_{\mathbb{F}_p[[\mathbf{H}]]}M.$$

By Propositopn 3.3(c),

$$\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\cong\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]}\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$$

as $(\mathbb{F}_p[[\mathbf{F}]],\mathbb{F}_p[[\mathbf{H}]])$-bimodules. Thus, there exists an isomorphism of $\mathbb{F}_p[[\mathbf{F}]]$-modules

$$(\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]}\mathcal{D}_{\mathbb{F}_p[[\mathbf{H})]]})\otimes_{\mathbb{F}_p[[\mathbf{H}]]}M\xrightarrow{\gamma}\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{H}]]}M.$$

We put $\varphi=\gamma\circ\beta\circ\alpha$ and apply Lemma 4.1. Since $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{H}]]}M$ is a $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-module and φ is an injective $\mathbb{F}_p[[\mathbf{F}]]$-homomorphism, $\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]}M$ is $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-torsion-free.

Another direction of the proposition is clear because M is a $\mathbb{F}_p[[\mathbf{H}]]$-submodule of $\mathbb{F}_p[[\mathbf{F}]]\otimes_{\mathbb{F}_p[[\mathbf{H}]]}M$. □

Lemma 4.6. *Let* $\mathbf{H}$ *be an open normal subgroup of* $\mathbf{F}$ *and assume that* $1\neq z\in\mathbf{H}$. *Let* $\mathbf{Z}$ *be the closed subgroup of* $\mathbf{H}$ *generated by* z. *Then the following are equivalent.*

(a) *The* $\mathbb{F}_p[[\mathbf{F}]]$*-module* $I^{\mathbf{F}}_{\mathbf{H}}/I^{\mathbf{F}}_{\mathbf{Z}}$ *is not* $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$*-torsion-free.*
(b) *The* $\mathbb{F}_p[[\mathbf{H}]]$*-module* $I_{\mathbf{H}}/I^{\mathbf{H}}_{\mathbf{Z}}$ *is not* $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$*-torsion-free.*
(c) *There are* $a\in I^{\mathbf{F}}_{\mathbf{H}}$ *and* $b\in I_{\mathbf{F}}$ *such that* $ba=z-1$.
(d) *There are* $a\in I^{\mathbf{F}}_{\mathbf{H}}$ *and* $b\in I_{\mathbf{F}}$ *such that* $ab=z-1$.

Proof. (a)$\Longleftrightarrow$(b): This follows from Lemma 4.5 and Lemma 2.3.

(c)$\Longrightarrow$(a): Put $N=\mathbb{F}_p[[\mathbf{F}]]a/I^{\mathbf{F}}_{\mathbf{Z}}$. Since, b is not invertible in $\mathbb{F}_p[[\mathbf{F}]]$,

$$N=\mathbb{F}_p[[\mathbf{F}]]a/I^{\mathbf{F}}_{\mathbf{Z}}=\mathbb{F}_p[[\mathbf{F}]]a/\mathbb{F}_p[[\mathbf{F}]](z-1)=\mathbb{F}_p[[\mathbf{F}]]a/\mathbb{F}_p[[\mathbf{F}]]ba\cong\mathbb{F}_p[[\mathbf{F}]]/\mathbb{F}_p[[\mathbf{F}]]b$$

is a non-trivial submodule of $I^{\mathbf{F}}_{\mathbf{H}}/I^{\mathbf{F}}_{\mathbf{Z}}$. Clearly $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}N=0$, and so (a) holds.

(a)$\Longrightarrow$(c): We put $M=I^{\mathbf{F}}_{\mathbf{H}}/I^{\mathbf{F}}_{\mathbf{Z}}$ and let $\phi:M\to\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}M$ be the canonical map. Let $\overline{L}=L/I^{\mathbf{F}}_{\mathbf{Z}}$ be the kernel of ϕ and $\overline{M}$ the image of ϕ. Hence we have the exact sequence

$$0\to L\to I^{\mathbf{F}}_{\mathbf{H}}\to\overline{M}\to 0.$$

After applying $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}$ we obtain the exact sequence

$$\operatorname{Tor}_1^{\mathbb{F}_p[[\mathbf{F}]]}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]},\overline{M})\to\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}L\to(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]})^{d(\mathbf{H})}\to\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}\overline{M}\to 0. \tag{16}$$

The $\mathbb{F}_p[\mathbf{F}]]$-module $I^{\mathbf{F}}_{\mathbf{H}}$ is free of rank $d(H)$ and the $\mathbb{F}_p[[\mathbf{F}]]$-module $I^{\mathbf{F}}_{\mathbf{Z}}$ is cyclic. Thus, by Lemma 4.3,

$$\dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}M)=d(\mathbf{H})-1.$$

Therefore,

$$\dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}\overline{M})=\dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}\otimes_{\mathbb{F}_p[[\mathbf{F}]]}M)=d(\mathbf{H})-1.$$

Observe also that by Proposition 3.1, $\mathrm{Tor}_1^{\mathbb{F}_p[[\mathbf{F}]]}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}, \overline{M}) = 0$. Therefore, (16) implies that

$$\dim_{\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}}(\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]} \otimes_{\mathbb{F}_p[[\mathbf{F}]]} L) = 1.$$

Since L is a free profinite $\mathbb{F}_p[[\mathbf{F}]]$-module (see, for example [26, Lemma 3.1]), L should be cyclic $\mathbb{F}_p[[\mathbf{F}]]$-module. We write $L = \mathbb{F}_p[[\mathbf{F}]]a$ for some $a \in I_{\mathbf{H}}^{\mathbf{F}}$. Then there exists $b \in \mathbb{F}_p[[\mathbf{F}]]$ such that $ba = z - 1$. By our assumption $L \neq I_{\mathbf{Z}}^{\mathbf{F}}$. Thus, b is not invertible, and so $b \in I_{\mathbf{F}}$.

(c)$\Longrightarrow$(d): The map $g \mapsto g^{-1}$ on $\mathbf{F}$ can be extended to a continuous anti-isomorphism $\alpha : \mathbb{F}_p[[\mathbf{F}]] \to \mathbb{F}_p[[\mathbf{F}]]$. If $z - 1 = ba$, then $z^{-1} - 1 = \alpha(z-1) = \alpha(a)\alpha(b)$ and so $z - 1 = (-z\alpha(a))\alpha(b)$. Now note that $-z\alpha(a) \in I_{\mathbf{H}}\mathbb{F}_p[[\mathbf{F}]]$ and $\alpha(b) \in I_{\mathbf{F}}$

(d)$\Longrightarrow$(a): It is proved in the same way as (c)$\Longrightarrow$(d). □

Now we are ready to prove Proposition 4.4.

Proof of Proposition 4.4. We can assume that $\mathbf{F}$ is not cyclic. There exists a normal open subgroup $\mathbf{N}$ of $\mathbf{F}$ such that $z\mathbf{N}$ is not a p-power in $F/\mathbf{N}$. We will prove the proposition by induction on $|\mathbf{F}/\mathbf{N}|$.

If $\mathbf{F}/\mathbf{N}$ is cyclic, then $z \notin \Phi(\mathbf{F})$ and so z is a member of a free generating system of $\mathbf{F}$. Thus $I_{\mathbf{F}}/I_{\mathbf{Z}}^{\mathbf{F}}$ is a free $\mathbb{F}_p[[\mathbf{F}]]$-module and we are done.

Assume now that $\mathbf{F}/\mathbf{N}$, and so $\mathbf{F}/\mathbf{N}\Phi(\mathbf{F})$ are not cyclic. Let $\mathbf{M}$ be the closed subgroup of $\mathbf{F}$ containing the commutator subgroup $[\mathbf{F}, \mathbf{F}]$ and the element z and such that $\mathbf{M}/([\mathbf{F}, \mathbf{F}]\mathbf{Z})$ is the torsion part of $\mathbf{F}/([\mathbf{F}, \mathbf{F}]\mathbf{Z})$. Observe that $\mathbf{M}\Phi(\mathbf{F})/\Phi(\mathbf{F})$ is cyclic or trivial and so, since $\mathbf{F}/\mathbf{N}\Phi(\mathbf{F})$ is not cyclic, $\mathbf{MN}$ is a proper subgroup of $\mathbf{F}$.

By the construction of $\mathbf{M}$, $\mathbf{F}/\mathbf{M} \cong \mathbb{Z}_p^k$ for some $k \geq 1$. Since $\mathbf{MN}$ is a proper subgroup of $\mathbf{F}$, $\mathbf{MN}/\mathbf{M}$ is a proper subgroup of $\mathbf{F}/\mathbf{M}$. Therefore, there exists a surjective map $\sigma : \mathbf{F} \to \mathbb{Z}_p$ such that $\mathbf{M} \leq \ker \sigma$ and $\mathbf{N} \ker \sigma \neq \mathbf{F}$. We put $\mathbf{H} = \mathbf{N} \ker \sigma$ and extend σ to the map $\widetilde{\sigma} : \mathbb{F}_p[[\mathbf{F}]] \to \mathbb{F}_p[[\mathbb{Z}_p]]$. Observe that $\ker \widetilde{\sigma} = I_{\ker \sigma}^{\mathbf{F}}$.

By way of contradiction, assume that $I_{\mathbf{F}}/I_{\mathbf{Z}}^{\mathbf{F}}$ is not $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-torsion-free. Then by Lemma 4.6, there are $a, b \in I_{\mathbf{F}}$ such that $ab = z - 1$. Applying $\widetilde{\sigma}$ we obtain that $\widetilde{\sigma}(a)\widetilde{\sigma}(b) = 0$. Since $\mathbb{F}_p[[\mathbb{Z}_p]]$ is a domain, either a or b lie in $\ker \widetilde{\sigma} = I_{\ker \sigma}^{\mathbf{F}} \subset I_{\mathbf{H}}^{\mathbf{F}}$. Applying again Lemma 4.6, we conclude that $I_{\mathbf{H}}/I_{\mathbf{Z}}^{\mathbf{H}}$ is not $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$-torsion-free.

However, observe that $\mathbf{N}$ is also a normal subgroup of $\mathbf{H}$, $z\mathbf{N}$ is not a p-power in $\mathbf{H}/\mathbf{N}$ and $|\mathbf{H}/\mathbf{N}| < |\mathbf{F}/\mathbf{N}|$. Thus, we can apply the inductive assumption and conclude that $I_{\mathbf{H}}/I_{\mathbf{Z}}^{\mathbf{H}}$ is $\mathcal{D}_{\mathbb{F}_p[[\mathbf{H}]]}$-torsion-free. We have arrived to a contradiction. Thus, $I_{\mathbf{F}}/I_{\mathbf{Z}}^{\mathbf{F}}$ is $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-torsion-free. □

4.3. **$\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free modules.** Let $\mathbf{F}$ be a finitely generated free pro-p group and let G be an (abstract) dense finitely generated subgroup of $\mathbf{F}$. First we prove the following analogue of Lemma 4.5.

Lemma 4.7. *Let H be a subgroup of G and let M be a $\mathcal{D}_{\mathbb{F}_p[H]}$-torsion-free left $\mathbb{F}_p[H]$-module. Then $\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} M$ is $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free.*

Proof. Let $\mathcal{D}_H$ be the division closure of $\mathbb{F}_p[H]$ in $\mathcal{D}_{\mathbb{F}_p[G]}$. Observe that $\mathcal{D}_H$ and $\mathcal{D}_{\mathbb{F}_p[H]}$ are isomorphic as $\mathbb{F}_p[H]$-rings (it follows, for example, from Proposition 3.4).

We have that the map $M \to \mathcal{D}_H \otimes_{\mathbb{F}_p[H]} M$ is injective. Then, since $\mathbb{F}_p[G]$ is a free right $\mathbb{F}_p[H]$-module, the map

$$\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} M \xrightarrow{\alpha} \mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} (\mathcal{D}_H \otimes_{\mathbb{F}_p[H]} M)$$

is also injective.

Consider the canonical isomorphism between tensor products

$$\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} (\mathcal{D}_H \otimes_{\mathbb{F}_p[H]} M) \xrightarrow{\beta} (\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} \mathcal{D}_H) \otimes_{\mathbb{F}_p[H]} M.$$

By [21], $\mathcal{D}_{\mathbb{F}_p[G]}$ is strongly Hughes-free. This means that the canonical map of $(\mathbb{F}_p[G], \mathbb{F}_p[H])$-bimodules

$$\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} \mathcal{D}_H \to \mathcal{D}_{\mathbb{F}_p[G]}$$

is injective. Moreover, the image of $\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} \mathcal{D}_H$ is a direct summand of $\mathcal{D}_{\mathbb{F}_p[G]}$ as a right $\mathcal{D}_H$-submodule (and so, it is also a direct summand as a right $\mathbb{F}_p[H]$-submodule). Thus, the following canonical map of $\mathbb{F}_p[G]$-modules

$$(\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} \mathcal{D}_H) \otimes_{\mathbb{F}_p[H]} M \overset{\gamma}{\hookrightarrow} \mathcal{D}_{\mathbb{F}_p[G]} \otimes_{\mathbb{F}_p[H]} M$$

is injective. We put $\varphi = \gamma \circ \beta \circ \alpha$ and apply Lemma 4.1. Since $\mathcal{D}_{\mathbb{F}_p[G]} \otimes_{\mathbb{F}_p[H]} M$ is a $\mathcal{D}_{\mathbb{F}_p[G]}$-module and φ is an injective $\mathbb{F}_p[G]$-homomorphism, $\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} M$ is $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free. □

Now we can present our first example of a $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free $\mathbb{F}_p[G]$-module.

Proposition 4.8. *Let H be a subgroup of G and A a maximal abelian subgroup of H. Then the $\mathbb{F}_p[G]$-module I_H^G/I_A^G is $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free.*

Proof. From Lemma 2.1 we know that

$$I_H^G/I_A^G \cong \mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} (I_H/I_A^H).$$

Thus, in view of Lemma 4.7, it is enough to show that I_H/I_A^H is $\mathcal{D}_{\mathbb{F}_p[H]}$-torsion-free.

Let $\mathbf{Z} = C_{\mathbf{F}}(A)$. Since $\mathbf{F}$ is a free pro-p group, $\mathbf{Z}$ is a maximal cyclic pro-p subgroup of $\mathbf{F}$.

Claim 4.9. *The canonical map $\mathbb{F}_p[H/A] \to \mathbb{F}_p[[\mathbf{F}/\mathbf{Z}]]$ is injective.*

Proof. Since A is maximal abelian in H, we have that $A = \mathbf{Z} \cap H$. Hence the obvious map $\mathbb{F}_p[H/A] \to \mathbb{F}_p[\mathbf{F}/\mathbf{Z}]$ in injective. The map $\mathbb{F}_p[\mathbf{F}/\mathbf{Z}] \to \mathbb{F}_p[[\mathbf{F}/\mathbf{Z}]]$ is also injective, because $\mathbf{Z}$ is closed in $\mathbf{F}$. This finishes the proof of the claim. □

Observe that $\mathbb{F}_p[H/A] \cong \mathbb{F}_p[H]/I_A^H$ and $\mathbb{F}_p[[\mathbf{F}]]/I_{\mathbf{Z}}^{\mathbf{F}} \cong \mathbb{F}_p[[\mathbf{F}/\mathbf{Z}]]$. Therefore, by Claim 4.9, I_H/I_A^H is a $\mathbb{F}_p[H]$-submodule of $I_{\mathbf{F}}/I_{\mathbf{Z}}^{\mathbf{F}}$. By Proposition 4.4, we can embed $I_{\mathbf{F}}/I_{\mathbf{Z}}^{\mathbf{F}}$ in a $\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-module. By Proposition 3.4, every

$\mathcal{D}_{\mathbb{F}_p[[\mathbf{F}]]}$-module is also a $\mathcal{D}_{\mathbb{F}_p[H]}$-module. Therefore, by Lemma 4.1, I_H/I_A^H is $\mathcal{D}_{\mathbb{F}_p[H]}$-torsion-free. □

The following proposition shows another example of a $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free $\mathbb{F}_p[G]$-module. This is the main result of this section.

Proposition 4.10. *Let $\mathbf{F}$ be a finitely generated free pro-p group and let G be an (abstract) dense finitely generated subgroup of $\mathbf{F}$. Let H be a subgroup of G and A a maximal abelian subgroup of H. Let B be an abelian subgroup of G containing A. We put*

$$J = \{(x, -x) \in I_H^G \oplus I_B^G : \ x \in I_A^G\}.$$

Then $M = (I_H^G \oplus I_B^G)/J$ is $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free and $\dim_{\mathbb{F}_p[G]} M = \dim_{\mathbb{F}_p[G]} I_H^G$.

Proof. Let $L = (I_A^G \oplus I_B^G)/J \le M$. Then $L \cong I_B^G$ is $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free. The $\mathbb{F}_p[G]$-module M/L is isomorphic to I_H^G/I_A^G, and so it is $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free by Proposition 4.8.

By (13), $\dim_{\mathbb{F}_p[A]} I_A = 1$. Therefore, by Proposition 3.3(b),

$$\dim_{\mathbb{F}_p[G]} I_A^G = \dim_{\mathbb{F}_p[A]} I_A = 1.$$

In the same way we obtain that $\dim_{\mathbb{F}_p[G]} I_B^G = 1$.

Since $\dim_{\mathbb{F}_p[G]}(I_H^G \oplus I_B^G) = \dim_{\mathbb{F}_p[G]} I_H^G + 1$, by Lemma 4.3,

$$\dim_{\mathbb{F}_p[G]} M = \dim_{\mathbb{F}_p[G]} I_H^G + 1 - 1 = \dim_{\mathbb{F}_p[G]} I_H^G \text{ and}$$
$$\dim_{\mathbb{F}_p[G]}(I_H^G/I_A^G) = \dim_{\mathbb{F}_p[G]} I_H^G - 1 = \dim_{\mathbb{F}_p[G]} M - 1.$$

Therefore,

$$\dim_{\mathbb{F}_p[G]}(M/L) = \dim_{\mathbb{F}_p[G]}(I_H^G/I_A^G) = \dim_{\mathbb{F}_p[G]} M - 1.$$

Thus, we have obtained that M/L and L are $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free and

$$\dim_{\mathbb{F}_p[G]} M = \dim_{\mathbb{F}_p[G]}(M/L) + \dim_{\mathbb{F}_p[G]} L.$$

Applying Lemma 4.2, we conclude that M is also $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free. □

5. Proof of main results

The following theorem is the main result of this section. Theorem 1.2 follows from it directly.

Theorem 5.1. *Let $\mathbf{F}$ be a finitely generated free pro-p group and let $H \hookrightarrow \mathbf{F}$ be a strong embedding of finitely generated group H. Let A be a maximal abelian subgroup of H and let B be an abelian finitely generated subgroup of $\mathbf{F}$ containing A. Then $G = \langle H, B\rangle$ is isomorphic to $H *_A B$, and the embedding $G \hookrightarrow \mathbf{F}$ is strong.*

Proof. In view of Proposition 2.4 we have to show that $I_H^G \cap I_B^G = I_A^G$ in $\mathbb{F}_p[G]$. Let

$$J = \{(x, -x) \in I_H^G \oplus I_B^G : \ x \in I_A^G\} \text{ and } M = (I_H^G \oplus I_B^G)/J.$$

Then by Proposition 4.10, $\dim_{\mathbb{F}_p[G]} M = \dim_{\mathbb{F}_p[G]} I_H^G$. Therefore,

$$\dim_{\mathbb{F}_p[G]} M = \dim_{\mathbb{F}_p[G]} I_H^G = \dim_{\mathbb{F}_p[H]} I_H = \beta_1^{\mathrm{mod}\ p}(H) + 1 = d(\mathbf{F}).$$

Since $I_G = I_H^G + I_B^G$, we have that the natural map $\alpha : M \to I_G$ is surjective. In particular

$$\beta_1^{\text{mod } p}(G) = \dim_{\mathbb{F}_p[G]} I_G - 1 \leq \dim_{\mathbb{F}_p[G]} M - 1 = d(\mathbf{F}) - 1.$$

Thus, using (15) we obtain that $\beta_1^{\text{mod } p}(G) = d(\mathbf{F}) - 1$ and $\dim_{\mathbb{F}_p[G]} I_G = \dim_{\mathbb{F}_p[G]} M$. This shows that the embedding $G \hookrightarrow \mathbf{F}$ is strong.

By Proposition 4.10, M is $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free. Therefore, by Lemma 4.3, for any proper quotient $\overline{M}$ of M, $\dim_{\mathbb{F}_p[G]} \overline{M} < \dim_{\mathbb{F}_p[G]} M$. This implies that α is an isomorphism, and so $I_H^G \cap I_B^G = I_A^G$. Hence, Proposition 2.4 implies that $G \cong H *_A B$ □

Another direct consequence of Theorem 5.1 is the following corollary.

Corollary 5.2. *Let $\mathbf{F}$ be a finitely generated free pro-p group and let $H \hookrightarrow \mathbf{F}$ be a strong embedding of finitely generated group H. Let A be a maximal abelian subgroup of H. Assume that A is finitely generated. Let B be a finitely generated torsion-free abelian group containing A and such that B/A has no p-torsion. Then there exists an embedding of $H *_A B$ into $\mathbf{F}$ that extends $H \hookrightarrow \mathbf{F}$. In particular, $H *_A B$ is SE(p).*

Proof. Let $H \hookrightarrow \mathbf{F}$ be a strong embedding. Since B/A has no p-torsion and B is torsion-free abelian, the embedding of A into $\mathbf{F}$ can be extended to an embedding of B into $\mathbf{F}$. Now, we can apply Theorem 5.1. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\phi : F^{\mathbb{Q}}(X) \to \mathbb{Q}\langle\langle Y \rangle\rangle$ be the Magnus homomorphism defined in the Introduction. Let H be a finitely generated subgroup of $F^{\mathbb{Q}}(X)$ and let ϕ_H be the restriction of ϕ on H. As we explained in the Introduction it is enough to show that ϕ_H is injective.

The group H is a subgroup of a group obtained from the free group $F(X)$ by adjoining progressively a n_1th root, a n_2th root, ..., and finally n_kth root. Take a prime p that does not divide the product $n_1 \cdots n_k$. Let $\mathbf{F}$ be the pro-p completion of $F(X)$. By iterated use of Corollary 5.2, we obtain that H can be embedded into $\mathbf{F}$ and this embedding extends the canonical embedding $F(X) \hookrightarrow \mathbf{F}$. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\mathbb{Z}_p$ its valuation ring. The ring $\mathbb{Z}_p\langle\langle y_1, \ldots, y_d \rangle\rangle$ is profinite and it is isomorphic to $\mathbb{Z}_p[[\mathbf{F}]]$. Therefore, there exists a unique continuous group homomorphism of $\mathbf{F}$ into $\mathbb{Z}_p\langle\langle y_1, \ldots, y_d \rangle\rangle^*$ that sends x_i to $1+y_i$. Denote by ψ the composition of this map with the embedding of $\mathbb{Z}_p\langle\langle y_1, \ldots, y_d \rangle\rangle^*$ into $\mathbb{Q}_p\langle\langle y_1, \ldots, y_d \rangle\rangle^*$. Since $\phi_H(x_i) = \psi(x_i)$ in $\mathbb{Q}_p\langle\langle y_1, \ldots, y_d \rangle\rangle^*$, and there is only one way to take roots in $\mathbb{Q}_p\langle\langle y_1, \ldots, y_d \rangle\rangle^*$, we obtain the following commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\phi_H} & \mathbb{Q}\langle\langle y_1, \ldots, y_d \rangle\rangle^* \\ \downarrow & & \downarrow \\ \mathbf{F} & \overset{\psi}{\hookrightarrow} & \mathbb{Q}_p\langle\langle y_1, \ldots, y_d \rangle\rangle^* \end{array}.$$

This implies that ϕ_H is injective as well. This proves Theorem 1.1. □

6. Linearity of free $\mathbb{Q}$-groups and free pro-p groups

We finish this paper with a discussion on another two well-known problems concerning linearity of free $\mathbb{Q}$-groups and free pro-p groups.

The problem of whether a free $\mathbb{Q}$-group $F^{\mathbb{Q}}(X)$ is linear appears in [8, Problem F13] and it is attributed to I. Kapovich (see also [35, Problem 13.39(b)]). The problem of whether a free pro-p group $\mathbf{F}$ is linear is usually attributed to A. Lubotzky (for example, we discussed this question in Jerusalem in November, 2001).

In the context of profinite groups, one can consider two kinds of linearity (see, for example, [25]). On one hand, we say that a profinite group G is **linear** if it is linear as an abstract group, that is it has a faithful representation by matrices of fixed degree over a field. On the other hand, the concept of **t-linear** profinite group takes into account the topology of G and means that G can be faithfully represented as a closed subgroup of the group of invertible matrices of fixed degree over a profinite commutative ring.

It is commonly believed that a non-abelian free pro-p group is not t-linear (see, [39, Conjecture 3.8] and [48, Section 5.3]). An equivalent reformulation of this statement is that a p-adic analytic pro-p group satisfies a non-trvial pro-p identity. A. Zubkov [53] proved that if $p > 2$, then a non-abelian free pro-p group cannot be represented by 2-by-2 matrices over a profinite commutative ring. E. Zelmanov announced that given a fixed n, a non-abelian free pro-p group cannot be represented by n-by-n matrices over a profinite commutative ring for every large enough prime $p >> n$ (see [51, 52]). Recently, D. El-Chai Ben-Ezra, E. Zelmanov showed that a free pro-2 group cannot be represented by 2-by-2 matrices over a profinite commutative ring of characteristic 2 [18].

Recall that by a result of A. Malcev [42, Theorem IV], a group can be represented by matrices of degree n over a field if and only if every its finitely generated subgroup has this property. Thus, in order to decide whether $F^{\mathbb{Q}}(X)$ or $\mathbf{F}$ are linear, we have to analyze the structure of their finitely generated (abstract) subgroups. In order to apply the Malcev criterion one should find a uniform n which does not depend on a finitely generated subgroup. We may ask a weaker question of whether $F^{\mathbb{Q}}(X)$ and $\mathbf{F}$ are locally linear. Using recent advances in geometric group theory one can answer this positively in the case of $F^{\mathbb{Q}}(X)$.

Theorem 6.1. *The groups $F^{\mathbb{Q}}(X)$ are locally linear over $\mathbb{Z}$.*

Proof. Let H be a finitely generated subgroup of $F^{\mathbb{Q}}(X)$. Then H is a subgroup of a group G obtained from the free group $F(X)$ by adjoining progressively several roots. Thus, from [9] we obtain that G is hyperbolic, and so [23, Theorem A] implies that G is virtually special. Thus, by [50], G, and so H, are linear over $\mathbb{Z}$. □

I would not be surprised if all groups H_k considered in Theorem 1.2 are linear over $\mathbb{Z}$. However, I am not so sure that there exists a universal upper bound on the minimal degree of faithful complex representations of these groups. Observe that, since a limit group is fully residually free, it can be embedded into $\mathrm{GL}_2(\mathbb{C})$. This shows that the main difficulty in showing that the groups from Theorem 1.2 are linear of bounded degree appears when we attach roots.

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Departamento de Matemáticas, Universidad Autónoma de Madrid and Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM

E-mail address: andrei.jaikin@uam.es