

THE HANNA NEUMANN CONJECTURE FOR SURFACE GROUPS

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ABSTRACT. The Hanna Neumann conjecture is a statement about the rank of the intersection of two finitely generated subgroups of a free group. The conjecture was posed by Hanna Neumann in 1957. In 2011, a strengthened version of the conjecture was proved independently by Joel Friedman and by Igor Mineyev. In this paper we show that the Strengthened Hanna Neumann conjecture holds not only in free groups but also in non-solvable surface groups.

In addition we show that a retract in a free group and in a surface group is inert. This implies the Dicks-Ventura inert conjecture for free and surface groups.

1. INTRODUCTION

Let G be a group. We say that G satisfies the **Howson property** if the intersection of two finitely generated subgroups of G is again finitely generated. This property was introduced by Howson [19] where he proved that it holds for free groups. In fact, Howson gave an effective bound for the number of generators of the intersection which was improved few years later by H. Neumann [39].

Let $d(G)$ denote the number of generators of a group G . H. Neumann showed that if U and W are non-trivial finitely generated subgroups of a free group, then

$$d(U \cap W) - 1 \leq 2(d(U) - 1)(d(W) - 1)$$

and she conjectured that, in fact, the factor 2 can be omitted. This conjecture became known as the **Hanna Neumann conjecture**.

In 1980, W. Neumann improved the result of H. Neumann. For a group G we put $\bar{d}(G) = \max\{0, d(G) - 1\}$. W. Neumann showed that if U and W are finitely generated subgroups of a free group F , then

$$\sum_{x \in U \setminus F/W} \bar{d}(U \cap xWx^{-1}) \leq 2\bar{d}(U)\bar{d}(W)$$

and he also conjectured that again the factor 2 can be omitted. This conjecture became known as the **Strengthened Hanna Neumann conjecture**. It was proved independently Friedman [11] and by Mineyev [37] in 2011. These were also the first proofs of the Hanna Neumann conjecture. Dicks presented two simplifications of the previous proofs (see Appendix to [11] and [7]).

In [21] the second author proved the Strengthened Hanna Neumann conjecture for free pro- p groups. The Friedman and Mineyev proofs used combinatorial and geometric aspects of free groups. This kind of techniques are not available (or probably not enough developed) in the world of pro- p groups and, therefore, the Jaikin-Zapirain proof used an homological approach. It turned out that this new method

(with suitable modifications) gave also a new proof of the original Strengthened Hanna Neumann conjecture for free groups. In [25] Jaikin-Zapirain and Shusterman developed further the pro- p part of [21] and showed that the Strengthened Hanna Neumann conjecture holds for non-solvable Demushkin pro- p groups (the Demushkin pro- p groups are the Poincaré duality pro- p groups of cohomological dimension 2).

By a **surface group** we will mean the fundamental group of a compact closed surface of negative Euler characteristic. In the orientable case, surface groups admits the presentations of the form $\langle x_1, \dots, x_n, y_1, \dots, y_n \mid [x_1 y_1] \cdots [x_n y_n] = 1 \rangle$ ($n \geq 2$); and in the non-orientable closed case it is $\langle x_1, \dots, x_n \mid x_1^2 \cdots x_n^2 = 1 \rangle$ ($n \geq 3$). Although free groups arise as fundamental groups of non-closed surfaces of negative Euler characteristic, we will not consider free groups as surface groups.

We note that all surface groups but $\langle a, b, c \mid a^2 b^2 c^2 = 1 \rangle$ are limit groups, and the latter has an index two subgroup that is a limit group. The class of virtually limit groups plays an important rôle though this work.

In this paper we will develop the discrete part of [21], and we will prove the Strengthened Hanna Neumann conjecture for surface groups.

Theorem 1.1. *Let G be a surface group. Then for any finitely generated subgroups U and W of G ,*

$$\sum_{x \in U \backslash G/W} \bar{d}(U \cap xWx^{-1}) \leq \bar{d}(U)\bar{d}(W).$$

In the context of the Hanna Neumann conjecture, the best previous bound when G an orientable surface group was obtained by Soma in [44, 45]: $\bar{d}(U \cap W) \leq 1161 \cdot \bar{d}(U)\bar{d}(W)$.

Theorem 1.1 is obtained from the following generalization of the Strengthened Hanna Neumann conjecture. Let Γ be a virtually FL -group. Then we define its Euler characteristic as

$$\chi(\Gamma) = \frac{1}{|\Gamma : \Gamma_0|} \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma_0, \mathbb{Q}),$$

where Γ_0 is a FL -subgroup of Γ of finite index.

Let $\bar{\chi}(\Gamma) = \max\{0, -\chi(\Gamma)\}$. Observe that for a non-trivial finitely generated free group Γ , $\bar{\chi}(\Gamma) = \bar{d}(\Gamma)$, for a surface group Γ we have $\bar{\chi}(\Gamma) = \bar{d}(\Gamma) - 1$ and for a finitely generated virtually abelian group Γ , $\bar{\chi}(\Gamma) = 0$.

Theorem 1.2. *Let G be a surface group. Then for every finitely generated subgroups U and W of G ,*

$$\sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1}) \leq \bar{\chi}(U)\bar{\chi}(W).$$

We conjecture that the previous theorem holds in a greater generality. Recall that the class of limit groups coincides with the class of constructible limit groups, and from that one can deduce that they are fundamental groups of finite CW -complexes [50, Corollary 4.11], and so, they are FL . Since finitely generated subgroups of limit groups are limit groups, and limit groups satisfy Howson's property [6] we see that the family of finitely generated subgroups of a limit group is a family

of FL -subgroups of closed under intersections and conjugations. We believe that the Theorem 1.2 can be extended further and we propose the following conjecture.

Conjecture 1 (The Geometric Hanna Neumann conjecture). *Let G be a limit group. Then for every finitely generated subgroups U and W of G ,*

$$\sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1}) \leq \bar{\chi}(U)\bar{\chi}(W).$$

The L^2 -independence and L^2 -Hall properties are two new technical notions that we introduce in this paper (see Section 4 for definitions). In this paper we prove that retracts in free and surface groups are L^2 -independent. In particular, this implies that surface groups are L^2 -Hall. Further understanding of these new concepts would help to make progress on Conjecture 1. For example, the proof of L^2 -Hall property for limit groups would lead to the solution of the conjecture in the case of hyperbolic limit groups.

Theorem 1.3. *Let G be a hyperbolic limit group. Assume that G satisfies the L^2 -Hall property. Then the Geometric Hanna Newman conjecture holds for G .*

Recall that a subgroup U of a group G is called **inert** if for every subgroup H of G , $d(H \cap U) \leq d(H)$. In addition to Theorem 1.1, the consideration of L^2 -independence helps us to show that a retract in a surface group is inert.

Theorem 1.4. *Let G be either a free or a surface group. Then any retract in G is inert.*

As a consequence we obtain the Dicks-Ventura inertia conjecture for free groups ([8, Problem 5],[47, Conjecture 8.1]) and the analagous result for surface groups. This conjecture has its origin in an influential paper of Bestvina and Handel [3], where a conjecture of Scott was proved: the subgroup of elements of a free group of rank n fixed by a given automorphism has rank at most n .

Corollary 1.5. *(The Dicks-Ventura inertia conjecture) Let G be either a free or a surface group and let \mathcal{F} be a finite collection of endomorphisms of G . Then*

$$\text{Fix}(\mathcal{F}) = \{g \in G : \phi(g) = g \text{ for all } \phi \in \mathcal{F}\}$$

is inert in G .

Proof. Assume first that G is a finitely generated free group. The fact that the inertia conjecture follows from inertia of retracts (i.e our Theorem 1.4) is well-known (see discussion of [47, Conjecture 81]) and we reproduce it for the convenience of the reader.

Since the intersection of inert subgroups is inert, without loss of generality, we can assume that \mathcal{F} consists of a single endomorphism ϕ . The case when ϕ is injective was proved in [8, Theorem IV.5.5]. Consider an arbitrary endomorphism ϕ . Let

$$\phi^\infty(G) = \bigcap_{i=0}^{\infty} \phi^i(G).$$

Then by [46, Theorem 1], $\phi^\infty(G)$ is a retract in G , and so, by Theorem 1.4, we only have to show that $\text{Fix}(\phi)$ is inert in $\phi^\infty(G)$. By [20, Theorem 1], the restriction of ϕ on $\phi^\infty(G)$ is an automorphism. Thus, [8, Theorem IV.5.5] gives us the desired result.

Enric Ventura has pointed out to us that the same reduction argument works in the case of a surface group G .

If ϕ is not an automorphism, then $\phi(G)$ has infinite index and hence it is free. In particular, $\phi^\infty(G)$ is still a retract of G and the argument applies verbatim. The only difference is when ϕ is an automorphism. However, this case was proved already by Wu and Zhang in [52, Corollary 1.5]. \square

Let us briefly describe the structure of the paper. In Section 2 we include main definitions and facts that we use in the paper. In Section 3 we introduce L^2 -Betti numbers $\beta_k^{K[G]}(M)$ for $K[G]$ -modules M with K a subfield of \mathbb{C} and explain the Atiyah and Lück approximation conjectures. The L^2 -independence and L^2 -Hall properties are discussed in Section 4. In Section 5 we prove Theorem 1.4. In Section 6 we introduce an auxiliary ring $L_\tau[G]$ which already played an important role in Dicks' simplification of Freidman's proof. We finish the proof of Theorem 1.2 and Theorem 1.1 in Section 7. In Section 8 we reformulate the Geometric Hanna Neumann conjecture in terms of an inequality for $\beta_1^{\mathbb{Q}[G]}$. A key step of our proof of Theorem 1.3 is to find a specific submodule of $K[G/U] \otimes K[G/W]$ with trivial $\beta_1^{K[G]}$. This is done in Section 11. But previously we present two auxiliary properties. In Section 9 we prove a generalization of Howson property for quasi-convex subgroups of hyperbolic groups and for subgroups of limit groups and in Section 10 we prove the Wilson-Zalesskii property for quasi-convex subgroups of hyperbolic virtually compact groups. We finish the proof of Theorem 1.3 in Section 12 and we describe also some limitations of our methods in order to extend them to more cases of Conjecture 1.

Remark 1.6. *Theorems 1.2, 1.4 and Corollary 1.5 hold also for fundamental groups of surfaces of non-negative Euler characteristic (i.e the trivial group, $\mathbb{Z}/2\mathbb{Z}$, \mathbb{Z}^2 and $\langle a, b \mid a^2b^2 \rangle$), the fundamental group of a Klein bottle). However the results are either trivial, or uses simple arguments specific for these cases. On the other hand, it is easy to produce a counter-example of Theorem 1.1 when G is virtually \mathbb{Z}^2 .*

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2. PRELIMINARIES

Although our main result is about surface groups, many steps of our proof hold in more general contexts of word hyperbolic, limit or virtually special compact groups. In this section we remind all the relevant definitions and facts about these groups.

A geodesic metric space Y is called **(Gromov) hyperbolic** if there exists a constant $\delta \geq 0$ such that for any geodesic triangle Δ in Y , any side of Δ is contained in the closed δ -neighborhood of the union of other sides. A subset $Z \subseteq Y$ is called **quasi-convex** if there exists $\epsilon \geq 0$ such that for any points $z_1, z_2 \in Z$, any geodesic joining these points is contained in the closed ϵ -neighborhood of Z .

A finitely generated group G is said to be **hyperbolic** if its Cayley graph with respect to some finite generating set is a hyperbolic metric space. Quasi-convex subgroups of G are very important in the study of hyperbolic groups. Such subgroups are themselves hyperbolic and are quasi-isometrically embedded in G (see, for example, [1]). Moreover, for finitely generated subgroups of hyperbolic groups, being quasi-isometrically embedded in G is equivalent to be quasi-convex. The intersection of two quasi-convex subgroups in a hyperbolic group is quasi-convex by a result of Short [43].

For a subgroup $H \leq G$, we write $H^g = gHg^{-1}$. A subgroup H of a group G is called **malnormal** if for every $x \in G \setminus H$, $H^x \cap H = \{1\}$.

A finitely generated group G is a **limit** group if, for any finite subset X of G , there exists a homomorphism $f: G \rightarrow F$ to a free group so that the restriction of f on X is injective. By a result of Wilton [49] a finitely generated subgroup of a limit group is a virtual retract. Therefore, in a limit group all finitely generated subgroups are quasi-isometrically embedded and, in particular, in hyperbolic limit groups finitely generated subgroups are quasi-convex.

A **right angled Artin group (RAAG)** is a group which can be given by a finite presentation, where the only defining relators are commutators of the generators. To construct such a group, one usually starts with a finite graph Γ with vertex set V and edge set E . One then defines the corresponding RAAG $A(\Gamma)$ by the following presentation:

$$A(\Gamma) = \langle V \mid uv = vu \text{ whenever } \{u, v\} \in E \rangle.$$

Special cube complexes were introduced in [15]. A group is called **(compact) special** if it is the fundamental group of a non-positively curved (compact) special cube complex. A nice group characterization of these groups is that a group is (compact) special if and only if it is a (quasi-convex) subgroup of a RAAG (see [14, 15]). By [14, Theorem F], quasi-convex subgroups of RAAGs are virtual retracts. Thus, we have the following that will be used several times.

Theorem 2.1. *Any quasi-convex subgroup of a virtually compact special group is a virtual retract.*

By a result of Wise [51, Corollary 16.11], a limit group is virtually compact special.

In this paper we will explore profinite properties of virtually compact special groups. Recall that if $\mathcal{S} = \{S_1, \dots, S_k\}$ is a family of disjoint subsets of a group G , we say that a normal subgroup N , of G **separates** \mathcal{S} if S_1N, \dots, S_kN are disjoint. The family \mathcal{S} is **separable** if there exists a normal subgroup of G of finite index that separates \mathcal{S} . A subset S of G is **separable** if for every $g \in G \setminus S$, the family $\{S, g\}$ is separable. A group is **residually finite** if the trivial element is separable. For example, since a quasi-convex subgroup H of a virtually compact special group G is a virtually retract and G is residually finite, any finite family $\{x_1H, \dots, x_kH\}$ of left classes of H are separable. For hyperbolic groups this is the property **GFERF** introduced in [36] and we will use the following.

Theorem 2.2. [36, Theorem 1.1. and Remark 2.2.] *Assume G is a virtually compact special hyperbolic group (or more generally a GFERF hyperbolic group), and H_1, \dots, H_s are quasiconvex subgroups of G , $s \in \mathbb{N}$ and $g_0, \dots, g_s \in G$. Then the product $g_0H_1g_1 \dots g_{s-1}H_s g_s$ is separable in G .*

3. L^2 -BETTI NUMBERS, THE STRONG ATIYAH CONJECTURE AND THE LÜCK APPROXIMATION

Let G be a discrete group and let $l^2(G)$ denote the Hilbert space with Hilbert basis the elements of G ; thus $l^2(G)$ consists of all square summable formal sums $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{C}$ and inner product

$$\langle \sum_{g \in G} a_g g, \sum_{h \in G} b_h h \rangle = \sum_{g \in G} a_g \overline{b_h}.$$

The left- and right-multiplication action of G on itself extend to left and right actions of G on $l^2(G)$. The ring $\mathcal{U}(G)$ is the ring of densely defined (unbounded) operators which commute with the left action of G . The right action of G on $l^2(G)$ extends to an action of $\mathbb{C}[G]$ on $l^2(G)$ and so we obtain that the group algebra $\mathbb{C}[G]$ acts faithfully as bounded linear operators on $l^2(G)$. We will often consider $\mathbb{C}[G]$ as a subalgebra of $\mathcal{U}(G)$.

The computations of L^2 -Betti numbers have been algebraized through the seminal works of Lück [33, 34]. The basic observation is that one can use a dimension function $\dim_{\mathcal{U}(G)}$, which is defined for all modules over $\mathcal{U}(G)$ and compute the k th L^2 -Betti number of a $\mathbb{C}[G]$ -module M using the following formula

$$\beta_k^{\mathbb{C}[G]}(M) = \dim_{\mathcal{U}(G)} \operatorname{Tor}_k(\mathcal{U}(G), M).$$

We recommend the book [35] for the definition of $\dim_{\mathcal{U}(G)}$ and its properties.

The ring $\mathcal{U}(G)$ is an example of a $*$ -regular ring. Already in the case $G = \langle t \rangle \cong \mathbb{Z}$ it is quite complicated as a ring (it is isomorphic to $L^1(S^1)$). Therefore, sometimes, it is more convenient to consider a smaller object $\mathcal{R}_{\mathbb{C}[G]}$ introduced by Linnell and Schick ([30]).

Let K be a subfield of \mathbb{C} . We define $\mathcal{R}_{K[G]}$ as the $*$ -regular closure of $K[G]$ in $\mathcal{U}(G)$, i. e. $\mathcal{R}_{K[G]}$ is the smallest $*$ -regular subring of $\mathcal{U}(G)$ that contains $K[G]$. We can also define a dimension function $\dim_{\mathcal{R}_{K[G]}}$ on $\mathcal{R}_{K[G]}$ -modules and use it in order to define the L^2 -Betti numbers (see [22, 23]). If M is a $K[G]$ -module then its L^2 -Betti numbers are computed using the formula

$$\beta_k^{K[G]}(M) = \dim_{\mathcal{R}_{K[G]}} \operatorname{Tor}_k^{K[G]}(\mathcal{R}_{K[G]}, M).$$

The object $\mathcal{R}_{K[G]}$ is much simpler than $\mathcal{U}(G)$. For example, in the case $G = \langle t \rangle \cong \mathbb{Z}$, $\mathcal{R}_{K[G]}$ is isomorphic to $K(t)$ and $\dim_{\mathcal{R}_{K[G]}}$ is the standard dimension over the field $K(t)$. More generally, the strong Atiyah conjecture (see [35]) predicts that if G is torsion-free, then all numbers $\beta_k^{K[G]}(M)$ are integers, $\mathcal{R}_{K[G]}$ is a division algebra and $\dim_{\mathcal{R}_{K[G]}}$ is the standard dimension over $\mathcal{R}_{K[G]}$.

In this paper we will use the solution of the strong Atiyah conjecture in the case where G is a torsion-free virtually compact special group.

Proposition 3.1. [9, 41, 22] *Let G be a torsion-free virtually compact special group, and let K be a subfield of \mathbb{C} . Then all numbers $\beta_k^{K[G]}(M)$ are integers and $\mathcal{R}_{K[G]}$ is a division algebra.*

Another important conjecture about L^2 -Betti numbers is the Lück approximation conjecture (see [35]). In this paper we will use the solution of this conjecture in the case of approximation by sofic groups.

Proposition 3.2. [32, 9, 10, 22] *Let G be a group and let $G > G_1 > G_2 > \dots$ be a chain of normal subgroups with trivial intersection such that G/G_i are sofic. Let K be a subfield of \mathbb{C} and let M be a finitely presented $K[G]$ -module. Then*

$$\dim_{\mathcal{R}_{K[G]}}(\mathcal{R}_{K[G]} \otimes_{K[G]} M) = \lim_{i \rightarrow \infty} \dim_{\mathcal{R}_{K[G/G_i]}}(\mathcal{R}_{K[G/G_i]} \otimes_{K[G]} M).$$

In this paper we will consider only the fields K which are subfields of \mathbb{C} . Let M be a $K[G]$ -submodule. By [22, Corollary 1.7], if G is sofic, then $\beta_k^{K[G]}(M)$ does not depend on the embedding of K into \mathbb{C} . Thus, in what follows if the group G is sofic, we will not indicate the embedding of K into \mathbb{C} .

Recall that the k th L^2 -Betti number of a group G is defined as the k th L^2 -Betti number of the trivial $\mathbb{C}[G]$ -module \mathbb{C} :

$$b_k^{(2)}(G) = \dim_{\mathcal{U}(G)} H_k(G; \mathcal{U}(G)) = \dim_{\mathcal{R}_{K[G]}} H_k(G; \mathcal{R}_{K[G]}) = \beta_k^{K[G]}(K).$$

In the case when G is a virtually limit group we have a good control of its L^2 -Betti numbers.

Proposition 3.3. *Let G be a virtually limit group and K a field. Then*

$$\beta_k^{K[G]}(K) = b_k^{(2)}(G) = 0 \text{ if } k \geq 2.$$

In particular,

$$\bar{\chi}(G) = b_1^{(2)}(G) = \beta_1^{K[G]}(K).$$

Proof. See, for example [4, Corollary C]. □

If U is a subgroup of a group G , then $\mathcal{R}_{K[G]}$ is a flat right $\mathcal{R}_{K[U]}$ -module and for every left $\mathcal{R}_{K[U]}$ -module M

$$\dim_{\mathcal{R}_{K[G]}} \mathcal{R}_{K[G]} \otimes_{\mathcal{R}_{K[U]}} M = \dim_{\mathcal{R}_{K[U]}} M.$$

This implies the following result.

Proposition 3.4. *Let U be a subgroup of a group G and let M be a left $K[U]$ -module, then for every k*

$$\beta_k^{K[U]}(M) = \beta_k^{K[G]}(K[G] \otimes_{K[U]} M).$$

4. THE L^2 -HALL PROPERTY FOR SURFACE GROUPS

Let U be a subgroup of G . The embedding of U into G induces the correstriction map

$$\text{cor} : H_1(U; \mathcal{U}(G)) \rightarrow H_1(G; \mathcal{U}(G)).$$

We say that U is L^2 -**independent** in G if

$$\dim_{\mathcal{U}(G)} \ker(\text{cor}) = 0.$$

We say that the group G is L^2 -**Hall**, if for every finitely generated subgroup U of G , there exists a subgroup H of G of finite index containing U such that U is L^2 -independent in H .

The L^2 -independence can also be characterized in terms of $\mathcal{R}_{K[G]}$.

Lemma 4.1. *Let G be a group and K a subfield of \mathbb{C} . Then a subgroup U of G is L^2 -independent if and only if*

$$\dim_{\mathcal{R}_{K[G]}} \ker(\text{cor}) = 0,$$

where $\text{cor} : H_1(U; \mathcal{R}_{K[G]}) \rightarrow H_1(G; \mathcal{R}_{K[G]})$ is the correstriction.

Proof. Since $\mathcal{R}_{K[G]}$ is von Neumann regular, $\mathcal{U}(G)$ is a flat $\mathcal{R}_{K[G]}$ -module and we are done. \square

If $\mathcal{R}_{K[G]}$ is a semi-simple algebra, Lemma 4.1 implies that in order to show that U is L^2 -independent in G , one has to prove that $\ker(\text{cor}) = \{0\}$. In the case of virtually limit groups we can give also the following description.

Proposition 4.2. *Let G be a virtually limit group and let $H_1 \leq H_2$ be two finitely generated subgroups of G . Let K be a subfield of \mathbb{C} . Consider the exact sequence.*

$$1 \rightarrow M \rightarrow K[G/H_1] \rightarrow K[G/H_2] \rightarrow 0.$$

Then H_1 is L^2 -independent in H_2 if and only if $\beta_1^{K[G]}(M) = 0$.

Proof. We have the following exact sequence of Tor-functors:

$$\begin{aligned} \text{Tor}_2^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_2]) \rightarrow \text{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, M) \rightarrow \\ \text{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_1]) \xrightarrow{\alpha} \text{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_2]). \end{aligned}$$

By Proposition 3.3, $\text{Tor}_2^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_2]) \cong \text{Tor}_2^{K[H_2]}(\mathcal{R}_{K[G]}, K) = 0$. Also the Shapiro lemma provides canonical isomorphisms

$$\gamma_i : \text{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_i]) \rightarrow \text{Tor}_1^{K[H_i]}(\mathcal{R}_{K[G]}, K) = H_1(H_i; R_{K[G]}) \quad (i = 1, 2)$$

such that $\text{cor} = \gamma_2 \circ \alpha \circ \gamma_1^{-1}$. Thus, $\dim_{\mathcal{R}_{K[G]}} \ker(\text{cor}) = 0$ if and only if $\dim_{\mathcal{R}_{K[G]}} \ker \alpha = 0$ if and only if $\beta_1^{K[G]}(M) = 0$. \square

Corollary 4.3. *Let U be a finitely generated subgroup of a virtual limit group G and K a subfield of \mathbb{C} . Then U is L^2 -independent in G if and only if $\beta_1^{K[G]}(I_G/I_U^G) = 0$.*

Here I_G (resp. I_U) is the augmentation ideal of $K[G]$ (resp. $K[U]$) and I_U^G is the left ideal of $K[G]$ generated by I_U .

Proof. Use Proposition 4.2 taking into account that $K[G/U] \cong K[G]/I_U^G$. \square

In this section we show that surface groups are L^2 -Hall.

Theorem 4.4. *Finitely generated free groups and surface groups are L^2 -Hall.*

Let P be a pro- p group. We denote by $d(P)$ the minimal cardinality of topological generating set of P . If P is finitely generated and L is a $\mathbb{F}_p[[P]]$ -module, then the functions $\beta_k^{\mathbb{F}_p[[P]]}(L)$ are defined in the following way. Fix a chain $P_1 > P_2 > P_3 > \dots$ of open normal subgroups of P with trivial intersection and we put

$$\beta_k^{\mathbb{F}_p[[P]]}(L) = \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{F}_p} H_k(P_i, L)}{|P : P_i|}.$$

The limit always exists and it does not depend on the chain (see [23, Proposition 11.2]).

A **Demushkin** pro- p group is a Poincare duality pro- p group of cohomological dimension 2. For the purposes of this paper it is enough to know that the fundamental group of a closed surface is residually-(finite 2-group) and its pro-2 completion is Demushkin. First let us present the following result whose proof is essentially contained in the proof of [25, Proposition 7.2].

Proposition 4.5. *Let P be a non-finite Demushkin pro- p group and let H be a proper closed subgroup of P such that the map $H_1(H; \mathbb{F}_p) \rightarrow H_1(P; \mathbb{F}_p)$ is injective. Let L be the kernel of the map $\mathbb{F}_p[[P/H]] \rightarrow \mathbb{F}_p$. Then*

$$\beta_1^{\mathbb{F}_p[[P]]}(L) = 0 \text{ and } \beta_0^{\mathbb{F}_p[[P]]}(L) = d(P) - d(H) - 1.$$

Proof. Since H is a proper subgroup of P and $H_1(H; \mathbb{F}_p) \rightarrow H_1(P; \mathbb{F}_p)$ is injective, H is of infinite index, and so, H is a free pro- p group. Moreover, since the map $H_1(H; \mathbb{F}_p) \rightarrow H_1(P; \mathbb{F}_p)$ is injective, H is finitely generated.

In the proof of [25, Proposition 7.2], it is shown that L is an one-relator $\mathbb{F}_p[[P]]$ -module. Thus, we can produce an exact sequence

$$0 \rightarrow C \rightarrow \mathbb{F}_p[[P]]^d \rightarrow L \rightarrow 0,$$

where C is a non-trivial cyclic $\mathbb{F}_p[[P]]$ -module. Since $\mathbb{F}_p[[P]]$ is a domain, $C \cong \mathbb{F}_p[[P]]$. By [25, Corollary 6.2], $\beta_1^{\mathbb{F}_p[[P]]}(L) = 0$. Hence, $\beta_0^{\mathbb{F}_p[[P]]}(L) = d - 1 = \chi_P(L)$, where $\chi_P(L)$ is the Euler characteristic of L as a $\mathbb{F}_p[[P]]$ -module.

On the other hand, using the exact sequence

$$0 \rightarrow L \rightarrow \mathbb{F}_p[[P/H]] \rightarrow \mathbb{F}_p \rightarrow 0,$$

we obtain that

$$\begin{aligned} \chi_P(L) &= \chi_P(\mathbb{F}_p[[P/H]]) - \chi_P(\mathbb{F}_p) = \chi_H(\mathbb{F}_p) - \chi_P(\mathbb{F}_p) = \\ &= 1 - d(H) - (2 - d(P)) = d(P) - d(H) - 1. \end{aligned}$$

In the penultimate equality we have used that H is free and P is Demushkin. \square

The previous proposition leads to a criterion for L^2 -independence of a subgroup of a free or a surface group.

Proposition 4.6. *Let G be a finitely generated free group or a surface group and U a retract of G . Then U is L^2 -independent in G .*

Proof. Without loss of generality we assume that U is non-trivial and proper. Thus, G is infinite and U is a free group. First consider the case where G is a surface group.

Since U is a retract, $H_1(U; \mathbb{F}_2) \rightarrow H_1(G; \mathbb{F}_2)$ is injective. Let P be the pro-2 completion of G . As we have mentioned, G is a Demushkin pro-2 group. Let $P_1 > P_2 > P_3 > \dots$ be a chain of open normal subgroups of P with trivial intersection. We put $G_i = G \cap P_i$. Let H be the closure of U in P . Since U is a retract of G , H is a free pro-2 group, and so, it is a proper subgroup of P .

The condition $H_1(U; \mathbb{F}_2) \rightarrow H_1(G; \mathbb{F}_2)$ is injective implies that $H_1(H; \mathbb{F}_2) \rightarrow H_1(P; \mathbb{F}_2)$ is injective and $d(H) = \dim_{\mathbb{F}_p} H_1(U; \mathbb{F}_2)$ (and so, $d(H) = d(U)$).

Consider two exact sequences

$$0 \rightarrow M \rightarrow \mathbb{Z}[G/U] \rightarrow \mathbb{Z} \rightarrow 0 \text{ and } 0 \rightarrow L \rightarrow \mathbb{F}_2[[P/H]] \rightarrow \mathbb{F}_2 \rightarrow 0.$$

Tensoring the first sequence with \mathbb{F}_2 over \mathbb{Z} , we obtain another exact sequence of $\mathbb{F}_2[G]$ -modules.

$$0 \rightarrow \mathbb{F}_2 \otimes M \rightarrow \mathbb{F}_2[G/U] \rightarrow \mathbb{F}_2 \rightarrow 0.$$

Since G is pro-2 good (see [13]), $H_1(G; \mathbb{F}_2[[P]]) = \text{Tor}_1^{\mathbb{F}_2[G]}(\mathbb{F}_2[[P]], \mathbb{F}_2) = 0$. Therefore, the sequence

$$0 \rightarrow \mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} M \rightarrow \mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} \mathbb{F}_2[G/U] \rightarrow \mathbb{F}_2 \rightarrow 0$$

is also exact. Since $\mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} \mathbb{F}_2[G/U] \cong \mathbb{F}_2[[P/H]]$, we obtain that $L \cong \mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} M$ as $\mathbb{F}_2[[P]]$ -modules. In particular

$$\dim_{\mathbb{F}_2} H_0(G_i; M) = \dim_{\mathbb{F}_2} H_0(P_i, \mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} M) = \dim_{\mathbb{F}_2} H_0(P_i, L).$$

Thus,

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{Q}} \operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q} \otimes_{\mathbb{Z}} M)}{|G : G_i|} &= \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{Q}} \operatorname{Tor}_0^{\mathbb{Z}[G_i]}(\mathbb{Q}, M)}{|G : G_i|} \leq \\ &= \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{F}_2} \operatorname{Tor}_0^{\mathbb{Z}[G_i]}(\mathbb{F}_2, M)}{|G : G_i|} = \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{F}_2} H_0(G_i; M)}{|G : G_i|} = \\ &= \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{F}_2} H_0(P_i, L)}{|P : P_i|} \stackrel{\text{Proposition 4.5}}{=} d(P) - d(H) - 1 = d(G) - d(U) - 1. \end{aligned}$$

Consider again the exact sequence $0 \rightarrow M \rightarrow \mathbb{Z}[G/U] \rightarrow \mathbb{Z} \rightarrow 0$. It induces the exact sequence

$$0 \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} M \rightarrow \mathbb{Q}[G/U] \rightarrow \mathbb{Q} \rightarrow 0.$$

The long exact sequences of Tor-functors implies that

$$\begin{aligned} \dim_{\mathbb{Q}} \operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q} \otimes_{\mathbb{Z}} M) &\leq \dim_{\mathbb{Q}} \operatorname{Tor}_2^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}) + \dim_{\mathbb{Q}} \operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}[G/U]) \\ &\quad - \dim_{\mathbb{Q}} \operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}) + \dim_{\mathbb{Q}} \operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q} \otimes_{\mathbb{Z}} M) \\ &\quad - \dim_{\mathbb{Q}} \operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}[G/U]) + \dim_{\mathbb{Q}} \operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}). \end{aligned}$$

Observe that

$$\dim_{\mathbb{Q}} \operatorname{Tor}_2^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}) = \dim_{\mathbb{Q}} \operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}) = 1,$$

$$\lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{Q}} \operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}[G/U])}{|G : G_i|} = d(U) - 1, \quad \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{Q}} \operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q})}{|G : G_i|} = d(G) - 2$$

and, since we assume that U is not trivial,

$$\lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{Q}} \operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}[G/U])}{|G : G_i|} = 0.$$

Putting all limits together, we obtain that

$$\beta_1^{\mathbb{Q}[G]}(\mathbb{Q} \otimes_{\mathbb{Z}} M) \stackrel{\text{Proposition 3.2}}{=} \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{Q}} \operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q} \otimes_{\mathbb{Z}} M)}{|G : G_i|} = 0.$$

By Proposition 4.2, U is L^2 -independent in G .

The remaining case is the case where G is a finitely generated free group. The proof works verbatim just bearing in mind that in Proposition 4.5 one has to change P to be a free pro- p -group in the hypothesis, and in the conclusion $\beta_0^{\mathbb{F}_p[[P]]}(L) = d(P) - d(H)$. Also in the proof of Proposition 4.6 one has that the groups G_i are free and hence $\dim_{\mathbb{Q}} \operatorname{Tor}_2^{\mathbb{Q}[G_i]}(\mathbb{Q}, \mathbb{Q}) = 0$. \square

Proof of Theorem 4.4. Let G be a finitely generated free group or a surface group and U a finitely generated subgroup of G . There exists a subgroup S of finite index in G , containing U and such that U is a retract of S (see [16, 42]). Now, we can apply Proposition 4.6. \square

5. THE PROOF OF THEOREM 1.4

In this section we prove Theorem 1.4. A similar argument will be used later in our proof of Theorem 1.2. A key observation is the following proposition.

Proposition 5.1. *Let G be a surface group or a free group and H a subgroup of G . Let K be a subfield of \mathbb{C} .*

- (1) *Any $K[G]$ -submodule of a $K[G]$ -module of projective dimension 1 is also of projective dimension 1.*
- (2) *If M is a $K[G]$ -module of projective dimension 1 and $\beta_1^{K[G]}(M) = 0$. Then $\beta_1^{K[H]}(M) = 0$.*

Proof. (1) is clear when G is free, because $K[G]$ is of global dimension 1. If G is not free, then $K[G]$ is of global dimension 2 and for such rings a submodule of a module of projective dimension 1 is also of projective dimension 1.

In order to show (2) we have to prove that $\text{Tor}_1^{K[H]}(\mathcal{R}_{K[H]}, M) = 0$. By Shapiro's lemma,

$$\text{Tor}_1^{K[H]}(\mathcal{R}_{K[H]}, M) \cong \text{Tor}_1^{K[G]}(\mathcal{R}_{K[H]} \otimes_{K[H]} K[G], M).$$

Observe that $\mathcal{R}_{K[H]} \otimes_{K[H]} K[G]$ is naturally embedded in $\mathcal{R}_{K[G]}$ (see, for example, the discussion after [29, Problem 4.5]). Since M is of projective dimension 1 and $\text{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, M) = 0$, $\text{Tor}_1^{K[G]}(\mathcal{R}_{K[H]} \otimes_{K[H]} K[G], M) = 0$ as well. \square

Proposition 5.2. *Let G be a free group or a surface group and U a L^2 -independent subgroup of G . If H is a finitely generated subgroup of G then $H \cap U$ is L^2 -independent in H . In particular $d(H \cap U) \leq d(H)$.*

Proof. Without loss of generality we may assume that $G \neq U$. Hence, G is infinite and U is free. By Corollary 4.3, since U is a L^2 -independent subgroup of G , $\beta_1^{K[G]}(I_G/I_U^G) = 0$.

Since U is free, $K[G/U] \cong K[G]/I_U^G$ is of projective dimension 1. By Proposition 5.1 (1), I_G/I_U^G is also of projective dimension 1. Therefore, by Proposition 5.1 (2), $\beta_1^{K[H]}(I_G/I_U^G) = 0$ as well.

Put $M = I_G/I_U^G$ and $L = I_H/I_{U \cap H}^H$. In the previous paragraph we have obtained that

$$\text{Tor}_1^{K[H]}(\mathcal{R}_{K[H]}, M) = 0.$$

Since $I_{U \cap H}^H = I_H \cap I_U^G$, L is a $K[H]$ -submodule of M . Let $T \subset G$ be a set of representatives of the double (H, U) -classes in G and assume that $1 \in T$. Consider the $K[H]$ -module M/L . Then we have that

$$M/L \cong K[G/U]/K[H/(U \cap H)] \cong \bigoplus_{t \in T \setminus \{1\}} K[H/(U^t \cap H)].$$

Since $U^t \cap H$ are free groups, M/L is of projective dimension 1 as $K[H]$ -module, and so,

$$\text{Tor}_2^{K[H]}(\mathcal{R}_{K[H]}, M/L) = 0.$$

Thus, from the exact sequence

$$\text{Tor}_2^{K[H]}(\mathcal{R}_{K[H]}, M/L) \rightarrow \text{Tor}_1^{K[H]}(\mathcal{R}_{K[H]}, L) \rightarrow \text{Tor}_1^{K[H]}(\mathcal{R}_{K[H]}, M)$$

we obtain that $\text{Tor}_1^{K[H]}(\mathcal{R}_{K[H]}, L) = 0$ and $\beta_1^{K[H]}(L) = \beta_1^{K[H]}(I_H/I_{U \cap H}^H) = 0$. Thus, $H \cap U$ is L^2 -independent in H by Corollary 4.3. \square

Proof of Theorem 1.4. Let U be a retract of G and H a subgroup of G . By Proposition 4.6, U is L^2 -independent in G . Thus, the theorem follows from Proposition 5.2. \square

A subgroup U of G is called **compressed** if $d(U) \leq d(H)$ for every subgroup H of G containing U . W. Dicks and E. Ventura conjectured that any compressed subgroup of a free group is also inert. We finish this section with the following natural question.

Question 2. *Is any compressed subgroup of a free group also L^2 -independent?*

6. THE STRUCTURE OF ACCEPTABLE $L_\tau[G]$ -MODULES

Let L be a field and let $\tau: G \rightarrow \text{Aut}(L)$ be a homomorphism. We denote by $L_\tau[G]$ the twisted group ring: its underlying additive group coincides with the ordinary group ring $L[G]$ but the multiplication is defined as follows

$$\left(\sum_{i=1}^n k_i f_i\right) \left(\sum_{j=1}^m l_j g_j\right) = \sum_{i=1}^n \sum_{j=1}^m k_i \tau(f_i)(l_j) f_i g_j, \quad k_i, l_j \in L, \quad f_i, g_j \in G.$$

The main advantage to work with $L_\tau[G]$ -modules instead of $L[G]$ -modules is showed in the following lemma.

Lemma 6.1. [21, Claim 6.3] *Let G be a group and L a field. Let $\tau: G \rightarrow \text{Aut}(L)$ and $H = \ker \tau$. Assume that H is of finite index in G . Then*

- (1) *L is an irreducible $L_\tau[G]$ -module if we define*

$$\left(\sum_{i=1}^k l_i f_i\right) \cdot l = \sum_{i=1}^k l_i \tau(f_i)(l) \quad (l, l_i \in L, \quad f_i \in G).$$

- (2) *Up to isomorphism, L is the unique irreducible $L_\tau[G]$ -module on which H acts trivially.*

Our next task is to prove a version of the strong Atiyah conjecture for $L_\tau[G]$ -module where G is a torsion-free virtually compact special group. We will use the fact that a torsion-free virtually compact special group G has the **factorization property**. This means that any map from G to a finite group factors through a torsion-free elementary amenable group. This was proved by K. Schreue (see Corollary 2.6, Lemma 2.2 and the proof of Theorem 1.1 in [41]).

Proposition 6.2. *Let G be a torsion-free virtually compact special group, L a subfield of \mathbb{C} and $\tau: G \rightarrow \text{Aut}(L)$. Assume that $H \leq \ker \tau$ is of finite index in G . Let M be a $L_\tau[G]$ -module with finite $\beta_k^{L[H]}(M)$. Then $|G : H|$ divides $\beta_k^{L[H]}(M)$.*

Remark. *In order to understand better the significance of this proposition, consider the case when τ sends all elements of G to the identity automorphism ($L_\tau[G] = L[G]$ in this case). Then, by the multiplicative property of L^2 -Betti numbers,*

$$\beta_k^{L[H]}(M) = |G : H| \cdot \beta_k^{L[G]}(M),$$

and so what we want to prove is that $\beta_k^{L[G]}(M)$ is an integer number. This is the strong Atiyah conjecture for G (see Proposition 3.1).

The idea of the proof of the proposition for general τ is to define $\beta_k^{L\tau[G]}(M)$ by

$$\beta_k^{L\tau[G]}(M) = \frac{\beta_k^{L[H]}(M)}{|G : H|},$$

and using the Lück approximation, show, in a similar way as in [9], that $\beta_k^{L\tau[G]}(M)$ is an integer number.

Proof. Recall that G is residually finite. Using the factorization property, we can construct a chain $G \geq H > T_1 > T_2 > \dots$ of normal subgroups of G with trivial intersection such that for each i , $A_i = G/T_i$ is torsion-free elementary amenable.

Since τ sends the elements of T_i to the trivial automorphism of L , abusing slightly the notation we can construct $L_\tau[A_i]$. By a result of J. Moody [38] (see also [26] and [28, Corollary 4.5]), $L_\tau[A_i]$ has no non-trivial zero-divisors. Since A_i is amenable and $L_\tau[A_i]$ is a domain, $L_\tau[A_i]$ satisfies the left Ore condition. Thus, $L_\tau[A_i]$ has the classical division ring of fractions $\mathcal{Q}(L_\tau[A_i])$.

Let $B_i = H/T_i$. Since, B_i is of finite index in A_i , $\mathcal{Q}(L_\tau[A_i])$ is isomorphic to the Ore localization of $L_\tau[A_i]$ with respect to non-zero elements of $L[B_i]$. Thus,

$$(1) \quad \mathcal{Q}(L_\tau[A_i]) \cong \mathcal{Q}(L[B_i]) \otimes_{L[B_i]} L_\tau[A_i]$$

as $(\mathcal{Q}(L[B_i]), L_\tau[G])$ -bimodules. Equivalently $\mathcal{Q}(L_\tau[A_i])$ is isomorphic to a crossed product $\mathcal{Q}(L[B_i]) * (A_i/B_i)$.

Let M be a finitely presented $L_\tau[G]$ -module and let

$$M_i = M/(T_i - 1)M \cong L_\tau[A_i] \otimes_{L_\tau[G]} M.$$

Then from (1) we obtain that

$$\mathcal{Q}(L[B_i]) \otimes_{L[B_i]} M_i \cong \mathcal{Q}(L[B_i]) \otimes_{L[B_i]} L_\tau[A_i] \otimes_{L_\tau[G]} M \cong \mathcal{Q}(L_\tau[A_i]) \otimes_{L_\tau[G]} M.$$

In particular, taking again into account (1), we conclude that

$$(2) \quad \dim_{\mathcal{Q}(L[B_i])}(\mathcal{Q}(L[B_i]) \otimes_{L[B_i]} M_i) = |G : H| \dim_{\mathcal{Q}(L_\tau[A_i])}(\mathcal{Q}(L_\tau[A_i]) \otimes_{L_\tau[G]} M).$$

The groups B_i are torsion-free elementary amenable groups, and so, they satisfy the strong Atiyah conjecture [27]. Hence the rings $\mathcal{R}_{L[B_i]}$ are division rings. Therefore, by Proposition 3.2, there exists i such that

$$\beta_0^{L[H]}(M) = \dim_{\mathcal{R}_{L[B_i]}}(\mathcal{R}_{L[B_i]} \otimes_{L[H]} M) = \dim_{\mathcal{R}_{L[B_i]}}(\mathcal{R}_{L[B_i]} \otimes_{L[B_i]} M_i).$$

Observe that $\mathcal{R}_{L[B_i]}$ is isomorphic to the classical division ring of fractions $\mathcal{Q}(L[B_i])$ of $L[B_i]$ as $L[B_i]$ -ring (see for example, [27] and [23, Corollary 9.4]). Therefore

$$\beta_0^{L[H]}(M) = \dim_{\mathcal{Q}(L[B_i])}(\mathcal{Q}(L[B_i]) \otimes_{L[B_i]} M_i) \stackrel{\text{by (2)}}{=} |G : H| \dim_{\mathcal{Q}(L_\tau[G])}(\mathcal{Q}(L_\tau[A_i]) \otimes_{L_\tau[G]} M).$$

This proves that $|G : H|$ divides $\beta_0^{L[H]}(M)$. Therefore, the proposition holds in the case $k = 0$ and M is finitely presented. In particular, the following Sylvester module rank function on $L_\tau[G]$ (see [23] for definitions)

$$\dim M := \frac{\dim_{\mathcal{R}_{L[H]}}(\mathcal{R}_{L[H]} \otimes_{L[H]} M)}{|G : H|} = \frac{\beta_0^{L[H]}(M)}{|G : H|}$$

is integer-valued. This Sylvester function is induced by the canonical embedding of $L_\tau[G]$ into $\text{Mat}_{|G:H|}(\mathcal{R}_{L[H]})$ (here the endomorphisms act on the right side):

$$L_\tau[G] \hookrightarrow \text{End}_{L[H]}(L_\tau[G]) \hookrightarrow \text{End}_{\mathcal{R}_{L[H]}}(\mathcal{R}_{L[H]} \otimes_{L[H]} L_\tau[G]) \cong \text{Mat}_{|G:H|}(\mathcal{R}_{L[H]}).$$

By an argument of Linnell (see [27, Lemma 3.7]), the division closure \mathcal{D}_G of $L_\tau[G]$ in $\text{Mat}_{|G:H|}(\mathcal{R}_{L[H]})$ is a division ring and

$$\dim M = \dim_{\mathcal{D}_G}(\mathcal{D}_G \otimes_{L_\tau[G]} M).$$

The division closure of $L[H]$ in $\text{Mat}_{|G:H|}(\mathcal{R}_{L[H]})$, and so in \mathcal{D}_G , is isomorphic to $\mathcal{R}_{L[H]}$ as $L[H]$ -ring. By [24, Proposition 2.7], the canonical map of $(\mathcal{R}_{L[H]}, L_\tau[G])$ -bimodules

$$(3) \quad \alpha : \mathcal{R}_{L[H]} \otimes_{L[H]} L_\tau[G] \rightarrow \mathcal{D}_G \text{ is bijective.}$$

This is an analog of the isomorphism (1). In particular, $\dim_{\mathcal{R}_{L[H]}} \mathcal{D}_G = |G : H|$.

Note that $L_\tau[G]$ is a free $L[H]$ -module. Thus every free resolution of $L_\tau[G]$ -module is also a free resolution of $L[H]$ -modules. Thus, (3) implies that for every $L_\tau[G]$ -module M we have that

$$\text{Tor}_k^{L[H]}(\mathcal{R}_{L[H]}, M) \cong \text{Tor}_k^{L_\tau[G]}(\mathcal{D}_G, M).$$

Therefore,

$$\beta_k^{L[H]}(M) = \dim_{\mathcal{R}_{L[H]}} \text{Tor}_k^{L[H]}(\mathcal{R}_{L[H]}, M) = |G : H| \dim_{\mathcal{D}_G} \text{Tor}_k^{L_\tau[G]}(\mathcal{D}_G, M),$$

and so, $|G : H|$ divides $\beta_k^{L[H]}(M)$ if it is finite. \square

We say that a $L_\tau[G]$ -module M is **acceptable** if there exists a $L_\tau[G]$ -submodule M_0 of M such that

- (1) $\dim_L(M/M_0) < \infty$.
- (2) $H = \ker(\tau) \leq C_G(M/M_0)$.
- (3) $\beta_k^{L[H]}(M_0) = 0$ for every $k \geq 1$.

In this paper acceptable $L_\tau[G]$ -modules will appear using the construction presented in the following lemma.

Lemma 6.3. *Let G be a group. Let M be a $\mathbb{Q}[G]$ -module, and let M_0 be a submodule of M such that*

- (1) $\dim_{\mathbb{Q}} M/M_0 < \infty$.
- (2) $H = C_G(M/M_0)$ is of finite index in G .
- (3) $\beta_k^{\mathbb{Q}[G]}(M_0) = 0$ for $k \geq 1$.

Put $F = G/H = \{f_1, \dots, f_t\}$ and let $L = \mathbb{Q}(x_f | f \in F)$ be the field of rational functions on t variables over \mathbb{Q} . Define $\tau : G \rightarrow \text{Aut}(L)$ via the formula

$$\tau(g)(p(x_{f_1}, \dots, x_{f_t})) = p(x_{gf_1}, \dots, x_{gf_t}), \quad p(x_{f_1}, \dots, x_{f_t}) \in L.$$

Put $\widetilde{M} = L_\tau[G] \otimes_{\mathbb{Q}[G]} M$. Then \widetilde{M} is acceptable $L_\tau[G]$ -module. Moreover if $\widetilde{M}_0 = L_\tau[G] \otimes_{\mathbb{Q}[G]} M_0$, then

- (1) $\dim_L(\widetilde{M}/\widetilde{M}_0)$ is finite.
- (2) $H \leq C_G(\widetilde{M}/\widetilde{M}_0)$.
- (3) $\beta_k^{L[H]}(\widetilde{M}_0) = 0$ for every $k \geq 1$.

Proof. (1) Since $L_\tau[G]$ is a flat $\mathbb{Q}[G]$ -module, we obtain that $\widetilde{M}_0 \cong L_\tau[G] \otimes_{\mathbb{Q}[G]} M_0$ and $\widetilde{M}/\widetilde{M}_0 \cong L_\tau[G] \otimes_{\mathbb{Q}[G]} (M/M_0)$. In particular, $\dim_L(\widetilde{M}/\widetilde{M}_0) = \dim_L(M/M_0) < \infty$.

(2) Since $H = \ker \tau$, $\ker \tau$ acts trivially on $\widetilde{M}/\widetilde{M}_0$.

(3) Observe that \widetilde{M}_0 as a $L[H]$ module is isomorphic to $L[H] \otimes_{\mathbb{Q}[H]} M_0$. Hence

$$\beta_k^{L[H]}(\widetilde{M}_0) = \beta_k^{L[H]}(L[H] \otimes_{\mathbb{Q}[H]} M_0) = \beta_k^{\mathbb{Q}[H]}(M_0) = |G : H| \beta_k^{\mathbb{Q}[G]}(M_0) = 0.$$

□

Proposition 6.4. *Let G be a torsion-free virtually limit group and let L be a subfield of \mathbb{C} . Let $\tau : G \rightarrow \text{Aut}(L)$ be a homomorphism with finite image. Put $H = \ker \tau$. Let M be an acceptable $L_\tau[G]$ -module. Then there exists a $L[H]$ -submodule M' of M such that*

$$\beta_1^{L[H]}(M') = 0, \dim_L M/M' \leq \frac{\beta_1^{L[H]}(M)}{|G : H|} \text{ and } H \leq C_G(M/M').$$

Proof. Let M_0 be from the definition of acceptable $L_\tau[G]$ -module. By induction on $\dim_L N/M_0$ we will prove that for every $L_\tau[G]$ -submodule N of M , satisfying $M_0 \leq N$, there exists a $L[H]$ -submodule $M_0 \leq N' \leq N$, such that

$$\beta_1^{L[H]}(N') = 0 \text{ and } \dim_L N/N' \leq \frac{\beta_1^{L[H]}(N)}{|G : H|}.$$

The base of induction, when $N = M_0$, is clear, because $\beta_1^{L[H]}(M_0) = 0$. Assume now that the proposition holds if $\dim_L N/M_0 < n$ and let us prove it in the case where $\dim_L N/M_0 = n$.

Let N_1 be a maximal $L_\tau[G]$ -submodule of N that contains M_0 . By Lemma 6.1, $N/N_1 \cong L$. Then, since $\dim_L N_1/M_0 < n$, there exists a $L[H]$ -submodule N'_1 of N_1 , containing M_0 , such that $\beta_1^{L[H]}(N'_1) = 0$ and

$$\dim_L(N_1/N'_1) \leq \frac{\beta_1^{L[H]}(N_1)}{|G : H|}.$$

Since G is a virtually limit group, by Proposition 3.3, $\beta_2^{L[H]}(L) = 0$. Therefore, by Proposition 8.1 (3),

$$\beta_1^{L[H]}(N_1) \leq \beta_1^{L[H]}(N).$$

By Proposition 6.2, $\beta_1^{L[H]}(N)$ and $\beta_1^{L[H]}(N_1)$ are divisible by $|G : H|$. Hence $\beta_1^{L[H]}(N) \geq \beta_1^{L[H]}(N_1) + |G : H|$ or $\beta_1^{L[H]}(N) = \beta_1^{L[H]}(N_1)$. In the first case we simply take $N' = N'_1$ and we have done. Thus, let us assume that $\beta_1^{L[H]}(N) = \beta_1^{L[H]}(N_1)$.

Take $a \in N \setminus N_1$ and let N' be the $L[H]$ -submodule generated by a and N'_1 . Since H acts trivially on N/M_0 , $\dim_L N'/N'_1 = 1$. Therefore, we have that

$$N_1 + N' = N \text{ and } N_1 \cap N' = N'_1.$$

This leads to the following exact sequence of $L[H]$ -modules:

$$0 \rightarrow N'_1 \rightarrow N_1 \oplus N' \rightarrow N \rightarrow 0.$$

Using the long exact sequence for Tor, we obtain that

$$\beta_1^{L[H]}(N_1) + \beta_1^{L[H]}(N') = \beta_1^{L[H]}(N_1 \oplus N') \leq \beta_1^{L[H]}(N) + \beta_1^{L[H]}(N'_1) = \beta_1^{L[H]}(N_1).$$

Thus, $\beta_1^{L[H]}(N') = 0$. The construction of N' implies also that

$$\dim_L(N/N') = \dim_L(N_1/N'_1) \leq \frac{\beta_1^{L[H]}(N_1)}{|G : H|} = \frac{\beta_1^{L[H]}(N)}{|G : H|}.$$

□

7. THE PROOFS OF THEOREM 1.2 AND THEOREM 1.1

In this section we finish the proof of Theorem 1.2 and deduce from it Theorem 1.1.

Proof of Theorem 1.2. Let G be a surface group and K a subfield of \mathbb{C} . Since

$$K[G/W] \cong \bigoplus_{x \in U \backslash G/W} K[U/U \cap xWx^{-1}]$$

as $K[U]$ -modules we have

$$\beta_1^{K[U]}(K[G/W]) = \sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1}).$$

Let $M = \mathbb{Q}[G/W]$. Using Theorem 4.4, we obtain that there exists a normal subgroup H of G of finite index such that if M_0 denotes the kernel of the map $\mathbb{Q}[G/W] \rightarrow \mathbb{Q}[G/WH]$, then

$$\beta_1^{\mathbb{Q}[G]}(M_0) = 0.$$

Define the ring $L_\tau[G]$ as in Lemma 6.3 and put

$$\widetilde{M} = L_\tau[G] \otimes_{\mathbb{Q}[G]} M.$$

Then, by Lemma 6.3, \widetilde{M} is an acceptable $L_\tau[G]$ -module. Thus, by Proposition 6.4 there exists a $L[H]$ -submodule \widetilde{M}' of \widetilde{M} such that

$$\beta_1^{L[H]}(\widetilde{M}') = 0, \dim_L(\widetilde{M}/\widetilde{M}') \leq \frac{\beta_1^{L[H]}(\widetilde{M})}{|G:H|} \text{ and } H \leq C_G(\widetilde{M}/\widetilde{M}').$$

Let us show that $\beta_1^{L[U \cap H]}(\widetilde{M}') = 0$. If W is of finite index in G , then $\widetilde{M}' = \{0\}$, so, we assume that W is of infinite index in G . Then $\mathbb{Q}[G/W]$ is of projective dimension 1 as $\mathbb{Q}[G]$ -module, and so \widetilde{M} is of projective dimension 1 as $L[H]$ -module. By Proposition 5.1, $\beta_1^{L[U \cap H]}(\widetilde{M}') = 0$. Therefore, we obtain

$$\begin{aligned} \beta_1^{\mathbb{Q}[U]}(\mathbb{Q}[G/W]) &= \frac{\beta_1^{\mathbb{Q}[H \cap U]}(M)}{|U:H \cap U|} = \frac{\beta_1^{L[H \cap U]}(\widetilde{M})}{|U:H \cap U|} \leq \\ &= \frac{\beta_1^{L[H \cap U]}(\widetilde{M}') + \beta_1^{L[H \cap U]}(\widetilde{M}/\widetilde{M}')}{|U:H \cap U|} \leq \frac{\beta_1^{L[H \cap U]}(L)\beta_1^{L[H]}(\widetilde{M})}{|U:H \cap U||G:H|} = \\ &= \frac{\beta_1^{\mathbb{Q}[H \cap U]}(\mathbb{Q})\beta_1^{\mathbb{Q}[H]}(M)}{|U:H \cap U||G:H|} = \beta_1^{\mathbb{Q}[U]}(\mathbb{Q})\beta_1^{\mathbb{Q}[G]}(M) = \bar{\chi}(U)\bar{\chi}(W) \end{aligned}$$

□

Proof of Theorem 1.1. First consider the case when U and W are of finite index. Let r be the number of the double (U, W) -classes in G . Observe that $r \leq |G:U| \leq \bar{\chi}(U)$. Therefore, we have

$$\begin{aligned} \sum_{x \in U \backslash G/W} \bar{d}(U \cap xWx^{-1}) &= r + \sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1}) \stackrel{\text{Theorem 1.2}}{\leq} \\ r + \bar{\chi}(U)\bar{\chi}(W) &= \bar{d}(U)\bar{d}(W) + r - 1 - \bar{\chi}(U) - \bar{\chi}(W) \leq \bar{d}(U)\bar{d}(W). \end{aligned}$$

If U or W is of infinite index, then $U \cap xWx^{-1}$ is free. Thus, we obtain

$$\sum_{x \in U \backslash G/W} \bar{d}(U \cap xWx^{-1}) = \sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1}) \stackrel{\text{Theorem 1.2}}{\leq} \bar{\chi}(U) \bar{\chi}(W) \leq \bar{d}(U) \bar{d}(W).$$

□

8. A MODULE THEORETIC REFORMULATION OF THE GEOMETRIC HANNA NEUMANN CONJECTURE FOR LIMIT GROUPS

Let G be a group and let K be a field. Let N and M be two left $K[G]$ -modules. Consider the tensor product $N \otimes_K M$. The diagonal action on $N \otimes_K M$:

$$g(n \otimes m) = (gn) \otimes (gm) \quad (g \in G, n \in N, m \in M)$$

defines on $N \otimes_K M$ a structure of left $K[G]$ -module.

For every $k \geq 1$ we put

$$\beta_k^{K[G]}(N, M) = \beta_k^{K[G]}(N \otimes_K M).$$

This definition is different from the one used in [21]. At the light of [5, Proposition III.2.2] and the Lück approximation (Proposition 3.2) one sees that two definitions are closely related. However, we do not claim that these two definitions define always the same invariant. We are very grateful to Mark Shusterman who suggested this new definition to us.

In the following proposition we collect the main properties of $\beta_k^{K[G]}(N, M)$.

Proposition 8.1. *Let G be a group and let K be a subfield of \mathbb{C} .*

(1) *Let N and M be left $K[G]$ -modules. Then*

$$\beta_1^{K[G]}(N, M) = \beta_1^{K[G]}(M, N) \text{ and } \beta_1^{K[G]}(N, K) = \beta_1^{K[G]}(N).$$

(2) *Let H be a subgroup of finite index in G . Let N and M be left $K[G]$ -modules. Then*

$$\beta_k^{K[G]}(N, M) = \frac{1}{|G : H|} \beta_k^{K[H]}(N, M).$$

(3) *Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be exact sequence of left $K[G]$ -modules. Assume that $\dim_K M_3 < \infty$ and $H = C_G(M_3)$ is of finite index in G . Then for any left $K[G]$ -module N and any $k \geq 1$ we have*

$$\beta_k^{K[G]}(N, M_1) - (\dim_K M_3) \beta_{k+1}^{K[G]}(N) \leq \beta_k^{K[G]}(N, M_2) \leq \beta_k^{K[G]}(N, M_1) + (\dim_K M_3) \beta_k^{K[G]}(N).$$

Proof. (1) This follows directly from the definitions.

(2) Observe that $\mathcal{R}_{K[H]} \otimes_{K[H]} K[G]$ is isomorphic to $\mathcal{R}_{K[G]}$ as a right $K[G]$ -module and $\dim_{\mathcal{R}_{K[G]}} = |G : H| \dim_{\mathcal{R}_{K[H]}}$. Let $L = N \otimes_K M$. Then we obtain

that

$$\begin{aligned} \beta_k^{K[G]}(N, M) &= \beta_k^{K[G]}(L) = \dim_{\mathcal{R}_{K[G]}} \operatorname{Tor}_k^{K[G]}(\mathcal{R}_{K[G]}, L) = \\ & \dim_{\mathcal{R}_{K[G]}} \operatorname{Tor}_k^{K[H]}(\mathcal{R}_{K[H]}, L) = \frac{1}{|G:H|} \dim_{\mathcal{R}_{K[H]}} \operatorname{Tor}_k^{K[H]}(\mathcal{R}_{K[H]}, L) = \\ & \frac{1}{|G:H|} \beta_k^{K[H]}(L) = \frac{1}{|G:H|} \beta_k^{K[H]}(N, M) \end{aligned}$$

(3) From the long exact sequence for the Tor functor, corresponding to the exact sequence

$$0 \rightarrow N \otimes_K M_1 \rightarrow N \otimes_K M_2 \rightarrow N \otimes_K M_3 \rightarrow 0,$$

it follows that

$$\begin{aligned} \beta_k^{K[G]}(N, M_1) - \beta_{k+1}^{K[G]}(N, M_3) &\leq \beta_k^{K[G]}(N, M_2) \leq \\ & \beta_k^{K[G]}(N, M_1) + \beta_k^{K[G]}(N, M_3). \end{aligned}$$

Observe that

$$\begin{aligned} \beta_k^{K[G]}(N, M_3) &= \frac{1}{|G:H|} \beta_k^{K[H]}(N, M_3) = \\ & \frac{\dim_K M_3}{|G:H|} \beta_k^{K[H]}(N, K) = (\dim_K M_3) \beta_k^{K[G]}(N). \end{aligned}$$

This finishes the proof of (3). \square

In the following proposition we give an algebraic reinterpretation of the sum which appears in Conjecture 1.

Proposition 8.2. *Let G be a limit group and let K a subfield of \mathbb{C} . Let U and W be two finitely generated subgroups of G . Then*

$$\beta_1^{K[G]}(K[G/U], K[G/W]) = \sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1}).$$

Proof. First let us show that

$$(4) \quad \beta_1^{K[G]}(K[G/U]) = \bar{\chi}(U).$$

Indeed, Proposition 3.4 implies that $\beta_1^{K[G]}(K[G/U]) = \beta_1^{K[U]}(K)$. Now, from Proposition 3.3, it follows that $\beta_1^{K[U]}(K) = \bar{\chi}(U)$.

Observe that

$$K[G/U] \otimes_K K[G/W] \cong \bigoplus_{x \in U \backslash G/W} K[G/(U \cap xWx^{-1})]$$

as $K[G]$ -modules. Therefore, we obtain

$$\begin{aligned} \beta_1^{K[G]}(K[G/U], K[G/W]) &= \sum_{x \in U \backslash G/W} \beta_1^{K[G]}(K[G/(U \cap xWx^{-1})]) \stackrel{\text{by (4)}}{=} \\ & \sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1}) \end{aligned}$$

\square

Corollary 8.3. *Conjecture 1 for a limit group G is equivalent to the following statement: for any finitely generated subgroups U and W of G*

$$\beta_1^{\mathbb{Q}[G]}(\mathbb{Q}[G/U], \mathbb{Q}[G/W]) \leq \beta_1^{\mathbb{Q}[G]}(\mathbb{Q}[G/U]) \cdot \beta_1^{\mathbb{Q}[G]}(\mathbb{Q}[G/W]).$$

9. THE STRENGTHENED HOWSON PROPERTY FOR HYPERBOLIC LIMIT GROUPS

The Howson property for limit groups was proved by Dahmani [6]. In the case of hyperbolic limit group we can prove the Strengthened Howson property. In fact, the Strengthened Howson property holds for the family of stable subgroups of a given group.

Let $f: \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a function. Let $H \leq G$ be finitely generated groups, and fix some word metrics. The **subgroup H is f -stable** in G if H is quasi-isometrically embedded (with respect the word metrics) and any (λ, c) -quasi-geodesic γ in G with end-points in H lies in the $f(\lambda, c)$ -neighbourhood of H .

A subgroup H is **stable** if it is f -stable for some f as above.

Examples of stable subgroups include quasi-convex subgroups of hyperbolic groups, subgroups quasi-isometrically embedded in the cone-off graph of relatively hyperbolic groups and convex cocompact subgroups of mapping class groups or $\text{Out}(F_n)$. See [2] and references therein for details.

Theorem 9.1. *Let U and W be two stable subgroups of a finitely generated group G . Then for almost all $x \in U \backslash G/W$ the subgroup $U \cap xWx^{-1}$ is finite. In particular, if U is torsion-free, the sum $\sum_{x \in U \backslash G/W} d(U \cap xWx^{-1})$ is finite.*

Proof. The theorem follows from [2, Lemma 4.2] which states that, under the hypothesis of our theorem, there is a constant $D \geq 0$ such that whenever $|U^{g_1} \cap W^{g_2}| = \infty$ for some $g_1, g_2 \in G$ then the cosets g_1U and g_2W have intersecting D -neighborhoods.

Suppose that $U \cap W^g$ is infinite. Then gW intersects the D -neighborhood of U . By multiplying g by an element of U on the left, we can assume that gW is a distance at most D from the identity. So, by multiplying g by an element of W on the right, we can assume that the length of g is less than D . Therefore, for all $UxW \in U \backslash G/W$, having no representative in the ball of radius D and center the identity, the subgroup $U \cap xWx^{-1}$ is finite. \square

Since hyperbolic limit groups are torsion-free and every finitely generated subgroup is quasi-convex (and hence stable), we obtain the following corollary.

Corollary 9.2. *Let G be a hyperbolic limit group and let U and W be two finitely generated subgroups of G . Then the sum $\sum_{x \in U \backslash G/W} d(U \cap xWx^{-1})$ is finite.*

The Strengthened Howson property is not true for limit groups that are not hyperbolic. A simple example can be constructed on abelian groups, since all conjugates of a subgroup are equal, regardless of conjugating by representatives of different cosets. This is essentially the only reason for which the strengthened Howson property fails for limit groups. For limit groups, or more generally relatively hyperbolic groups, one has a similar statement to Theorem 9.1 if one restricts to non-parabolic intersections.

Let G be a group and $\mathbb{H} = \{H_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups. Let $\mathcal{H} = \sqcup_{\lambda \in \Lambda} H_\lambda$. A group G is **hyperbolic relative to a family of subgroups \mathbb{H}** if it admits a

finite relative presentation with linear relative isoperimetric inequality. The group has a **finite relative presentation with respect to \mathbb{H}** if

$$G = (\langle X \mid \rangle * (*_{\lambda \in \Lambda} H_\lambda)) / \langle\langle R \rangle\rangle,$$

with X and R finite. Let \mathcal{R} be all the words over \mathcal{H} that represent trivial elements. The relative presentation has **linear isoperimetric inequality**, if there is a constant C such that for every $w \in (X \cup \mathcal{H})^*$ representing 1 in G , then w is equal in $\langle X \mid \rangle * (*_{\lambda \in \Lambda} H_\lambda)$ to a product of conjugates of elements of $R \cup \mathcal{R}$ using at most $C\ell(w) + C$ conjugates of R . Here $\ell(w)$ denotes the length of w .

An important property that will be used is that the Cayley graph of G with respect to $X \cup \mathcal{H}$, denoted $\Gamma(G, X \cup \mathcal{H})$, is hyperbolic.

A subgroup $U \leq G$ is **relatively quasi-convex** if U is a quasi-convex set in $\Gamma(G, X \cup \mathcal{H})$. Being relatively quasi-convex is independent of the generating set X .

Conjugates of elements of \mathbb{H} are called **parabolic**. Non-parabolic infinite order elements are called **loxodromic** and indeed, they act as a loxodromic isometry of $\Gamma(G, X \cup \mathcal{H})$. Note that parabolic subgroups are bounded subsets of $\Gamma(G, X \cup \mathcal{H})$ and therefore, they are relatively quasi-convex.

Connecting with the previous notion of stability, if a subgroup of a relatively hyperbolic group is quasi-convex and has no non-trivial parabolic elements then it is stable.

Lemma 9.3. *Let U and W be two relatively convex subgroups of a finitely generated, relatively hyperbolic group G . Then for almost all $x \in U \backslash G/W$ the subgroup $U \cap xWx^{-1}$ does not contain a loxodromic element.*

Proof. The key arguments of this proof are contained in [17, Lemma 8.4] whose proof we follow closely. We assume that X is a finite generating set of G .

Suppose that $U \cap gWg^{-1}$ does contain a loxodromic element f . Since f is a loxodromic isometry of $\Gamma = \Gamma(G, X \cup \mathcal{H})$, the subgroup $\langle f \rangle$ has two different accumulation points $\{f^\infty, f^{-\infty}\} \in \partial\Gamma$, the Gromov boundary of Γ . Since $f \in gWg^{-1}$, we have that $\langle f \rangle g \in gW$ and note that $\langle f \rangle g$ has also $\{f^\infty, f^{-\infty}\}$ as accumulation points since it is at finite Hausdorff X -distance from $\langle f \rangle$. Thus, the accumulation points of $\langle g^{-1}fg \rangle \leq W$ in $\partial\Gamma$ are $\{g^{-1}f^\infty, g^{-1}f^{-\infty}\}$.

By [17, Lemma 8.3] there are bi-infinite geodesics γ_U and γ_W in Γ from $f^{-\infty}$ to f^∞ and $g^{-1}f^{-\infty}$ to $g^{-1}f^\infty$ respectively, and they are at a finite Hausdorff X -distance from $\langle f \rangle$ and $\langle g^{-1}fg \rangle$ respectively. Finally, the vertices of γ_U lie in the σ -neighborhood of U and the vertices of γ_W lie on the σ -neighborhood of W , where σ is the quasi-convexity constant of U and W .

Note that $g\gamma_W$ has the same end points at infinite than γ_U . Now, by [17, Lemma 8.2], there is a constant L , only depending on Γ , such that the vertices of the geodesics γ_U and $g\gamma_W$ are at most L Hausdorff X -distance of each other. Thus, $\langle f \rangle$ and $\langle f \rangle g$ are at Hausdorff X -distance at most $L + 2\sigma$. This implies that U and gW have intersecting $D = L + 2\sigma$ neighbourhoods. By multiplying g by an element of U on the left, we can assume that gW is a distance at most D from the identity. So, by multiplying g by an element of W on the right, we can assume that the length of g is less than D . Therefore, for all $UxW \in U \backslash G/W$, having no representative in the X -ball of radius D and center the identity, the subgroup $U \cap xWx^{-1}$ does not contain loxodromic elements. \square

Theorem 9.4. *Let G be a limit group and let U and W be two finitely generated subgroups of G . Then for almost all $x \in U \backslash G/W$ the subgroup $U \cap xWx^{-1}$ is abelian. In particular, the sum $\sum_{x \in U \backslash G/W} \bar{\chi}(U \cap xWx^{-1})$ is finite.*

Proof. The case where G is hyperbolic follows from Theorem 9.2.

If G is a non-hyperbolic limit group, then G is finitely generated and hyperbolic relative to the family \mathcal{H} of maximal abelian non-cyclic subgroups (see [6, Theorem 4.5]).

Recall that by [6, Proposition 4.6] finitely generated subgroups of limit groups are relatively quasi-convex. In particular, U and W are relatively quasi-convex. By Lemma 9.3, for almost all $x \in U \backslash G/W$ the subgroup $U \cap xWx^{-1}$ does not contain a loxodromic element. As limit groups are torsion-free, this implies that for almost all $x \in U \backslash G/W$, the subgroup $U \cap xWx^{-1}$ is contained in a parabolic subgroup and hence it is abelian. Moreover, since limit groups have the Howson property [6, Theorem 4.7], each $U \cap xWx^{-1}$ is finitely generated and hence a limit group. Hence $\bar{\chi}(U \cap xWx^{-1})$ is well-defined and zero when $U \cap xWx^{-1}$ is abelian. \square

Theorem 9.1 implies also that a quasi-convex subgroup of a hyperbolic virtually compact special group is virtually malnormal.

Corollary 9.5. *Let G be a hyperbolic virtually compact special group and H a quasi-convex subgroup of G . Then H is virtually malnormal and a virtual retract.*

Proof. By Theorem 2.1, H is a virtual retract.

Since G is virtually a subgroup of a RAAG, G is virtually torsion-free and residually finite. By passing to a finite index subgroup of G , we can assume that H is torsion-free. By Theorem 9.1, there is only a finite number of double classes HxH such that $H^x \cap H \neq \{1\}$. By Theorem 2.2, each of these double classes is separable. Since a finite collection of disjoint separable sets is separable, there exists a normal subgroup N of G of finite index such that separates these double classes. Hence H is malnormal in HN . \square

10. THE WILSON-ZALESSKII PROPERTY IN VIRTUALLY COMPACT SPECIAL HYPERBOLIC GROUPS

Let G be a residually finite group and let U and W two subgroups of G . We say that U and W satisfy the **Wilson-Zaleskii property** if

$$\overline{U} \cap \overline{W} = \overline{U \cap W}.$$

Here \overline{U} denotes the closure of U in the profinite completion \widehat{G} of G . When G is virtually free, the Wilson-Zaleskii property for G was proved by Wilson and Zaleskii in [48, Proposition 2.4] for every pair of finitely generated subgroups (see also [40, Lemma 3.6] for the case when U and W are cyclic). In this section we will show that a pair of quasi-convex subgroups of a virtually compact special hyperbolic group satisfies the Wilson-Zaleskii property. Our argument essentially follows the original argument of Wilson and Zaleskii. It uses a beautiful idea of double trick that goes back to the work of Long and Niblo [31]. First let us start with the following useful lemma.

Lemma 10.1. *Let G be a residually finite group and let U and W be two finitely generated subgroups of G . Let H be a subgroup of \widehat{G} of finite index. Assume that*

$$\overline{(U \cap H)(W \cap H)} \cap G = (U \cap H)(W \cap H) \text{ and } \overline{U \cap H} \cap \overline{W \cap H} = \overline{U \cap W \cap H}.$$

Then $\overline{U \cap W} = \overline{U \cap \overline{W}}$.

Proof. Note that one always have that $\overline{U \cap W} \subseteq \overline{U \cap \overline{W}}$. Let $v \in \overline{U \cap W}$. Then we can write $v = u_1 u_2 = w_1 w_2$, where $u_1 \in U$, $u_2 \in \overline{U \cap H}$, $w_1 \in W$ and $w_2 \in \overline{W \cap H}$. Thus,

$$k = u_1^{-1} w_1 = u_2 w_2^{-1} \in \overline{(U \cap H)(W \cap H)} \cap G = (U \cap H)(W \cap H).$$

Therefore, there are $u_3 \in U \cap H$ and $w_3 \in W \cap H$ such that $k = u_3 w_3^{-1}$. Hence,

$$u_1 u_3 = w_1 w_3 \in U \cap W \text{ and } u_3^{-1} u_2 = w_3^{-1} w_2 \in \overline{U \cap H} \cap \overline{W \cap H} = \overline{U \cap W \cap H}.$$

Thus,

$$v = u_1 u_2 = (u_1 u_3)(u_3^{-1} u_2) \in \overline{U \cap W}.$$

□

Lemma 10.2. *Let G be a hyperbolic virtually compact special group, U a malnormal retract of G and W a quasi-convex subgroup of G . Put $K = W \cap U$. Then $G *_U G$ is hyperbolic virtually compact special and $W *_K W$ is quasi-convex in $G *_U G$.*

Proof. Since U is a retract in G , U is quasi-isometrically embedded in G and hence quasi-convex. By [12, Lemma 5.2], $G *_U G$ is hyperbolic and by [18, Theorem A], it is virtually compact special.

For the sake of the proof, let G' denote a copy of G and U' , W' and K' the corresponding copies of U , W and K in G' . Let $B = G *_U G'$ and $A = \langle W, W' \rangle \leq B$. From Bass-Serre theory (or normal forms on amalgamated free products), it follows easily that the natural map $W *_K W' \rightarrow \langle W, W' \rangle \leq G *_U G'$ is injective, and hence an isomorphism.

Fix finite generating sets Y and Y' of G and G' respectively and let $X = Y \cup Y'$ a finite generating set of B . In [12] terminology, a path p on the Cayley graph of $\Gamma(B, X)$ is in normal form if p is the concatenation of subpaths

$$p \equiv p_1 p_2 \dots p_n,$$

such that the label of each p_i is the label of a geodesic word either on $\Gamma(G, Y)$ or on $\Gamma(G', Y')$, no two labels of consecutive subpaths p_i and p_{i+1} lie in the same set Y or Y' , and finally no label represents an element of $U = U'$ except maybe the label of p_1 . Now, [12, Lemma 4.1] in view of [12, Lemma 5.2] claims that there is a constant C such that any geodesic path in $\Gamma(B, X)$ is in the C -neighbourhood of any path in normal form with the same endpoints. From this, it easily follows that A is quasi-convex in B . Indeed, let q be any geodesic path in $\Gamma(B, X)$ with end-points in A . By the equivariance of the action, we can assume that q goes from 1 to $a \in A$. Since $A \cong W *_K W'$, there is a path p in normal form $p \equiv p_1 \dots p_n$ from 1 to a where the labels of each p_i represents an element of W or W' . By the mentioned result of Gitik, q is in the C -neighbourhood of p . Let σ denote the quasi-convexity constant of W and W' as subspaces of $\Gamma(G, Y)$ and $\Gamma(G', Y')$ respectively. We claim that each p_i is in the σ -neighbourhood of A . Indeed, let $a_0 = 1$ and $a_i \in A$ be the element represented by the label of $p_1 p_2 \dots p_{i-1}$. Suppose that the label of p_i is a word in Y . The case where the label of p_i is a word in Y' is analogous. Then $a_i^{-1} p_i$ is a geodesic path in $\Gamma(G, Y)$ with endpoints in W , and therefore lies in the σ -neighbourhood of W . So p_i lies in the σ -neighbourhood of $a_i W \subseteq A$ as claimed. Therefore p is in the σ -neighbourhood of A . □

Theorem 10.3. *Let G be a virtually compact special hyperbolic group and U and W quasi-convex subgroups. Then U and W satisfy the Wilson-Zaleskii property.*

Proof. Put $K = U \cap W$. We want to show that $\overline{U} \cap \overline{W} = \overline{K}$. By Corollary 9.5, there exists a subgroup H of G of finite index, containing U , such that U is retract and malnormal in H . Moreover, by Lemma 10.1, it is enough to prove that $(\overline{U \cap H})(\overline{W \cap H}) \cap G = (U \cap H)(W \cap H)$ and $\overline{U} \cap \overline{W} \cap \overline{H} = \overline{K \cap H}$. The first condition follows from Theorem 2.2. Thus, without loss of generality we may assume that $H = G$.

Let $G' = \{g' : g \in G\}$ be a group isomorphic to G such that the map $g \mapsto g'$ is an isomorphism between G and G' . Put $F = G *_U G'$. In the following we will identify U and U' in F . Let P be a subgroup of F generated by W and W' . Since $G \cap G' = U$, we obtain that $W \cap W' = W \cap W' \cap U = K$, and so $P \cong W *_K W'$.

Observe that G is a retract of F . Hence the closure \widehat{G} of G in \widehat{F} is isomorphic to \widehat{G} . Thus, the closures of U , W and K in \widehat{F} are isomorphic to the closures of U , W and K in \widehat{G} , respectively. In particular, \widehat{F} is isomorphic to the profinite amalgamated product $\widehat{G} \widehat{*}_U \widehat{G}'$, and so $\widehat{G} \cap \widehat{G}' = \widehat{U}$.

Since W is quasi-convex in G , it is virtually a retract (Theorem 2.1). Hence $\overline{W} = \widehat{W}$. Thus the closures of K in \widehat{W} and \widehat{G} are isomorphic. Therefore, \widehat{P} is isomorphic to the profinite amalgamated product $\widehat{W} \widehat{*}_K \widehat{W}'$.

By Lemma 10.2, P is quasi-convex in F and F is virtually compact special. Therefore, by Theorem 2.1, P is virtually retract in F , and so the closure \overline{P} of P in \widehat{F} is isomorphic its profinite completion $\widehat{W} \widehat{*}_K \widehat{W}'$. Hence $\overline{W} \cap \overline{W}' = \overline{K}$. On the other hand,

$$\overline{W} \cap \overline{W}' = \overline{U} \cap \overline{W} \cap \overline{W}' = \overline{U} \cap \overline{W}.$$

We conclude that $\overline{U} \cap \overline{W} = \overline{K}$. \square

Corollary 10.4. *Let G be a hyperbolic limit group and let U and W two finitely generated subgroups of G . Then for every normal subgroup T of G of finite index there exists a finite-index normal subgroup H of G such that*

$$UH \cap WH \leq (U \cap W)T.$$

Proof. Assume that for every normal subgroup H of G of finite index there exists $x_H \in (UH \cap WH) \setminus (U \cap W)T$. Let $G > H_1 > H_2 > \dots$ be a chain of normal subgroups of G of finite index that form a base of neighbors of 1 in the profinite topology of G . Without loss of generality we may assume that there exists

$$x = \lim_{i \rightarrow \infty} x_{H_i} \in \widehat{G}.$$

Clearly, $x \in \overline{U} \cap \overline{W}$. By Theorem 10.3, U and W satisfy the Wilson-Zaleskii property. Hence $x \in \overline{U \cap W}$. Therefore there exists n such that if $i \geq n$, $x_{H_i} \in (U \cap W)T$. We have arrived to a contradiction. \square

11. CONSTRUCTIONS OF SUBMODULES WITH TRIVIAL $\beta_1^{K[G]}$

In this section we assume that G is a L^2 -Hall hyperbolic limit group. For example, by Theorem 4.4, G can be the fundamental group of a closed orientable surface. Let K be a subfiled of \mathbb{C} . Let W be a finitely generated subgroup of G . Then, since G is L^2 -Hall, there exists a normal subgroup H of G of finite index such that W is

L^2 -independent in WH . Let N_0 be the kernel of the map $K[G/W] \rightarrow K[G/HW]$ then by Proposition 4.2, $\beta_1^{K[G]}(N_0) = 0$.

Now, let U be another finitely generated subgroup of G . The main result of this section is the proposition which generalizes the result described in the previous paragraph (it is a particular case corresponding to $U = G$).

Proposition 11.1. *Let G be a L^2 -Hall hyperbolic limit group and U, W two finitely generated subgroups of G . There exists a normal subgroup H of G of finite index such that if L_0 denotes the kernel of the map*

$$K[G/U] \otimes_K K[G/W] \rightarrow K[G/U] \otimes_K K[G/WH],$$

then $\beta_1^{K[G]}(L_0) = 0$.

Remark. *Let N_0 denote the kernel of the map $K[G/W] \rightarrow K[G/WH]$. Then $L_0 \cong K[G/U] \otimes_K N_0$.*

Proof. By Corollary 9.2, there are only finitely many double classes UxW such that $U \cap xWx^{-1}$ is non-trivial. By a result of Minasyan (Theorem 2.2), each of these double classes is separable. It is easy to see that a finite family of disjoint separable sets is separable. Therefore, there exists a normal subgroup H_0 of G of finite index that separates these classes.

Let UsW be a double class with $U \cap sWs^{-1}$ non-trivial. Remind that

$$K[G](1U \otimes sW) \cong K[G/(U \cap sWs^{-1})]$$

as $K[G]$ -modules. Let T_s be a normal subgroup of G of finite index such that $U \cap sWs^{-1}$ is L^2 -independent in $(U \cap sWs^{-1})T_s$. By Corollary 10.4, there exists a normal subgroup H_s of G of finite index such that $U \cap sWH_s s^{-1} \leq (U \cap sWs^{-1})T_s$.

Now we put $H = H_0 \cap (\cap_s H_s)$, where the intersection is over double classes UsW with $U \cap sWs^{-1}$ non-trivial.

Let \tilde{S} be a set of representatives of the (U, HW) -double classes and extend it to \tilde{S} , $S \subset \tilde{S}$, a set of representatives of the (U, W) -classes. Observe that if $x \in \tilde{S} \setminus S$, then $U \cap xWx^{-1}$ is trivial. Define the map $\pi: \tilde{S} \rightarrow S$ in such way that $U\pi(x)WH = UxWH$. Then we obtain the following decomposition of $K[G/U] \otimes_K K[G/W]$:

$$(5) \quad K[G/U] \otimes_K K[G/W] = (\oplus_{s \in S} K[G](1U \otimes sW)) \oplus (\oplus_{x \in \tilde{S} \setminus S} K[G](1U \otimes (xW - \pi(x)W))).$$

Moreover, if $s \in S$ and $x \in \tilde{S} \setminus S$, then

$$K[G](1U \otimes sW) \cong K[G/(U \cap sWs^{-1})] \quad \text{and} \quad K[G](1U \otimes (xW - \pi(x)W)) \cong K[G].$$

Observe also that

$$K[G/U] \otimes K[G/WH] = \oplus_{s \in S} K[G](1U \otimes sWH).$$

Moreover, $K[G](1U \otimes sWH) \cong K[G/U \cap sWHs^{-1}]$.

For each $s \in S$ we denote by I_s the kernel of the map

$$K[G/U \cap sWs^{-1}] \rightarrow K[G/U \cap sWHs^{-1}].$$

Then, from the decomposition (5), we obtain that

$$L_0 \cong (\oplus_{s \in S} I_s) \oplus (\oplus_{x \in \tilde{S} \setminus S} K[G](1U \otimes (xW - \pi(x)W))).$$

If $x \in \tilde{S} \setminus S$, then $K[G](1U \otimes (xW - \pi(x)W)) \cong K[G]$. Hence

$$\beta_1^{K[G]}(K[G](1U \otimes (xW - \pi(x)W))) = 0.$$

If $s \in S$, then I_s is a submodule of the kernel J_s of the map

$$K[G/U \cap sWs^{-1}] \rightarrow K[G/(U \cap sWs^{-1})T_s].$$

Since $U \cap sWs^{-1}$ is L^2 -independent in $(U \cap sWs^{-1})T_s$, $\beta_1^{K[G]}(J_s) = 0$. Hence, by Proposition 8.1(3) (applied with $N = K$, $M_1 = I_s$ and $M_2 = J_s$), $\beta_1^{K[G]}(I_s) = 0$. Thus, $\beta_1^{K[G]}(L_0) = 0$. \square

12. THE GEOMETRIC HANNA NEUMANN CONJECTURE

12.1. The proof of Theorem 1.3. Let G be a hyperbolic limit group, and assume that G is L^2 -Hall.

Let $N = \mathbb{Q}[G/U]$ and $M = \mathbb{Q}[G/W]$. Using Proposition 11.1, we obtain that there exists a normal subgroup H of G of finite index such that

(1) If N_0 denotes the kernel of the map $\mathbb{Q}[G/U] \rightarrow \mathbb{Q}[G/UH]$, then

$$\beta_1^{\mathbb{Q}[G]}(N_0) = 0.$$

(2) If M_0 denotes the kernel of the map $\mathbb{Q}[G/W] \rightarrow \mathbb{Q}[G/WH]$, then

$$\beta_1^{\mathbb{Q}[G]}(N, M_0) = 0.$$

Observe that $H \leq C_G(M/M_0) \cap C_G(N/N_0)$.

Define the ring $L_\tau[G]$ as in Lemma 6.3 and put

$$\tilde{M} = L_\tau[G] \otimes_{\mathbb{Q}[G]} M \text{ and } \tilde{N} = L_\tau[G] \otimes_{\mathbb{Q}[G]} N.$$

Then, by Lemma 6.3, \tilde{N} is an acceptable $L_\tau[G]$ -module. Thus, by Proposition 6.4 there exists a $L[H]$ -submodule \tilde{N}' of \tilde{N} such that

$$\beta_1^{L[H]}(\tilde{N}') = 0, \dim_L(\tilde{N}/\tilde{N}') \leq \frac{\beta_1^{L[H]}(\tilde{N})}{|G:H|} \text{ and } H \leq C_G(\tilde{N}/\tilde{N}').$$

Let us show first that $\beta_1^{L[H]}(\tilde{N}', \tilde{M}) = 0$,

$$\begin{aligned} \beta_1^{L[H]}(\tilde{N}', \tilde{M}) &\stackrel{\text{Proposition 8.1(3)}}{\leq} \beta_1^{L[H]}(\tilde{N}', \tilde{M}_0) \stackrel{\text{Proposition 8.1(3)}}{\leq} \\ &\beta_1^{L[H]}(\tilde{N}, \tilde{M}_0) = \beta_1^{\mathbb{Q}[H]}(N, M_0) = \frac{\beta_1^{\mathbb{Q}[G]}(N, M_0)}{|G:H|} = 0. \end{aligned}$$

Thus

$$\begin{aligned} \beta_1^{L[H]}(\tilde{N}, \tilde{M}) &\stackrel{\text{Proposition 8.1(3)}}{\leq} \dim_L(\tilde{N}/\tilde{N}')\beta_1^{L[H]}(\tilde{M}) + \beta_1^{L[H]}(\tilde{N}', \tilde{M}) = \\ &\dim_L(\tilde{N}/\tilde{N}')\beta_1^{L[H]}(\tilde{M}) \leq \frac{\beta_1^{L[H]}(\tilde{N})\beta_1^{L[H]}(\tilde{M})}{|G:H|}. \end{aligned}$$

Therefore,

$$\begin{aligned} \beta_1^{\mathbb{Q}[G]}(N, M) &\stackrel{\text{Proposition 8.1(2)}}{=} \frac{\beta_1^{\mathbb{Q}[H]}(N, M)}{|G : H|} = \frac{\beta_1^{L[H]}(\tilde{N}, \tilde{M})}{|G : H|} \leq \\ &\frac{\beta_1^{L[H]}(\tilde{N})\beta_1^{L[H]}(\tilde{M})}{|G : H|^2} = \frac{\beta_1^{\mathbb{Q}[H]}(N)\beta_1^{\mathbb{Q}[H]}(M)}{|G : H|^2} = \beta_1^{\mathbb{Q}[G]}(N)\beta_1^{\mathbb{Q}[G]}(M). \end{aligned}$$

By Corollary 8.3, we are done.

12.2. The Geometric Hanna Neumann conjecture beyond the surface groups. As we have shown in order to settle the case of hyperbolic limit groups of Conjecture 1, it is enough to prove the L^2 -Hall property for these groups. We strongly believe that L^2 -Hall property holds for arbitrary limit groups.

In the case of limit groups the generalized Howson property holds if one replaces d by $\bar{\chi}$ (Theorem 9.4) since the reduced Euler characteristic for the abelian finitely generated abelian groups is zero. However, if G is a limit group we do not know whether the double classes with respect to two finitely generated subgroups are separated (in the hyperbolic case this follows from [36]) and we do not know whether the Wilson-Zalesskii property holds for every pair of finitely generated subgroups of a limit group.

Since quasi-convex subgroups of hyperbolic virtually compact special groups satisfy the Howson property we may ask whether they satisfy also the conclusion of the Geometric Hanna Neumann conjecture. This is not true. Fix a natural number d . In a 2-generated free group F we can find a finitely generated malnormal subgroup H with $d(H) = d$. Then, by [12, Corollary 5.3] and [18, Corollary B], $G = F *_H F$ is hyperbolic and virtually special compact. The two copies of F are quasi-convex in G , and their intersection has rank d .

Also we want to mention that a hyperbolic virtually special compact group is not always L^2 -Hall with respect to a quasi-convex subgroup. For example, take a free non-abelian retract in the fundamental group of a compact hyperbolic 3-manifold group.

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